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On Zero-Divisor Graphs of Z_n, when n Is Square Free

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Abstract: In this article, some properties of the zero divisor graph $\Gamma(Z_n)$ are investigated when n is a square free positive integer and it is shown that the zero divisor graph $\Gamma(Z_n)$ of the ring Z_n is a $(2^k - 2)$-partite graph, when the prime decomposition of n contains k distinct square free primes using the method of congruence relation. We present some examples, accompanied by pictorial visualizations, to achieve the desired results. Since Z_n is a semisimple ring when n is square-free, the results can be generalized to characterize semisimple rings and modules, as well as rings satisfying Artinian and Noetherian conditions, through the properties of their zero-divisor graphs.

Keywords: Graphs; Zero-divisor graphs; Bi-partite graph; k-partite graph; Congruence relation; Semi-simple rings

MSC: 13A70, 05C25, 05E40

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1. Introduction

Graph theory is regarded as one of the most intriguing topics, offering a powerful means to visualize the interconnections between objects. Its applications span various fields in both social and natural sciences, leveraging its inherent visual representation to simplify complex properties. For a more comprehensive exploration of graph theory, we refer readers to [1,11,18,21]. Moreover, graph theory serves as a valuable tool in numerous branches of mathematics, including algebra.

The zero-divisor graph of a commutative ring was first introduced by Beck in 1988 [9] to explore the structure of the underlying ring. Since then, this concept has been extensively studied by various researchers [5,13,15,20]. The study of zero-divisor graphs has also been extended to noncommutative rings [2]. Redmond [19] further generalized this concept to arbitrary rings, demonstrating that for any ring R, any two vertices in $\Gamma(R)$) are connected by a path of length at most 3. It has also been shown that for any ring R, if $\Gamma(R)$ contains a cycle, the length of the shortest cycle is at most 4. Anderson introduced various structural properties of zero-divisor graphs, proving that $\Gamma(R)$ is always connected with $\mathrm{diam}(\Gamma(R)) \leq 3$. Additionally, they discussed conditions under which $\Gamma(R)$ becomes a complete graph or a star graph [6].

Beck [9] defined zero-divisors graph $\Gamma(R)$ of a ring R as a graph whose vertex set $V(R)$ is the set of all the elements of R and the edge sets $E(R)$ is defined by the relation on the elements of R as $x, y \in V(R)$ are adjacent if and only if $xy = 0$ for distinct $x, y \in R$, and it is noted that the relation in a zero-divisor graph $\Gamma(R)$ is always symmetric while R is a commutative ring. If the ring is commutative, then $\Gamma(R)$ is a simple graph and $\Gamma(R)$ is an empty graph while R is an integral domain with a slight modification in the definition. The zero divisor graph is slightly different from the definition given by Beck [9] is defined by considering the non-zero zero divisors as vertex sets instead of taking all the elements of the ring as vertex sets [6]. The intention of Anderson and Livingston [6] was to simplify $\Gamma(R)$ so that it can be visualized in a simple way without losing the relevant pieces of information contained in it. Later on, many properties of $\Gamma(R)$ were studied, for instance, diameter, girth, and chromatic number (see [6,7]).

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A ring R is called simple if it has no proper nonzero two-sided ideals, and an R module M is called simple if M is nontrivial and M has no proper nonzero submodules. A simple ring or module can be treated as the prime number in number theory as Z_n is simple while n is a prime. Like primes are the buliding blocks of numbers, the simple submodules are the building blocks of the module. A module is called a semisimple (or completely reducible) module if it is the sum of simple (irreducible) submodules. A ring R with unity is semisimple if R_R is a semisimple module. A square-free prime decomposition of a positive integer n is as simple as the semisimple finite ring Z_n, when n is square free. The ring $R = Z_p \times Z_p \times ... \times Z_p$ is also a semisimple ring for any prime p. This motivated us to study properties of the zero divisor graphs of Z_n while n is square free. As Z_n is a semi-simple ring while n is square-free, our investigation becomes a bit interesting. We would like to know the ring-theoretic properties of the semisimple ring in terms of its zero divisor graphs. Our aim is to study the interplay of ring-theoretic properties of semisimple ring R with the graph theoretic properties of $\Gamma(R)$ connecting through number theoretic aspects. We could able to find a fine relationship between R and $\Gamma(R)$ while $R = Z_n$. The Rest of the work is out of the scope of this article and is under investigation and may be an outcome as a sequel of this work.

In this article, effort is given to show that $\Gamma(R)$ is a partite graph with a certain condition on n and it is also shown that $\Gamma(R)$ is a complete graph while $n = p^2$ for a prime p. To prove the results, we took the help of number theoretic congruence relation which grabbed our attention and made the investigation more interesting. We present some illustrative examples to accomplish the results. It is to be noted that $\Gamma(Z_n)$ is always connected, and this graph's diameter and girth are small. Some examples of zero divisor graphs Z_n are given below (Figure 1).

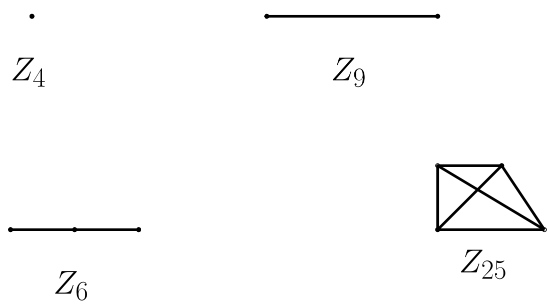

Figure 1. Zero divisor graphs.

2. Key Definitions and Notations

Definition 1. *A ring is a non-empty set R together with two binary operation's addition (+) and multiplication (.) for which $< R, + >$ is a commutative ring, multiplication is associative and multiplication is distributive over addition from both sides. If multiplication is commutative, then R is called a commutative ring.*

Definition 2. *[21] A graph G is a collection of objects in which the objects are pairwise related in some sense. The objects are called vertices and the related or linked vertices are called edges. Formally, it consists of two sets V and E, where E is a set of unordered pairs $\{u, v\}$ for some $u, v \in V$ called the edges set of G, and it is denoted by $G = (V, E)$. If V is a finite set, then G is called a finite graph.*

Definition 3. *Two integers a and b are congruent modulo m iff they have the same remainder when divided by m [18], denoted by $a \equiv b(\mod m)$.*

Property 1. *[18] The linear congruence equation*

$$ax \equiv b(mod\ m)$$

has a solution if and only if $d|b$, where $d = gcd(a, m)$.

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A path in a graph is an alternating sequence of distinct vertices and edges such that consecutive vertices in the sequence are adjacent to the graph. For an unweighted graph, the length of a path is the number of edges on the path. The eccentricity of a vertex is defined as the maximum distance of a vertex to all other vertex and it is denoted by $e(V)$. If a path is closed, i.e., starting and end vertices are the same, then the path is called a cycle. A graph is called connected, if always finds a path between any two vertices. A cycle free connected graph is called a tree. A star graph S_n of order n, sometimes simply known as n-star is a tree on n nodes with one node having vertex degree $n-1$ and the other $n-1$ having vertex degree 1.

Definition 4. *[21] The center of a graph is the set of all vertices of minimum eccentricity and the radius of a graph is the minimum eccentricity of any graph vertex in a graph.*

Definition 5. *[6] Let R be a commutative ring with 1 and let $Z(R)$ be its set of non-zero zero-divisors. We associated a simple graph $\Gamma(R)$ to R with vertex set $V = Z(R)$, the set of non-zero zero divisors of R, and for distinct $x, y \in Z(R)$, the vertices x and y are adjacent if and only if $xy = 0$.*

A graph is called a bi-partite graph if the vertex set can be partitioned into two sets V_1 and V_2 for which no two vertices from the same set can be adjacent. A graph is called a k-partite graph if vertices can be partitioned into k disjoint sets so that no two vertices within the same set are adjacent. It is to be noted that if $k = 2$, then the graph is called a bipartite graph.

A complete k- partite graph is a k-partite graph such that every pair of graph vertices in the k sets are adjacent. If there are $p, q, \ldots, r$ graph vertices in the k sets, the complete k-partite graph is denoted by $K_{p,q,\ldots,r}$. If $n = 2$, then the graph is said to be a complete bi-partite graph.

A graph that is complete k-partite for some k is called a complete multipartite graph. A Turán graph is a complete multipartite graph whose partite sets are as nearly equal in cardinality as possible.

3. Zero Divisor Graph of Z_n, for Square Free n

In this section, we the two main results are proved and discussed some properties.

Theorem 1. *If $n = p_1p_2$, the product of two distinct primes, then*

1. *$\Gamma(Z_{p_1p_2})$ is a complete bi-partite graph K_{p_1-1,p_2-1}. In particular, if $p_1 = 2$, then $\Gamma(Z_{p_1p_2})$ is a star graph and the centre of $\Gamma(Z_{p_1p_2})$=p_2.*
2. *diameter, i.e., diam($\Gamma(Z_{p_1p_2})) = 2$.*
3. *radius, rad($\Gamma(Z_{p_1p_2})) = 1$, if $p_1 = 2$ and otherwise 2.*

Proof. 1. Let us consider

$$V = \{n_1, n_2, n_3, ..., n_t\}$$

be the set of all non-zero zero divisors of $Z_{p_1p_2}$. Then by the properties of zero divisor we know that for each i,

$$(n_i, p_1p_2) \neq 1.$$

Since 0 is always a zero divisor element,

$$t = p_1p_2 - 1 - \phi(p_1p_2),$$

where ϕ is a Euler's totient function. It follows that

$$t = p_1p_2 - 1 - (p_1 - 1)(p_2 - 1)$$

$$t = p_1 + p_2 - 2.$$

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Therefore,

$$t = |V(\Gamma Z_{p_1 p_2})| = p_1 + p_2 - 2,$$

where V is the set of vertices of $\Gamma(Z_{p_1 p_2})$. Now, one of the way to determine the zero-divisors elements of Z_{p_1, p_2} is to solve for the incongruent solution of the congruent equation.

$$kx \equiv 0(\mod p_1 p_2) \tag{1}$$

where k is a non-zero zero divisor element of $Z_{p_1 p_2}$.
Now, equation (1) has a solution as

$$gcd(p_1 p_2, k) \mid 0.$$

We may assume that

$$k = p_1 \text{ or } k = p_2.$$

Then

$$kx \equiv 0(\mod p_1 p_2)$$

can be written as

$$p_1 x \equiv 0(\mod p_1 p_2) \tag{2}$$

$$or$$

$$p_2 x \equiv 0(\mod p_1 p_2). \tag{3}$$

Now, from equation (2)

$$p_1 p_2 \mid p_1 x.$$

Therefore,

$$p_1 x = p_1 p_2 l,$$

for some l. We note that for incongruent solution, l must be less than $p_1 p_2$. It follows that

$$x = p_2 l.$$

Thus the incongruent solutions of equation (2) are given by those integers which are multiples of p_2 and less than $p_1 p_2$. Similarly, the incongruent solutions of equation (3) are given by those integers which are less than $p_1 p_2$ and multiples of p_1. For instance, if p_1=3 and p_2=7, then the solutions of these two congruence equations (2) and (3) will be multiples of 7 and 3, i.e., $\{7, 14\}$ and $\{3, 6, 9, 12, 15, 18\}$, respectively. Now, let us consider V_1= all the incongruent solutions of equation (2) and V_2= all the incongruent solutions of equation (3). Then

$$|V(\Gamma(Z_{p_1 p_2}))| = |V_1 \cup V_2| = t.$$

Since p_1 and p_2 are distinct primes,

$$V_1 \cap V_2 = \phi.$$

Moreover, no two vertices in V_i are connected for each i=1, 2, and every vertices of V_1 are connected to every vertices of V_2. If possible, assume

$$v_1, v_2 \in V_1$$

are connected, then, we have

$$v_1 v_2 \equiv 0(\mod p_1 p_2).$$

Since $v_1, v_2 \in V_1$

$$v_1 = p_2 l_1 \text{ and } v_2 = p_2 l_2,$$

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where $l_1, l_2 < p_1$. It follows that

$$p_1 p_2 \mid p_1 l_1 p_2 l_2 \Rightarrow p_2 l_1 l_2 = p_1 m.$$

Since p_1 is prime and $l_1, l_2 < p_1$, we have $p_1 \mid p_2$, which is a contradiction that p_1 and p_2 are distinct primes. Since

$$|V_1| = p_1 - 1 \text{ and } |V_2| = p_2 - 1,$$

we have $\Gamma(Z_{p_1 p_2})$ is a complete bi-partite graph, i.e., $\Gamma(Z_{p_1 p_2}) = K_{p_1 - 1, p_2 - 1}$.
If any of the prime p_1 or p_2 is 2, say, $p_1 = 2$, then $|V_1| = 1$ and the only vertex in V_1 must have an edge with every vertex in V_2 and so $\Gamma(Z_{p_1 p_2})$ is a star graph.

2. To prove this we consider an example as shown in the Figure 2. Since $\Gamma(Z_{p_1 p_2})$=$\Gamma(Z_{21})$ is a bi-partite graph diameter $\Gamma(Z_{p_1 p_2}) = 2$ and chromatic number,

$$\chi(Z_{p_1 p_2}) = 2.$$

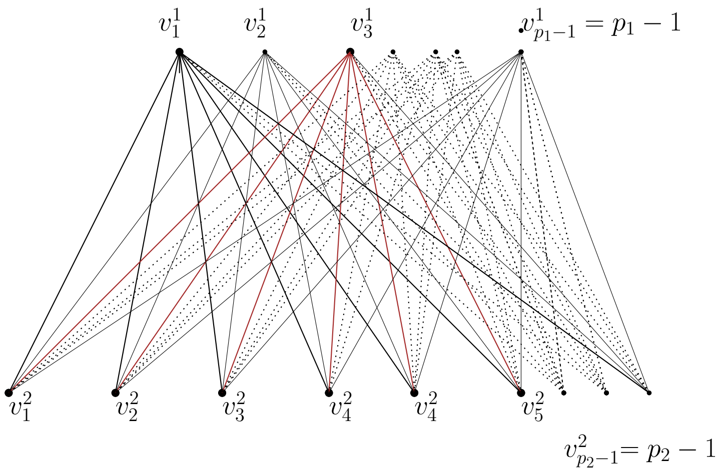

Figure 2. Example of bi-partite graph of $\Gamma(Z_{p_1-1, p_2-1})$, where p_1 and p_2 are distinct primes.

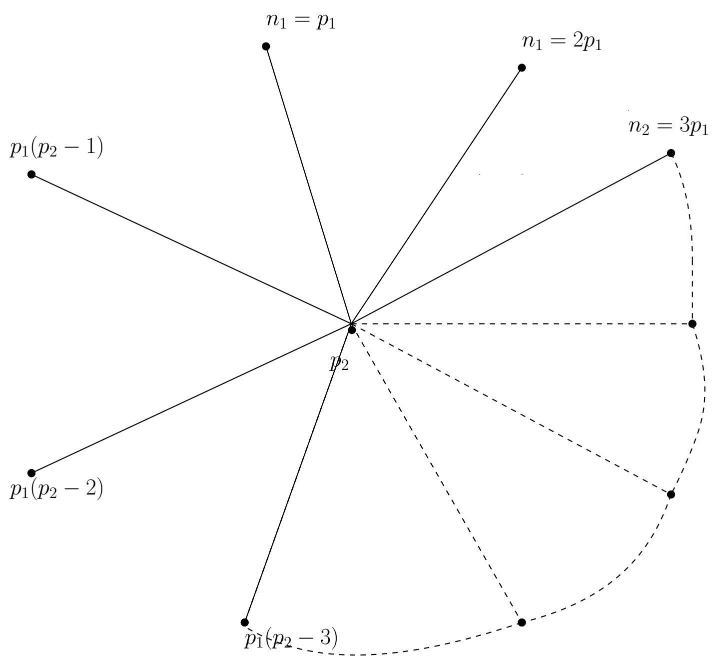

Figure 3. Figure showing the schematic star graph for $p_1 = 2$ and centre $\Gamma(Z_{p_1 p_2})$ as p_2.

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3. Radius, rad$(\Gamma(Z_{p_1p_2})) = \Gamma(Z_{21})$ is the minimum eccentricity of all the vertex of $\Gamma(Z_{p_1p_2})$ which is 2.

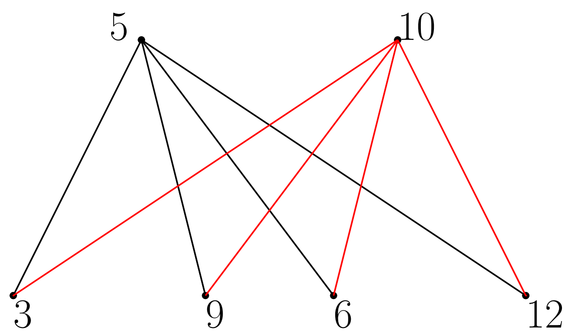

Figure 4. Example of bi-partite graph of $\Gamma(Z_{15})$, where $p_1 = 3$ and $p_2 = 5$.

□

Theorem 2. *If $n = p_1p_2p_3...p_k$ is a square free prime factorisation, then the non-zero zero divisor graph, $\Gamma(Z_n)$, is a $(2^k - 2)$-partite graph.*

Proof. If n is a prime, i.e., $n = p_1$, then the non-zero zero divisor graph $\Gamma(Z_n)$ is a null graph. Hence, in this case, the result is true.

If n=$p_1.p_2$, then by Theorem 1, the non-zero zero divisor graph, $\Gamma(Z_n)$ is a bipartite graph, i.e., $(2^2 - 2)$-partite graph. Thus the result is true in this case also.

In the proof of Theorem 1, it can be observed that the partition of the vertex set is obtained by taking the incongruent solution of the two congruent equations

$$p_1x \equiv 0(\mod n) \text{ and } p_2x \equiv 0(\mod n),$$

respectively.

Now, we assume that n=$p_1p_2p_3$. Let $V = \{n_1, n_2, n_3, ..., n_t\}$ be the set of non-zero zero divisor elements of $Z_{p_1p_2p_3}$. As we know that for each i

$$(n_i, n) \neq 1.$$

It follows that the number of non-zero zero divisor which is also the number of vertices in $\Gamma(Z_{p_1p_2p_3})$ can be obtained by

$$\begin{aligned} |V| &= n - \phi(n) - 1, \text{ where } \phi \text{ denotes the Euler totient function.} \\ &= p_1p_2p_3 - \phi(p_1p_2p_3) - 1 \\ &= p_1(p_2 - 1) + p_2(p_3 - 1) + p_3(p_1 - 1). \end{aligned}$$

For any $m \in V$, we have m is a zero divisor of Z_n. It follows that there exists $x, 1 \leq x < n$ such that $n \mid mx$. Therefore to list all such zero divisors of Z_n, we must solve for the incongruent solution of the congruent equation

$$mx \equiv 0(\mod n). \tag{4}$$

Equation (4) is solvable as $gcd(m, n) \mid 0$. Since $n = p_1p_2p_3$ is a product of three distinct primes and $0 < m < n$, some possible values of m which the m may be any one of the primes or product of primes by taking two primes at a times, namely,

$$p_1, p_2, p_3, p_1p_2, p_1p_3, p_2p_3.$$

For our convenience, depending upon the choices of m, we may consider the following cases to solve the congruence Equation (4).

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Case I: For each i, let $m = p_i$. In this case, we shall have three linear congruence equations, namely,

$$p_i x \equiv 0(\bmod n),$$

where $i = 1, 2, 3$. These three congruence equations can be thought as ${}^3C_1$ number of choices. Now for each i, $n \mid p_i x$, whence $xp_i = (p_1 p_2 p_3) \times r, \quad r = 1, 2, 3, ...$ In this way we shall get 3 sets of solution for each i, i=1, 2, 3. As we are looking for non-zero zero divisors of Z_n only, so x must be greater than 0 and less than n, i.e., $0 < x < n$. Now, for a fixed i, $(i = 1, 2, 3)$, the equation $xp_i = p_1 p_2 p_2 \times r$ can be reduced to

$$x = p_2 p_3 r \text{ or } x = p_1 p_3 r \text{ or } x = p_1 p_2 r.$$

Since $0 < x < p_1 p_2 p_3$, we have $r = 1, 2, 3, ..., p_i - 1$ for each corresponding to choice of $i = 1, 2, 3$. Let us denote these solutions sets by S_{m_i} corresponds to each i. We note that then three sets of solution can be thought as 3 = ${}^3C_1$ sets of solutions (number of combination of picking 1 member at a time from the set of 3 members).

Case II:

$m = p_i p_j \, , i \neq j.$

In this case, we shall have ${}^3C_2$ sets of solution as we can choose two members from a set of 3 members by $3C_2$ ways. Thus the congruence equations reduces to

$$p_i p_j x \equiv 0(\bmod n), i \neq j$$

whence,

$$p_i p_j x = p_1 p_2 p_3 r, r = 0, 1, 2, 3, ...$$

For fixed i and j,

$$x = p_l r, \text{ where } l = 1, 2, 3.$$

We note that $i \neq j \neq l$.

Since $0 < x < p_1 p_2 p_3$, we get $r = 1, 2, 3, ..., p_i p_j$, respectively. That is any set of solutions of these ${}^3C_2$ congruence equations must be multiples of p_l. Let us denote these sets of solutions by $S_{m_{ij}}$ corresponds to p_i and p_j. It is to be noted that there are

$${}^3C_1 +^3 C_2 = (2^k - 2)$$

sets of solution which may not be mutually disjoint. This process may be continued for n as product of 4, 5,... distinct primes. Suppose

$$n = p_1.p_2.p_3...p_k.$$

Let,

$$V = \{n_1, n_2, n_3, n_5, ..., n_q\}$$

be set of non-zero zero divisors elements of $\Gamma(Z_n)$. Then

$$(n_i, n) \neq 1.$$

Thus, the number of non-zero zero divisors can be obtained by

$$|V(\Gamma Z_n)| = p_1 p_2 p_3 ... p_k - 1 - \phi(p_1 p_2 p_3 ... p_k).$$

Let $m \in V$. Since m is a zero divisor, there exists $x, 1 \leq x < n$ such that

$$n \mid mx.$$

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Therefore to list all such zero divisors of Z_n, we must solve the congruence equation

$$mx \equiv 0(\text{mod } n) \tag{5}$$

Since m and x are both variables, we have to fixed one. For our convenience , we are fixing m by choosing some particular values of m. We would like to note that we are choosing m in such a way so that the totality of such solutions for x provides us the complete set of non-zero zero divisors of the ring Z_n and which further leads to a partition to the vertex set of the non-zero zero divisor graph $\Gamma(Z_n)$ in the formation of a *partite graph*.

Now, Equation (5) is solvable as $gcd(m,n) \mid 0$. Since $n = p_1p_2p_3...p_k$ is a product of k distinct primes and $(0 < m < n)$. To fix m, we choose those zero-divisors which are products of primes by taking at most $K-1$ primes from $p_1, p_2, p_3, ..., p_k$ at a time. In this way, we shall have the following cases.

1. $m = p_i, i = 1,2,...,k($ picking one prime at a time).
2. $m = p_ip_j, 1 < i \neq j < k($ picking two distinct primes at a time)
   and so on. After $k-2$ steps, we have
3. $m = p_1p_2...p_{i-1}p_{i+1}...p_k($ picking k-1 distinct primes at a time).

In this way, we shall have

$$^kC_1 +^k C_2 + ... +^k C_{k-1} = 2^k - 2$$

numbers of equations corresponding to which we shall have $2^k - 2$ sets of solution. Now, we solve the $(2^k - 2)$ equations to get the complete list of non-zero zero divisors of Z_n.

Case(1):

If $m = p_i$ for some p_i (picking one prime at a time), then the Equation (5) reduces to

$$p_ix \equiv 0(\text{ mod } n), \text{ for fixed } i.$$

It follows that

$$p_1p_2p_3...p_k \mid p_ix.$$

Whence,

$$x = (p_1p_2p_3p_{i-1}p_{i+1}...p_k) \times r, r = 0,1,2,3,...$$

Since $0 < x < n$, we have $0 < r < p_i$, that is , $r = 1,2,3,...,p_i - 1$. In this way we shall get k sets of solution for each i, $i = 1,2,3,...,k$. Let us denote these solutions set by S_{m_i} corresponds to p_i, We note that k = $^kC_1$ (Number of combination of picking 1 member at a time from the sets of k members).

Case (2): $m = p_ip_j, i \neq j($ picking two prime at a time). In this case, we shall have $^kC_2$ sets of solution as we can choose two members from a set of k members by $^kC_2$ ways. Thus the congruence equation reduces to

$$p_ip_jx \equiv 0(\text{ mod n}), i \neq j.$$

Whence, $p_ip_jx = p_1p_2p_3...p_kr, r = 0,1,2,3...$

For fixed, i and j , $i \neq j$ (without loss of generality, we assume, $i < j$), we have

$$x = p_1p_2...p_{i-1}p_{i+1}...p_{j-1}p_{j+1}...p_kr.$$

Since $0 < x < p_1p_2...p_ip_{i+1}..p_jp_{j+1}...p_k.$, we get $r = 1,2,3,...,p_ip_j - 1$. Let us denote these set of solution by $S_{m_{i,j}}$ corresponds to p_i and p_j. Continue in this manner, we have the following case.

$(K-1)^{th}$ **case (picking** $(k-1)$ **primes at a time)**:

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In the $(k-1)^{th}$ case, we shall have ${}^kC_{k-1}$ sets of solutions corresponding to the congruence equations

$$m_i x \equiv 0(\text{mod n}),$$

where $m_i = p_1.p_2p_3...p_{i-1}.p_{i+1}...p_k$. Whence,

$$x \equiv 0(\text{ mod } p_i),$$

where

$$0 < x < p_1p_2...p_{i-1}p_ip_{i+1}...p_k.$$

Let us denote this ${}^kC_{k-1} = k$ sets of solutions by

$$S_{m_{1,2,...,i-1,i+1,...,k}}, \; i = 1,2,3,4,...,k.$$

For each i, the solution set $S_{m_{1,2,...,i-1,i+1,...,k}}$ is given by

$$x = p_i r, r = 1,2,...,p_1p_2p_3...p_{i-1}.p_{i+1}...p_k - 1.$$

Therefore in total, we shall have

$${}^kC_1 +^k C_2 +^k C_3 + ... +^k C_{k-1} = (2^k - 2)$$

sets of solution of the congruence equation,

$$mx \equiv 0(\text{mod n}).$$

Construction of $(2^k - 2)$- partite graph:

We note that these $2^k - 2$ sets of solution may not be a pairwise disjoint but exhausted the set of all non-zero zero divisors, i.e., union of all such sets of solutions give the set of all non-zero zero divisors. In our next step, we would like to re-structure the $2^k - 2$ sets of solutions in such a way that they are pairwise disjoint and their union is the set of all non-zero zero divisors of the ring Z_n which infact forms a partition of the vertex set of the zero divisor graph $\Gamma(Z_n)$. Now, we shall show that this $2^k - 2$ sets of solution will act as a partition of the vertex set and forms a $(2^k - 2)-$ *partite graph*. These $2^k - 2$ sets are splitted into $k - 1$ number of levels. We shall start with the inner most level. The inner most level corresponds to sets of solutions

$$S_{m_i}, i = 1,2,3,4,5,...,k.$$

Let us denote them by

$$V_i^1 = S_{m_i},$$

where

$$S_{m_i} = \{p_1p_2...p_{i-1}p_{i+1}...p_k, 2p_1p_2...p_{i-1}p_{i+1}...p_k, ..., (p_i - 1)p_1p_2...p_{i-1}p_{i+1}...p_k\}.$$

That is, it is the collection of integers which are multiples of $p_1p_2...p_{i-1}p_{i+1}...p_k$ (mod n). In this way, we shall have

$$V_i^1, i = 1,2,3,4,...,k$$

number of sets of vertices and since $p_i's$ are distinct primes, these sets of vertices are pairwise disjoint. In the next outer level, i.e., $k-2$(level), we consider those sets of solutions for which each solution m

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is a multiples of the primes $p_1, p_2, p_3, ..., p_k$ by removing a pair of primes at a time and it is denoted by $S_{m_{i,j}}$ and constructed the next level of vertices as follows:

$$\begin{aligned} V_t^2 &= S_{m_{i,j}} - S_{m_i} - S_j \\ &= S_{m_{i,j}} - S_{m_i} \cup S_{m_j},\ t = 1,2,3,4,...,^k C_{k-2}, \end{aligned}$$

where '$-$' stands for the difference of sets and repeated use of '$-$' is for cumulative difference in the sense that if $X - A - B$, then we mean that it is $X - (A \cup B)$. Then it can be verified that V_t^{k-2}'s are pair wise disjoint and

$$\left(\bigcup_t V_t^2\right) \cup \left(\bigcup_t V_t^1\right) = \left(\bigcup_i S_{m_i}\right) \cup \left(\bigcup_{i,j} S_{m_{i,j}}\right).$$

Now, the next outer level should be

$$V_t^3, t = 1,2,...,^k C_{k-3}$$

by considering those sets of solutions for which each solution m is a multiples of the primes $p_1, p_2, p_3, ..., p_k$ by removing three primes at a time. Then

$$\begin{aligned} V_t^3 &= S_{m_{i,j,k}} - S_{m_{i,j}} - S_{m_{i,k}} - S_{m_{j,k}} - S_{m_i} - S_{m_j} - S_{m_k} \\ &= S_{m_{i,j,k}} - S_{m_{i,j}} \cup S_{m_{i,k}} \cup S_{m_{j,k}} - S_{m_i} \cup S_{m_j} \cup S_{m_k} \end{aligned}$$

where,

$$t = 1,2,3,4,...,^k C_{k-3}.$$

It can be observed that V_t^{k-3}'s are pairwise disjoint and

$$\left(\bigcup_t V_t^1\right) \cup \left(\bigcup_t V_t^2\right) \cup \left(\bigcup_t V_t^3\right) = \left(\bigcup_{i,j,k} S_{m_{i,j,k}}\right) \cup \left(\bigcup_{i,j} S_{m_{i,j}}\right) \cup \left(\bigcup_i S_{m_i}\right).$$

Continuing these process, the outermost level would be $(k-1)$ level for which corresponding sets are V_t^1, where $t = 1,2,3,4,...,^k C_{k-(k-1)}$. Now, we consider a sets of solutions for which each solution m is a multiple of the primes $p_1, p_2, ..., p_k$ by removing $k-1$ of primes at a time, and construct the next level of vertices as follows:

$$\begin{aligned} V_t^{k-1} &= S_{m_{1,2,...,i-1,i+1,...,k}} - \Sigma S_{m_{1,2,...,i_{r-1},i_{r+1},...,i_{s-1},i_{s+1},...,k}} - ... - S_{m_{i_1}} - S_{m_{i_2}} - ... - S_{m_{i_{k-1}}} \\ &= S_{m_{1,2,...,i-1,i+1,...,k}} - \left(\bigcup S_{m_{1,2,...,i_r-1,i_r+1,...,i_s-1,i_s+1,...,k}}\right) \cup ... \cup \left(\bigcup_l S_{m_{i_l}}\right), \end{aligned}$$

where $t = 1,2,3,...,k$ and

$$\begin{aligned} \left(\bigcup_t V_t^1\right) \cup \left(\bigcup_t V_t^2\right) \cup ... \cup \left(\bigcup_t V_t^{k-1}\right) &= \left(\bigcup_{i_1,i_2,...,i_{k-1}} S_{m_{i_1,i_2,...,i_{k-1}}}\right) \\ &\quad \cup \left(\bigcup_{i'_1,i'_2,...,i'_{k-2}} S_{m_{i'_1,i'_2,...,i'_{k-2}}}\right) \\ &\quad \cup ... \cup \left(\bigcup_i S_{m_i}\right), \end{aligned}$$

where the sum $\Sigma S_{m_{1,2,...,i_{r-1},i_{r+1},...,i_{s-1},i_{s+1},...,k}}$ runs over all combinations by picking $k-2$ distinct elements from $k-1$ elements from $i_1, i_2, i_3 ... i_{k-1}$.

Next we claim that no two vertices in the same set of the above partition of the vertex set are connected by an edge.

From the construction of level sets, it can be observed that for an arbitrary V_t^m, say, we can always find a prime p_i such that $p_i \nmid v$ for all $v \in V_t^m$. For instances, if

$$v \in V_t^1 = S_{m_i},$$

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then $p_i \nmid v$. And if
$$v \in V_t^2 = S_{m_{i,j}} - S_{m_i} - S_{m_j},$$
then $p_i \nmid v$ and $p_j \nmid v$ and so on.

Now, if $v_1, v_2 \in V_i^j$ are connected, then
$$v_1 v_2 = 0(\mod n).$$

It follows that $n | v_1 v_2$. Since $p_i | n$, we have $p_i | v_1 v_2$, which is a contradiction that $p_i \nmid v_1$ and $p_i \nmid v_2$. Therefore, any two vertices are non-adjacent whenever they are from the same level set.

Edge formation:

If $u \in V_i^1, v \in V_j^1, i \neq j$, then all p_l's are divisor of u except p_i and divisor of v except p_j. Since $i \neq j$, it follows that all p_l's divides uv, that is, $uv \equiv 0(\mod n)$. Thus $\{u, v\}$ is an edge of the zero-divisor graph.

Hence, every vertex of an inner most level V_i^1 is adjacent to all vertices of another same level set V_j^1.

Again, every vertex $v \in V_j^1 = S_{m_i}$ must have edge with every vertex of the next outer level set
$$V_t^2 = S_{m_{p,q}} - S_{m_p} - S_{m_q}$$
for which i is different from p and q, and no vertex of V_i^1 can have an edge with any vertex of
$$V_t^2 = S_{m_{p,q}} - S_{m_p} - S_{m_q}$$
for which i is either coincides with p or q.

Similarly, it can be observed that every vertex of inner most level $v \in V_i^1 = S_{m_i}$ must have edge with every vertex of the next outer level set
$$V_t^3 = S_{m_{p,q,r}} - S_{m_{p,q}} - S_{m_{p,r}} - S_{m_{q,r}} - S_{m_p} - S_{m_q} - S_{m_r}.$$
for which i is different from p, q and r, and no vertex of V_i^{k-1} can have an edge with any vertex of V_t^{k-3} for which any one i is either coincides with p or q or r.

Continuing in this process, we get every vertex $v \in V_i^1 = S_{m_p}$ must have an edge with every vertex of the outer most level set
$$v \in V_t^{k-1} = S_{m_{1,2,\ldots,i-1,i+1,\ldots,k}} - \Sigma S_{m_{1,2,\ldots,i_{r-1},i_{r+1},\ldots,i_{s-1},i_{s+1},\ldots,k}} - \ldots - S_{m_{i_1}} - S_{m_{i_2}} - \ldots - S_{m_{i_{k-1}}}.$$
only.

Moreover, every vertex $v \in V_s^2 = S_{m_{i,j}} - S_{m_i} - S_{m_j}$ must have edge with every vertex of an another set of the same level
$$V_t^2 = S_{m_{p,q}} - S_{m_p} - S_{m_q}$$
for which i and j is different from p and q, and no vertex of V_s^2 can have an edge with any vertex of V_t^2 for which any one i or j is either coincides with p or q.

Similarly, every vertex of $v \in V_s^2 = S_{m_{i,j}} - S_{m_i} - S_{m_j}$ must have edge with every vertex of the next outer level set
$$V_t^3 = S_{m_{p,q,r}} - S_{m_{p,q}} - S_{m_{p,r}} - S_{m_{q,r}} - S_{m_p} - S_{m_q} - S_{m_r}.$$
for which i and j is different from p, q and r, and no vertex of V_s^{k-2} can have an edge with any vertex of $V_t^{k-3} = V_t^{k-3}$ for which i or j is either coincides with p or q or r. Continuing in this way, it can be obtained that every vertices of each set of second most inner level have an edge with every vertices of only one set of second outer most level, and no vertex of a set of second inner most level can have an edge to any vertex of any set from the outer most level.

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This process will be continued till the outer most level is reached. In the most outer level, vertices of V_i^{k-1} are non-adjacent to vertices of V_j^{k-1}, for $i \neq j$.

In this way, we obtain a $(2^k - 2)$-partite structure of the non-zero zero divisor graph $\Gamma(Z_n)$.

□

4. Algorithm to Determine $\Gamma(Z_n)$, when n Is Square Free

Step 1:
Find the number of non-zero zero divisors of Z_n, which is given by

$$|V(Z_n)| = p_1 p_2 p_3 .. p_n (\text{say}) - 1 - \phi(p_1 p_2 p_3 ... p_n).$$

Step 2:
Find out the number of partitions, which is given by

$$^nC_1 +^n C_2 +^n C_3 + ... +^n C_{n-1} = 2^k - 2$$

Step 3:
List all the congruence equations in order to get the non-zero zero divisors by using the equation

$$p_1 x \equiv 0(\text{mod n}).$$

$$p_1 p_2 x \equiv 0(\text{mod n}).$$

$$p_1 p_2 p_3 x \equiv 0(\text{mod n}).$$

$$p_1 p_2 p_3 p_4 x \equiv 0(\text{mod n}).$$

$$...........$$

$$...........$$

$$...........$$

$$p_1 p_2 p_3 p_4 ... p_{n-1} x \equiv 0(\text{mod n}).$$

Step 4: Find the incongruent solution for all the congruence equations listed in **Step 3**.
Step 5: Partitioning of the solution set into level sets V_j^i.
Step 6: Formation of edges, i.e., connecting of vertices to get edges.

5. Illustrated Examples

Example 1. *(3-distinct primes) For 3-distinct primes, p_1, p_2 and p_3, if*

$$n = p_1 p_2 p_3,$$

then, Z_n is a ring in which n is square free. For instance, if $p_1 = 2, p_2 = 3, p_3 = 5$, then, using the above algorithm, we can construct a non-zero zero divisor graph of Z_{30} (see Figure 5).

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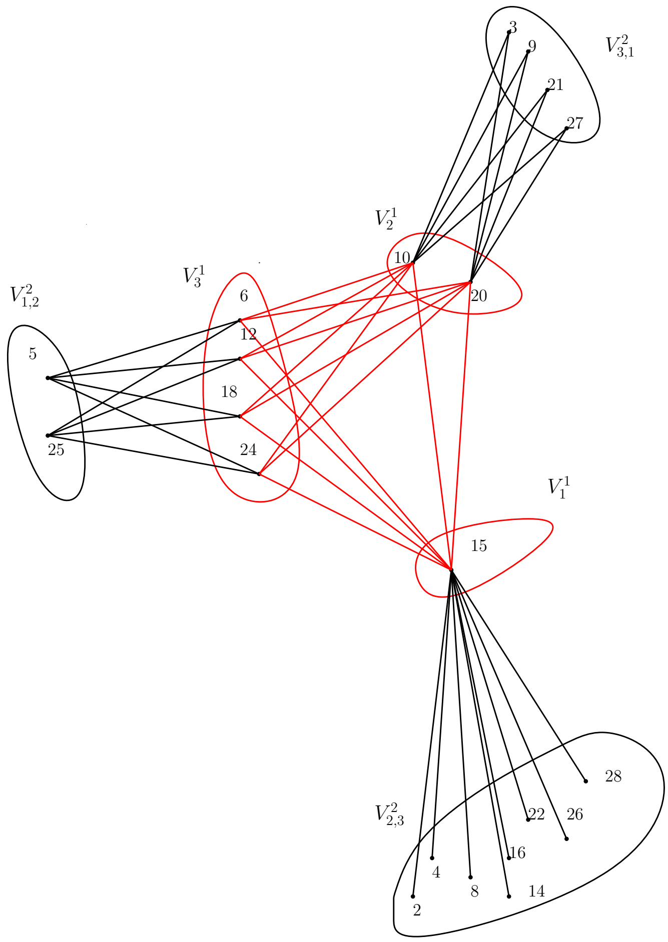

Figure 5. Zero divisor graph of Z_{30}.

The number of non-zero zero divisor elements will be given by

$$\begin{aligned} |V| &= 2.3.5 - 1 - \phi(2.3.5) \\ &= 30 - 1 - \phi(2)\phi(3)\phi(5) \\ &= 29 - 1.2.4 \\ &= 29 - 8 \\ &= 21. \end{aligned}$$

According to step 1 of the algorithm, the number of partitions should be

$$^3C_1 +^3 C_2 = 6.$$

Thus, the graph must be a 6 *- partite graph. Let us denote V be the set of non-zero zero divisor elements of* Z_{30}*, Then*

$$V = \{2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28\}.$$

The vertex set V can be obtained using the algorithm in terms of solution sets using the method described above. The number of incongruent equations will be $2x \equiv 0(\mod 30), 3x \equiv 0(\mod 30)$ *and* $5x \equiv 0(\mod 30)$ *by picking* 1 *member at a time. Similarly* $(2.3)x \equiv 0(\mod 30)$, $(2.5)x \equiv 0(\mod 30)$ *and*$(3.5)x \equiv 0(\mod 30)$

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by picking 2 *member at a time. Solving the incongruent solution of the given congruence equations, we get* 6 *disjoint subsets of the vertices which can be denoted by* $V_1^1, V_2^1, V_3^1, V_{1,2}^2, V_{2,3}^2, V_{3,1}^2$.

$$V_1^1 = 15x$$

$$V_2^1 = 10x$$

$$V_3^1 = 6x$$

$$V_{1,2}^2 = 5x - 10x - 15x$$

$$V_{2,3}^2 = 2x - 6x - 10x$$

$$V_{3,1}^2 = 3x - 6x - 15x,$$

i.e., solutions of the above equations are given by

$$V_1^1 = \{15\}.$$

$$V_2^1 = \{10, 20\}.$$

$$V_3^1 = \{6, 12, 18, 24\}.$$

$$V_{1,2}^2 = \{5, 25\}.$$

$$V_{2,3}^2 = \{2, 4, 8, 14, 16, 22, 26, 28\}.$$

$$V_{3,1}^2 = \{3, 9, 21, 27\}.$$

Now, edges can be formed using step 6. Once the edges are formed, we get a 6*-partite graph Figure 5. It can be verified that*

$$V_1^1 \cup V_2^1 \cup V_3^1 \cup V_{1,2}^2 \cup V_{2,3}^2 \cup V_{1,3}^2 = V.$$

Remark 1. *For the convenience, the zero-divisor graph* $\Gamma(Z_{30})$ *has been represented by single connected edge among two level sets in the sense that if elements of a level set* V_i^k *is connected to elements of another level set* V_j^l*, then the corresponding two level set is connected by a edge. In this way, we can minimize the complexity of drawing the zero divisor graph and it can take the form as shown in (Figure 6). This phenomenon can be thought as merging of vertices and edges.*

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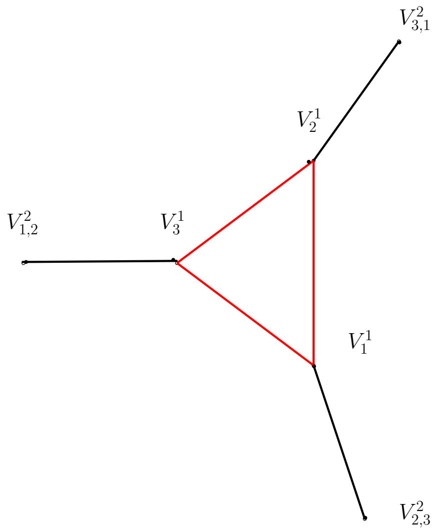

Figure 6. Simplified form of $\Gamma(Z_{30})$ by merging vertices and edges.

Example 2. *(4-distinct primes) Let*

$$n = p_1 p_2 p_3 p_4,$$

$$p_1 = 2, p_2 = 5, p_3 = 7, p_4 = 11.$$

Then, Z_n=Z_{770}. *The total number of non-zero zero divisor elements of* Z_{770} *is given by-*

$$\begin{aligned} |V| &= 2.5.7.11 - 1 - \phi(2.5.7.11) \\ &= 770 - 1 - \phi(2)\phi(5)\phi(7)\phi(11) \\ &= 769 - 1.4.6.10 \\ &= 769 - 240 \\ &= 529 \textit{ elements}. \end{aligned}$$

Here, $K = 4$, *the number of partitions must be*

$$^4C_1 +^4 C_2 +^4 C_3 = 4 + 6 + 4 = 14.$$

The graph must be a $14-$ *partite graph. Now, to list out all the vertices of* $\Gamma(Z_{770})$, *we must find all the incongruent solutions of the congruence equations according to step 3 and 4. Using step 5, we can construct 14 disjoint subsets of the vertices which partitioned the vertices. Finally, using step 6, a* $14-$ *partite graph can be constructed which is depicted in the simplified form (see Figure 7).*

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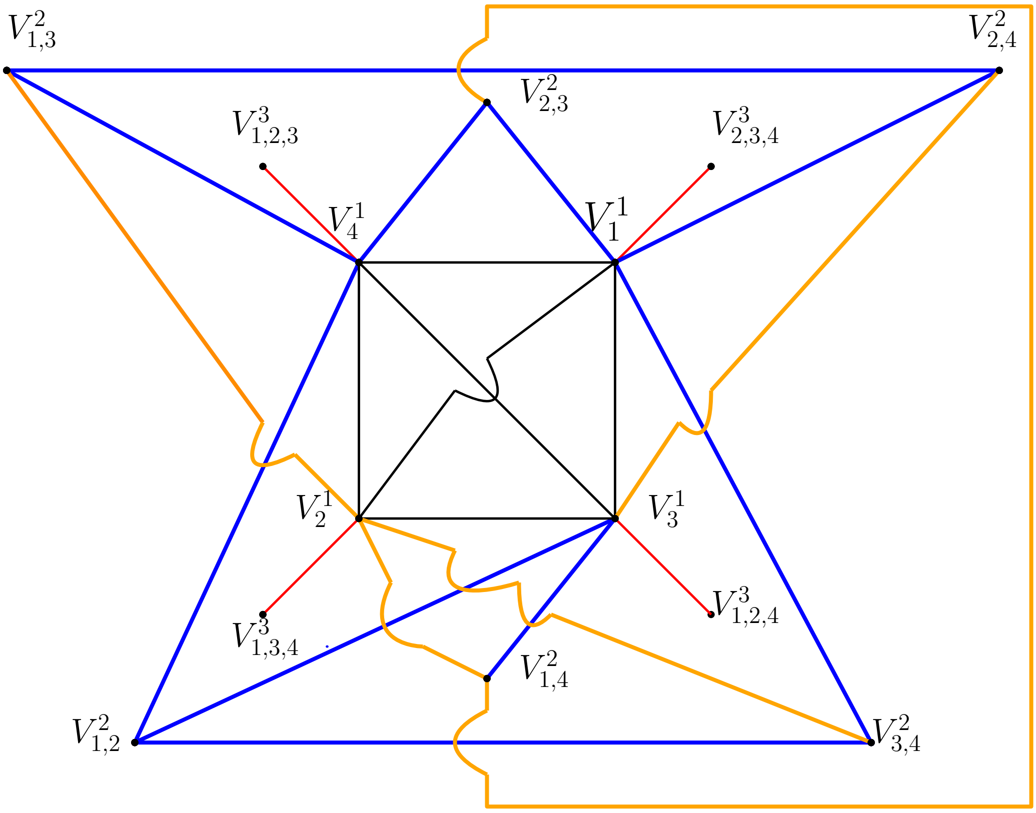

Figure 7. Simplified form of the zero-divisor graph of Z_{770}.

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The partitioning the vertex as level subsets of $\Gamma(Z_{770})$ *are as follows:*

$$V_1^1 = \{385, 770\}.$$

$$V_2^1 = \{154, 308, 462, 610, 770\}.$$

$$V_3^1 = \{110, 220, 330, 440, 550, 660, 770\}.$$

$$V_4^1 = \{70, 140, 210, 280, 350, 420, 490, 560, 630, 700, 770\}.$$

$$V_{1,2}^2 = \{77, 231, 539, 693\}.$$

$$V_{1,3}^2 = \{55, 165, 275, 495, 605, 715\}.$$

$$V_{1,4}^2 = \{35, 105, 175, 245, 315, 455, 525, 595, 665, 735\}.$$

$$V_{2,3}^2 = \{22, 44, 66, 88, 132, 176, 198, 242, 264, 286, 352, 374, 396, 418, 484, 506, 528, 572, 594, 638, 682, 704, 726, 748\}.$$

$$V_{2,4}^2 = \{14, 28, 42, 56, 84, 98, 112, 126, 168, 182, 196, 224, 238, 252, 266, 294, 322, 336, 364, 378, 392, 406, 434, 448, 476, 504, 518, 532, 546, 574, 588, 602, 644, 658, 672, 686, 714, 728, 742, 756\}.$$

$$V_{3,4}^2 = \{10, 20, 30, 40, 50, 60, 80, 90, 100, 120, 130, 150, 160, 170, 180, 190, 200, 230, 240, 250, 260, 270, 290, 300, 310, 320, 340, 360, 370, 380, 390, 400, 410, 430, 450, 460, 470, 480, 500, 510, 520, 530, 540, 570, 580, 590, 600, 610, 620, 640, 650, 670, 680, 690, 710, 720, 730, 740, 750, 760\}.$$

$$V_{1,2,3}^3 = \{11, 33, 99, 121, 143, 187, 209, 253, 297, 319, 341, 363, 407, 429, 451, 473, 517, 561, 583, 627, 649, 671, 737, 759\}.$$

$$V_{1,4,3}^3 = \{5, 15, 25, 45, 65, 75, 85, 95, 115, 125, 135, 145, 155, 185, 195, 205, 215, 225, 235, 255, 265, 285, 295, 305, 325, 335, 345, 355, 365, 375, 395\ 405, 415, 425, 435, 445, 465, 475, 485, 505, 515, 535, 454, 555, 565, 575, 585, 615, 625, 635, 645, 655, 675, 685, 695, 705, 725, 745, 755, 765\}.$$

$$V_{1,2,4}^3 = \{7, 21, 49, 63, 91, 119, 133, 147, 161, 189, 203, 217, 259, 273, 287, 301, 329, 343, 357, 371, 385, 413, 427, 441, 469, 483, 497, 511, 553, 567, 581, 609, 623, 637, 651, 679, 707, 721, 749, 763\}.$$

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$$V^3_{2,3,4} = \{2,4,6,8,12,16,18,24,26,32,34,36,$$
$$38,46,48,52,54,58,62,64,68,72,74,76,78,$$
$$82,86,92,94,96,102,104,106,108,114,116,$$
$$118,122,124,128,134,136,138,142,144,146,$$
$$148,152,156,158,162,164,166,172,174,178,$$
$$184,186,188,194,202,204,206,208,212,214,$$
$$216,218,222,226,228,232,234,236,244,246,$$
$$248,254,256,258,262,268,272,274,276,278,$$
$$282,284,288,292,296,298,302,304,306,312,$$
$$314,316,318,324,326,328,332,334,338,342,$$
$$344,,346,348,354,356,358,362,366,368,372,376,$$
$$382,384,386,388,394,398,402,404,408,412,$$
$$414,416,422,424,426,428,432,436,438,442,$$
$$444,446,452,454,456,458,464,466,468,472,$$
$$474,478,482,486,488,492,494,496,498,502,$$
$$508,512,514,516,522,524,526,534,536,538,$$
$$542,544,548,552,554,556,558,562,568,576,$$
$$578,582,584,586,592,596,598,604,606,608,$$
$$612,614,618,622,624,626,628,632,634,636,$$
$$642,646,648,652,654,656,662,664,666,668,$$
$$674,676,678,684,688,692,694,696,698,702,$$
$$706,708,712,716,718,722,724,732,734,736,$$
$$738,744,746,752,754,758,762,764,766,768\}.$$

Remark 2. *For our convenience, we denote the level sets as*

$$V^r_{\text{suffixes of primes } p_i\text{'s separated by comma}},$$

where r represents the levels and p_i's denote the family of distinct primes whose product appears in the congruence relation which particularly means that $V^1_i = S_{m_i}$ and $V^2_{1,2} = S_{m_{p,q}}$ and it is to be noted that, $S_i = S_{m_i} - \{0\}$. For illustration, V^1_i is a partition of the vertex set corresponding to solution set S_i (a simple component of Z_n) obtained corresponding to the congruence relation $p_i x \equiv 0(\bmod n)$, and $V^2_{i,j}$ is a partition of the vertex set corresponding to solution set $S_i \bigoplus S_j$ (direct sum of two simple components Z_i and Z_j of Z_n) obtained corresponding to the congruence relation $p_i p_j x \equiv 0(\bmod n)$ in level 2. This notation conveniently allow us find the edges in such a way that if any suffixes are missing in a particular level set, then vertices in that set must be connected to every vertices of another level set containing only the missing suffixes. In this way, the graph edges can be drawn with the help of Table 1 and 2. A similar kind of table can be established in determining the edges and vertices of non-zero zero-divisors graph of a semi-simple ring R as a semi-simple ring is a direct sum of its simple components.

Table 1. Formation of partite graphs of Z_{30}.

Equations	Solution sets	Partition sets	Edges formations
$2x \equiv 0(\text{mod } 30)$	$S_1 = S_{m_1} \cup \{0\} = \{0,15\}$	$V^1_1 = S_1 - \{0\}$	$V^2_{2,3}, V^1_2, V^1_3$
$3x \equiv 0(\text{ mod } 30)$	$S_2 = S_{m_2} \cup \{0\} = \{0,10,20\}$	$V^1_2 = S_2 - \{0\}$	$V^1_3, V^1_1, V^2_{3,1}$
$5x \equiv 0(\text{ mod } 30)$	$S_3 = S_{m_2} \cup \{0,6,12,18,24\}$	$V^1_3 = S_3 - \{0\}$	$V^1_2, V^1_1, V^2_{1,2}$
$6x \equiv 0(\text{ mod } 30)$	$S_1 \bigoplus S_2$	$V^2_{1,2} = S_1 \bigoplus S_2 - S_1 \cup S_2$	V^1_3
$10x \equiv 0(\text{ mod } 30)$	$S_1 \bigoplus S_3$	$V^2_{2,3} = S_1 \bigoplus S_3 - S_1 \cup S_3$	V^1_1
$15x \equiv 0(\text{ mod } 30)$	$S_2 \bigoplus S_3$	$V^2_{3,1} = S_2 \bigoplus S_3 - S_2 \cup S_3$	V^1_2

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Table 2. Formation of partite graphs of Z_{770}.

Equations	Solution sets	Partition sets	Edges formations
$2x \equiv 0(\mathrm{mod}\ 770)$	$S_1 = S_{m_1} \cup \{0\}$	$V_1^1 = S_{m_1} = S_1 - \{0\}$	$V_2^1, V_3^1, V_4^1, V_{2,3}^2, V_{3,4}^2, V_{2,4}^2, V_{2,3,4}^3$
$5x \equiv 0(\mathrm{mod}\ 770)$	$S_2 = S_{m_2} \cup \{0\}$	$V_2^1 = S_{m_2} = S_2 - \{0\}$	$V_1^1, V_3^1, V_4^1, V_{1,3}^2, V_{3,4}^2, V_{1,4}^2, V_{1,3,4}^3$
$7x \equiv 0(\mathrm{mod}\ 770)$	$S_3 = S_{m_3} \cup \{0\}$	$V_3^1 = S_{m_3} = S_3 - \{0\}$	$V_1^1, V_2^1, V_4^1, V_{1,2}^2, V_{2,4}^2, V_{2,3}^2, V_{1,2,4}^3$
$11x \equiv 0(\mathrm{mod}\ 770)$	$S_4 = S_{m_4} \cup \{0\}$	$V_4^1 = S_{m_4} = S_4 - \{0\}$	$V_1^1, V_2^1, V_3^1, V_{2,3}^2, V_{1,2}^2, V_{1,3}^2, V_{1,2,3}^3$
$2.5x \equiv 0(\mathrm{mod}\ 770)$	$S_1 \bigoplus S_2 = S_{m_{1,2}} \cup \{0\}$	$V_{1,2}^2 = S_1 \bigoplus S_2 - S_1 \cup S_2$	$V_3^1, V_4^1, V_{3,4}^2$
$2.7x \equiv 0(\mathrm{mod}\ 770)$	$S_1 \bigoplus S_3$	$V_{1,3}^2 = S_1 \bigoplus S_3 - S_1 \cup S_3$	$V_4^1, V_2^1, V_{2,4}^2$
$2.11x \equiv 0(\mathrm{mod}\ 770)$	$S_1 \bigoplus S_4$	$V_{1,4}^2 = S_1 \bigoplus S_4 - S_1 \cup S_4$	$V_2^1, V_3^1, V_{2,3}^2$
$5.7x \equiv 0(\mathrm{mod}\ 770)$	$S_2 \bigoplus S_3$	$V_{2,3}^2 = S_2 \bigoplus S_3 - S_2 \cup S_3$	$V_1^1, V_4^1, V_{1,4}^2$
$5.11x \equiv 0(\mathrm{mod}\ 770)$	$S_2 \bigoplus S_4$	$V_{2,4}^2 = S_2 \bigoplus S_4 - S_2 \cup S_4$	$V_1^1, V_3^1, V_{1,3}^2$
$7.11x \equiv 0(\mathrm{mod}\ 770)$	$S_3 \bigoplus S_4$	$V_{3,4}^2 = S_3 \bigoplus S_4 - S_3 \cup S_4$	$V_1^1, V_2^1, V_{1,2}^2$
$2.5.7x \equiv 0(\mathrm{mod}\ 770)$	$S_1 \bigoplus S_2 \bigoplus S_3$	$V_{1,2,3}^3 = S_1 \bigoplus S_2 \bigoplus S_3 - [\{S_1 \bigoplus S_2\} \cup \{S_1 \bigoplus S_3\} \cup \{S_2 \bigoplus S_3\}]$	V_4^1
$2.7.11x \equiv 0(\mathrm{mod}\ 770)$	$S_1 \bigoplus S_3 \bigoplus S_4$	$V_{1,3,4}^3 = S_1 \bigoplus S_3 \bigoplus S_4 - [\{S_1 \bigoplus S_3\} \cup \{S_1 \bigoplus S_4\} \cup \{S_3 \bigoplus S_4\}]$	V_2^1
$5.7.11x \equiv 0(\mathrm{mod}\ 770)$	$S_2 \bigoplus S_3 \bigoplus S_4$	$V_{2,3,4}^3 = S_2 \bigoplus S_3 \bigoplus S_4 - [\{S_2 \bigoplus S_3\} \cup \{S_2 \bigoplus S_4\} \cup \{S_3 \bigoplus S_4\}]$	V_1^1
$5.11.2x \equiv 0(\mathrm{mod}\ 770)$	$S_1 \bigoplus S_2 \bigoplus S_4$	$V_{2,4,1}^3 = S_1 \bigoplus S_2 \bigoplus S_4 - [\{S_1 \bigoplus S_2\} \cup \{S_1 \bigoplus S_4\} \cup \{S_2 \bigoplus S_4\}]$	V_3^1

Theorem 3. *[12] For a prime p, $\Gamma(Z_{p^2})$ is a complete graph.*

Proof. Let

$$V = \{n_1, n_2, n_3 ... n_k\}$$

be the set of vertices of non-zero zero divisors of (Z_{p^2}). Then, for each i, $(n_i, p^2) \neq 1$. Since 0 is always a zero divisor elements of Z_{p^2}. If k is arbitrary elements V, then $(k, p^2) \neq 1$. Therefore,

Example 3. *The zero divisor graphs of $z_{2.2}$, $z_{3.3}$, $z_{5.5}$ and $z_{7.7}$.*

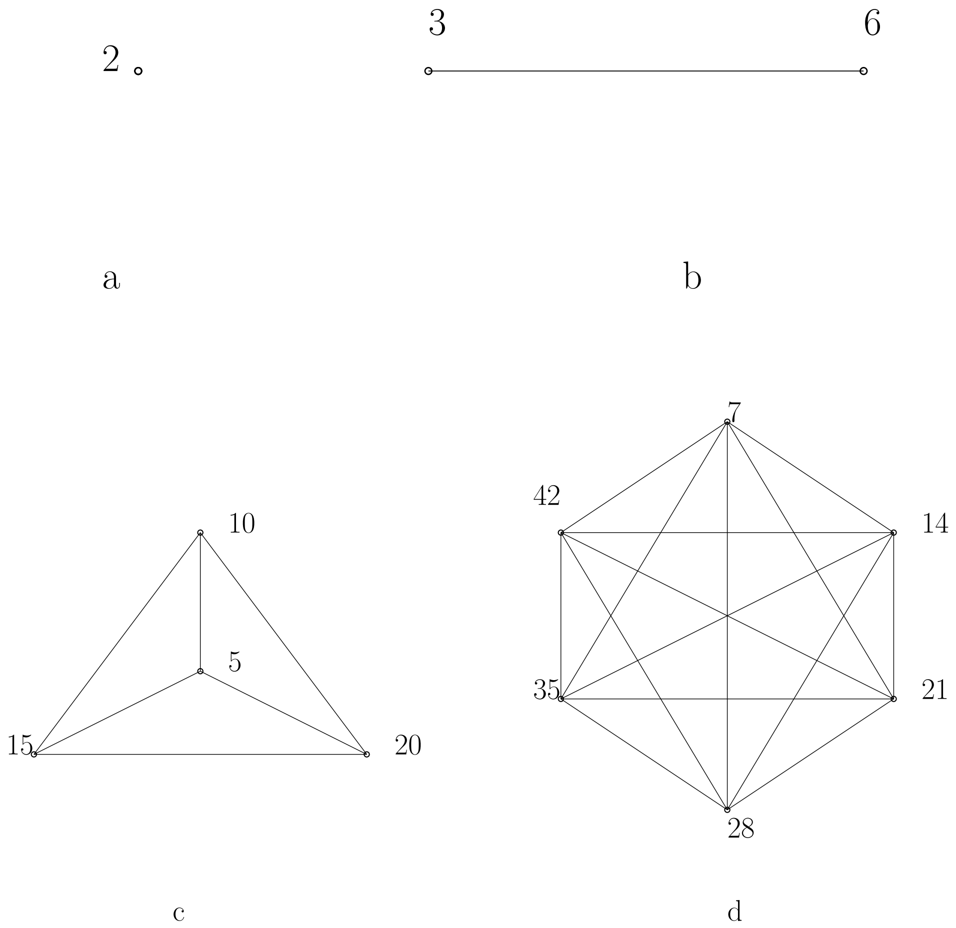

Figure 8. Zero divisor graphs (a). $\Gamma(Z_4)$ (b). $\Gamma(Z_9)$ (c). $\Gamma(Z_{25})$ and (d). $\Gamma(Z_{49})$.

$$\begin{aligned} V &= p^2 - 1 - \phi(p^2) \\ &= p^2 - 1 - p^{2-1}(p-1) \\ &= p^2 - 1 - p(p-1) \\ &= p^2 - 1 - p^2 + p \\ &= (p-1) \text{ elements.} \end{aligned}$$

That is, total number of vertices in $\Gamma(Z_{p^2})$ is $p-1$. Now, in order to list all the vertices of $\Gamma(Z_{p^2})$, we shall find all the incongruent solutions of the congruent equation

$$kx \equiv 0(\bmod p^2), \tag{6}$$

where k is a non-zero zero divisor of Z_{p^2}.

Now, the congruence linear equation has solution as

$$gcd(kx, p^2) \mid 0.$$

Since k is element of Z_{p^2}, we have

$$0 < k < p^2$$

From equation (6), we get,

$$p^2 \mid (kx - 0), \text{ It follows that}$$

$$p \mid x \text{ or } p \mid k.$$

If $p \mid x$, we must get $k < p$. Since, $|V| = p-1$, the zero divisors are $p, 2p, 3p, ..., (p-1)p$. Since every vertex is a multiple of p, the product of any two vertices is divisible by p^2. Hence each vertex is connected to every vertex of $\Gamma(Z_{p^2})$. Therefore $\Gamma(Z_{p^2})$. is a complete graph K_{p-1}, where p is a prime number. □

Remark 3. *It is obtained that $\Gamma(Z_n)$ when n is square-free is a partite graph. It can be observed that Zpq is a complete bipartite graph, while Z_{p^2} is a complete graph. On the other hand, the semisimple ring $\Gamma(Z_p \times Z_p)$ is not a complete graph. In particular, it is complete bi- partite graph. In fact, $\Gamma(Z_p \times Z_p)$ is a Turán graph. This observation leads us to a conjecture that for any finite semisimple ring, the zero divisor graph of them is a partite graph and some of it's subgraphs are Turán graph. We will take it as a sequel to our current study.*

6. Conclusion

The zero-divisor graph $\Gamma(Z_n)$ of Z_n for which n is linearly factored are investigated and it is shown that $\Gamma(Z_n)$ of the ring Z_n is a $r = (2^k - 2)$-partite graph, when $n = p_1 p_2 ... p_k$ ($p_1, p_2, ..., p_k$ are k distinct primes). We note that Z_n, for $n = p_1 p_2 ... p_k$, ($p_1, p_2, ..., p_k$ distinct primes) are semi-simple ring. We believe the results obtained can be used as tools to investigate the nature and properties of the zero divisor graphs of semi-simple rings and modules. It is to be noted that the results obtained are solely applicable to the ring Z_n, but may not be applicable to an arbitrary ring of order n. For example, the ring Z_4 and $Z_2 \times Z_2$ are both of order $4 = 2^2$, but $\Gamma(Z_4)$ and $\Gamma(Z_2 \times Z_2)$ are two different graphs.

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