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Convergence of Implicit Iterative Processes for Semigroups of Nonlinear Operators Acting in Regular Modular Spaces

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Abstract: Let C be subset of a vector space, and consider a semigroup of nonlinear mappings $T_t : C \to C$, where $t \in [0, +\infty)$. The common fixed points of this semigroup can be interpreted as stationary points of a dynamic system defined by the semigroup, meaning they remain unchanged during the transformation T_t at any given time t. This paper focuses on semigroups of ρ-nonexpansive mappings in an abstract modular space X_ρ, where ρ is a regular convex modular. By employing recent results on the existence of such stationary points, we demonstrate that, under specific conditions, the sequence $\{x_k\}$ generated by the implicit iterative process

$$x_{k+1} = c_k T_{t_{k+1}}(x_{k+1}) + (1 - c_k)x_k$$

is ρ-convergent to a common fixed point of the semigroup. Our findings extend existing convergence results for semigroups of operators from Banach spaces and modular function spaces to a broader class of regular modular spaces.

Keywords: common fixed point; modular space; semigroup of nonlinear operator; implicit iterative process; convergence of fixed point approximation process

MSC: 47H09; 47H10; 47H20

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1. Introduction

Let C be subset of a vector space X. We consider a semigroup of nonlinear mappings $T_t : C \to C$ such that $T_0(x) = x$ and $T_{s+t}(x) = T_s(T_t(x))$, with the parameter t ranges over $[0, +\infty)$. Thus, the parameter t can be interpreted as representing continuous time. The common fixed points of this semigroup can be viewed as the stationary points of the system, implying that they remain invariant during the state space transformation T_t at any given time t. Given that the state space X may be infinite-dimensional, it is reasonable to use these results to both deterministic and stochastic dynamical systems. In this context, the search for algorithms capable of constructing common fixed points for such semigroups is closely related to the challenge of solving stochastic evolution equations, as discussed in references [1–3].

According to the Browder Theorem [4], if X is a uniformly convex Banach space, C is nonempty, closed, bounded, convex subset of X, then every nonexpansive mapping $T : C \to C$ has at least one fixed point. This leads to a pressing question: how can we construct such a fixed point? One effective method for fixed point construction is based on the observation that for any number $0 < c < 1$ and any given point $x_0 \in C$, the equation $x = cT(x) + (1 - c)x_0$ has a unique solution $x_c \in C$, as guaranteed by the Banach Contraction Principle [5]. This solution can be obtained as the strong limit of the corresponding Picard iterates. Furthermore, Browder [6] proved that as $c \to 1$, x_c converges strongly to a fixed point of T in a Hilbert space. This result was generalized by Reich [7] to the case of uniformly smooth and uniformly convex Banach spaces. Kozlowski [8] extended this fixed point construction method, often known as an implicit iterative process, to achieve strong convergence to a common fixed point of a semigroup of nonexpansive operators in a uniformly convex Banach space. Additionally, this extension addresses the weak convergence case in a Banach space that is both uniformly convex and uniformly smooth [9].

Since the pioneering works [10–12] it has been widely acknowledged that the modular equivalents of norm concepts frequently arise in fixed point theory and approximation theory, due to their practicality in various applications. Furthermore, it is well established that modular techniques yield results that are not attainable within the constraints of normed spaces. Consequently, fixed point theory in modular function spaces has emerged as a vibrant area of research, as evidenced by [13] and its extensive references, along with a plethora of studies published thereafter. In relation to the existence and construction of common fixed points for semigroups of nonlinear operators operating in modular function spaces, several pertinent works can be highlighted [14–18]. Notably, similarities can be observed between these fixed point results and those derived for semigroups in Banach spaces. However, given that not all Banach spaces qualify as modular function spaces, there is a need for a framework that encompasses both normed and modular function spaces. To address this, we adopt the framework of regular modular spaces recently introduced in [19], with key assumptions limited to the convexity of the modular, the closedness of all modular balls, and the modular completeness of the space. These conditions are satisfied in Banach spaces, where norms are exemplary instances of regular modulars, as well as in modular function spaces that posses the Fatou property. It is important to note that the class of regular modular spaces also includes spaces that are neither Banach spaces nor modular function spaces, as illustrated in our Example 1, see also [20] (Example 2.1). This latter paper presents also an existence theorem that plays a crucial role in our current work by ensuring the existence of common fixed points for the specified semigroups of operators. This existence result is referenced in our paper as Theorem 2.

The paper is organized as follows: Section 1 offers a brief introduction to the theory of regular modular spaces. Section 2, titled "Results", is divided into three subsections. The first subsection presents essential auxiliary results of a technical nature, followed by the second subsection, which discusses a specific version of the Banach Contraction Principle necessary for the main fixed point theorems presented in the final subsection. Section 3 reviews the paper's findings, explores their implications within a broader historical context, and highlights potential directions for future research.

2. Preliminaries

In this paper, we adopt the terminology and notation from references [19,20]. Throughout, X will denote a nontrivial real vector space.

Definition 1. *[21] A functional $\rho : X \to [0, \infty]$ is called a convex modular if*

1. $\rho(x) = 0$ *if and only if* $x = 0$
2. $\rho(-x) = \rho(x)$
3. $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ *for any* $x, y \in X$*, and* $\alpha, \beta \geq 0$ *with* $\alpha + \beta = 1$

The vector space $X_\rho = \{x \in X : \rho(\lambda x) \to 0,\ as\ \lambda \to 0\}$ *is called a modular space.*

The concepts outlined in Definition 2 below, introduced in [19] within the framework of regular modular spaces, parallel the equivalent definitions established for modulated topological vector spaces [22] and modular function spaces in [10,13], as well as in other related literature.

Definition 2. *[19] (Def. 2.2) Let ρ be a modular defined on a vector spaces X.*

1. *We say that* $\{x_n\}$*, a sequence of elements of* X_ρ *is* ρ*-convergent to* x *and write* $x_n \xrightarrow{\rho} x$ *if* $\rho(x_n - x) \to 0$.
2. *A sequence* $\{x_n\}$ *where* $x_n \in X_\rho$ *is called* ρ*-Cauchy if* $\rho(x_n - x_m) \to 0$ *as* $n, m \to \infty$.
3. X_ρ *is called* ρ*-complete if every* ρ*-Cauchy is* ρ*-convergent to an* $x \in X_\rho$.
4. *A set* $B \subset X_\rho$ *is called* ρ*-closed if for any sequence of* $x_n \in B$*, the convergence* $x_n \xrightarrow{\rho} x$ *implies that* x *belongs to* B.

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5. *A set $B \subset X_\rho$ is called ρ-bounded if its ρ-diameter $diam_\rho(B) = \sup\{\rho(x-y) : x \in B, y \in B\}$ is finite.*
6. *A set $K \subset X_\rho$ is called ρ-compact if for any sequence $\{x_n\}$ in K, there exists a subsequence $\{x_{n_k}\}$ and an $x \in K$ such that $\rho(x_{n_k} - x) \to 0$.*
7. *Let $x \in X_\rho$ and $C \subset X_\rho$. The ρ-distance between x and C is defined as $d_\rho(x, C) = inf\{\rho(x-y) : y \in C\}$.*
8. *A ρ-ball $B_\rho(x,r)$ is defined by $B_\rho(x,r) = \{y \in X_\rho : \rho(x-y) \le r\}$.*

The concept of ρ-convergence is widely utilized in fixed point theory and approximation theory within modular spaces. It is evident that if the ρ-limit of a sequence in a modular space exists, it is uniquely defined. Furthermore, every subsequence of a ρ-convergent sequence also ρ-converges to the same limit. Additionally, it follows that if $x_n \xrightarrow{\rho} x$, then $x_n - y \xrightarrow{\rho} x - y$. It can also be shown that if $x_n \xrightarrow{\rho} x$, $y_n \xrightarrow{\rho} y$, and $x_n - y_n \xrightarrow{\rho} 0$, then $x = y$ (see [20] (Prop. 2.1)). Moreover, similar to the case in metric spaces, ρ-compactness of a set $C \subset X$ implies that C is ρ-closed.

However, it is important to exercise caution, as some of the standard properties of convergence in topological vector spaces do not directly extend to the case of ρ-convergence. For example, $x_n \xrightarrow{\rho} x$ does not generally imply thatl $\lambda x_n \xrightarrow{\rho} \lambda x$ for $\lambda > 1$. Additionally, the ρ-compactness of a set does not necessarily ensure its ρ-boundedness. Moreover, ρ-balls are not always ρ-closed. However, in many interesting cases these ρ-balls are indeed ρ-closed, which highlights the significance of regular modular spaces, where ρ-balls are guaranteed to be ρ-closed.

Definition 3. *[19] (Def. 2.3) A convex modular is called BC-regular if every ρ-ball $B(x,r)$, where $x \in X_\rho$, and $r > 0$, is ρ-closed. In this context, we also refer to the modular space X_ρ as possessing property (BC).*

Definition 4. *[19] (Def. 2.4) A modular space X_ρ is called a regular modular space if ρ is a convex, (BC)-regular modular, and X_ρ is ρ-complete.*

The following result offers an important characterization of BC-regularity.

Proposition 1. *[20] (Prop. 2.2) The following two conditions are equivalent*

1. *ρ is BC-regular;*
2. *$\rho(x) \le \liminf_{n\to\infty} \rho(x_n)$ provided $x_n \xrightarrow{\rho} x$.*

Modulars that satisfy condition (2) of Proposition 1 are commonly referred to as ρ-lower semicontinuous, or equivalently as possessing the Fatou property. In this paper, we will always assume the regularity of modular spaces. The class of regular modular spaces includes all real Banach spaces (where ρ denotes a norm) and all modular function spaces that possess the Fatou property, such as Lebesgue spaces, Orlicz spaces, Musielak-Orlicz spaces, variable exponent Lebesgue spaces, and, more generally, modulated LTI-spaces, [22].

It is important to note that every regular modular space can be endowed with a norm, often referred to as the Luxemburg norm, which is defined by the following formula

$$\|x\|_\rho = \inf\{\alpha > 0 : \rho\left(\frac{x}{\alpha}\right) \le 1\}.$$

It is straightforward to demonstrate that convergence of a sequence in this norm implies its ρ-convergence; however, the reverse is generally not the case. This norm is not directly defined and can often be difficult to compute, whereas modulars are typically represented by explicit formulas, simplifying calculations. Moreover, it has been established in [11] that while nonexpansiveness with respect to this norm guarantees nonexpansiveness in the modular sense, there exist mappings that are nonexpansive in the modular sense but not with respect to the corresponding Luxemburg norm. Due to these and similar considerations, this paper adopts the standard approach in fixed point theory

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within modular spaces by expressing all conditions imposed on operators—such as different forms of nonexpansiveness or uniform convexity—exclusively in modular terms.

It is noteworthy that some regular modular spaces are neither normed spaces nor modular function spaces. Example 1 below describing the concept of φ-variation, which was introduced in [23] as a generalization of the classical quadratic variation established by Wiener a century ago [24], illustrates this point. Convergence in φ-variation has found numerous applications.

Example 1. *Let $\varphi : [0, \infty) \to [0, \infty)$ be a convex function such that $\varphi(t) = 0$ if and only if $t = 0$. Let X be a space of all real-valued functions defined in the interval $[a, b]$ and vanishing at $t = a$. Musielak and Orlicz introduced in [23] a notion of a φ-variation of a function $x \in X$ defined by the following formula*

$$\rho(x) = \sup_{\Pi} \sum_{i=1}^{\infty} \varphi(|x(t_i) - x(t_{t-1})|),$$

where the supremum is taken over all partitions $\Pi : a = t_0 < t_1 < \ldots < t_m = b$ of the interval $[a, b]$. It is easy to see that ρ is a convex modular on X and that the value of $\rho(x)$ may be infinite. Using results of [23] it is straightforward to demonstrate that X_ρ is a regular modular space, see also [21]. The associated Luxemburg norm leads to a convergence that is equivalent to ρ-convergence only under certain additional assumptions (see [23]). The space X_ρ is not a modular function space either because φ-variation is not monotone.

The literature presents multiple non-equivalent definitions of modular uniform convexity (see, for example, [13,25,33]). In the context of regular modular spaces, the strongest version - often referred to as the $(UUC1)$ property - is commonly employed because it is equivalent to the standard definition of uniform convexity in Banach spaces. In this paper, we adopt this perspective, as outlined in the following definition.

Definition 5. *[19] (Def. 5.1) Let X_ρ be a regular modular space. Let $r > 0, \varepsilon > 0$. Set*

$$D_1(r, \varepsilon) = \{(x, y) : x, y \in X_\rho, \rho(x) \le r, \rho(y) \le r, \rho(x - y) \ge \varepsilon r\}. \tag{1}$$

Let

$$\delta_1(r, \varepsilon) = \inf \left\{1 - \frac{1}{r}\, \rho\left(\frac{x + y}{2}\right) : (x, y) \in D_1(r, \varepsilon)\right\}, if D_1(r, \varepsilon) \ne \emptyset, \tag{2}$$

$\delta_1(r, \varepsilon) = 1$ if $D_1(r, \varepsilon) = \emptyset$. We say that X_ρ is uniformly convex if, for every $s \ge 0, \varepsilon > 0$, there exists

$$\eta_1(s, \varepsilon) > 0,$$

depending only on s and ε such that

$$\delta_1(r, \varepsilon) > \eta_1(s, \varepsilon) > 0 \text{ for } r > s. \tag{3}$$

The significance of the aforementioned definition of modular uniform convexity is exemplified by the following example drawn from the theory of Orlicz spaces.

Example 2. *It is known that in Orlicz spaces, the Luxemburg norm is uniformly convex if and only φ is uniformly convex and Δ_2 property holds; this result can be traced to [26–29]. Furthermore, it is recognized that, under appropriate conditions, modular uniform convexity in Orlicz spaces is equivalent to the very convexity of the Orlicz function [30,31]. Note that a function φ is termed very convex if, for every $\varepsilon > 0$ and any $x_0 > 0$, there exists a $\delta > 0$ such that*

$$\varphi\left(\frac{1}{2}(x - y)\right) \ge \frac{\varepsilon}{2}\left(\varphi(x) + \varphi(y)\right) \ge \varepsilon\varphi(x_0),$$

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implies

$$\varphi\left(\frac{1}{2}(x+y)\right) \leq \frac{1}{2}(1-\delta)\left(\varphi(x)+\varphi(y)\right).$$

Typical examples of Orlicz functions that do not satisfy the Δ_2 *condition but are very convex are:* $\varphi_1(t) = e^{|t|} - |t| - 1$ *and* $\varphi_2(t) = e^{t^2} - 1$, *[27,32]. Therefore, these are the examples of regular modular spaces that are not uniformly convex in the Luxemburg norm sense and hence the classical Banach space fixed point theorems cannot be easily applied. However, these spaces are uniformly convex in the modular sense, and respective modular fixed point results can be applied.*

The following elementary result will be utilized in our main fixed point result.

Lemma 1. *[34] Let* $\{t_n\}_{n\in\mathbb{N}}$ *be a sequence of real numbers satisfying*

$$\lim_{n\to\infty}(t_{n+1}-t_n) = 0. \tag{4}$$

Let $\tau \in \mathbb{R}$ *be such that*

$$\liminf_{n\to\infty} t_n \leq \tau \leq \limsup_{n\to\infty} t_n. \tag{5}$$

Then τ *is a cluster point of the sequence* $\{t_n\}_{n\in\mathbb{N}}$.

To conclude this preliminary section, we will define ρ-Lipschitzian mappings, ρ-contractions, and ρ-nonexpansive mappings in the framework of regular modular spaces.

Definition 6. *Let* X_ρ *be a regular modular space and let* $C \subset X_\rho$ *be convex, nonempty,* ρ*-closed and* ρ*-bounded. A mapping* $T : C \to C$ *is called*

(i) ρ*-Lipschitzian if there exists* $\alpha > 0$ *such that*

$$\rho(T(x)-T(y)) \leq \alpha\rho(x-y) \; for \; any \; x, y \in C.$$

(ii) a ρ*-contraction if it is* ρ*-Lipschitzian with* $alpha < 1$.
(iii) ρ*-nonexpansive if it is* ρ*-Lipschitzian with* $\alpha = 1$.

An element $x \in C$ *is called a fixed point of* T *whenever* $T(x) = x$*. The set of fixed points of* T *will be denoted by* $F(T)$.

3. Results

3.1. Auxiliary results

Given $t \in (0,1)$, we define

$$\delta_1^t(r,\varepsilon) = \inf\left\{1 - \frac{1}{r}\rho\Big(tx+(1-t)y\Big) : (x,y) \in D_1(r,\varepsilon)\right\}, if D_1(r,\varepsilon) \neq \emptyset, \tag{6}$$

where $D_1(r,\varepsilon)$ is the set described by (1).

Lemma 2. X_ρ *is a uniformly convex regular modular space if and only if for every* $t \in (0,1)$, $s > 0$, $\varepsilon > 0$ *there exists* $\eta_1^t(s,\varepsilon) > 0$ *depending only on* t, s *and* ε *such that*

$$\delta_1^t(r,\varepsilon) > \eta_1^t(s,\varepsilon) > 0 \; for \; any \; r > s. \tag{7}$$

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Proof. Assume that (7) holds for all $t \in (0,1)$. The uniform convexity of X_ρ will be established by (7) with $t = \frac{1}{2}$ in conjunction with Definition 5. To prove the converse, assume that X_ρ is a uniformly convex and for a given t use (3) with z and w such that $tx + (1-t)y = \dfrac{z+w}{2}$ and $z - w = x - y$. □

The following property of modular uniform convexity will be crucial for proving our result on approximate sequences, as stated in Lemma 5.

Lemma 3. *Let X_ρ be uniformly convex. Let $\{t_k\}_{k\in\mathbb{N}}$ be a sequence of real numbers from $(0,1)$, which is bounded away from 0 and 1, that is, there exist $0 < a < b < 1$ such that $a \le t_k \le b$ for all natural k. Assume $u_k, v_k \in X_\rho$ for every $k \in \mathbb{N}$. If there exists $R > 0$ such that*

$$\limsup_{k\to\infty} \rho(u_k) \le R, \limsup_{k\to\infty} \rho(v_k) \le R, \tag{8}$$

and

$$\lim_{k\to\infty} \rho\Big(t_k u_k + (1-t_k)v_k\Big) = R, \tag{9}$$

then

$$\lim_{k\to\infty} \rho(u_k - v_k) \to 0. \tag{10}$$

Proof. Assume, for the sake of contradiction, that (10) does not hold. Fix an arbitrary $0 < \gamma < 1$. We may assume, passing to a subsequence if necessary, that $\lim_{k\to\infty} t_k = t_0$ for some number $t_0 \in [a,b]$, and that for every natural number k,

$$\rho(u_k) \le R,\ \rho(v_k) \le R, \tag{11}$$

and also that there exists an $\varepsilon > 0$ for which

$$\rho(u_k - v_k) \ge (R+1)\varepsilon \tag{12}$$

holds for every k. By Lemma 2 there exists $\eta_1^{t_0}(R,\varepsilon) > 0$ satisfying

$$0 < \eta_1^{t_0}(R,\varepsilon) \le \delta_1^{t_0}(R+\gamma,\varepsilon). \tag{13}$$

Recall that $\eta_1^{t_0}(R,\varepsilon)$ depends only on R and ε, and, importantly, it is independent of the choice of γ. Note that, since $0 < \gamma < 1$, it follows from (12) and (11) that

$$\rho(u_k - v_k) \ge (R+\gamma)\varepsilon, \tag{14}$$

and

$$\rho(u_k) \le R+\gamma,\ \rho(v_k) \le R+\gamma. \tag{15}$$

For every $t \in [0,1]$ and $u, v \in D_1(R+\gamma,\varepsilon)$, define $\varphi_{u,v}(t) = \rho(tu + (1-t)v)$. Note that the function $\varphi_{u,v} : [0,1] \to [0, R+\gamma]$ is a bounded convex function, which implies that

$$\varphi(t) = \sup\{\varphi_{u,v}(t) : u, v \in D_1(R+\gamma,\varepsilon)\}$$

is also a bounded convex real-valued function on $[0,1]$. Consequently $\varphi(t)$ is a continuous function of $t \in [a,b]$. Noting that

$$\delta_1^t(R+\gamma,\varepsilon) = 1 - \frac{1}{r}\varphi_{u,v}(t),$$

we conclude that $\delta_1^t(R+\gamma,\varepsilon)$ is a continuous function of $t \in [a,b]$, which implies that

$$\lim_{k\to\infty} \delta_1^{t_k}(R+\gamma,\varepsilon) = \delta_1^{t_0}(R+\gamma,\varepsilon). \tag{16}$$

By (14), (15), and the definition of $\delta_1^{t_k}$,

$$\delta_1^{t_k}(R+\gamma,\varepsilon) \leq 1 - \frac{1}{R+\gamma}\rho(t_k u_k + (1-t_k)v_k). \tag{17}$$

Taking the limit as $k \to \infty$ in (17) and applying (16) and (9) we obtain

$$\delta_1^{t_0}(R+\gamma,\varepsilon) \leq \frac{\gamma}{R+\gamma}. \tag{18}$$

By combining (18) with (13), we obtain the following inequality

$$0 < \eta_1^{t_0}(R,\varepsilon) \leq \frac{\gamma}{R+\gamma}.$$

Recalling that the value of $\eta_1^{t_0}(R,\varepsilon)$ is independent of the choice of γ, and taking the limit as $\gamma \to 0$, we arrive at a contradiction. Thus, the proof is complete. □

Note that for the special case $t_k = \frac{1}{2}$ for every $k \in \mathbb{N}$, the above result was given in [19, Lemma 5.4]. The parallel results for normed spaces were proved in [35,36], and for modular function spaces in [25,37]. Our proof followed the path of [37] (Lemma 3.2).

3.2. Banach Contraction Principle for ρ-Contractions Acting in Regular Modular Spaces

Variants of the renowned Banach Contraction Principle are widely recognized as essential tools for establishing the existence and uniqueness of solutions identified as fixed points of self-mappings defined on metric, normed, and modular function spaces (see, for example, the discussions in [13,38]). These theorems also offer constructive methods for finding such fixed points. We require a version of this celebrated theorem that is adapted to the context of regular modular spaces. Although the proof is quite classical and follows the approach used for the modular function space result [13] (Theorem 5.1), we present the Principle along with a complete proof due to its significance for our methodology and its broad applicability in addressing fixed point problems in regular modular spaces.

Theorem 1. *Let X_ρ be a regular modular space, and let $C \subset X_\rho$ be nonempty, ρ-closed and ρ-bounded. Let $T : C \to C$ be a ρ-contraction. Then, T has a unique fixed point $\bar{x} \in C$. Moreover, for any $x \in C$, $\rho(T^n(x) - \bar{x}) \to 0$ as $n \to \infty$, where T^n is the n-th iterate of T.*

Proof. By the definition of the ρ-contraction, there exists $\alpha < 1$ such that

$$\rho(T(x) - T(y)) \leq \alpha\, \rho(x-y), \quad \text{for all } x, y \in C.$$

Fix $x_0 \in C$. Since C is ρ-bounded,

$$\delta_\rho(C) = \sup\{\rho(x-y) : \ x, y \in C\} < \infty.$$

Observe that

$$\begin{aligned}
\rho\left(T^{n+k}(x_0) - T^n(x_0)\right) &\leq \alpha\, \rho\left(T^{n+k-1}(x_0) - T^{n-1}(x_0)\right) \\
&\leq \alpha^n\, \rho\left(T^k(x_0) - x_0\right) \\
&\leq \alpha^n\, \delta_\rho(C),
\end{aligned}$$

for any $n, k \geq 1$. Since $\alpha < 1$ and $\delta_\rho(C) < \infty$, we conclude that $\{T^n(x_0)\}$ is ρ-Cauchy. The ρ-completeness of X_ρ implies the existence of $\bar{x} \in X_\rho$ such that $\lim_{n\to\infty} \rho\left(T^n(x_0) - \bar{x}\right) = 0$. Note that $\bar{x} \in C$ because C is ρ-closed. Since

$$\begin{aligned} \rho\left(\frac{\bar{x} - T(\bar{x})}{2}\right) &\leq \rho\left(\bar{x} - T^n(x_0)\right) + \rho\left(T^n(x_0) - T(\bar{x})\right) \\ &\leq \rho\left(\bar{x} - T^n(x_0)\right) + \alpha\, \rho\left(T^{n-1}(x_0) - \bar{x}\right) \to 0, \text{ as } n \to \infty, \end{aligned}$$

it follows that $T(\bar{x}) = \bar{x}$. To prove that $\bar{x}$ is the unique fixed point for T, observe first that if $T(x_1) = x_1$ and $T(x_2) = x_2$, then

$$\rho(x_1 - x_2) = \rho\left(T(x_1) - T(x_2)\right) \leq \alpha\, \rho(x_1 - x_2). \tag{19}$$

Since $\alpha < 1$ and the right hand side is finite due to the ρ-boundedness of C, (19) can only hold if $x_1 = x_2$. □

3.3. Nonexpansive semigroups in regular modular spaces

Let us present the definition of a ρ-nonexpansive semigroup of operators acting within a regular modular space.

Definition 7. *Let X_ρ be a regular modular space and let $C \subset X_\rho$ be convex, nonempty, ρ-closed and ρ-bounded. A one-parameter family $\mathcal{T} = \{T_t : t \geq 0\}$ of mappings from C into itself is called a ρ-nonexpansive semigroup on C if $\mathcal{T}$ satisfies the following conditions:*

1. *$T_0(x) = x$ for $x \in C$;*
2. *$T_{t+s}(x) = T_t(T_s(x))$ for $x \in C$ and $t, s \geq 0$;*
3. *for each $t \geq 0$, T_t is a ρ-nonexpansive mapping, i.e., such that for every $x, y \in C$,*

$$\rho(T_t(x) - T_t(y)) \leq \rho(x - y); \tag{20}$$

4. *for each $x \in C$, the mapping $t \to T_t(x)$ is ρ-continuous at every $t \geq 0$, which means that,*

$$\rho(T_{t_n}(x) - T_t(x)) \to 0,$$

whenever $0 \leq t_n \to t$.

For each $t \geq 0$, let $F(T_t)$ denote the set of its fixed points. Define then the set of all common fixed points set for mappings from $\mathcal{T}$ as the intersection

$$F(\mathcal{T}) = \bigcap_{t \geq 0} F(T_t).$$

Note that ρ-nonexpansive semigroups are a specific instance of asymptotic pointwise nonexpansive semigroups discussed in [20]. It is known that in uniformly convex regular modular spaces every ρ-nonexpansive semigroup has common fixed points, as outlined in the following result.

Theorem 2. *[20] (Theorem 3.4) Let X_ρ be uniformly convex regular modular space. Let $\mathcal{T}$ be a ρ-nonexpansive semigroup on C, where $C \subset X_\rho$ is convex, nonempty, ρ-closed and ρ-bounded. Then there exists a common fixed point for $\mathcal{T}$, that is, $F(\mathcal{T})$ is nonempty. Moreover, $F(\mathcal{T})$ is convex and ρ-closed.*

As mentioned in the Introduction, our objective is to define an iterative process for constructing a common fixed point, which exists according to Theorem 2. We will begin with a precise definition of the implicit iteration process and its related concepts.

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Definition 8. *Given a ρ-nonexpansive semigroup $\mathcal{T} = \{T_t : t \in [0, \infty)\}$ on C, the implicit iteration process $P(C, \mathcal{T}, x_0, \{c_k\}, \{t_k\})$ is defined by the following formula:*

$$\begin{cases} x_0 \in C \\ x_{k+1} = c_k T_{t_{k+1}}(x_{k+1}) + (1 - c_k)x_k, \ for\ k \in \mathbb{N}^0, \end{cases} \tag{21}$$

where $\mathbb{N}^0 = \mathbb{N} \cup \{0\}$, the sequence $\{c_k\}_{k\in\mathbb{N}^0}$ of real numbers from $(0, 1)$ is bounded away from 0 and 1, and $\{t_k\}_{k\in\mathbb{N}}$ is a sequence of positive real numbers. We will also say that the sequence $\{x_k\}_{k\in\mathbb{N}^0}$ is generated by the process $P(C, \mathcal{T}, x_0, \{c_k\}, \{t_k\})$ and write

$$\{x_k\}_{k\in\mathbb{N}^0} = P(C, \mathcal{T}, x_0, \{c_k\}, \{t_k\}). \tag{22}$$

For $k \in \mathbb{N}^0$, $u \in C$, $w \in C$ let us introduce the following notation

$$P_{k,w}(u) = c_k T_{t_{k+1}}(u) + (1 - c_k)w. \tag{23}$$

Since each $P_{k,w}(u) : C \to C$ is a ρ-contraction, it follows from the Banach Contraction Principle (Theorem 1) that each x_{k+1} in (21) is uniquely defined.

The following result describes the general behavior of implicit iterative processes in regular modular spaces.

Lemma 4. *Let X_ρ be a regular modular space and let $C \subset X_\rho$ be convex, nonempty, ρ-closed and ρ-bounded. Let $\mathcal{T}$ be a ρ-nonexpansive semigroup on C, $w \in F(\mathcal{T})$ and $\{x_k\}_{k\in\mathbb{N}^0} = P(C, \mathcal{T}, x_0, \{c_k\}, \{t_k\})$ be an implicit iteration process. Then there exists $r \geq 0$ such that $\lim_{k\to\infty} \rho(x_k - w) = r$.*

Proof. Let us calculate

$$\begin{aligned} \rho(x_{k+1} - w) &= \rho(c_k T_{t_{k+1}}(x_{k+1}) + (1 - c_k)x_k - w) \\ &\leq c_k \rho(T_{t_{k+1}}(x_{k+1}) - T_{t_{k+1}}(w)) + (1 - c_k)\rho(x_k - w) \\ &\leq c_k \rho(x_{k+1} - w) + (1 - c_k)\rho(x_k - w). \end{aligned} \tag{24}$$

From (24) it follows that $\rho(x_{k+1} - w) \leq \rho(x_k - w)$ which implies that there exists $r \geq 0$ with $\lim_{k\to\infty} \rho(x_k - w) = r$. □

Lemma 5 below demonstrates that the sequence generated by the implicit iterative process in a uniformly convex regular modular space represents an approximate common fixed point sequence.

Lemma 5. *Let X_ρ be a regular modular space and let $C \subset X_\rho$ be convex, nonempty, ρ-closed and ρ-bounded. Let $\mathcal{T}$ be a ρ-nonexpansive semigroup on C and $\{x_k\}_{k\in\mathbb{N}^0} = P(C, \mathcal{T}, x_0, \{c_k\}, \{t_k\})$ be an implicit iteration process. Then*

$$\lim_{k\to\infty} \rho\left(\frac{T_{t_k}(x_k) - x_k}{2}\right) = 0. \tag{25}$$

Proof. By Theorem 2 there exists $w \in F(\mathcal{T})$. It follows from Lemma 4 that there exists a real number $r \geq 0$ such that $\lim_{k\to\infty} \rho(x_k - w) = r$. Thus,

$$\limsup_{k\to\infty} \rho\Big(T_{t_k}(x_k) - w\Big) = \limsup_{k\to\infty} \rho\Big(T_{t_k}(x_k) - T_{t_k}(w)\Big) \leq \limsup_{k\to\infty} \rho(x_k - w) = r. \tag{26}$$

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Denote $v_k = x_{k-1} - w$, $u_k = T_{t_k}(x_k) - w$ and observe that $\lim_{k\to\infty} \rho(v_k) = r$. It follows from (26) that $\limsup_{k\to\infty} \rho(u_k) \le r$. By utilizing the definition of the implicit process (21) and the right-hand side of (26), we obtain the following equalities:

$$\begin{aligned}\lim_{k\to\infty} \rho(c_k u_k + (1-c_k)v_k) &= \lim_{k\to\infty} \rho\Big(c_k(T_{t_k}(x_k) - w) + (1-c_k)(x_{k-1} - w)\Big) \\ &= \lim_{k\to\infty} \rho(x_k - w) = r.\end{aligned}$$

It follows from Lemma 3 that

$$\lim_{k\to\infty} \rho\Big(T_{t_k}(x_k) - x_{k-1}\Big) = \lim_{k\to\infty} \rho(u_k - v_k) = 0. \tag{27}$$

Observe that

$$\rho(x_{k+1} - x_k) = \rho\Big(c_k T_{t_{k+1}}(x_{k+1}) + (1-c_k)x_k - x_k\Big) \le c_k \rho\Big(T_{t_{k+1}}(x_{k+1}) - x_k\Big). \tag{28}$$

By (28) in combination with (27),

$$\lim_{k\to\infty} \rho(x_{k+1} - x_k) = 0. \tag{29}$$

By utilizing (27) and (29), we obtain

$$\lim_{k\to\infty} \rho\left(\frac{T_{t_k}(x_k) - x_k}{2}\right) \le \frac{1}{2} \lim_{k\to\infty} \rho\Big(T_{t_k}(x_k) - x_{k-1}\Big) + \frac{1}{2} \lim_{k\to\infty} \rho(x_{k-1} - x_k) = 0,$$

which completes the proof. □

The following theorem is the main result of this paper.

Theorem 3. *Let X_ρ be a uniformly convex regular modular space, and let $C \subset X_\rho$ be convex, nonempty, ρ-compact and ρ-bounded. Assume that $\mathcal{T}$ is a ρ-nonexpansive semigroup on C. Let $\{x_k\}_{k\in\mathbb{N}^0} = P(C, \mathcal{T}, x_0, \{c_k\}, \{t_k\})$ be an implicit iteration process, where*

(i) $t_n > 0$ *for every* $n \in \mathbb{N}$
(ii) $\liminf_{n\to\infty} t_n = 0$
(iii) $\limsup_{n\to\infty} t_n > 0$
(iv) $\lim_{n\to\infty} (t_{n+1} - t_n) = 0$,

then there exists a common fixed point $x \in F(\mathcal{T})$ such that $\rho(x_k - x) \to 0$ when $k \to \infty$.

Proof. Let us fix arbitrarily $0 < t < \limsup_{n\to\infty} t_n$. By Lemma 1, we can choose a subsequence $\{t_{k_n}\}$ of $\{t_n\}$ such that $t_{k_n} \to t$ as $n \to \infty$. By Lemma 5,

$$\lim_{n\to\infty} \rho\left(\frac{T_{t_{k_n}}(x_{k_n}) - x_{k_n}}{2}\right) = 0. \tag{30}$$

Since C is ρ-compact, there exists $\{x_{k_{n_i}}\}$, a subsequence of $\{x_{k_n}\}$, and an element $x \in C$ such that, denoting $w_i = x_{k_{n_i}}$, we get

$$\lim_{i\to\infty} \rho(w_i - x) = 0. \tag{31}$$

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Denote $s_i = t_{k_{n_i}}$. By applying the convexity of ρ and ρ-nonexpansiveness of every T_t, we find that

$$\begin{aligned}
\rho\left(\frac{T_{s_i}(x)-x}{6}\right) &= \rho\left(\frac{1}{3}\frac{T_{s_i}(x)-T_{s_i}(w_i)}{2}+\frac{1}{3}\frac{T_{s_i}(w_i)-w_i}{2}+\frac{1}{3}\frac{w_i-x}{2}\right)\\
&\le \rho\Big(T_{s_i}(x)-T_{s_i}(w_i)\Big)+\rho\left(\frac{T_{s_i}(w_i)-w_i}{2}\right)+\rho(w_i-x) \qquad (32)\\
&\le \rho(w_i-x)+\rho\left(\frac{T_{s_i}(w_i)-w_i}{2}\right)+\rho(w_i-x).
\end{aligned}$$

The right-hand side of (32) tends to zero as $i \to \infty$ because of (31) and (30). From the continuity of $\mathcal{T}$ (condition 4 in Definition 7) combined with (32), we infer that

$$\rho\left(\frac{T_t(x)-x}{12}\right) \le \frac{1}{12}\rho\Big(T_t(x)-T_{s_i}(x)\Big)+\frac{1}{2}\left(\frac{T_{s_i}(x)-x}{6}\right) \to 0$$

as $i \to \infty$, which implies that $T_t(x) = x$. Recall that t was an arbitrary number satisfying $0 < t < \limsup_{n\to\infty} t_n$. Let us take now any $s > 0$ and observe that there exist numbers $0 < t < \limsup_{n\to\infty} t_n$, $0 < u < \limsup_{n\to\infty} t_n$ and $k \in \mathbb{N}^0$ such that $s = t + ku$. Thus, we conclude that

$$T_s(x) = T_{ku}(T_t(x)) = T_{ku}(x) = T_{u+\ldots+u}(x) = x,$$

which means that $x \in F(\mathcal{T})$. We know from (31) that $\lim_{i\to\infty} \rho(x_{k_{n_i}} - x) = 0$. On the other hand, by Lemma 4, the limit $\lim_{k\to\infty} \rho(x_k - x)$ exists, because, as we demonstrated above, x is a common fixed point of $\mathcal{T}$. By bringing these two points together, we conclude that $\lim_{k\to\infty} \rho(x_k - x) = 0$, thereby completing the proof. □

It is a valid inquiry regarding the applicability of Theorem 3. We have previously highlighted that the class of regular modular spaces is extensive, including all Banach spaces and modular function spaces. As a result, it encompasses various important spaces such as L^p, l^p, variable exponent versions of these spaces, Orlicz spaces, Musielak-Orlicz spaces, and many other important function and sequence spaces. It is elementary to construct a sequence $\{t_k\}$ satisfying conditions $(i) - (iv)$ from Theorem 3. A more pertinent question concerns the existence of natural examples of ρ-nonexpansive semigroups in regular modular spaces. In [39], the author examined the following initial value problem defined by a differential equation for an unknown function $u : [0, A] \to C$:

$$\begin{cases} u(0) = x \\ u'(t) + (I - T)u(t) = 0, \end{cases} \qquad (33)$$

where C represents a subset of a modular function space, $x \in C$ and $A > 0$ are given. Let $T : C \to C$ be a ρ-nonexpansive mapping. It can be shown that, under certain technical assumptions, equation (33) has a solution u_x for every initial values of x (see [40]). Denote $S_t(x) = u_x(t)$. Theorem 5.1 in [39] states that, under natural assumptions, $\{S_t\}_{t\ge 0}$ constitutes a ρ-nonexpansive semigroup of nonlinear operators on C. The same paper applies this result to the the scenario where T is a nonlinear integral Urysohn operator, and the modular ρ is defined by this operator itself.

4. Discussion

Theorem 3 is the principal result of this paper. It specifies conditions under which an implicit iteration process ρ-converges to a common fixed point of a ρ-nonexpansive semigroup $\{T_t\}_{t\ge 0}$ of

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operators acting within a regular modular space X_ρ, which is defined as a modular space where ρ-balls are ρ-closed. Consequently, this result offers a constructive and computationally practical method for approximating common fixed points, which, as mentioned in the Introduction, relates to the challenge of identifying stationary points in potentially stochastic dynamic systems and solving evolutionary equations. This novel result simultaneously generalizes analogous findings established for Banach spaces [8,9] and modular function spaces [18]. We also explored applicability of our results to regular modular spaces that are neither Banach nor modular function spaces. Notably, in Example 1, we discussed regular modular spaces defined by φ-variations that emerged from Wiener's quadratic variation. By utilizing the examples presented at the end of the previous section, we demonstrated how our results can be applied to identify stationary points for systems defined by nonlinear differential equations, where the unknown functions take values in complex infinite-dimensional function spaces. It is also important to reaffirm our explanation provided in Section 2 regarding why, within the context of this paper, modular methods in fixed point theory are typically simpler and more general compared to the application of normed space results to the Luxemburg norms associated with these modulars. The findings of this paper highlight the advantages of further advancing fixed point theory in regular modular spaces, which provide a highly general yet minimalist framework.

Given that the foundations of fixed point theory in regular modular spaces were established quite recently, there are numerous potential avenues for further advancement. In regular modular spaces, as defined in both this paper and earlier works [19,20], there is no counterpart to the weak topology of Banach spaces, nor is there an equivalent to ρ-almost everywhere convergence that plays a similar role in the theory of modular function spaces. Since both concepts are fundamental to fixed point theory in their respective spaces, future research on the existence of fixed points in regular modular spaces, as well as on the convergence of fixed point approximation processes, will require the development of an analogous framework.

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