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Vector Representations of Euler's formula and Riemann's Zeta Function

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Abstract

Just as Gauss's interpretation of complex numbers as points in a number plane in the form of a suitably formulated axiom found its way into the vector representation of Fourier transforms, this is the case with Euler's formula and Riemann's Zeta function considered here. The description of the connection between variables through complex numbers as it is given in Euler's formula and emphasized by Riemann, is reflected here with great flexibility in the introduction of non-classically generalized complex numbers and the vector representation of the generalized Zeta function based on them. For describing such dependencies of two variables with the help of generalized complex numbers, we introduce manifolds underlying certain Lie groups as level sets of norms, antinorms or semi-antinorms. No undefined or "imaginary" quantities are used for this. In contrast to the approach of Hamilton and his numerous successors, the vector-valued vector product of non-classically generalized complex numbers is commutative and the whole number system satisfies a weak distributivity property as considered by Hankel, but not the strong one.

Keywords: vector representation; dependent variables; non-classically generalized complex numbers; Lie group; antinorm and semi-antinorm; vector-valued vector product and vector power; vector-valued exponential function; vector-valued vector logarithm; Euler type vector formula

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1. Introduction

The operations of addition and subtraction as well as multiplication and division are each explained for elements from a well-defined number system such as that of real numbers or that of complex numbers. But is it reasonable to divide a number from one system by a number from another system? For example, what are the consequences of dividing a real number by an ordered pair of real numbers? From a scientific history perspective, answers to such and similar questions may depend on the time at which they are asked or answered.

When Euler [7] solved the Basel problem in 1735, he calculated the value of the series

$$\zeta(x) = 1 + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + ...$$

for $x = 2$. The great importance of his results for number theory lies in the agreement of this so-called Zeta function with Euler's product

$$\zeta(x) = \prod_p \frac{1}{1 - \frac{1}{p^x}}, x > 1$$

which extends over all prime numbers p.

Starting with Riemann [22], the Zeta function $z \to \zeta(z)$ for complex z is considered, initially with a real part of z greater than one. Thus, this function was already studied for complex arguments, even before the character of complex numbers was finally and completely explored. To substantiate this, we would like to recall that Gauss [10], Cauchy [3], Dedekind [5] and Hankel [15] had self-skeptical and

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critical opinions about the final introduction and representation of the system of complex numbers. To be more specific, Crowe [4] states that Gauss did not accept his geometric interpretation as the foundation of complex numbers. Bedürftig and Murawski [1] share this view with reference to the following words of Gauss [12]: "However, the representation of imaginary magnitudes in the relations of points in plano is not so much their essence itself, which must be understood in a higher and more general way, but rather the purest, or perhaps the only completely pure, example of their applications for us humans."

A corresponding fundamental problem with many of today's representations is that a real number x is equated with the vector $(x,0)^T$ therein, which is often taken as identification but lacks rigorous justification. In this context, real numbers are often unfortunately considered as special complex numbers. But the field of complex numbers is, strictly speaking, not an extension field of the real numbers. Some authors say that they "identify" the vector $(x,0)^T$ with the scalar x or "interpret" it as such, without giving these words a precise and acceptable mathematical content. Conclusions for Euler's formula and for Riemann's Zeta function, which result from author's updating and generalizing complex numbers in [18] and a series of subsequent papers including those on (p_1, p_2, p_3)-complex numbers, complex numbers in higher dimensions and complex numbers related to semi-antinorms, ellipses or matrix-homogeneous functionals, will be presented here.

Just as Gauss's ingenious interpretation of complex numbers as points in a number plane [10,11] in the form of a suitably formulated axiom found its way into the vector representation of Fourier transforms of probability densities [19], this is the case with Euler's formula and Riemann's Zeta function considered here. The description of the connection between variables through complex numbers, reflected in Euler's formula [8] and emphasized by Riemann [22], is reflected here with great flexibility in the introduction of non-classically generalized complex numbers and the vector representation of the Zeta function based on them. At the same time, it should be emphasized that no undefined or "imaginary" quantities are needed or routinely used for this.

In contrast to the approach of Hamilton [13] and his numerous successors, the vector product of our non-classically generalized complex numbers is commutative and associative and the whole number system satisfies a weak distributivity property as considered by Hankel [15], but not the usual strong one.

Although or because the true nature of the so-called "imaginary quantity" i has been unknown for centuries, the following transformation has become common practice in the literature on complex analysis:

$$\frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}. \tag{1}$$

It may not have led to any prominent contradiction since the times of Euler and Gauss. Can it therefore be considered sufficiently proven from a strict mathematical point of view? That would be the case if i were a real number (satisfying $i^2 = -1$), but that is exactly what i is not supposed to be. Using this transformation and the subsequent interpretation adapting Gauss

$$x - iy = \begin{pmatrix} x \\ -y \end{pmatrix},$$

one arrives at

$$\frac{1}{x+iy} = \frac{\begin{pmatrix} x \\ -y \end{pmatrix}}{x^2+y^2}. \tag{2}$$

However, according to the correct definitions of scalar and complex vector-valued vector multiplication as well as vector-valued vector division as in [18],[19],

$$\frac{1}{x^2+y^2}\begin{pmatrix} x \\ -y \end{pmatrix} \circledast \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

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differently from (2),

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \oslash \begin{pmatrix} x \\ y \end{pmatrix} = \frac{\begin{pmatrix} x \\ -y \end{pmatrix}}{x^2 + y^2}. \tag{3}$$

In this light, the starting point of transformation (1) may appear to be unmotivated from a rigorous point of view: why should one divide the scalar 1 by the complex number $x + iy$ or ordered pair (x, y) or vector $\begin{pmatrix} x \\ y \end{pmatrix}$ instead of dividing vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by vector $\begin{pmatrix} x \\ y \end{pmatrix}$ in a well defined way? The reader might not find the differences between (1) and (3) dramatic, but that would not touch the core of the situation.

Classical generalizations of complex numbers often begin by determining the generally non-commutative product for every two or more basic elements. A closed formula or a geometric interpretation of this product within the whole space is generally not a motivating starting point, but has emerged as an interesting aspect in some individual cases. For example, the so-called scalar and vectorial parts of a certain quaternion product reflect a certain Euclidean scalar product and a usual vector product, respectively [6]. ***In another upcoming work [21], some of the ideas of this work are transferred to the area of quaternions but using non-commutative products instead of the commutative ones here. In particular, the three imaginary units usually considered in quaternion theory are replaced by three linear independent vectors, whereby the area of the quaternions is also freed from any alchemical approach.***

In contrast, the non-classical generalization of complex numbers that are used in the present work begins with the consideration of geometric locations that represent level lines of a norm, antinorm or semi-antinorm $||.||$ in R^2, and with movements along such level lines as well as transitions between these lines. The latter are described based on the situation with complex numbers by varying an angle variable or a generalized radius variable and thus motivate the definitions of generalized coordinates and of a vector-valued vector product of generalized complex numbers as well as its inverse operation of a vector-valued vector division. The manifolds and product operations formed in this way build the Lie groups that we will be talking about. The manifolds generated in this work by certain phs-functionals contain the manifolds arising in earlier considerations on hypercomplex numbers and contain the manifolds generated in author's corresponding work by semi-antinorms, in particular l_p-functionals for $p < 0$, or $(a, b)-$elliptical functionals as special cases. For numerous figures illustrating these manifolds, we refer to author's corresponding work. It should be noted that complex numbers in the context of classical coordinate geometry have already been studied in [16].

Our vector approach to usual and generalized complex numbers which was first described in [18] for the l_p-case turns Gauss's interpretation of complex numbers as points in a number plane into a suitably defined axiom and just starts from a two-dimensional vector space, which we choose R^2 in this paper for the sake of simplicity. In addition, the unexplained classical "imaginary unit" i from the theory of usual complex numbers is replaced with a well-defined element of the complex plane, that is a vector $\boldsymbol{I} \in R^2$. The reader is encouraged to always strictly distinguish between the two quantities i and $\boldsymbol{I}$. While $\boldsymbol{I} \in R^2$ is a well defined vector, it is only said with respect to the "imaginary unit" i that

$$\bullet \text{ it is not a real number} \tag{4}$$

and

$$\bullet \text{ it allows squaring such that } i^2 = -1. \tag{5}$$

But what kind of squaring could achieve this if, for example, $i =$ my best friend, or if we had $i =$ the moon? And what kind of addition could be considered in this context to form expressions like $3 + 4i$? The classical theory of complex numbers raises more questions than it answers!

The rest of the paper is organized as follows. We introduce a generalized complex algebraic vector structure and a Lie group in Section 2. Section 3 deals with a vector representation of Euler's formula

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and Section 4 with one of Riemann's Zeta function and a generalization of this. Finally we close this paper by a short discussion.

2. Revised and Non-Classically Generalized Complex Numbers

Let $(R^2, \oplus, \cdot)$ be the two-dimensional vector space of columns $(x, y)^T$ of real numbers where $\oplus : R^2 \times R^2 \to R^2$ and $\cdot : R \times R^2 \to R^2$ denote the usual componentwise vector addition and scalar multiplication, respectively. The vector $\mathfrak{o} = (0, 0)^T$ is the additive neutral element of this space. Suppose that $||.|| : R^2 \to [0, \infty)$ is a **p**ositively **h**omogeneous functional such that the set $B = \{z \in R^2 : ||z|| \leq 1\}$ is **s**tar-shaped with respect to the inner point $\mathfrak{o}$. We call such functional a phs-functional. A particular element of this class of functionals can be a norm, an antinorm or a semi-antinorm. Furthermore, we call B the unit disc with respect to the functional $||.||$ and its boundary

$$S = \{z \in R^2 : ||z|| = 1\}$$

the corresponding unit circle. Based upon the classical complex multiplication of $x_1 + y_1 i$ and $x_2 + y_2 i$, we introduce the notion of the vector-valued complex product of the vectors $z_l = (x_l, y_l)^T, l = 1, 2$ from R^2 by

$$z_1 \circledast z_2 = \begin{pmatrix} x_1 x_2 - y_1 y_2 \\ x_1 y_2 + x_2 y_1 \end{pmatrix}. \tag{6}$$

Note that

$$\begin{pmatrix} x \\ y \end{pmatrix} \circledast \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}.$$

This notion is now far-reaching generalized.

Definition 1. *The vector-valued generalized complex product of the vectors $z_l = (x_l, y_l)^T, l = 1, 2$ from R^2 with respect to the phs-functional $||.||$ is defined by*

$$z_1 \odot_{||.||} z_2 = \frac{||z_1|| \cdot ||z_2||}{||z_1 \circledast z_2||} \cdot z_1 \circledast z_2, z_l \neq \mathfrak{o}, l = 1, 2 \tag{7}$$

where it is assumed that all $||.||$-values are declared and finite. For short, $z_1 \odot_{||.||} z_2 = z_1 \odot z_2$.

We refer in particular to Figures 3 and 4 in author's work on complex numbers related to semi-antinorms, ellipses or matrix homogeneous functionals where the level lines for different p-values of l_p-circles are shown. Obviously,

$$||z_1 \odot z_2|| = ||z_1|| ||z_2||.$$

Further note that if the functional $||.||$ denotes the Euclidean norm $|.|_2$, then the multiplication operation $\odot$ agrees with the operation $\circledast$ and is called the vector-valued vector multiplication with respect to the functional $|.|_2$, thus $\odot_{|.|_2} = \circledast$. In addition, the generalized product with respect to the arbitrary phs-functional $||.||$ is commutative and associative, but addition and multiplication are not distributive, in general. Moreover,

$$z_1 \odot z_2 = \mathfrak{o} \text{ if and only if } z_1 = \mathfrak{o} \text{ or } z_2 = \mathfrak{o},$$

$$\frac{z_1}{||z_1||} \odot \frac{z_2}{||z_2||} \in S \text{ if } z_l \neq \mathfrak{o}, l = 1, 2$$

as well as

$$(\lambda z_1) \odot (\mu z_2) = (\lambda \mu) z_1 \odot z_2 \text{ for all } z_1 \text{ and } z_2 \text{ from } R^2 \text{ and real numbers } \lambda \text{ and } \nu.$$

As usual, for simplicity, we write λz and $\frac{z}{\lambda}$ instead of $\lambda \cdot z$ and $\frac{1}{\lambda} \cdot z$.

 doi:10.20944/preprints202508.1567.v1

Definition 2. *The vector-valued vector division of vector z_1 by vector z_2 is defined by*

$$z_1 \oslash_{|.|_2} z_2 = \frac{|z_1|_2}{|z_2|_2} \frac{z_1 \div z_2}{|z_1 \div z_2|_2}, z_l \neq \mathfrak{o}, l = 1,2$$

with the l_2-complex division according to [18],

$$z_1 \div z_2 = \begin{pmatrix} x_1x_2 + y_1y_2 \\ y_1x_2 - x_1y_2 \end{pmatrix}, z_2 \neq \mathfrak{o}. \tag{8}$$

The vector-valued generalized vector division of z_1 by z_2 with respect to the phs-functional $||.||$ is defined by

$$z_1 \oslash_{||.||} z_2 = \frac{||z_1||}{||z_2||} \frac{z_1 \div z_2}{||z_1 \div z_2||}, z_l \neq \mathfrak{o}, l = 1,2$$

or

$$z_1 \oslash_{||.||} z_2 = z_1 \oslash z_2, z_2 \neq \mathfrak{o}$$

for short.

The following table provides an overview of the symbols used here for the different vector-valued operations and the places where they first appear.

Table 1. Vector -valued operations.

$\oplus$	First line of Section 2
$\cdot$	First line of Section 2
$\circledast$	(6)
$\odot_{\|\|.\|\|}$	(7)
$\odot$	Definition 1
$\div$	(8)
$\oslash_{\|\|.\|\|}$	Definition 2
$\oslash$	Definition 2

Obviously,

$$||z_1 \oslash z_2|| = \frac{||z_1||}{||z_2||}, z_2 \neq \mathfrak{o}.$$

Let further linear independent vectors $\mathbb{1}$ and $\boldsymbol{I}$ from R^2 satisfy

$$||\mathbb{1}|| = ||\boldsymbol{I}|| = 1, \tag{9}$$

$$\mathbb{1} \odot z = z \tag{10}$$

and

$$\boldsymbol{I} \odot \boldsymbol{I} = -\mathbb{1}. \tag{11}$$

Example 1. *As one of the simplest examples, the Euclidean norm $||.|| = |.|_2$, the product $\odot = \circledast$ with respect to the Euclidean norm $|.|_2$ and the vectors $\mathbb{1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{I} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ satisfy the assumptions (9)-(11). In this case,*

$$\mathfrak{C}_{|.|_2} = (R^2, |.|_2, \oplus, \circledast, \cdot, \mathfrak{o}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix})$$

replaces the classical complex plane.

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Note that there is no need to use the "formal" or mystical notion of an "imaginary unit" which in classical complex number theory is explained more poorly than well by the assumptions (4) and (5).

The vectors $\mathbb{1}$ and $\boldsymbol{I}$ are called the vector-multiplicative neutral element and the anti-clockwise quarter rotation multiplier, respectively. The latter is well motivated by the obvious relations

$$\begin{pmatrix} x \\ y \end{pmatrix} \circledast \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$
$$= \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \varphi = -90^o.$$

Example 2. *Let us write Gauss's interpretation of complex numbers as points in a number plane as*

$$"\lambda + it \rightsquigarrow " \begin{pmatrix} \lambda \\ t \end{pmatrix} = \lambda \cdot \mathbb{1} + t \cdot \boldsymbol{I}$$

where "$\xi \rightsquigarrow$ "η means "interpret ξ as" η. Then

$$(1+ix)\cdot(1-ix) \rightsquigarrow \begin{pmatrix} 1 \\ x \end{pmatrix} \circledast \begin{pmatrix} 1 \\ -x \end{pmatrix} = \begin{pmatrix} 1+x^2 \\ 0 \end{pmatrix}.$$

However, in the literature one can often read the unproven or "formal" statement

$$\frac{1}{1+x^2} = \frac{1}{(1+ix)(1-ix)} \tag{12}$$

which then serves as the basis for further statements. Note that the division expression on the right side of (12) combines a real number in the numerator and two complex numbers in the denominator instead of combining only complex numbers or only real numbers. However, the correct way to look at the right side of (12) is

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \oslash \left(\begin{pmatrix} 1 \\ x \end{pmatrix} \circledast \begin{pmatrix} 1 \\ -x \end{pmatrix} \right).$$

Remark 1. *One can easily check that this quantity can be evaluated according to the general rule*

$$z_1 \oslash (z_2 \circledast z_3) = (z_1 \oslash z_2) \oslash z_3.$$

Definition 3. *We call*

$$\mathfrak{C}_{||.||} = (R^2, ||.||, \oplus, \odot, \cdot, \mathfrak{o}, \mathbb{1}, \boldsymbol{I})$$

the complex algebraic structure with respect to the phs-functional $||.||$, $\mathfrak{o} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

This non-classically generalized complex algebraic structure was first introduced for the particular l_p-case, that is for $||\begin{pmatrix} x \\ y \end{pmatrix}|| = (|x|^p + |y|^p)^{1/p}$, in [18].

Remark 2. *Regardless of possible isomorphic considerations commonly used in the literature, there is obviously no true justification for setting the vector $\mathbb{1}$ equal to the scalar 1 in equation (11) as is done by many authors.*

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Generalized trigonometric functions with respect to the unit circle S are defined by

$$\cos_S(\varphi) = \frac{\cos\varphi}{N(\varphi)}, \quad \sin_S(\varphi) = \frac{\sin\varphi}{N(\varphi)}$$

where

$$N(\varphi) = ||(\cos\varphi)\mathbb{1} + (\sin\varphi)\boldsymbol{I}||$$

and a corresponding generalized polar coordinate transformation

$$Pol_S : [0,\infty) \times [0,2\pi) \to R^2$$

by

$$\begin{pmatrix} x \\ y \end{pmatrix} = Pol_S[r,\varphi] = r[(\cos_S(\varphi))\mathbb{1} + (\sin_S(\varphi))\boldsymbol{I}].$$

In what follows we will make use of the equation

$$N(\pi) = ||\mathbb{1}||.$$

If $z = Pol_S[r,\varphi]$ then

$$r = ||z|| \text{ and } (\cos_S(\varphi))\mathbb{1} + (\sin_S(\varphi))\boldsymbol{I} = \frac{z}{||z||}.$$

The following theorem is a geometric reflection of the analytical definition (7).

Theorem 1. *Let $z_l = Pol_S[r_l,\varphi_l], l = 1,2$. The product $z_1 \odot z_2$ satisfies*

$$Pol_S[r_1,\varphi_1] \odot Pol_S[r_2,\varphi_2] = Pol_S[r_1 r_2, \varphi_1 + \varphi_2].$$

Proof. To see this, we first recall that $||Pol_S[r,\varphi]|| = r$. Moreover,

$$\begin{aligned} Pol_S[r_1 r_2, \varphi_1 + \varphi_2] &= \frac{r_1 r_2}{N(\varphi_1+\varphi_2)} \begin{pmatrix} \cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 \\ \sin\varphi_1\cos\varphi_2 + \sin\varphi_2\cos\varphi_1 \end{pmatrix} \\ &= \frac{||z_1||||z_2||}{N(\varphi_1+\varphi_2)} \frac{z_1 \circledast z_2}{|z_1|_2|z_2|_2}. \end{aligned}$$

Thus,

$$r_1 r_2 = ||Pol_S[r_1 r_2, \varphi_1+\varphi_2]|| = \frac{||z_1||||z_2||}{N(\varphi_1+\varphi_2)} \frac{||z_1 \circledast z_2||}{|z_1|_2|z_2|_2}$$

and

$$N(\varphi_1+\varphi_2) = \frac{||z_1 \circledast z_2||}{|z_1|_2|z_2|_2}.$$

Finally,

$$\begin{aligned} Pol_S[r_1 r_2, \varphi_1+\varphi_2] &= |z_1|_2|z_2|_2 \frac{||z_1||||z_2||}{||z_1 \circledast z_2||} \frac{z_1 \circledast z_2}{|z_1|_2|z_2|_2} \\ &= z_1 \odot z_2 \end{aligned}$$

□

Remark 3. *The manifold $S = Pol_S(\{1\} \times [0,2\pi))$ together with the vector-valued vector product $\odot : R^2 \times R^2 \to R^2$ build the Lie group $(S,\odot)$ of the non-classically generalized complex algebraic structure $\mathfrak{C}_{||.||} = (R^2, ||.||, \oplus, \odot, \cdot, \mathfrak{o}, \mathbb{1}, \boldsymbol{I})$.*

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Remark 4. *This structure is weakly distributive in the sense of Hankel [15] because for all real α and β, $\alpha \cdot \boldsymbol{I} \oplus \beta \cdot \boldsymbol{I} = (\alpha + \beta) \cdot \boldsymbol{I}$. Their assumption of the indispensability of strong distributivity $z_1 \odot (z_2 \oplus z_3) = z_1 \odot z_2 \oplus z_1 \odot z_3$, however, led Hamilton and his successors into a completely different mathematical area than the one considered here.*

Remark 5. *Associativity of multiplication $\odot$, that is $z_1 \odot (z_2 \odot z_3) = (z_1 \odot z_2) \odot z_3$, obviously follows from that of addition of angles and multiplication of radii according to Theorem 1 .*

We now turn to power and exponential functions. To this end, let $||.||$ continue to denote an arbitrary phs-functional, x and y be real numbers and $z = x\mathbb{1} + y\boldsymbol{I}$.

Definition 4. *The* n*th power of z with respect to the phs-functional $||.||$ is defined by*

$$z^{\odot n} = z^{\odot(n-1)} \odot z,\, n = 1, 2, ..., z^{\odot 0} = \mathbb{1}.$$

Definition 5. *The exponential function with respect to the phs-functional $||.||$, $\exp_{||.||} : R^2 \to R^2$, as well as the exponential-projection function $z \to e^z_{||.||}, z \in R^2$ are defined by*

$$\exp_{||.||}(z) = \sum_{k=0}^{\infty} \frac{z^{\odot k}}{k!}$$

and

$$e^z_{||.||} = \frac{\exp_{||.||}(z)}{||\exp_{||.||}(z)||},$$

respectively.

3. Vector Representation of Euler's Formula

We recall that the well-known Euler formula relates the so-called "imaginary unit" i from usual complex number theory with the Euclidean unit circle. The following theorems suitably update and far-reaching generalize Euler's formula with respect to circles which are generated by any phs-functionals. Note that, in the sense of mathematical rigor, no non-declared so-called "imaginary unit" i appears in the present results. Instead, the anti-clockwise quarter rotation multiplier $\boldsymbol{I}$ will be used. The easiest formula of such type is the following vectorial update of Euler's famous formula which is an immediate consequence from the latter definition :

$$e^{\pi \boldsymbol{I}}_{||.||} + \frac{\mathbb{1}}{||\mathbb{1}||} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{13}$$

Theorem 2. *(`Vectorial Euler type formula`) The particular value $exp_{||.||}(y\boldsymbol{I})$ of the complex exponential function does not depend on the functional $||.||$ and allows the representation*

$$exp_{||.||}(y\boldsymbol{I}) = (\cos y)\mathbb{1} + (\sin y)\boldsymbol{I},\ y \in R.$$

Proof. The theorem follows by rearranging the terms in the series expansion of $\exp_{||.||}(y\boldsymbol{I})$:

$$\exp_{||.||}(y\boldsymbol{I}) = \sum_{k=0}^{\infty} \frac{y^{2k}\boldsymbol{I}^{\odot(2k)}}{(2k)!} + \sum_{k=0}^{\infty} \frac{y^{2k+1}\boldsymbol{I}^{\odot(2k+1)}}{(2k+1)!}$$

□

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We now present a particular modification of the vectorial Euler type formula wherein $\boldsymbol{I}$ is replaced with $\frac{\sqrt{2}}{2}\begin{pmatrix}1\\1\end{pmatrix}$ and the phs-functional $||.||$ is the Euclidean norm $|.|_2$.

Theorem 3. *For the vector* $z = \frac{\sqrt{2}}{2}\begin{pmatrix}1\\1\end{pmatrix}$,

$$\exp_{|.|_2}(y\cdot z) = e^{y/\sqrt{2}}((\cos\frac{y}{\sqrt{2}})\cdot\mathbb{1} + (\sin\frac{y}{\sqrt{2}})\cdot\boldsymbol{I}),\ y\in R.$$

Proof. We start from

$$z^{\circledast 2} = \boldsymbol{I}, z^{\circledast 3} = \frac{\sqrt{2}}{2}\begin{pmatrix}-1\\1\end{pmatrix}, \dots, z^{\circledast 7} = \frac{\sqrt{2}}{2}\begin{pmatrix}1\\-1\end{pmatrix}, z^{\circledast 8} = \mathbb{1}.$$

Thus, for $k = 0,1,2,...,$

$$z^{\circledast(8k)} = \mathbb{1}, z^{\circledast(8k+1)} = \frac{\sqrt{2}}{2}\mathbb{1}, z^{\circledast(8k+2)} = \boldsymbol{I}, z^{\circledast(8k+3)} = \frac{\sqrt{2}}{2}\begin{pmatrix}-1\\1\end{pmatrix},$$

$$z^{\circledast(8k+4)} = -\mathbb{1}, z^{\circledast(8k+5)} = -z, z^{\circledast(8k+6)} = -\boldsymbol{I}, z^{\circledast(8k+7)} = \frac{\sqrt{2}}{2}\begin{pmatrix}1\\-1\end{pmatrix}.$$

It follows that

$$\begin{aligned}\exp_{|.|_2}(y\cdot z) = &[(\frac{y^0}{0!}+\frac{y^8}{8!}+\frac{y^{16}}{16!}+\dots)-(\frac{y^4}{4!}+\frac{y^{12}}{12!}+\frac{y^{20}}{20!}+\dots)]\mathbb{1}\\ &+[(\frac{y^2}{2!}+\frac{y^{10}}{10!}+\frac{y^{18}}{18!}+\dots)-(\frac{y^6}{6!}+\frac{y^{14}}{14!}+\frac{y^{22}}{22!}+\dots)]\boldsymbol{I}\\ &+[(\frac{y^1}{1!}+\frac{y^9}{9!}+\frac{y^{17}}{17!}+\dots)-(\frac{y^5}{5!}+\frac{y^{13}}{13!}+\frac{y^{21}}{21!}+\dots)]\frac{\sqrt{2}}{2}(\mathbb{1}+\boldsymbol{I})\\ &+[(\frac{y^3}{3!}+\frac{y^{11}}{11!}+\frac{y^{19}}{19!}+\dots)-(\frac{y^7}{7!}+\frac{y^{15}}{15!}+\frac{y^{23}}{23!}+\dots)]\frac{\sqrt{2}}{2}(\boldsymbol{I}-\mathbb{1})\end{aligned}$$

and, with the notation

$$S(y,q) = \sum_{k=0}^{\infty}(-1)^k\frac{y^{4k+q}}{(4k+q)!},\ q = 0,1,2,3$$

we arrive at

$$\exp_{|.|_2}(y\cdot z) = [S(y,0)+\frac{\sqrt{2}}{2}\{S(y,1)-S(y,3)\}]\mathbb{1} + [S(y,2)+\frac{\sqrt{2}}{2}\{S(y,1)+S(y,3)\}]\boldsymbol{I}.$$

As H.Kösters and D.Müller kindly informed me, WolframAlpha [23] applies; it provides

$$S(y,q) = \frac{y^q}{q!}\,{}_1F_4(1;\frac{q+1}{4},\frac{q+2}{4},\frac{q+3}{4},\frac{q}{4}+1;-\frac{y^4}{256})$$

where ${}_1F_r(1;a_1,...,a_l;b_1,...,b_r;z)$ denotes the generalized hypergeometric function. In particular,

$$S(y,0) = \cos\frac{y}{\sqrt{2}}\cosh\frac{y}{\sqrt{2}}, S(y,2) = \sin\frac{y}{\sqrt{2}}\sinh\frac{y}{\sqrt{2}},$$

$$S(y,1) = \frac{1}{\sqrt{2}}(\sin\frac{y}{\sqrt{2}}\cosh\frac{y}{\sqrt{2}}+\cos\frac{y}{\sqrt{2}}\sinh\frac{y}{\sqrt{2}}),$$

$$S(y,3) = \frac{1}{\sqrt{2}}(\sin\frac{y}{\sqrt{2}}\cosh\frac{y}{\sqrt{2}}-\cos\frac{y}{\sqrt{2}}\sinh\frac{y}{\sqrt{2}}).$$

It follows that
$$\exp_{|.|_2}(y \cdot z) = \begin{pmatrix} \cos\frac{y}{\sqrt{2}}\cosh\frac{y}{\sqrt{2}} + \cos\frac{y}{\sqrt{2}}\sinh\frac{y}{\sqrt{2}} \\ \sin\frac{y}{\sqrt{2}}\sinh\frac{y}{\sqrt{2}} + \sin\frac{y}{\sqrt{2}}\cosh\frac{y}{\sqrt{2}} \end{pmatrix},$$
which finally proves the theorem □

Instead of considering further particular modifications of the Euler type vector formula, we present the following generalization of the latter theorem.

Theorem 4. *The complex exponential function with respect to the Euclidean norm $|.|_2$ allows the representation*
$$\exp_{|.|_2}(x\mathbb{1} + y\boldsymbol{I}) = e^x \cdot \exp_{|.|_2}(y\boldsymbol{I}).$$

Proof. We recall that $\mathbb{1}^{\circledast n} = \mathbb{1}, \boldsymbol{I}^{\circledast(2k)} = (-1)^k\mathbb{1}, \boldsymbol{I}^{\circledast(2k+1)} = (-1)^k\boldsymbol{I}$ and
$$\cos y = \sum_{n=0}^{\infty}(-1)^n\frac{y^{2n}}{(2n)!} \text{ and } \sin y = \sum_{n=0}^{\infty}(-1)^n\frac{y^{2n+1}}{(2n+1)!}.$$
The statement of the theorem now follows by using properties of distributivity and of Cauchy's product series:
$$\begin{aligned}\exp_{|.|_2}(z) &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\nu=0}^{n}\begin{pmatrix} n \\ \nu \end{pmatrix}(x\mathbb{1})^{\circledast(n-\nu)} \circledast (y\boldsymbol{I})^{\circledast\nu} \\ &= \sum_{n=0}^{\infty}\sum_{\nu=0}^{n}\frac{(x\mathbb{1})^{\circledast(n-\nu)}}{(n-\nu)!} \circledast \frac{(y\boldsymbol{I})^{\circledast\nu}}{\nu!} \\ &= \left(\sum_{n=0}^{\infty}\frac{(x\mathbb{1})^{\circledast n}}{n!}\right) \circledast \left(\sum_{n=0}^{\infty}\frac{(y\boldsymbol{I})^{\circledast n}}{n!}\right) \\ &= (e^x\mathbb{1}) \circledast \exp_{|.|_2}(y\boldsymbol{I}) = e^x \exp_{|.|_2}(y\boldsymbol{I})\end{aligned}$$
□

Lemma 1. *The nth vector-valued vector power allows the representation*
$$z^{\odot n} = \frac{||z||^n}{||z^{\circledast n}||}z^{\circledast n}.$$

Proof. This follows immediately by induction □

Theorem 5. *(`Vector exponential function and Euler type formula`) The vector-valued non-classically generalized complex exponential function with respect to the phs-functional $||.||$ allows for $z = \varrho\begin{pmatrix} \cos\varphi \\ \cos\varphi \end{pmatrix}$ the representation*
$$\exp_{||.||}(z) = \sum_{k=0}^{\infty}\frac{||z||^k}{k!}\Theta(k) \textit{ where } \Theta(k) = (\cos_S(k\varphi))\mathbb{1} + (\sin_S(k\varphi))\boldsymbol{I}$$
and for $y \in R$ the non-classically generalized vectorial Euler type representation
$$e^{yI}_{||.||} = (\cos_S(y))\mathbb{1} + (\sin_S(y))\boldsymbol{I}.$$

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Proof. With Lemma 1, the first statement follows from

$$\exp_{||.||}(z) = \sum_{k=0}^{\infty} \frac{z^{\odot k}}{k!} = \sum_{k=0}^{\infty} \frac{||z||^k z^{\circledast k}}{k!||z^{\circledast k}||} = \sum_{k=0}^{\infty} \frac{||z||^k}{k!} \frac{\begin{pmatrix} \cos k\varphi \\ \sin k\varphi \end{pmatrix}}{||\begin{pmatrix} \cos k\varphi \\ \sin k\varphi \end{pmatrix}||}$$

and the second statement follows by extending the proof of Theorem 2 □

Example 3. *Let $a > 0, b > 0, p \in R$ with $p \neq 0$ and*

$$||\begin{pmatrix} x \\ y \end{pmatrix}|| = |\begin{pmatrix} x \\ y \end{pmatrix}|_p = (|\frac{x}{a}|^p + |\frac{y}{b}|^p)^{1/p} \text{ with } x \neq 0, y \neq 0 \text{ if } p < 0.$$

This functional is a norm, antinorm or semi-antinorm if respectively $p \geq 1$, $0 < p \leq 1$ or $p < 0$. The vector-valued vector product with respect to the phs-functional $||.||$ is defined by

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \odot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \frac{(|\frac{x_1}{a}|^p + |\frac{y_1}{b}|^p)^{1/p}(|\frac{x_2}{a}|^p + |\frac{y_2}{b}|^p)^{1/p}}{(|\frac{x_1x_2 - y_1y_2}{a}|^p + |\frac{x_1y_2 + x_2y_1}{b}|^p)^{1/p}} \begin{pmatrix} x_1x_2 - y_1y_2 \\ x_1y_2 + x_2y_1 \end{pmatrix}$$

and is suitable for describing or causing movements of points along $l_{a,b;p}-$circles or ellipses and changes between such. The vectors

$$\mathbb{1} = \frac{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{|\begin{pmatrix} 1 \\ 0 \end{pmatrix}|_p} = \begin{pmatrix} a \\ 0 \end{pmatrix} \text{ and } \boldsymbol{I} = \frac{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}{|\begin{pmatrix} 0 \\ 1 \end{pmatrix}|_p} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

satisfy the equations (9)-(11), that is

$$|\mathbb{1}|_p = |\boldsymbol{I}|_p = 1, \begin{pmatrix} a \\ 0 \end{pmatrix} \odot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ b \end{pmatrix} \odot \begin{pmatrix} 0 \\ b \end{pmatrix} = -\begin{pmatrix} a \\ 0 \end{pmatrix}.$$

The exponential function with respect to the functional $|.|_p$ and the corresponding exponential projection function satisfy the representations

$$\exp_{|.|_p}(\varrho \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}) = \sum_{k=0}^{\infty} \frac{\varrho^k}{k!} \Theta(k) \text{ with } \Theta(k) = \cos_p(k\varphi)\mathbb{1} + \sin_p(k\varphi)\boldsymbol{I}$$

and

$$e^{yI}_{|.|_p} = \cos_p(k\varphi)\mathbb{1} + \sin_p(k\varphi)\boldsymbol{I},$$

respectively, where the generalized trigonometric functions $\cos_S = \cos_p$ and $\sin_S = \sin_p$ with respect to the unit circle $S = S_p = \{z \in R^2 : |z|_p = 1\}$ are

$$\cos_p y = \frac{\cos y}{(|\cos y|^p + |\sin y|^p)^{1/p}} \text{ and } \sin_p = \frac{\sin y}{(|\cos y|^p + |\sin y|^p)^{1/p}}$$

and satisfy the equation

$$|\cos_p y|^p + |\sin_p y|^p = 1.$$

Finally,

$$e^{\pi I}_{|.|_p} + \mathbb{1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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and

$$|e^{yI}_{|.|_p}|_p = 1.$$

Definition 6. *We call* $(R^2, |.|_p, \oplus, \odot, \cdot, \mathfrak{o}, \begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b \end{pmatrix})$ *an* $(a, b; p)-$*elliptic complex algebraic structure.*

Remark 6. *The symmetry properties of the vectorial Euler type formula (13) can be seen particularly impressively in the special case of Example 3.*

Our next considerations are devoted to the study of complex powers of positive real numbers. For every $z \in R^2$ there is a unique $y = y(z) \in [0, 2\pi)$ such that

$$\frac{z}{||z||} = e^{y\boldsymbol{I}}_{||.||}.$$

Consequently, every vector allows the representation

$$z = ||z|| e^{y\boldsymbol{I}}_{||.||}. \tag{14}$$

Definition 7. *For every positive real* a *and any* z *from the general complex algebraic structure* $\mathfrak{C} = (R^2, ||.||, \oplus, \odot, \cdot, \mathfrak{o}, \mathbb{1}, \boldsymbol{I})$, *we define the z-power of a with respect to the phs-functional* $||.||$ *by*

$$a^z_{||.||} = \exp_{||.||}((\ln a)z). \tag{15}$$

Remark 7. *In the spirit of the discussion in Section 1, one might believe, based on the notation* $a^z_{||.||}$ *with* $a \in R$ *and* $z \in \mathbb{C}$, *that here too there is a not well-defined combination of a real and a complex number within this power operation. However, that this is actually not the case follows from the actual definition of the term on the right side of (15).*

Corollary 1. *For every positive real a, the particular* $(y\boldsymbol{I})$*-power of a with respect to the functional* $||.||$ *does not depend on the choice of* $||.||$ *and allows the representation*

$$a^{y\boldsymbol{I}}_{||.||} = \begin{pmatrix} \cos(y \ln a) \\ \sin(y \ln a) \end{pmatrix}, y \in R.$$

Proof. This statement follows immediately from Definition 5 and the expansions of the trigonometric functions as in the proof of Theorem 2 □

An obvious conclusion from this is that

$$\frac{a^{y\boldsymbol{I}}_{||.||}}{||a^{y\boldsymbol{I}}_{||.||}||} = \begin{pmatrix} \cos_S(y \ln a) \\ \sin_S(y \ln a) \end{pmatrix}, y \in R. \tag{16}$$

Corollary 2. *For every positive real a, the particular* $(x\mathbb{1})$*-power of a with respect to the functional* $||.||$ *does not depend on the choice of* $||.||$ *and allows the representation*

$$a^{x\mathbb{1}}_{||.||} = \sum_{n=0}^{\infty} \frac{(\ln a)^n}{n!} x^n \mathbb{1} = a^x \mathbb{1}.$$

Remark 8. *Norms, antinorms and semi-antinorms* $||.||$ *used to generate the manifolds underlying the Lie groups considered in this paper do not change the "shape" of the* $||.||-$*related circles* rS *when changing* $r > 0$ *and could therefore be called shape preserving or static functionals. For the use of shape changing or dynamic functionals when introducing another type of Lie groups we refer to author's work on* (p_1, p_2, p_3)*-complex numbers.*

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4. Vector Representation of the Zeta Function

Riemann's Zeta function was introduced 1859 in [22] and studied in detail and extensively in [14], [17] and by many other authors. Let us first restrict our considerations to the classical complex number plane. The Zeta function is usually considered there for the unfortunately not very well defined symbolic quantities $z = x + iy$ with $x > 1$ as the series

$$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + ... = \sum_{n=1}^{\infty} \frac{1}{n^z} \tag{17}$$

where it is said that the symbolic quantity i together with a non-declared squaring rule, which is also assumed to apply to real numbers, satisfies $i^2 = -1$ and that i is not a real number. That is, assumptions (4) and (5) are assumed to be satisfied. These partially unclear or incomplete assumptions make it difficult to handle this function with greatest possible mathematical rigor.

With regard to our discussion in Section 1, it turns out that one of the most famous mathematical hypotheses concerns the zeros of this ominous function $\zeta(.)$ and that the formula manipulations carried out to substantiate corresponding statements can only be judged with considerable efforts.

The following considerations are therefore carried out in the completely defined algebraic structure $(R^2, |.|_2, \oplus, \circledast, \cdot, \mathfrak{o}, \mathbb{1}, \boldsymbol{I})$ in which real numbers are not considered to be special complex numbers but first components of vectors whose second component is zero and the phs-functional $|.|_2$ denotes the Euclidean norm.

Theorem 6. *(`Vectorially revised Zeta function`) On making use of the z-power function and vector division, with* $z = \begin{pmatrix} x \\ y \end{pmatrix}$ *the vectorially updated Zeta function*

$$\zeta(z) = \zeta_{|.|_2}(z) = \sum_{n=1}^{\infty} \mathbb{1} \oslash n^z_{|.|_2} \tag{18}$$

can be written as the vector-valued vector function

$$\zeta(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2^x}\begin{pmatrix} \cos \ln 2^y \\ -\sin \ln 2^y \end{pmatrix} + \frac{1}{3^x}\begin{pmatrix} \cos \ln 3^y \\ -\sin \ln 3^y \end{pmatrix} + ... \tag{19}$$

with the series converging in the half-plane $x > 1$.

Proof. This can be seen as follows. By Definition 5 and Theorem 2,

$$n^z_{|.|_2} = \exp_{|.|_2}((\ln n)z) = n^x \exp_{|.|_2}((y \ln n)\boldsymbol{I}) = n^x \begin{pmatrix} \cos(y \ln n) \\ \sin(y \ln n) \end{pmatrix}.$$

Finally, complex division results in

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \oslash (n^x \begin{pmatrix} \cos(y \ln n) \\ \sin(y \ln n) \end{pmatrix}) = \frac{1}{n^{2x}} \begin{pmatrix} n^x \cos(y \ln n) \\ -n^x \sin(y \ln n) \end{pmatrix}$$

□

We note that a zero of the revised Zeta function is a vector $z_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ with the property $\zeta(z_0) = \mathfrak{o}$, without pursuing the determination of such zeros here. Instead, we consider the following conclusions from Theorem 6.

Corollary 3. *The following two particular equations are conclusions from equation (19):*

$$\zeta(\begin{pmatrix} x \\ 0 \end{pmatrix}) = (\frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \ldots)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{20}$$

and

$$\zeta(\begin{pmatrix} 0 \\ y \end{pmatrix}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos(y\ln 2) \\ -\sin(y\ln 2) \end{pmatrix} + \begin{pmatrix} \cos(y\ln 3) \\ -\sin(y\ln 3) \end{pmatrix} + \ldots \tag{21}$$

We now continue our considerations in the non-classically generalized complex algebraic structure $(R^2, ||.||, \oplus, \odot, \cdot, \mathbf{o}, \mathbf{1}, \boldsymbol{I})$ in which, as already said earlier, the well defined vector $\boldsymbol{I}$ is to be distinguished from the unknown or "imaginary" quantity i and where the vector product $\odot$ is related to a phs-functional which can be in particular a norm, antinorm or semi-antinorm, and the vectors $\mathbf{1}$ and $\boldsymbol{I}$ still satisfy (9)-(11).
Known statements about the Zeta function also rise the question of whether the vector-valued function in (19) can be continued analytically. The foundation of a first step in this direction were laid in author's work on complex numbers in higher dimensions, thank you very much for where in Appendix A2 the concept of differentiability of vector-valued functions was defined with the help of vector-valued division.

Instead of pursuing this question, we now introduce the following notation.

Definition 8. *The generalization of the vectorially revised Zeta function with respect to the functional* $||.||$ *is defined by*

$$\zeta_{||.||}(z) = \sum_{n=1}^{\infty} \mathbf{1} \oslash n^z_{||.||}, \; z = x\mathbf{1} + y\boldsymbol{I}.$$

Based upon this generalized definition of the vectorially revised Zeta function, the validity of equations (20) and (21) can be extended as follows.

Theorem 7. *For the particular values* $z \in \{\begin{pmatrix} x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix}\}$ *from* R^2*, the generalization of the vectorially revised Zeta function with respect to the functional* $||.||$ *does actually not depend on the functional* $||.||$ *and allows the representations*

$$\zeta_{||.||}(\begin{pmatrix} x \\ 0 \end{pmatrix}) = (\frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \ldots)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{22}$$

and

$$\zeta_{||.||}(\begin{pmatrix} 0 \\ y \end{pmatrix}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos(y\ln 2) \\ -\sin(y\ln 2) \end{pmatrix} + \begin{pmatrix} \cos(y\ln 3) \\ -\sin(y\ln 3) \end{pmatrix} + \ldots \tag{23}$$

Proof. We first recall that

$$\zeta_{||.||}(\begin{pmatrix} x \\ 0 \end{pmatrix}) = \sum_{n=1}^{\infty}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \oslash \exp_{||.||}(\ln n \begin{pmatrix} x \\ 0 \end{pmatrix})\right)$$

$$= \sum_{n=1}^{\infty}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \oslash \sum_{k=0}^{\infty} \frac{(\ln n \begin{pmatrix} x \\ 0 \end{pmatrix})^{\odot k}}{k!}\right).$$

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By Lemma 1 and the homogeneity of the phs-functional $||.||$,

$$\zeta_{||.||}(\begin{pmatrix} x \\ 0 \end{pmatrix}) = \sum_{n=1}^{\infty}(\mathbf{1} \oslash (\sum_{k=0}^{\infty} \frac{(\ln n)^k ||x\mathbf{1}||^k}{k!||\mathbf{1}||}\mathbf{1}))$$

$$= \sum_{n=1}^{\infty}(\mathbf{1} \oslash (\sum_{k=0}^{\infty} \frac{(x \ln n)^k}{k!}\mathbf{1})) = \sum_{n=1}^{\infty}(\mathbf{1} \oslash n^x \mathbf{1}) = (\sum_{n=1}^{\infty} \frac{1}{n^x})\mathbf{1}.$$

Similarly,

$$\zeta_{||.||}(\begin{pmatrix} 0 \\ y \end{pmatrix}) = \sum_{n=1}^{\infty}\left(\mathbf{1} \oslash (\sum_{k=0}^{\infty} \frac{(y \ln n\ \boldsymbol{I})^{\odot k}}{k!})\right)$$

$$= \sum_{n=1}^{\infty}(\mathbf{1} \oslash \begin{pmatrix} \cos(y \ln n) \\ \sin(y \ln n) \end{pmatrix}) = \sum_{n=1}^{\infty}\begin{pmatrix} \cos(y \ln n) \\ -\sin(y \ln n) \end{pmatrix}.$$

□

Theorem 8. *For* $\begin{pmatrix} x \\ y \end{pmatrix} = \tau \begin{pmatrix} \cos_S \varphi \\ \sin_S \varphi \end{pmatrix}$ *with* $\tau = ||\begin{pmatrix} x \\ y \end{pmatrix}||$*, the generalization of the vectorial revised Zeta function with respect to the functional* $||.||$ *allows the representation*

$$\zeta_{||.||}(\begin{pmatrix} x \\ y \end{pmatrix}) = \sum_{n=1}^{\infty}(\mathbf{1} \oslash (\sum_{k=0}^{\infty} \frac{(||\begin{pmatrix} x \\ y \end{pmatrix}|| \ln n)^k}{k!} \begin{pmatrix} \cos_S(k\varphi) \\ \sin_S(k\varphi) \end{pmatrix})).$$

Proof. This can be seen as follows:

$$\zeta_{||.||}(\begin{pmatrix} x \\ y \end{pmatrix}) = \sum_{n=1}^{\infty}(\mathbf{1} \oslash (\exp_{||.||}((\ln n)\begin{pmatrix} x \\ y \end{pmatrix}))) = \sum_{n=1}^{\infty}(\mathbf{1} \oslash (\sum_{k=0}^{\infty} \frac{1}{k!}((\ln n)\begin{pmatrix} x \\ y \end{pmatrix})^{\odot k})).$$

By Lemma 1 and the homogeneity of the phs functional $||.||$,

$$\zeta_{||.||}(\begin{pmatrix} x \\ y \end{pmatrix}) = \sum_{n=1}^{\infty}(\mathbf{1} \oslash (\sum_{k=0}^{\infty} \frac{1}{k!}||\begin{pmatrix} x \\ y \end{pmatrix}||^k (\ln n)^k \frac{\begin{pmatrix} \cos(k\varphi) \\ \sin(k\varphi) \end{pmatrix}}{||\begin{pmatrix} \cos(k\varphi) \\ \sin(k\varphi) \end{pmatrix}||})).$$

□

We conclude this section with the brief remark that each point $\begin{pmatrix} \cos_S(k\varphi) \\ \sin_S(k\varphi) \end{pmatrix}$ belongs to the $||.||$-generalized unit circle S.

5. Discussion

Because of the apparent lack of mathematical rigor in the usual way of introducing complex numbers, the consistent vector space approach was established in [18]. In this approach, known statements regarding complex numbers occasionally need to be adjusted slightly, but others require noticeable change. This was shown for the area of Fourier transformations of probability densities in [19] and is also evident in the present work. For example, the real number 1 is not a complex number, so that the expression $\frac{1}{z}, z \in R^2$ is strictly speaking unexplained. But the slightly adjusted ratio of complex numbers

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \oslash \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{||\begin{pmatrix} x \\ y \end{pmatrix}||^2} \begin{pmatrix} 1 \cdot x + 0 \cdot y \\ x \cdot 0 - y \cdot 1 \end{pmatrix} = \frac{\begin{pmatrix} x \\ -y \end{pmatrix}}{||\begin{pmatrix} x \\ y \end{pmatrix}||^2}.$$

is well defined, which has been used in Section 3. Sometimes similar adjustments do not have a significant impact but sometimes they do and it is not always possible to predict in advance which of the two situations will occur.

Some authors encourage their readers to deal with usual complex numbers on the grounds that this is possible without contradiction. Quite often, however, they do not add whether this statement is mathematically proven or empirically observed. Every reader knows, of course, that measuring ten diameters of a Reuleaux triangle does not prove that a Reuleaux triangle is a circle. So why not look for a proof of the above statement? If one tries to provide such a proof, one seems to be forced to make use of the inadmissible equation of $(0,1)^T$ with i.

Another message from some authors is that some users of "imaginary numbers" "simply use them formally without thinking about further justifications". This, however, should not tempt any user to deal with mathematical problems according to the "Third Cologne Basic Law": "it still went well".

Lie groups have been widely studied in the literature, particularly for matrices. In connection with the generalization of complex numbers, groups acting in general vector spaces may be of interest. However, we limit ourselves here to groups that act in generalized circles belonging to R^2. Several results from the literature now appear as particular cases of the present general result for circles with respect to arbitrary phs-functionals such like norms, antinorms or semi-antinorms.

It may be useful to consider and mathematically describe various applied problems in different algebraic structures. Experts in science, technology and other fields will appropriately contribute their experience to the modeling. Occasionally a specific application or even a large non-mathematical theory will simply make use of existing mathematical methods, but in other cases the application will stimulate the development of new techniques. The knowledge of the large variety of complex algebraic structures and their corresponding Lie groups considered in the present work may motivate, for example, to give a broader meaning to the well-known notion of Riemann's Zeta function. In the best case scenario, the present results can stimulate further consideration. Finally, we recall Hankel's [15] statement that it was the authority of Gauss that gave complex numbers full rights to exist. Here we have followed the Gaussian interpretation [11] of complex numbers as points in a number plane, but even in the sense of a suitably adopted axiom.

6. In Commemoration of the 100th Anniversary of Gottlob Frege's Death

If one were to call I an imaginary unit, then Example 3 would show that this term would not be uniquely defined. The use of the definite article in "the imaginary unit" would then be wrong. Authors in the field of ordinary complex numbers, on the other hand, usually suggest the impression of the unique determination of an "imaginary number i". The following quotations are intended to prove this and the following examples illustrate it once again impressively. The very deserving basic researcher Gottlob Frege from Wismar in Mecklenburg-Vorpommern commented on such questions already in the middle of the last century, which is highlighted here on the above-mentioned reason.

"... It is not immaterial to the cogency of our proof whether '$a + bi$' has a sense or is nothing more than printer's ink. It will not get us anywhere simply to require that it have sense, or to say that it is to have the sense of the sum of a and bi, when we have not previously defined what 'sum' means in this case and when we have given no justification for the use of the definite article."[9]

Is mathematics in 2019 further ahead on this question
than Frege was?

"...Frege never developed an account of complex numbers... As I will show, we can be quite confident from what little he does say that Frege intended his logicist program to extend to complex numbers... Frege is famous for his logicism. This is not a doctrine about mathematics generally, but only about one part of it: arithmetic, the science that studies numbers... Frege endorsed a very special form of logicism, what Dummett calls platonistic logicism. This is the thesis that numbers are purely logical objects. To call something a 'logical object' in Frege's sense is to say that it is an object whose ***existence and uniqueness can be proven***..."[2].

To discuss existence and uniqueness in the area of complex numbers, let V be a two-dimensional vector space, $\oplus : V \times V \to V$ the vector space addition, $\otimes : V \times V \to V$ another binary operation, $\cdot : R \times V \to V$ multiplication of a vector by a real scalar, $\mathfrak{o}$ and $\mathbb{1}$ the elements being neutral with respect to the operations $\oplus$ and $\otimes$, respectively, and let there hold $\boldsymbol{I} \otimes \boldsymbol{I} = (-1) \cdot \mathbb{1}$. We call $\mathfrak{C} = (V, \oplus, \otimes, \cdot, \mathfrak{o}, \mathbb{1}, \boldsymbol{I})$ an abstract complex algebraic structure.

Example 4. *Complex plane, revisited: Let* $V = R^2$, $\mathfrak{o} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\mathbb{1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\boldsymbol{I} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ *and* $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \otimes \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1x_2 - y_1y_2 \\ x_2y_1 + x_1y_2 \end{pmatrix}$.

The mathematical existence of the abstract complex algebraic structure is already proven by the indication of this example. The next example shows that there is not only one realization of the general complex algebraic structure $\mathfrak{C}$.

Example 5. *Let* $V = \{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, a \in R, b \in R\}$ *and* $\oplus, \otimes, \cdot$ *denote usual matrix addition and multiplication as well as multiplication of a matrix by a scalar, respectively,* $\mathfrak{o} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\boldsymbol{I} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Note that we avoid the common "identification" error of the complex analysis literature, which states that "obviously $\boldsymbol{I} \otimes \boldsymbol{I} = -1$."

If one wanted to call $\boldsymbol{I}$ "imaginary unit", then our examples show that it can take different forms. One may now miss the usual relationship $i^2 = -1$, which is deliberately called alchemical or mystical, here. Let us recall that nothing is said in the mathematical literature about the existence of i, nothing about its uniqueness and nothing about its possible diversity. Exchanging such a nebulous quantity for a tangible one should be one of the concerns of mathematics.

7. Conclusion

If one considers mathematical formulations using single or even multiple imaginary units as constructive in principle and yet alchemical, then the results obtained by the author since 2020 show that and how the alchemical aspects can be eliminated from the considered algebraic structures. This opens up a challenging and far-reaching mathematical work program.

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