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Article

Fractal Geometry: Historical and Conceptual Background

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Abstract

Presently, fractal geometry serves as a framework for studies of complex systems of diverse nature. One of the most fundamental geometric conceptions is the concept of symmetry. Different geometries can be classified according to the group of transformations under which their propositions remain true. In particular, the key symmetries of the fractal geometry are the scale and conformal invariance. Another key paradigm in the fractal geometry is that different properties of a fractal pattern are associated with different dimension numbers, at least one of which differs from the topological dimension. Accordingly, the inherent features of a fractal pattern are characterized by a set of generally independent dimension numbers. These numbers allow for the classification of fractal patterns. In this review we briefly survey the historical background and the conceptual foundations of fractal geometry.

Keywords: fractals; scale invariance; conformal invariance; dimension numbers; degrees of freedom

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1. Introduction

Geometry provides a formal representation of shapes in space. Fractal geometry deals with objects called fractals. The notion of fractals was put forward by a Polish-born French-American mathematician Benoit Mandelbrot in reference to irregular but scale invariant patterns that he studied [1]. The word *fractal* came from a Latin word *fractus* meaning broken or fractured. Although there is no canonical definition of fractals, the fractal can be defined as a scale and/or conformally invariant pattern whose dimension linked to a suitable defined measure D strictly exceeds the topological dimension d, which is defined regarding the way how the pattern can be divided into parts of arbitrary sizes [2].

The topological dimension d is a topological invariant, whereas the dimension number D, commonly called the fractal dimension, is not necessary invariant under homeomorphic transformations. For Euclidean patterns $D = d$, the fractals are characterized by $D > d$ per definition [3]. However, the dimension number D along is insufficient for fractal characterization. Indeed, the fractal patterns having the same D can possess very different topological, morphological, and topographical properties [2–5]. Nonetheless, the inherent features of fractals can be specified by a set of generally independent dimension numbers [2,6]

Another constitutive property of fractals is their invariance under suitable defined scale and/or conformal transformations. In Figure 1 we present the corresponding classification of fractal patterns. The scale invariant fractals look the same under some sort of scaling operations. Specifically, a self-similar fractal is invariant under an appropriate similarity transformation, whereas a self-affine fractal is invariant under a suitable affine transformation (see, for instance, illustrations in Figure 1). Besides there are the hierarchical fractals which are only locally scale invariant. The difference between the scale invariant and locally scale-invariant fractals is illustrated by Figure 2. Finally, a self-conformal fractal is invariant under a special conformal transformation which locally looks as the scale one (see, for example, the illustration in Figure 1). Accordingly, the self-conformal fractals are also scale-independent, but in a stronger sense than the scale invariant fractals.

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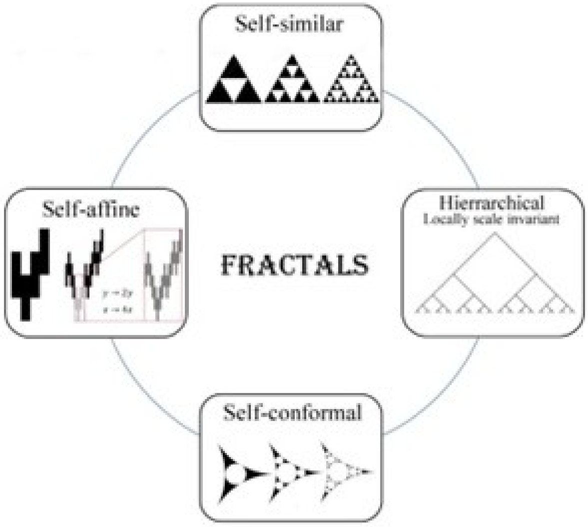

Figure 1. Classification of fractals according to the type of invariance.

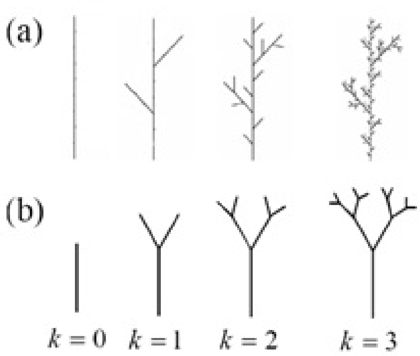

Figure 2. Iterative construction of: (a) self-similar fractal and (b) hierarchical, locally self-similar fractal.

Although the term fractal was appeared only in 1975, a recursive self-similarity was already studied in the 17th century by a German mathematician and philosopher Gottfried von Leibniz (see Ref. [7] and references therein). In his celebrated letter to Guillaume de L'Hôpital von Leibniz introduced the notion of fractional exponent associated with the recursive self-similarity. Furthermore, the mathematical foundations of fractal geometry and most of paradigmatic fractals were created a long time before than the word fractal was invented by Mandelbrot (see, for review, Refs. [6–11] and references therein). Namely, several important concepts of fractal geometry have their origins in analytical and geometric constructions of nineteenth or early to mid-twentieth century.

In this review we briefly survey the historical background and conceptual foundations of fractal geometry. The rest of the paper is organized as follows. Section 2 is devoted to the history of fractal geometry creation. The conceptual foundations of the fractal geometry are highlighted in Section 3. Some key issues are outlined in Section 4.

2. Brief Excursion into the History of Fractal Geometry

In 1872 a German mathematician Karl Weierstrass has presented a function

$$f(x) = \sum_{k=0}^{\infty} b^k \cos\left(\pi a^k x\right),$$

where a is an odd integer, and $ab > 1 + 3\pi/2$. This function is everywhere continuous but nowhere differentiable, if $b \in (0,1)$. In his seminal paper [12] Weierstrass also stated that Georg Friedrich Bernhard Riemann was the first (in 1861) to assert that the infinite series $\sum_{k=0}^{\infty} \sin(k^2 x)/k^2$ is continuous but not differentiable. However, it is not clear whether the non-differentiability was proved by Riemann or not. The Weierstrass work shocked the mathematical community, but the continuous nowhere differentiable functions remained considered as rare pathologies for a long time. Presently, the existence of continuous nowhere differentiable functions is crucial to our proper understanding of mathematical analysis [13]. In particular, the nowhere-differentiability in the conventional sense is inherent feature of at almost all fractals. The graph of Weierstrass function is shown on in Figure 3). One can see that it exhibits a fine structure at all scales of observation, such that a smaller scale blow-up looks broadly similar to the wider plot. So, the Weierstrass graph possesses some form of approximate self-symmetry which is presently known as the self-affinity [3,14–16].

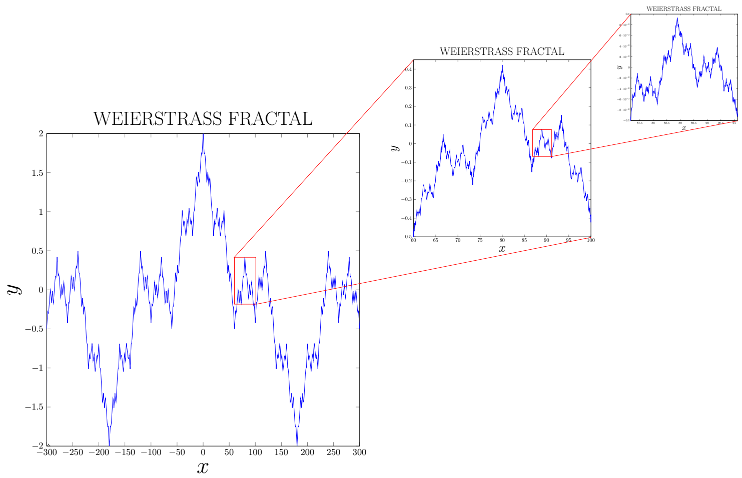

Figure 3. The graph of Weierstrass function and its affine transformations.

One of the students attending the Weierstrass'lectures was Georg Cantor who later becomes a famous mathematician. In 1883 Georg Cantor [17] explored a set of points lying on a single line segment which can be presented as

$$C = c_1/3 + \cdots + c_k/3^k + \cdots,$$

where c_k is 0 or 2 for each integer k. Although the same set was early studied by Henry John Stephen Smith [18], it was named as the Cantor set since through consideration of this set Cantor helped to lay the foundations of modern point-set topology [19]. The Cantor set is an example of uncountable bounded sets that have zero one-dimensional Lebesgue measure (length). It has been also recognized that the Cantor set is a subset of Euclidean space which is perfect (a closed set that contains no isolated point) but nowhere dense in any interval, regardless of how small the interval is taken to be (closure has empty interior). Geometrically, the middle third Cantor set is constructed by repeatedly removing middle third open intervals from the unit interval $[0,1] \subset \mathbb{R}$ as it is shown in Figure 4.

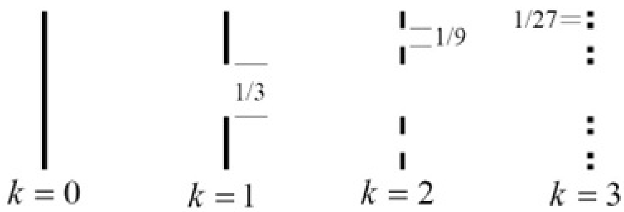

Figure 4. Three steps of iterative construction of the middle third Cantor set.

Due to its remarkable properties, the Cantor set plays a vital role in many branches of mathematics. In particular, its special place in the theory of compact spaces is coined by the Hausdorff–Alexandroff theorem which states that any compact metric space is a continuous image of the Cantor set [20]. In the fractal geometry the Cantor set constitutes the class of totally disconnected fractals (see, for review, Refs. [21–23] and references therein).

In the last part of 19 century, Felix Klein and Henri Poincaré introduced a category of objects that we now call the self-inverse fractals (see, for example, the construction of self-conformal fractal in Figure 1) and form a part of the self-conformal fractals (see, for review, Refs. [24–27]). Furthermore, self-inverse fractals have become a special topic of the theory of automorphic functions (see Ref. [24] and references therein).

In 1890, an Italian mathematician and glottologist Giuseppe Peano [28] has discovered the first example of curves that passes through every point of a two-dimensional region having a positive Jordan area (see in Figure 5a). Immediate1y after that a German mathematician David Hilbert [29] replaced the purely arithmetic definition proposed by Peano on a more descriptive geometric construction of graphs and thus built up a celebrated variant of Peano curve (see in Figure 5b). Later there were constructed non-intersecting continuous curves filling the n-dimensional spaces. Presently the class of space-filling curves is also called the Peano curves [30–32].

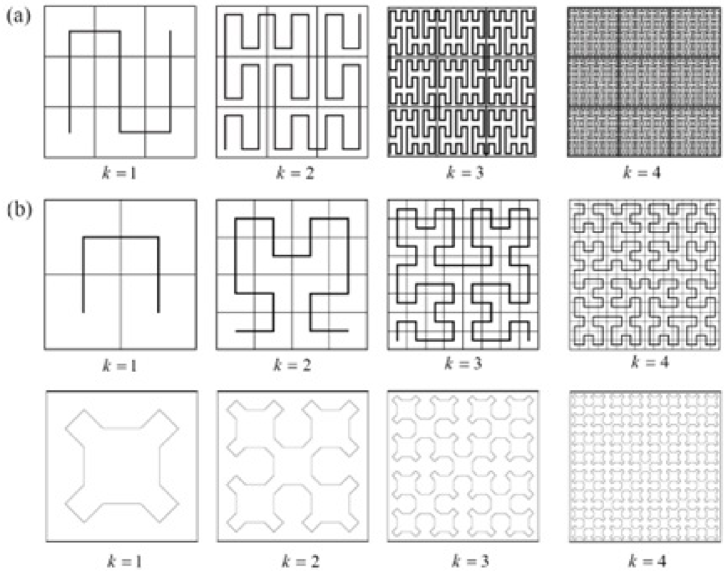

Figure 5. Iterative construction of the space-filling curves: (a) Peano curve; (b) Hilbert curve; and (c) Sierpiński square snowflake.

In 1904, a Swedish mathematician Niels Fabian Helge von Koch [33] has devised the simplest variant of continuous curve without tangents (see Fig. 6), which is now called the Koch curve. Koch has showed that there exists a parameterization of the curve $x = f(t)$, $y = g(t)$ for $t \in [0, 1]$, where both functions $f(t)$ and $g(t)$ are continuous but nowhere differentiable. The Koch curve challenges our notion of a curve. While the curve is connected and contained in a bounded region of the plane its length is infinite. Moreover, any part of the Koch curve has infinite length (see Figure 6a). Fitting together three suitably rotated copies of the curve produces a snowflake curve (see Figure 6b), sometimes also called the Koch island [34]. In the fractal geometry Koch curves constitute the class of self-avoiding self-similar fractals (see, for review, Refs. [34–36] and references therein). Figure 6c shows the construction of Koch curve starting from a triangle.

In 1912 by a Polish mathematician Wacław Sierpiński [37] has presented the most famous space-filling curve nowadays called the Sierpiński square snowflake (see Figure 5c). So it was revealed a continuous crossover between the Koch and Peano curves [38]. In 1918 two French mathematicians Pierre Fatou and Gaston Julia found a self-similar behavior associated with mapping complex numbers and iterative functions [39–41]. The idea of self-similar curves was taken further by Paul Pierre Lévy, who, in 1938, created a self-similar curve, presently known as the Lévy C curve [42]. In 1915 Sierpiński [43] has devised a self-similar curve having almost all points as the branching points. This

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curve, known as the Sierpiński gasket was constructed by deleting an open middle triangle from a closed equilateral triangle of unit side and by repeating this step for the remaining sub-triangles ad infinitum (see Figure 7a). Later Sierpiński [44,45] also proposed to create the same shape as the limit of the Koch-like arrowhead curve (see Figure 7b). However, in contrast to the Sierpiński gasket with loops at all scales, the Sierpiński arrowhead curve remains loopless even in the limit of infinite number of iterations [46]. Further there were suggested many other ways to create the Sierpiński triangle shape (see, for review, Refs. [47–49] and references therein). Nowadays the Sierpinski triangle is one of the most studied fractal shapes.

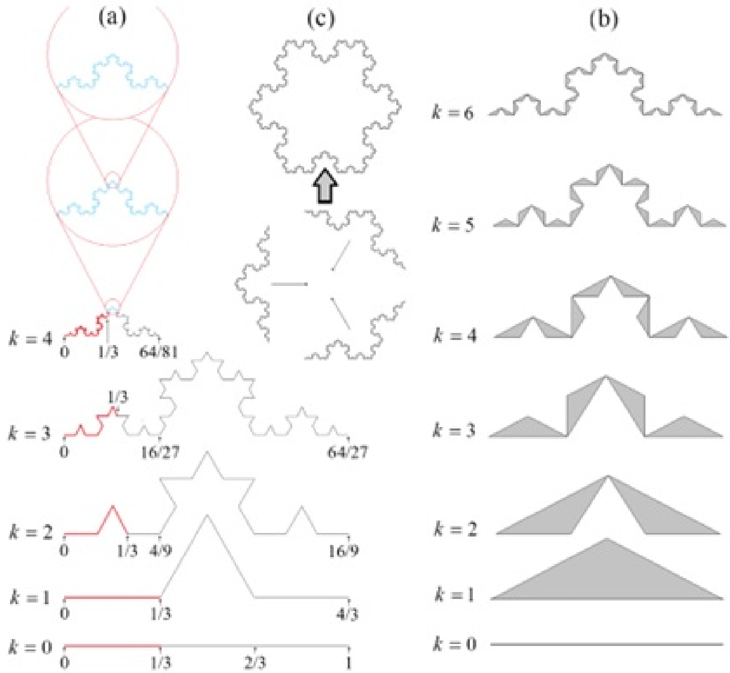

Figure 6. (a), (b) Two ways to construct the Koch curve and (c) building the Koch snowflake

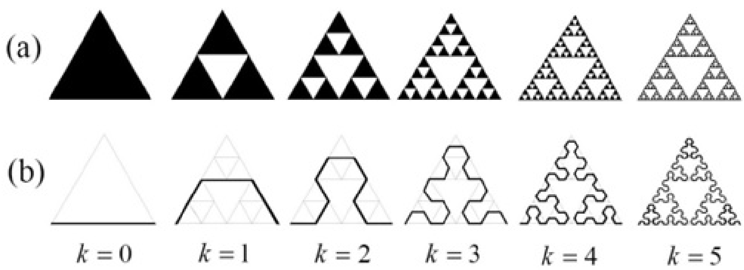

Figure 7. Two ways to construct the Sierpinski triangle shape as the limits of the iterations of the pre-fractal: (a) Sierpinski gasket and (b) Sierpinski arrowhead curve.

In 1916 Sierpiśki [50] constructed a curve forming an infinitely ramified network (see Figure 8) nowadays known as the Sierpiński carpet. Analogous curve in three dimensions was presented by Karl Menger in 1926 [51]. Notice that the Sierpiński carpet and the Menger sponge can be viewed as the analogs of Cantor set on the plane and in the three dimensional space, respectively (see, for review, Ref. [21]).

On the other hand, in 1918 Felix Hausdorff [52] has introduced a new definition of covering measure based on the set size variations with the scale of measurements. The dimension number D associated with that measure, presently called the Hausdorff dimension, can be fractional. In particular, Hausdorff proved that the middle-third Cantor set is characterized by fractional dimension $D = \ln 2/\ln 3$. Further, the conceptual and technical aspects regarding the Hausdorff measure and dimension were disused by Besicovitch [53,54]. In 1968 a biologist Aristid Lindenmayer [55]

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invented an approach for simulating the development of multicellular organisms, subsequently named L-systems [56]. The central concept of the Lindenmayer system is that of re-writing. This approach was specifically created for the description of natural growth processes and so enables us to see in more detail how a fractal grows (see, for example, Figure 2b). Accordingly, the L-systems represent a large class of real-world fractals in a mathematical way [57].

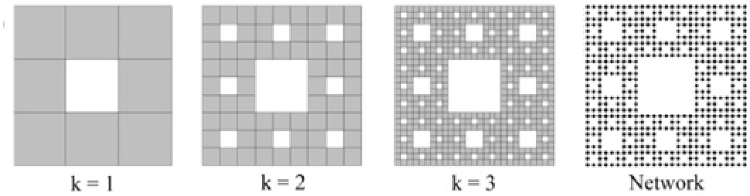

Figure 8. Three first iterations of Sierpiński carpet and corresponding Sierpiński network.

In this background Benoit Mandelbrot published his celebrated paper [58] in which he resolved the Steinhaus Paradox that the measured length of geographic features increases with increasing map scales. Sometime afterward, Mandelbrot [1] coined the notion of fractals to define a large class of mathematical and natural objects that possess the property of scale invariance whose covering dimension strictly exceeds the topological one. In a unified way, the scale-invariant fractals were created by John Hutchinson [59] using the method of Iterated Function Systems (IFS). Further, the IFS method was popularized by Barnsley [60] as a generalization of the Banach contraction principle. In the late 20th century fractals became a topic of rising interest for researchers specializing in diverse areas of mathematics, physics, and natural sciences (see, for instance, Refs. [61–72] and references therein). Presently, the fractal geometry still remains a burgeoning area of research (see, for review, Refs. [73–103] and references therein).

3. Conceptual Foundations of Fractal Geometry

The fractal geometry deals with intricate patterns and irregular shapes possessing the scale and/or conformal symmetry. In this regard, it is pertinent to note that the scale symmetry is an inherent feature not only of fractals, but also of the small-world and scale-free networks, as well as of the objects with a power-law size distribution of the building blocks (see Figure 9). Geometrically, the scale invariance is associated with the notion of self-similarity or self-affinity, which can be exact (deterministic) or approximate (statistical). Besides, there are self-conforming fractals that are invariant under special conformal transformations.

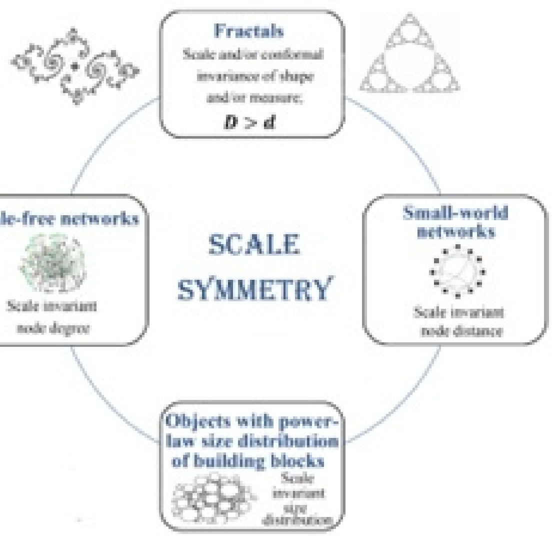

Figure 9. Classification of objects with different kinds of scale symmetry.

The scale invariance can be characterized by the similarity dimension D_S defined via the Hutchinson–Moran formula

$$\sum_{i}^{m} c_i^{D_S} = 1, \tag{1}$$

where $0 < c_i < 1$ are the contraction ratios and m is the number of contractions at each iteration step [59]. For scale invariant Euclidean patterns the similarity dimension is equal to the topological one (that is $D_S = d$), whereas for fractals $D_S > d$ per definition. The topological dimension is defined inductively as follows. Let us an empty space has the topological dimension $d(\varnothing) = -1$. A pattern has the topological dimension zero if for any point of it there exist arbitrarily small neighborhoods whose boundary is empty. A pattern $\mathcal{P}$ has the topological dimension $d = n$ if there are arbitrarily small neighborhoods of any point $p \in \mathcal{P}$ whose boundary is of dimension $d \leq n - 1$, where n is a natural number.

An equivalent recursive definition of topological dimension reads as:

$$d = dim_t\mathcal{P} = \min\{s : \exists\mathcal{S} \subset \mathcal{P} \text{ such that } dim_t\mathcal{S} \leq d - 1 \text{ and } dim_t(\mathcal{P}\backslash\mathcal{S}) \leq 0\}, \tag{2}$$

while it is assumed that $dim_t(\phi) = -1$ [104]. So, the topological dimension stipulates a way to divide an object into parts of arbitrary sizes. Furthermore, in the Euclidean geometry, the topological dimension d also:

1) handles the scale invariance of the Euclidean object ($D_S = d$);
2) characterizes the object connectivity;
3) establishes the object ramification;
4) sets the maximum number of mutually orthogonal vectors in the object;
5) governs the Lebesgue measure and other Borel measures on Euclidean space;
6) determines the numbers of spatial and dynamic degrees of freedom of a point walker in the object
7) rules the statistics of thomogeneous Poisson point processes;
8) controls the vibrational dynamics of the object;
9) manages the information flow;
10) settles the values of universal exponents associated with critical phenomena.

Conversely, in the fractal geometry, the above features are associated with a set of different dimensional numbers, some of which are topological invariants and others are not. Specifically, the similarity dimension defined by Eq. (2) can be linked to a suitable defined covering measure, e.g. the Hausdorff, box-counting, packing, or Assouad measure (see, for review, Table 1 and Figure 10. In this way, it was recognized that the similarity and Hausdorff dimensions are equivalent when the IFS satisfies the open set condition [64]. Therefore the fractal dimension can be thought of as a measure of a pattern's ability to fill the space in which it resides. The scale invariance implies that the number of boxes needed to cover a fractal pattern scales with the box size ϵ as $N \propto \epsilon^{-D}$. Accordingly, the fractal dimension is frequently defined through the scaling relation $N(\lambda\epsilon) \propto \lambda^{-D}N(\epsilon)$, where $\lambda > 0$ is the scale factor [17].

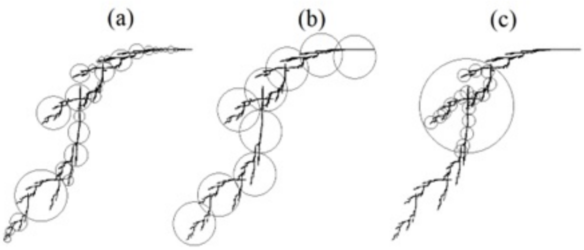

Figure 10. Covering schemes associated with: (a) Hausdorff, (b) box-counting, and (c) Assouad measures.

In contrast to the topological dimension, the fractal dimension is not a topological invariant. Accordingly the topological features of fractals (see Figure 11) can be characterized by a set of dimension numbers which generally differ from the fractal and topological dimensions [2].

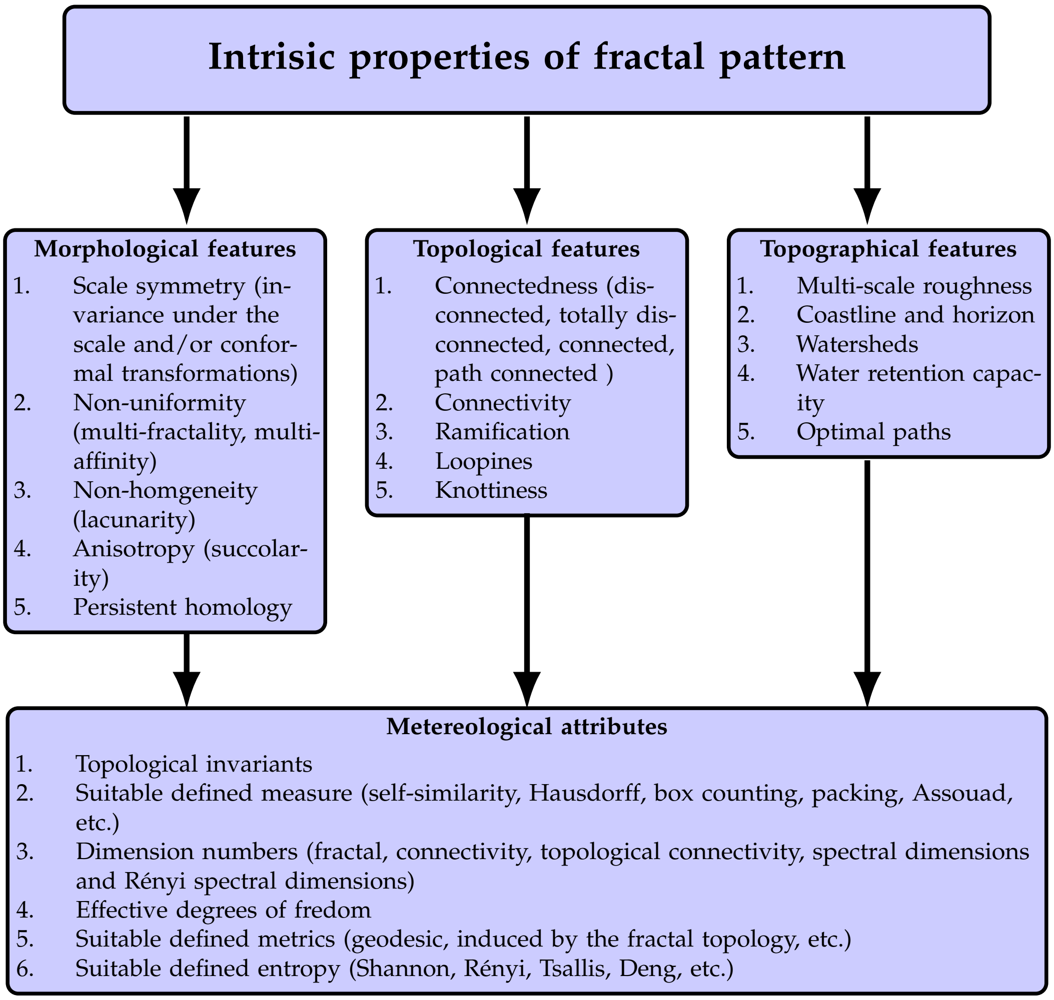

Figure 11. Inherent features and key attributes of fractal patterns.

Specifically, the pattern's connectivity can be quantified by the connectivity dimension defined as:

$$d_\ell = [lim_{\ell\to\infty} \ln \mathcal{N}(\ell) / \ln \ell], \tag{3}$$

where $N(\ell)$ is the number of pattern's points connected with an arbitrary point inside of the d_ℓ-ball of diameter l [105]. Accordingly, the connectivity dimension of the fractal network is equal to

$$d_\ell = [lim_{\ell\to\infty} \ln \mathcal{N}(\ell) / \ln \mathcal{D}], \tag{4}$$

where $\mathcal{D} = W/\mathcal{N}^2$ is the network diameter defined as the maximum geodesic distance between two sites on the network, while $W(F) = \frac{1}{2}\sum_{x,y\in F} \ell(x,y)$ is the Wiener index and $\ell(x,y)$ is the minimum number of steps needed to go from site x to site y on the network F [97]. It is a straightforward matter to understand that the ratio of the fractal and connectivity dimensions is equal to the fractal dimension of geodesic paths [2]. That is the fractal dimension of geodesic paths is equal to

$$d_g = D/d_\ell. \tag{5}$$

Accordingly, fractals can be either of the metric origin, if $d = d_\ell < D$ (see, for instance, Figure 12a), or of the topological origin, if $d < d_\ell = D$ (see Figure 12b). Furthermore, fractals can have combined origin, such that $d < d_\ell < D$ (see, for example, Figure 12c).

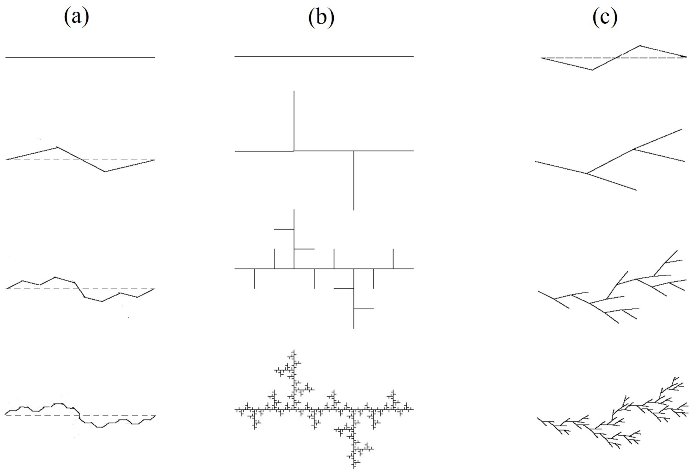

Figure 12. Three first iterations of fractals having: (a) metric, (b) topological, and (c) combined origin.

A point-like random walker on a Euclidean pattern has equal numbers of spatial and dynamical degrees of freedom. Both are equal to the topological dimension of the pattern. In contrast to this, for fractal patters the number of effective dynamical degrees of freedom d_s can be equal to or less than the number of effective spatial degrees of freedom n_γ [106]. The number of effective dynamical degrees of freedom is defined via the scaling asymptotic behavior of the probability that the random walker returns to its origin point after t steps ($P \propto t^{-d_s/2}$), while the number of effective spatial degrees of freedom can be viewed as the number of independent directions under a constrain imposed by the fractal topology. In Ref. [106] it has been established that

$$n_\gamma + d_s = 2d_\ell. \tag{6}$$

while the number of effective dynamical degrees of freedom of the point-like random walker is equal to the spectral dimension of the fractal pattern. The Einstein law relating the drift and diffusion of charge implies that

$$d_s = 2(D_W - \zeta)/D_W, \tag{7}$$

where D_W is the random walk dimension and ζ is the electrical resistance exponent defined via the scaling relations $\varrho \propto L^\zeta$ if $\zeta > 0$, or $\varrho = \varrho_\infty - cL^\zeta$, if $\zeta < 0$ [107]. Furthermore, many kinds of fractals obey the Alexander-Orbach relation $d_s = 2D/D_W$ and so the electrical resistance exponent is equal to $\zeta = D_W - D$ [61–63].

Scale invariant fractals can be loopless (see, for example, fractals in Figure 12) or have loops at all scales (see, for example, fractals in Figs. 7 and 8). For the loopless fractals $\zeta = d_g$. Consequently, the spectral dimension of a loopless fractal is equal to

$$d_s = 2D/(D + d_g) = 2d_\ell/(d_\ell + 1), \tag{8}$$

whereas for fractals with loops at all scales the spectral dimension is in the range of

$$2d_\ell/(d_\ell + 1) \leq d_s \leq d_\ell, \tag{9}$$

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Accordingly, the fractal loopiness can be characterized by the loopiness index which was defined in Ref. [2] as

$$\Lambda = (d_s/n_\gamma) - 1/d_\ell, \tag{10}$$

while for the loopless fractals ($\Lambda = 0$) the spectral dimension is equal to $d_s = n_\ell$ and loopy fractals ($0 < \Lambda < 1$) have $d_s > n_\gamma/d_\ell$.

Another important property of patterns is their ramification. Quantitatively, the order of ramification R_j at point j on the pattern $\mathcal{P}$ is defined as the smallest number of bonds that should be cut in order to isolate an arbitrarily large bounded set of points $C_j \subset \mathcal{P}$ connected to the point j. Then the order of pattern ramification is defined as [108]:

$$\mathcal{R} = \min_{j\in\mathcal{P}}\{\mathcal{R}_j(C_j) : C_j \subset \mathcal{P}\}.$$

Finitely ramified patterns have the finite orders of ramification, whereas the order of ramification of an infinitely ramified pattern grows with the size L of $C_j \subset \mathcal{P}$ as $R_j \propto L^{Q_j}$, such that $R_j \to \infty$, as $L \to \infty$. Accordingly the order of ramification is characterized by the ramification exponent

$$Q = \min_{j\in\mathcal{P}}\{Q(C_j) : C_j \subset \mathcal{P}\},$$

while for the finitely ramified patterns $Q = 0$. In Ref. [109] it was recognized that the ramification exponent can be related to the topological Hausdorff dimension D_{tH} which was introduced in [104] via a combination of the definitions of the topological and Hausdorff dimensions:

$$D_{tH} = \min\{s : \text{ there is a subset} \mathcal{S} \subset \mathcal{P} \text{ such that } dim_H\mathcal{S} = d-1, \text{ while } dim_t(\mathcal{P}\backslash\mathcal{S}) \leq 0\}, \tag{11}$$

Further, it was argued that this definition can be generalized to define the topological fractal dimension D_{tF} with the use of any suitable fractal dimension (see, for instance, Table 1) instead of the Hausdorff one. The topological fractal dimension is related to the ramification exponent as $D_{tF} = 1 + Q$ [109]. Generally, $d \leq D_{tF} \leq D$, while for Euclidean patterns $d = D_{tF} = D$ [2].

Table 1. Some different definitions for fractal dimension.

Dimension	Definition	Measure/Comments
Hausdorff-Besicovitch dimension	$dim_H F = \inf\{S : H^S(F) = 0\}$ $= \sup\{S : H^S(F) = \infty\}$	$\{U_i\}$is a cover of F i.e. $F \subset \cup_{i=1}^{\infty}$ with $0 \leq\mid U_i \mid\leq \delta$ where U is any non-empty subset of n-dimensional Euclidean space, $\Re^n$. Hausdorff meausre is $H^s(F) = \lim_{\delta\to\infty} H^S_\delta(F)$, where $H^S_\delta(F) = \inf\{\sum_{i=0}^{\infty} \mid U_i \mid^S : \{\}U_i\}$, and diameter of U is $\mid U \mid = \sup\{\mid x-y \mid : x,y \in U\}$
Minkowski -Bouligand dimension	$\Delta(F) = \overline{\lim}_{\gamma\to 0}\{\log N_\delta(F)/\log(1/\delta)\}$ $\Delta(F) \geq dim_H F$	Let $N_\delta(F)$ denotes the least number of balls in a covering of F by balls of radius ϵ. It is follows from the definition of H^S_δ that $H^S_\delta(F) \leq (2\delta)^S \times N_\delta(F)$
Minkowski dimension	$dim^M_B(F) = n - \lim_{\delta\to 0}\left\{\frac{\log vol^n(F_\delta)}{\log\delta}\right\}$	F_δ is the parallel body to F: $F_\delta = \{x \in \Re^n : \mid x-y \mid\leq \delta$, for some $y \in F\}$, where n is the topological dimension
Kolmogorov-Schirelman-Potjrajin	$D_K = \lim_{\epsilon\to 0}\{\sup\ \log N(\epsilon)/\log(1/\epsilon)\}$	$N(\epsilon)$ is the smallest number of balls of diameter less or equal to ϵ which are needed to cover fractal
Mandelbrot-Schirelman-Kolmogorov	$D_F = \lim_{\epsilon\to 0}\{\sup\ \ln N_x(\epsilon)/\ln(1/\epsilon)\}$ $= \inf\left\{d \geq 0, \lim_{\epsilon\to 0}\left[\sup \epsilon^d \times N_x(\epsilon) = 0\right]\right\}$	$N_x(\epsilon)$ is the least number of balls of radius less than ϵ which are needed to cover fractal

Table 1. *Cont.*

Dimension	Definition	Measure/Comments
Upper box-counting dimension	$\overline{dim}_B F = \overline{\lim}_{\delta\to 0}\left\{\frac{\log N_\delta(F)}{\log \delta}\right\} \geq dim_H F$	F is non-empty subset of $\Re^n$. $N_\delta(F)$ is any of the following: A) The smallest number of (1) closed balls of radius δ, (2) cubes of side δ and (3) sets of diameter at most δ, that cover F B) The largest number of disjoint ball of radius δ with center in F C) The number of δ-mesh cubes that intersect of F.
Lower box-counting dimension	$\underline{dim}_B F = \underline{\lim}_{\delta\to 0}\left\{\frac{\log N_\delta(F)}{\log \delta}\right\} \geq dim_H F$	
box-counting dimension	$dim_B F = \lim_{\delta\to 0}\left\{\frac{\log N_\delta(F)}{\log \delta}\right\} \geq dim_H F$	
upper modified box-counting dimension	$\overline{dim}_{MB} F = \inf\left\{\sup \overline{dim}_B F_i \,:\, F \subset \cup_{i=1}^{\infty} F_i\right\}$	If F can be decomposed into a countable number of pieces F_i in such a way that the largest piece has a small a dimension as possible.
Lower modified box-counting dimension	$\underline{dim}_{MB} F = \inf\{\sup \underline{dim}_B F_i \,:\, F \subset \cup_{i=1}^{\infty} F_i\}$	
Packing dimension	$dim_P F = \inf\{S : P^S(F) = 0\} = \sup\{S : P^S(F) = \infty\}$, $dim_H F \leq dim_P F \leq \Delta(F)$	B_i is a collection a disjoint balls of radius at most δ with center in F. Packing measures is: $P^S(F) = \inf\{\sum_i P_0^S(F_i) \,:\, F \subset \cup_{i=1}^{\infty} F_i\}$, where $P_0^S = \lim_{\delta\to 0} P_\delta^S$, $\P_\delta^S = \sup\{\mid B_i \mid^S : \{B_i\}\}$.
Assouad dimension	$D_A = \{s > 0 \,:\, C > 0$ such that $N_r[B(x,R) \cap F] \leq C(R/r)^2$ for all $x \in F$ and $0 < r < R\}$	$B(x,R)$ denote the covering balls (see for illustration Figure 10c).
Divider dimension (of Jordan curves)	$D_D = \lim_{\delta\to 0}\{\log M_\delta(C)/\log(1/\delta)\}$	$M_\delta(C)$-maximum number of points $x_0, x_1, \ldots x_m$, on the curve C, in that order, such that $\mid x_k - x_{k-1} \mid = \delta$, $k_1, 2, \ldots, m$.

Likewise the fractal dimension the topological fractal dimension is not topologically invariant. The topological invariant associated with the fractal ramification was introduced in Ref. [81] and named the topological connectivity dimension. It is defined as

$$d_{t\ell} = \inf\{s : \exists \mathcal{S} \subset \mathcal{P} \text{ such that } D_{F\ell}(\mathcal{S}) \leq s - 1 \text{ and } d_t(\mathcal{P}\backslash\mathcal{S}) \leq 0\}, \tag{12}$$

where $D_{F\ell} = D(\mathcal{S})/d_g(\mathcal{P})$ is the fractal dimension of the subset S with respect to the geodesic metric on the pattern $\mathcal{P}$. The infinitely ramified patterns are characterized by $d_{t\ell} > 1$, whereas the finitely ramified patterns have $d_{t\ell} = d = 1$ [2].

The classification of scale invariant patterns from a topological viewpoint is given in Figure 13. The fractal attributes of some self-invariant patterns are summarized in Table 2.

Table 2. Fractal attributes of some self-similar patterns.

	Figure	D	d	$d_{t\ell}$	d_ℓ	d_s	n_γ	Λ
Cantor set C	4	$\frac{\ln 2}{\ln 3}$	0	0	See discussion in Ref. [110]	See discussion in Ref. [110]	See discussion in Ref. [110]	See discussion in Ref. [110]
Cartesian product $C \times [0,1]$	See Ref. [4]	$\frac{\ln 6}{\ln 3}$	1	$\frac{\ln 6}{\ln 3}$	See discussion in Ref. [110]	See discussion in Ref. [110]	See discussion in Ref. [110]	See discussion in Ref. [110]

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Table 2. *Cont.*

	Figure	D	d	$d_{t\ell}$	d_ℓ	d_s	n_γ	Λ
Cartesian tartan $C \times [0,1] \cup C \times [1,0]$	See Ref. [111]	$\frac{\ln 6}{\ln 3}$	1	$\frac{\ln 6}{\ln 3}$	$\frac{\ln 6}{\ln 3}$	$\frac{\ln 6}{\ln 3}$	$\frac{\ln 6}{\ln 3}$	0.387
Koch curve	6	$\frac{\ln 4}{\ln 3}$	1	1	1	1	1	0
Sierpiński arrowhead curve	7b	$\frac{\ln 3}{\ln 2}$	1	1	1 See [112]	1 See [112]	1	0
Koch curve	12a	1.99	1	1	1	1	1	0
Tree	12b	$\frac{\ln 5}{\ln 3}$	1	1	$\frac{\ln 5}{\ln 3}$	1.188...	1.741	0
Leaf	12c	1.756	1	1	$\frac{\ln 5}{\ln 3}$	1.188...	1.741	0
Tree	2a	$\frac{\ln 5}{\ln 3}$	1	1	$\frac{\ln 5}{\ln 3}$	1.188...	1.741	0
Diamond fractal	13(6)	$\frac{\ln 6}{\ln 3}$	1	1	$\frac{\ln 6}{\ln 4}$	1.137	1.448	0.012
Sierpiński gasket	7a	$\frac{\ln 3}{\ln 2}$	1	1	$\frac{\ln 3}{\ln 2}$	$\frac{\ln 9}{\ln 5}$	1.805	0.635
Sierpiśki carpet SC	8	$\frac{\ln 8}{\ln 3}$	$\frac{\ln 6}{\ln 3}$	$\frac{\ln 8}{\ln 3}$	1.806	1.979	0.384	0.635
Sierpiśki cube	See Ref. [92]	$\frac{\ln 26}{\ln 3}$	2	$\frac{\ln 24}{\ln 3}$	$\frac{\ln 26}{\ln 3}$	2.933	2.998	0.41
Sierpiśki waveguide $SC \times [0,1]$	See Ref. [2]	$\frac{\ln 24}{\ln 3}$	2	$\frac{\ln 24}{\ln 3}$	$\frac{\ln 24}{\ln 3}$	2.806	2.98	0.596
Menger sponge	13(4)	$\frac{\ln 20}{\ln 3}$	2	$\frac{\ln 12}{\ln 3}$	$\frac{\ln 20}{\ln 3}$	2.52	2.94	0.49
Complement of Menger sponge	13(1)	3	3	3	3	3	3	2/3
Percolation cluste in E^2	See Ref. [81]	91/94	1	1.6574	1.6617	1.317	2	0.053
Percolation cluste in E^3	See Ref. [97]	2.52293	1	1.828	1.834	1.327	2.341	0.022

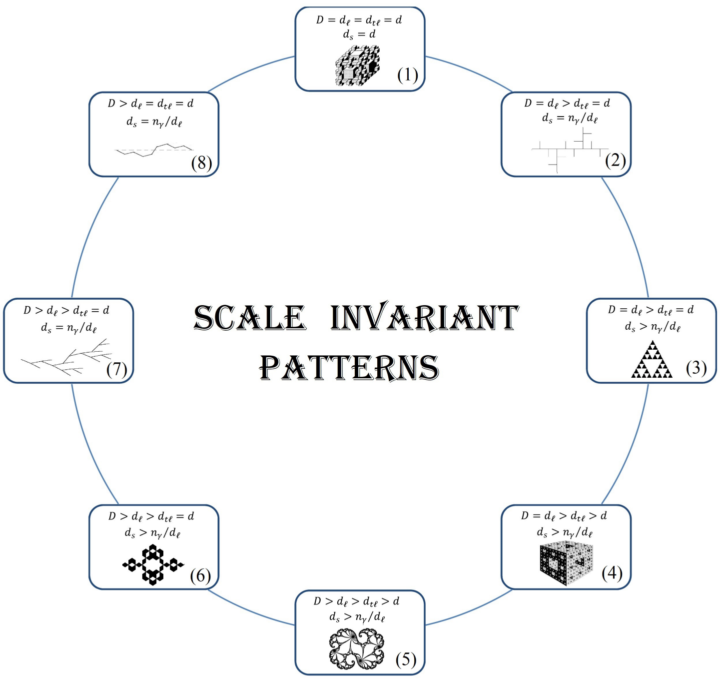

Figure 13. Classification of scale invariant patterns from a topological viewpoint: (1) Euclidean pattern; (2) - (4) fractals of topological origin; (8) fractal of metric origin; and (5) – (7) fractals of combined origin. (1), (4), (5) - infinitely ramified; (2), (3), (6) – (8) – finitely ramified. Illustrations: Euclidean compliment of Menger sponge (1); fractal tree (2); Sierpiński gasket (3); Menger sponge (4); Julia-Mandelbrot set (5); diamond fractal (6); fractal leaf (7); fractal Koch-like curve (8).

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4. Final Remarks

Fractal geometry is a relatively new branch of mathematics that was launched in the 1970's. Since then, it has been linked to many branches of mathematics, including number theory, topology, differential geometry, statistics, operator algebras, potential theory, mathematical physics, harmonic analysis, and the theory of dynamical systems. Presently, the fractal geometry remains an active area of development. In particular, many works were focused on the inherent features of fractal patterns. The morphological properties of scale invariant patterns were reviewed in Ref. [5]. The topographical attributes of fractals were discussed in details in Ref. [2]. In this review we discuss the topological features of fractal patterns. The pattern's topology can be characterized by calculating so-called topological invariants. The topological invariants fix certain topological features such as the number of connected components, the pattern connectivity and ramification, the existence and distribution of holes and knots in the studied pattern. We state that the fractal loopiness can be characterized by the ratio between the numbers of effective spatial and effective dynamical degrees of freedom on the fractal. However, an appropriate characterization of the fractal knottedness remains open (see Refs. [113–115] and references therein).

In nature, complex irregular and fragmented patterns arise in a wide variety of systems. The main appeal of fractal geometry is its ability to describe complex patterns that traditional Euclidean geometry is unable to analyze. Accordingly, fractal geometry has become one of the powerful tools for image analysis in many fields of science, including computer science, physics, mechanical and electrical engineering, biophysics, medicine and economics, and others.

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