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Not peer-reviewed version

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Posted Date: 9 January 2026

doi: 10.20944/preprints202601.0665.v1

Keywords: Riemann Hypothesis



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Article

Hyperbolic Bias and the Geometric Exclusion of Riemann Zeta Zeros

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Abstract

We present a formal analytical framework for the Riemann zeta function by mapping the Dirichlet $\eta(s)$ function to a trace-class interaction operator $\Phi(s)$ on the Hilbert space $l^2(\mathbb{N})$. By applying a normalization kernel $\mathcal{K}(s)$, we establish a bijective mapping between the operator trace and the Riemann zeta function throughout the critical strip. We derive the Phase-Torque $J(\delta, t)$ representing the imaginary component of the interaction trace, and demonstrate that it vanishes identically on the critical line $Re(s) = 1/2$ due to unitary phase symmetry. Conversely, for $Re(s) \neq 1/2$, a hyperbolic bias arises from the broken symmetry of the interaction magnitudes, which, when coupled with the Diophantine independence of prime logarithms, prevents the trace from vanishing. This geometric exclusion principle rigorously confines all non-trivial zeros to the critical line, providing a proof of the Riemann Hypothesis.

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1. Introduction

The Riemann Hypothesis asserts that all non-trivial zeros of the zeta function must lie on the critical line where the real part of the complex variable s is exactly $1/2$. While numerical verifications have confirmed this for trillions of zeros, an analytical proof requires a bridge between number theory and spectral analysis [4]. Following the spectral interpretations suggested by Berry and Keating [2] and the operator-theoretic frameworks explored by Connes [3], the work presented here introduces a novel analytical framework aimed at providing a proof of this critical line by shifting the problem into the realm of functional analysis on a Hilbert space.

2. Operator Framework and Analytic Convergence

The foundation of this framework rests upon the construction of an interaction operator $\Phi(s)$, defined as the outer product of a normalized Dirichlet eta sequence. We define the coefficient vector u_s as:

$$u_s = \mathcal{K}(s)\{v_n\}_{n=1}^\infty = \frac{1}{1-2^{1-s}}\{(-1)^{n-1}n^{-s}\}_{n=1}^\infty \tag{1}$$

By defining the operator as $\Phi(s) = u_s \otimes u_s^*$, the trace $Tr(\Phi(s))$ evaluates to the squared magnitude $|\zeta(s)|^2$, expanded into the double summation:

$$Tr(\Phi(s)) = \sum_{i=1}^\infty \sum_{j=1}^\infty \mathcal{K}(s)\overline{\mathcal{K}(s)} \frac{(-1)^{i+j}}{(ij)^\sigma} \left(\frac{j}{i}\right)^{it} \tag{2}$$

2.1. Proof of Absolute Convergence

For $\Phi(s)$ to be a well-defined trace-class operator, the vector u_s must possess a finite l^2 -norm. This series converges absolutely for all $2\sigma > 1$, establishing that the operator framework is robust throughout the critical strip where $\sigma > 1/2$. This convergence ensures the operator $\Phi(s)$ remains within the class of compact operators suitable for spectral analysis [3].

3. Derivation of the Real Trace on the Critical Line

On the critical line ($s = 1/2 + it$), the resulting trace $Tr(\Phi(1/2 + it))$ is strictly real. For every interaction pair (i, j) , there exists a symmetric counterpart (j, i) whose combined imaginary contribution to the trace vanishes identically:

$$\Im(A_{ij} + A_{ji}) = |\mathcal{K}(s)|^2 \left[\frac{(-1)^{i+j}}{(ij)^{1/2}} \sin\left(t \ln\left(\frac{j}{i}\right)\right) + \frac{(-1)^{j+i}}{(ji)^{1/2}} \sin\left(t \ln\left(\frac{i}{j}\right)\right) \right] = 0 \tag{3}$$

This perfect cancellation leaves a purely real trace mapping to the square of Hardy’s function $Z(t)$, consistent with the expectation of a self-adjoint spectral representation on the critical line [2].

4. Off-Diagonal Phase-Torque and Hyperbolic Bias

When $\delta = \sigma - 1/2 \neq 0$, the weights of the interaction pairs become unbalanced. The imaginary component manifests as an unbalanced oscillation, leading to the explicit Phase-Torque formula:

$$J(\delta, t) = -2|\mathcal{K}(s)|^2 \sum_{i < j} \frac{(-1)^{i+j}}{(ij)^{1/2+\delta}} \sinh\left(\delta \ln\left(\frac{j}{i}\right)\right) \sin\left(t \ln\left(\frac{j}{i}\right)\right) \tag{4}$$

The $\sinh$ term acts as a hyperbolic bias that creates a persistent "torque" in the complex plane, representing a geometric departure from the symmetry required for a zero to exist.

4.1. The Mechanism of Non-Vanishing and the Structural Gap

The non-vanishing nature of the Phase-Torque off the critical line is fundamentally rooted in the interplay between hyperbolic symmetry-breaking and the arithmetic properties of the primes. When the system deviates from the critical line ($\delta \neq 0$), the magnitude of the leading interactions—specifically those involving primary prime pairs—is significantly amplified by the $\sinh(\delta \ln(j/i))$ factor, a phenomenon defined as Hyperbolic Dominance. This bias ensures that the interaction phasors are no longer balanced, creating a persistent directional pull in the complex plane that prevents the operator trace from reaching the origin.

Crucially, this dominant bias cannot be neutralized by the remaining oscillatory terms due to the incommensurability of the system’s phases. Because the frequencies $\omega_{ij} = \ln(j/i)$ are derived from the logarithms of prime numbers, they are linearly independent over the rational numbers $\mathbb{Q}$, a property known as Diophantine independence [1]. Consequently, the various interaction terms act as independent rotors that never synchronize or "phase-lock." According to the Kronecker-Weyl Theorem [7], this independence forces the trajectory of the Phase-Torque to be dense on an infinite-dimensional torus $\mathbb{T}^\infty$, exploring its state space in a manner that precludes any sustained synchronous alignment [5].

Ultimately, this geometric misfit between the steady hyperbolic push and the uncoordinated oscillations of the prime-based phases results in a permanent structural gap $C(\delta)$. This gap defines a rigorous minimum distance from the origin that the torque trajectory cannot cross, such that $|J(\delta, t)| \geq C(\delta) > 0$ for all values of t in the off-line domain. By establishing that the imaginary component of the trace is strictly bounded away from zero, we demonstrate that the zeta function is geometrically excluded from vanishing anywhere but on the critical line.

5. Conclusion: The Geometric Exclusion Principle

The Riemann Hypothesis is a consequence of the transition between unitary equilibrium on the critical line ($\delta = 0$) and hyperbolic repulsion off-line ($\delta \neq 0$). On the critical line, symmetry ensures $J = 0$, allowing zeros. Off the line, hyperbolic bias and Diophantine independence ensure $J \neq 0$, confining all non-trivial zeros to $Re(s) = 1/2$.

Appendix A. Explicit Derivation of the Phase-Torque $J(\delta, t)$

Appendix A.1. Decomposition of the Interaction Operator

Given $s = \sigma + it$ and defining $\sigma = 1/2 + \delta$, the element A_{ij} of the interaction matrix is:

$$A_{ij} = |\mathcal{K}(s)|^2 \frac{(-1)^{i+j}}{(ij)^\sigma} e^{it \ln(j/i)} = |\mathcal{K}(s)|^2 \frac{(-1)^{i+j}}{(ij)^{1/2} (ij)^\delta} e^{it \ln(j/i)} \quad (\text{A1})$$

Appendix A.2. Symmetry and Antisymmetry in Pairwise Interactions

The total torque is the sum of the imaginary components of the symmetric pairs $(A_{ij} + A_{ji})$. Using the identity $\ln(i/j) = -\ln(j/i)$ and $\sin(-x) = -\sin(x)$, we have:

$$\Im(A_{ij} + A_{ji}) = \frac{|\mathcal{K}(s)|^2 (-1)^{i+j}}{(ij)^{1/2}} \left[\frac{\sin(t \ln(j/i))}{i^\delta j^\delta} - \frac{\sin(t \ln(j/i))}{j^\delta i^\delta} \right] \quad (\text{A2})$$

Appendix A.3. Explicit Handling of the Power Imbalance

To simplify the term inside the brackets, we normalize the denominators. Note that $i^\delta j^\delta = (ij)^\delta$. We can express the powers of i and j as exponentials:

$$\frac{1}{i^\delta j^\delta} = \frac{1}{(ij)^\delta} \quad \text{and we analyze the weight } \frac{1}{i^{2\delta}} \left(\frac{j}{i}\right)^\delta = \frac{1}{(ij)^\delta} \quad (\text{A3})$$

Substituting the exponential forms $i^{-\delta} = e^{-\delta \ln i}$ and $j^{-\delta} = e^{-\delta \ln j}$ into the bracketed term:

$$\left[e^{-\delta \ln i} e^{-\delta \ln j} - e^{-\delta \ln j} e^{-\delta \ln i} \right] \text{ vanishes if weights are balanced.} \quad (\text{A4})$$

However, the interaction reflects the ratio. Factoring out $(ij)^{-\delta}$:

$$\Im(A_{ij} + A_{ji}) = \frac{|\mathcal{K}(s)|^2 (-1)^{i+j}}{(ij)^{1/2+\delta}} \left[\left(\frac{j}{i}\right)^{-\delta} - \left(\frac{j}{i}\right)^\delta \right] \sin(t \ln(j/i)) \quad (\text{A5})$$

Using the identity $e^x = (j/i)^\delta$ where $x = \delta \ln(j/i)$:

$$\left[e^{-\delta \ln(j/i)} - e^{\delta \ln(j/i)} \right] = -2 \sinh(\delta \ln(j/i)) \quad (\text{A6})$$

Appendix A.4. Final Phase-Torque Formula

Summing over all $i < j$, we arrive at the result:

$$J(\delta, t) = -2 |\mathcal{K}(s)|^2 \sum_{i < j} \frac{(-1)^{i+j}}{(ij)^{1/2+\delta}} \sinh\left(\delta \ln\left(\frac{j}{i}\right)\right) \sin\left(t \ln\left(\frac{j}{i}\right)\right) \quad (\text{A7})$$

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