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Not peer-reviewed version

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Posted Date: 25 February 2025

doi: 10.20944/preprints202411.1062.v4

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Article

On the Method for Proving the RH Using the Alcantara-Bode Equivalence

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Abstract: Among the equivalents of the Riemann Hypothesis (RH), a millennium unsolved problem ([7]), the Alcantara-Bode equivalent ([2]) is of a particular interest due to its formulation: the Riemann Hypothesis holds iff the null space of the integral operator on $L^2(0, 1)$ having the kernel function the fractional part of the ratio (y/x) , contains only the element 0 ([2]), i.e. iff the operator is injective. This equivalent formulation allows us to use techniques outside of the number theory field in order to prove it and so, to show that RH holds. We introduced in this article a functional analysis - numerical method for investigating the injectivity of the linear bounded operators on separable Hilbert spaces, updating and extending the result from [1]. The method is built around the result obtained (Theorem 1) saying that if a linear bounded operator is strict positive on a dense set, then it is injective. When the operator - or its associated Hermitian replacing it if needed, is only positive definite on a dense set, additional operator properties should be considered. Dealing with this case, two are the directions we choose for obtaining the corresponding criteria, using the operator approximations over finite dimension subspaces in $L^2(0, 1)$ whose union is dense: · involving the operator finite rank approximations on subspaces or, · involving its adjoint restrictions on subspaces. (Injectivity Criteria [1]). Applying both versions of the method on the dense set of indicator interval functions, we proved the Alcantara-Bode equivalent is true so, that RH holds. This solution for RH is not one in pure math. field as seems to have been expected since 1859. However, it is in line with the Clay Math Inst. principle that has been expressed (citing [7]) by: "A proof that it is true for every interesting solution would shed light on many of the mysteries surrounding the distribution of prime numbers."

Keywords: integral operators; hilbert spaces; approximation methods; multi-level discretizations; riemann hypothesis

MSC: 31A10; 45P05; 47G10; 65R99; 11M26

1. Introduction

For those who need an answer to RH

The Riemann Hypothesis (RH) claims (1859) that the Riemann Zeta function defined by the infinite sum $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$ has its non trivial zeros on the vertical line $\sigma = 1/2$ ([7], [11]). The Hilbert-Schmidt integral operator T_ρ defined on $L^2(0, 1)$ having the kernel function $\rho(y, x) = \{y/x\}$, the fractional part of the quantity between brackets,

$$(T_\rho u)(y) = \int_0^1 \rho(y, x) u(x) dx, \quad u \in L^2(0, 1) \quad (1)$$

has been used by Alcantara-Bode ([2], pg. 151) in his theorem of the equivalent formulation of the RH. Denoting by N_{T_ρ} the null space of this operator, the equivalent formulation consists in:

The Riemann Hypothesis holds if and only if $N_{T_\rho} = \{0\}$.

A connection between the Zeta function and the integral operator T_ρ could be observed ([4], pg. 313) reformulating the left term in the equality:

$$\int_0^1 \rho(\theta/x) x^{s-1} dx = \theta/(s-1) - \theta^s \zeta(s)/s, \quad \sigma > 0, s = \sigma + it$$

as $(T_\rho x^{s-1})(\theta)$. The equivalent formulation of RH given by Beurling in [4] has been used by Alcantara-Bode in his equivalence ([2]). Both outstanding equivalences reduced RH, a problem from the pure math still unsolved since 1859 and listed on *Millennium Problems* ([7], [16]), to a concrete problem: the injectivity of the Hilbert-Schmidt integral operator T_ρ defined in (1). Its kernel function $\rho \in L^2(0, 1)^2$ defined by the fractional part of the ratio (y/x) is continue almost everywhere, the discontinuities in $(0, 1)^2$ consisting in a set of numerable one dimensional lines of the form $y = kx, k \in \mathbb{N}$ and so, being of the Lebesgue measure zero. The integral operator is Hilbert-Schmidt ([2]) and so compact, allowing us to consider its approximations on finite dimension subspaces ([3]).

2. Two Injectivity Theorems and Approximations on Subspaces

Let H be a separable Hilbert space and T be a linear bounded operator on H positive on a dense set $S \subset H$: $\langle Tv, v \rangle > 0 \forall v \in S$. Then it has no zeros in the dense set otherwise, if there exists $w \in S$ such that $Tw = 0$ then $\langle Tw, w \rangle = 0$ contradicting the positivity of T on S . Follows: its 'eligible' zeros are all in the difference set $E := H \setminus S$, i.e. $N_T \subset E$. In order to avoid any misunderstanding, we mention that on finite dimension subspaces the positivity and strict positivity of an operator coincide. On a infinite dimension (sub)space S , the strict positivity involves a constant $C > 0$ for which $\langle Tv, v \rangle \geq C\|v\|^2, \forall v \in S$ and T is only positive when $\langle Tv, v \rangle > 0, \forall v \in S$. The norm on H is the norm induced by the inner product.

The next theorem (introduced in [11]) proves that if T is strict positive definite on the dense set it has no zeros in the difference set either, obtaining $N_T = \{0\}$.

Theorem 1. *A linear bounded operator T strict positive definite on a dense set of a separable Hilbert space is injective, equivalently $N_T = \{0\}$.*

Proof. Let's take in consideration only the set of eligible zeros that are on the unit sphere without restricting the generality, once for an element $w \in H, w \neq 0$ and for $w/\|w\|$ both are or both are not in N_T . The set $S \subset H$ is dense if its closure coincides with H . Then, if $w \in E$, for every $\varepsilon > 0$ there exists $u_{\varepsilon,w} \in S$ such that $\|w - u_{\varepsilon,w}\| < \varepsilon$. If $\|w\| \geq \|u_{\varepsilon,w}\|$:

$$0 \leq \|w\| - \|u_{\varepsilon,w}\| = \|w - u_{\varepsilon,w} + u_{\varepsilon,w}\| - \|u_{\varepsilon,w}\| \leq \|w - u_{\varepsilon,w}\| + \|u_{\varepsilon,w}\| - \|u_{\varepsilon,w}\| < \varepsilon.$$

If $\|u_{\varepsilon,w}\| \geq \|w\|$ instead, then: $0 \leq \|u_{\varepsilon,w} - w + w\| - \|w\| < \varepsilon$ to obtain:

given $w \in E$, for every $\varepsilon > 0$ there exists $u_{\varepsilon,w} \in S$ such that

$$\| \|w\| - \|u_{\varepsilon,w}\| \| < \varepsilon \quad (2)$$

Let w be an eligible element from the unit sphere, $\|w\| = 1$ and take $\varepsilon_n = 1/n$. Then there exists at least one element in the dense set $u_{\varepsilon_n,w} \in S$ such that $\|u_{\varepsilon_n,w} - w\| < \varepsilon_n$ holds. Follows from (2), $1 - \|u_{\varepsilon_n,w}\| < 1/n$ showing that, for any choices of a sequence approximating w , $u_{\varepsilon_n,w} \in S, n \geq 1$, it verifies $\|u_{\varepsilon_n,w}\| \rightarrow 1$.

If T is a linear bounded operator on H strict positive on S , then there exists $\alpha > 0$ such that $\forall u \in S$, $\langle Tu, u \rangle \geq \alpha\|u\|^2$.

Suppose that there exists $w \in E, \|w\| = 1$ a zero of T , i.e. $w \in N_T$ and consider a sequence of approximations of w , $u_{\varepsilon_n,w} \in S, n \geq 1$ that, as we showed, has its normed sequence converging in norm to 1. From the positivity of T on the dense set S , follows:

$$\alpha\|u_{\varepsilon_n,w}\|^2 \leq \langle Tu_{\varepsilon_n,w}, u_{\varepsilon_n,w} \rangle = \langle T(u_{\varepsilon_n,w} - w), u_{\varepsilon_n,w} \rangle < \varepsilon_n \|T\| \|u_{\varepsilon_n,w}\| \quad (3)$$

With $c = \|T\|/\alpha$, we obtain $\|u_{\varepsilon_n, w}\| \leq c/n$. Then, $\|u_{\varepsilon_n, w}\| \rightarrow 0$ with $n \rightarrow \infty$, in contradiction with its convergence $\|u_{\varepsilon_n, w}\| \rightarrow 1$ with $n \rightarrow \infty$.

Or, this happen for any choice of the sequence of approximations of w , verifying $\|w - u_{\varepsilon_n, w}\| < \varepsilon_n$, $n \geq 1$, when $Tw = 0$.

Thus $w \notin N_T$, valid for any $w \in E$, $\|w\| = 1$, proving the theorem because no zeros of T there are in S either. $\square$

From now on, we suppose that:

- The dense set S is the result of an union of finite dimension subspaces of a family F : $S = \bigcup_{n \geq 1} S_n$, $\overline{S} = H$. It is not mandatory but will ease our proofs considering that the subspaces are including: $S_n \subset S_{n+1}$, $n \geq 1$.

- The linear operator on H is positive definite on the dense set S . A such operator has no zeros in the dense set, otherwise if $u \in S$ then there exists a subspace containing it such that $Tu = 0$ attracting the violation of its positivity: $\langle Tu, u \rangle = 0$.

Observation 1. Let $\beta_n(u) := \|u - u_n\|$ be the normed residuum of the eligible element $u \in E$ after its orthogonal projection on S_n . Then, $\beta_n(u) \rightarrow 0$ with $n \rightarrow \infty$.

Proof. Given $\epsilon > 0$, from the density of the set S in H there exists $u_\epsilon \in S$ verifying $\|u - u_\epsilon\| < \epsilon$, as per the observations made in the proof of the Theorem 1. Let S_{n_ϵ} be the coarsest subspace, i.e. with the smallest dimension, from the family of subspaces containing u_ϵ . Because the best approximation of u in S_{n_ϵ} is its orthogonal projection, we obtain

$$\beta_{n_\epsilon}(u) := \|u - P_{n_\epsilon}u\| \leq \|u - u_\epsilon\| < \epsilon,$$

inequality valid for every $\epsilon > 0$, proving our assertion. $\square$

Theorem 2. Let T be a linear bounded operator on H , positive definite on a dense set S result of the union of including finite dimension subspaces of a family F . Consider $\{T_n, n \geq 1\}$ be a sequence of its approximations on the family F . If

$$i) \langle T_n v, v \rangle \geq \alpha_n \|v\|^2, \forall v \in S_n, \text{ with } \alpha_n \geq \alpha > 0, n \geq 1;$$

$$ii) \epsilon_n := \|T - T_n\| \rightarrow 0 \text{ with } n \rightarrow \infty,$$

then $N_T = \{0\}$.

Proof. Being positive on S , the operator has no zeros in the dense set. Showing now, that T has no zeros in the difference set. For $u \in E := H \setminus S$, $\|u\| = 1$ having the not null orthogonal projections $P_n u, n \geq n_0 := n_0(u)$, from Observation 1 we have its residuum sequence on the approximation subspaces $\beta_n := \beta_n(u) = \|u - P_n u\| \rightarrow 0$.

If there exists $u \in N_T \cap E$, then:

$$\begin{aligned} \alpha_n \|P_n u\|^2 &\leq \langle T_n P_n u, P_n u \rangle \leq \|T_n P_n u\| \|P_n u\| \\ &\leq (\|T_n P_n u - T P_n u + T P_n u - Tu\|) \|P_n u\| \\ &\leq (\|T - T_n\| \|P_n u\| + \|T\| \|u - P_n u\|) \|P_n u\|, \end{aligned}$$

obtaining

$$\alpha \leq (\epsilon_n + \|T\| \beta_n / \sqrt{1 - \beta_n^2}) \rightarrow 0.$$

The inequality is violated from an $n_1 \geq n_0$, $n \geq n_1$ involving $u \notin N_T$, valid for any supposed zero of T in E . Once T has no zeros in the dense set, $N_T = \{0\}$. $\square$

Lemma 1 (Finite rank approximations). Let T be positive definite on the dense set S . If $\{P_n^r, n \geq 1\}$ is a sequence of finite rank operator orthogonal projections converging strong to identity operator such that:

$$\langle P_n^r(T) v, v \rangle \geq \alpha_n \|v\|^2, \alpha_n \geq \alpha > 0 \forall v \in S_n, \text{ for every } S_n \in F,$$

then T is injective.

Proof. Taking $T_n = P_n^r(T), n \geq 1$ then

$\|T - T_n\| = \|(I - P_n^r)T\| \leq \|T\|\|I - P_n^r\| \rightarrow 0$. Follows: both requests of the Theorem 2 are fulfilled proving our assumption.

In Note 2(Appendix) we showed that a such operator satisfies the requests of the generic Theorem 1 being strict positive on the dense set. $\square$

3. Dense Sets in $L^2(0,1)$

The intervals of equal lengths $h = 2^{-m}, m \in N, nh = 1, \Delta_{h,k} = ((k-1)/2^m, k/2^m], k = 1, n-1$ together with $\Delta_{h,n}$ are defining for $m \geq 1$ a partition of $(0,1), k=1,n, n = 2^m, nh = 1$. Consider the interval indicator functions having the supports these intervals $(k=1,n), nh=1$:

$$I_{h,k}(t) = 1 \text{ for } t \in \Delta_{h,k} \text{ and } 0 \text{ otherwise} \quad (4)$$

The family F of finite dimensional subspaces S_h that are the linear spans of interval indicator functions of the h -partitions defined by (4) with disjoint supports, $S_h = \text{span}\{I_{h,k}; k = 1, n, nh = 1\}$, built on a multi-level structure, are including $S_h \subset S_{h/2}$ by halving the mesh h . In fact, due to (4) any $I_{h,k} \in S_h$ is contained in $S_{h/2}$ by $I_{h,k}(t) = I_{h/2,2k-1}(t) + I_{h/2,2k}(t)$ for any $t \in (0,1), k=1,n, nh=1$ (or $(2n)(h/2)=1$). So, $S_h \ni u_h = \sum_{k=1,n} c_k^h I_{h,k} = \sum_{j=1,2n} c_j^{h/2} I_{h/2,j} \in S_{h/2}$ with $c_k^h := c_{2k-1}^{h/2} = c_{2k}^{h/2}, k = 1, n$; equivalently, $c_{j-1}^{h/2} = c_j^{h/2} = c_j^h$, for $j/2 = 1, n, nh = 1$.

We now consider the indicator open-interval functions $I_h^k, k = 1, n, nh = 1$ having the supports $\Delta_h^k = ((k-1)/2^m, k/2^m), k = 1, n$ that differ from the semi-open intervals $\Delta_{h,k}$ of the defined partitions only by the end-points $kh, k = 1, (n-1)$. Denoting with F^0 the family of subspaces S_n^0 spanned by the open interval indicator functions $\{I_h^k, k = 1, n, nh = 1\}$, and denoting with $S^0 = U_{n \geq 2} S_n^0$, then from $\int_0^1 I_h^k(t) dt = \int_0^1 I_{h,k}(t) dt, k = 1, n, n \geq 2, nh = 1$, we obtain $\|I_h^k - I_{h,k}\| = 0$ in $L^2(0,1)$; moreover:

Remark 2. If one of the sets S and S^0 is dense, then another is dense.

Proof. Let S^0 be dense, known in literature. If f is orthogonal on any $I_{h,k} \in S$ then:

$|\langle f, I_h^k \rangle| = |\langle f, I_h^k - I_{h,k} \rangle| \leq \|f\| \|I_h^k - I_{h,k}\| = 0, k=1,n, \forall n, nh = 1$, proving that f is orthogonal to any I_h^k and so f should be 0 because S^0 is dense. So, S is dense. Now, the reverse statement is also true, if S is dense then S^0 is dense using the same arguments. $\square$

Note 1. Because on every level of discretization any pair of indicator functions in each dense set have disjoint supports, then on S (or on S^0):

- i) for any $\varphi \in L^2(0,1)^2, I_{h,k}(y)\varphi(y,x)I_{h,j}(x) = 0$, for every $k \neq j$;
- ii) because $\sum_{i=1,n} I_{h,i}(t) = 1$ for every $t \in (0,1), \int_0^1 \int_0^1 I_{h,k}(y)\varphi(y,x)I_{h,j}(x) = 0$ for $k \neq j$, and $\int_0^1 \int_0^1 \varphi(y,x)I_{h,j}(x) = \int_{\Delta_{h,j}} \int_{\Delta_{h,j}} \varphi(y,x)I_{h,j}(x)$, valid also for replacing the arguments of φ .
- iii) Let $M_h^0(T_\varphi)$ and $M_h(T_\varphi)$ be the matrix representations of the integral operator on the finite dimension subspaces in F^0 and F spanned by the corresponding orthogonal indicator interval functions.

Then both matrices M_h^0 and M_h are identical and sparse diagonal matrices with the entries verifying

$$d_{kj}^h := \langle T_\varphi I_{h,k}, I_{h,j} \rangle = \int_{\Delta_{h,k}} \int_{\Delta_{h,j}} I_{h,k}(x)\varphi(y,x)I_{h,j}(y) dx dy = 0 \text{ for } k \neq j \text{ and,}$$

Remark 3. If $d_{kk}^h := \int_{\Delta_{h,k}} \int_{\Delta_{h,k}} \rho(y,x) dx dy > 0, k = 1, n, nh = 1$ then

- on every level, the matrix representations of its restrictions on S_h and S_h^0 are diagonal, identical and strict positive definite;

- the integral operator T_ρ defined in (1) is positive definite on the dense sets S and S^0 .

Proof. From Note 1-ii), both matrices are sparse diagonal on every level h , identical and strict positive.

Now, we have for $v = \sum_{k=1,n} c_k I_{h,k} \in S_h, n \geq 2, nh = 1$ from $\langle T_\rho I_{h,k}, I_{h,k} \rangle = \langle T_\rho I_h^k, I_h^k \rangle := d_{kk}^h$:

$$\langle T_\rho v, v \rangle = \sum_{k=1,n} c_k \bar{c}_k d_{kk}^h = (c_1, \dots, c_n) M_h(T_\rho) (\bar{c}_1, \dots, \bar{c}_n)^T$$

$= (c_1, \dots, c_n)[\text{diag}(d_{kk}^h)](\bar{c}_1, \dots, \bar{c}_n)^T \geq h^{-1} \min_{k=1,n}(d_{kk}^h) \|v\|^2 > 0$
 Then, T_ρ is (strict) positive definite on S_h with the positivity parameter:

$$\alpha_h(T_\rho) = h^{-1} \min_{k=1,n} d_{kk}^h; \quad n \geq 2, nh = 1. \quad (5)$$

and so, $\forall v \in S_h, \langle T_\rho v, v \rangle > 0$, following: $\langle T_\rho v, v \rangle > 0, \forall v \in S$.

Because on $S_h^0, M_h^0(T_\rho) = M_h(T_\rho)$, T_ρ has its restriction strict positive definite with the same parameter $\alpha_h(T_\rho)$. So, if $d_{kk}^h > 0, k = 1, n, nh = 1$, then T_ρ is positive definite on the dense sets S and S^0 . $\square$

4. The Proof of the RH Equivalent

The entries in the diagonal matrix representation $M_h(T_\rho)$ of the integral operator T_ρ restriction to $S_h \in F, d_{kk}^h = \int_{\Delta_{h,k}} \int_{\Delta_{h,k}} \rho(y, x) dx dy$, are valued (see [1]) as follows:

$$d_{11}^h = h^2(3 - 2\gamma)/4$$

$$d_{kk}^h = \int_{\Delta_{h,k}} \left[\int_{\Delta_{h,k}} (y/x) dx - \int_{(k-1)h}^y dx \right] dy = \frac{h^2}{2} \left(-1 + \frac{2k-1}{k-1} \ln\left(\frac{k}{k-1}\right) \right)^{k-1} \quad (6)$$

for $k \geq 2$, where γ is the Euler-Mascheroni constant ($\simeq 0.5772156$). The sequence

$f(k) := h^{-2} d_{kk}^h = (-1 + \frac{2k-1}{k-1} \ln(\frac{k}{k-1}))/2$ is monotone decreasing for $k \geq 2$ and converges to 0.5 for $k \rightarrow \infty$. Then: $\forall h, d_{kk}^h > h^2/2 > d_{11}^h, k \geq 2$.

So, the Remark 3 is fulfilled T_ρ being positive definite on the dense set S with the positivity parameters of the integral operator restrictions on S_h in (5) given by

$\alpha_h(T_\rho) = h(3 - 2\gamma)/4 > 0, n \geq 2, nh = 1$. Then we will take in consideration the finite rank approximations of the integral operator on the same dense set in order to apply Lemma 1.

Citing [5], (pg 986), the integral operator $P_h^r, n \geq 1$ having the kernel function:

$$r_h(y, x) = h^{-1} \sum_{k=1,n} I_{h,k}(y) I_{h,k}(x) \quad (7)$$

is an finite rank integral operator orthogonal projection having the spectrum $(\{0, 1\})$ with the eigenvalue 1 of the multiplicity n ($nh=1$) corresponding to the orthogonal eigenfunctions $I_{h,k}, k = 1, n$.

Thus, $\|I - P_h^r\| \rightarrow 0$ for $n \rightarrow \infty, nh = 1$. As result, the integral operator approximation of T_ρ on S_h is an finite rank operator approximation, T_{ρ_h} , having the kernel function ([5])

$$\rho_h(y, x) = h^{-1} \sum_{k=1,n} I_{h,k}(y) \rho(y, x) I_{h,k}(x) := h^{-1} \sum_{k=1,n} \rho_h^k(y, x) \quad (8)$$

From $\langle T_{\rho_h} I_{h,k}, I_{h,k} \rangle = h^{-1} d_{kk}^h$, we have $M_h(T_{\rho_h}) = h^{-1} M_h(T_\rho) := h^{-1} \text{diag}[d_{kk}^h]; k = 1, n, nh = 1$, and the positivity parameter of the finite rank approximation of T_{ρ_h} is a result of: for $v_h = \sum_{k=1,n} c_k I_{h,k} \in S_h$,

$$\begin{aligned} \langle T_{\rho_h} v_h, v_h \rangle &= h^{-1} \sum_{k=1,n} c_k \bar{c}_k \int_{\Delta_{h,k}} \int_{\Delta_{h,k}} I_{h,k} \rho(y, x) I_{h,k} dx dy = h^{-1} \sum_{k=1,n} c_k \bar{c}_k d_{kk}^h \\ &\geq h^{-2} (\min_{k=1,n} d_{kk}^h) \|v_h\|^2 := \alpha_h(T_{\rho_h}) \|v_h\|^2 \end{aligned} \quad (9)$$

Follows:

$$\alpha_h(T_{\rho_h}) := h^{-2} d_{11}^h = (3 - 2\gamma)/4, \text{ a constant mesh independent.} \quad (10)$$

Theorem 3. (Finite Rank Approximations): The Alcantara-Bode equivalent holds and, the Riemann Hypothesis is true.

Proof. From (10), the sequence of the positivity parameters corresponding to the finite rank operator approximations T_{ρ_h} on the dense family F is inferior bounded, obtaining from Lemma 1 that $N_{T_\rho} = \{0\}$.

Now, obtaining $N_{T_\rho} = \{0\}$ half from equivalent formulation of the Alcantara-Bode is holds. Then, the other half should be true i.e.: the Riemann Hypothesis is true. $\square$

5. Appendix

In [1] we introduced the Injectivity Criteria method on the discretizations of the operator like in [5]. At that time, we did not have the Theorem 1 nor the finite rank operator approximations defined for our scope. Here we will use this method considering the operator restrictions on the subspaces F^0 taking advantage of the correlations between both dense sets S and S^0 obtained previously.

Next, with Note 2 we will show that the finite rank approximations is a method that satisfies the requests of the generic Theorem 1.

Lemma 2. (Injectivity Criteria for operator restrictions.) Let T be an bounded linear operator positive definite on the dense set S in the separable Hilbert space, obtained by the union of finite dimension including subspaces of the family F having its positivity parameters sequence corresponding to the operator restrictions on the family subspaces inferior unbounded, i.e.:

$$\langle Tv, v \rangle \geq \alpha_n(T) \|v\|^2 \quad \forall v \in S_n, \text{ with } \alpha_n(T) \rightarrow 0 \text{ and, consider}$$

$$\mu_n := \alpha_n(T) / \omega_n \text{ where } \omega_n \text{ verifies } \|T^*v\| \leq \omega_n \|v\|, \forall v \in S_n, n \geq 1.$$

If there exists $C > 0$ such that $\mu_n \geq C$, for every $n \geq 1$, then $N_T = \{0\}$.

Proof. Suppose that there exists $u \in (H \setminus S) \cap N_T$, $\|u\| = 1$ and let u_n its orthogonal projection on S_n , $n \geq 1$. Then, from the strict positivity of T on the subspaces S_n , $n \geq 1$ (see (3)):

$$\alpha_n(T) \|u_n\|^2 \leq \langle Tu_n, u_n \rangle = \langle T(u_n - u), u_n \rangle = \langle (u_n - u), T^*u_n \rangle \leq \beta_n \omega_n \|u_n\|$$

Then, from

$C \leq \mu_n \leq \beta_n / \sqrt{1 - \beta_n^2} \rightarrow 0$ where $\beta_n := \beta_n(u) = \|u - u_n\|$, we obtain a contradiction. Thus, $u \notin N_T$ for any $u \in H \setminus S$. Follows: $N_T = \{0\}$. $\square$

With $H := L^2(0, 1)$ and taking the family F^0 in place of S , on S_h^0 the positivity parameter in (5) is valued above by $\alpha_h(T_\rho) = h(3 - 2\gamma)/4$, we could go directly by invoking the adjoint operator of T_ρ .

In order to apply Lemma 2 we have to invoke the adjoint operator whose kernel function is defined by $\rho^*(y, x) = \overline{\rho(x, y)} = \rho(x, y)$. For $v_h = \sum_{k=1,n} c_k I_h^k \in S_h^0$ and from Note 1-i):

$$T_\rho^* v_h = \sum_{k=1,n} c_k \int_{\Delta_h^k} \rho(x, y) I_h^k(y) dy = \sum_{k=1,n} c_k \int_{\Delta_h^k} \rho_h^k(x, y) dy,$$

where $\rho_h^k = I_h^k(x) \rho(x, y) I_h^k(y)$. Follows:

$$\begin{aligned} \|T_\rho^* v_h\|^2 &= \langle \sum_{k=1,n} c_k \int_{\Delta_h^k} \rho_h^k(x, y) dy, \sum_{j=1,n} c_j \int_{\Delta_h^j} \rho_h^j(x, y) dy \rangle \\ &= \sum_{k=1,n} c_k \overline{c_k} \left(\int_{\Delta_h^k} \left[\int_{\Delta_h^k} \rho(x, y) I_h^k(y) dy \right]^2 I_h^k(x) dx \right). \end{aligned}$$

Because $\rho(x, y)$ is valued in $[0, 1]$, $\rho(y, x) < 1$ for every $x, y \in (0, 1)$, we obtain:

$$\|T_\rho^* v_h\|^2 \leq \sum_{k=1,n} c_k \overline{c_k} h^3 = h^2 \|v_h\|^2 \text{ and, } \|T_\rho^* v_h\| \leq h \|v_h\|.$$

Taking $\omega_h(T_\rho^*) = h$, the injectivity parameter of T on S_h^0 is given by:

$$\mu_h = (3 - 2\gamma)/4, \quad \text{a mesh independent constant } \forall n, nh = 1 \quad (11)$$

Theorem 4. (Injectivity Criteria): The Alcantara-Bode equivalent of RH holds.

Proof. Because μ_h is a constant (see (11)) for any $h, nh = 1$, applying Lemma 2 we obtain $N_{T_\rho} = \{0\}$. $\square$

Observation 2. We could obtain the positivity parameters for the restriction operators as follows. Denoting with $P_h : E \subset H \rightarrow S_h^0 \subset H$ the orthogonal projection onto S_h^0 , then:

$$E \ni u \rightarrow u_h := P_h u = \sum_{k=1,n} I_h^k \frac{1}{|\Delta_h^k|} \int_{\Delta_h^k} u(s) ds \in S_h^0 \text{ holds. Or,}$$

$$u_h(t) := (P_h u)(t) = h^{-1} \sum_{k=1,n} c_k I_h^k(t); \quad c_k = \int_{\Delta_h^k} u(s) ds \quad (12)$$

For obtaining the positivity parameters, we have to evaluate the inner product between $(T_\rho u_h) \notin S_h^0$ and $u_h \in S_h^0$ applying the formula (12), where u_h is the not null orthogonal projection of an eligible $u \in E := H \setminus S$. Doing it, we will retrieve (5) and from (6) on S^0 the positivity parameters of T_ρ on $S_h^0, n \geq 2, nh = 1$ are:

$$\alpha_h(T_\rho) = h^{-1}d_{11}^h = h(3 - 2\gamma)/4 \rightarrow 0 \quad (13)$$

showing that T_ρ is (only) positive definite on the dense set S^0 .

Note 2. An operator satisfying Lemma 1 is strict positive on the dense set and so, being one from the class of the linear operators satisfying the Theorem 1.

Proving it as follows. From the convergence to zero of the sequence $\epsilon_n, n \geq 1$ there exists ϵ_0 a 'compactness' parameter verifying $\epsilon_0 := \max_n \{\epsilon_n; \epsilon_n < \alpha\}$ corresponding to a subspace $S_{n_0}, n_0 < \infty$. The parameter ϵ_0 is independent from any $v \in S$.

For an arbitrary $v \in S$ there exists a coarser subspace (i.e. with a smaller dimension) $S_n, n \geq n_0 := n_0(v)$, for which $v \in S_n$. We have:

$\langle Tv, v \rangle = \langle T_n v, v \rangle - \langle (T_n - T)v, v \rangle > 0$. Because T and T_n are positive definite on S both inner products being real valued, $\langle T_n v, v \rangle > \langle (T_n - T)v, v \rangle$.

Reminding that from subspaces including property, $v \in S_m, \forall m \geq n_0$ and $\langle T_m v, v \rangle$ is bounded by $\alpha \|v\|^2$ for any $m \geq n_0$. Meantime, $|\langle (T_n - T)v, v \rangle| \rightarrow 0$ telling us that there exists an index n_1 such that for $n \geq n_1, \alpha \|v\|^2 \geq |\langle (T_n - T)v, v \rangle|$. Then for $n \geq n_1$:

$$\langle Tv, v \rangle \geq \alpha \|v\|^2 - |\langle (T - T_n)v, v \rangle| \geq (\alpha - \epsilon_n) \|v\|^2 \geq (\alpha - \epsilon_0) \|v\|^2$$

Again, from the subspaces including property, the previous inequality is valid for any $n \geq n_0$, just "moving" v from S_m , where $n_0 \leq m \leq n_1$ in the right subspace where the inequality holds. Moreover, because v has been arbitrary taken from the dense set, follows with $\alpha(T) := (\alpha - \epsilon_0)$ a constant:

$$\langle Tv, v \rangle \geq \alpha(T) \|v\|^2 \quad \forall v \in S, \text{ showing the strict positivity of } T \text{ on } S. \quad \square$$

If $T := T_\rho$ is an Hilbert-Schmidt integral operator, it is compact and so, it can be approximated by finite rank integral operators. These finite rank approximations denoted by $T_n, n \geq 1$ verify $\epsilon_n := \|T - T_n\| \rightarrow 0$ for $n \rightarrow \infty$ meaning, the second property in the Theorem 2 is automatically fulfilled.

Observation 3. A kernel discretization like in (8), has been used in [5], [6] for dealing with the decay rate of convergence to zero of eigenvalues corresponding to the Hermitian integral operators having the kernels like the Mercer kernels ([9]).

If one wants to use a finite element discretization in which to every node kh is associated a linear 'trial' function with the compact support $[(k-1)h, (k+1)h]$, valued to 1 in the middle node and zero in the endpoint nodes, then the resulting matrix on every level $n, nh = 1$, will be a sparse 3- or 5-diagonal one. But, we have to deal in this case at least with the density of the union of finite element subspaces.

If there exists a basis in H , let it be $\{e_i, i \in \mathbb{N}\}$ then the finite dimension subspaces could be defined as $S_n = \text{span}_{i=1,n}(e_i, i = 1, n)$ and now, we are dealing probably with full associated matrices **because** the basis orthogonality in H is not induced by their supports like in (4) or like in finite element discretizations nor, eigenvectors of the operator.

The references [13-16] are related to other RH equivalents, [8] to exotic integrals and [12] to multi-level discretizations on separable Hilbert spaces.

References

1. Adam, D., (2022) "On the Injectivity of an Integral Operator Connected to Riemann Hypothesis", *J. Pure Appl Math.* 2022; 6(4):19-23, DOI: 10.37532/2752-8081.22.6(4).19-23 (crossref:) (2021) <https://doi.org/10.21203/rs.3.rs-1159792/v9>
2. Alcantara-Bode, J., (1993) "An Integral Equation Formulation of the Riemann Hypothesis", *Integr Equat Oper Th*, Vol. 17 pg. 151-168, 1993.

3. Atkinson, K., Bogomolny, A., (1987) "The Discrete Galerkin Method for Integral Equations", *Mathematics of Computation*, Vol. 48. Nr 178, pg. 595-616 1987.
4. Beurling, A., (1955) "A closure problem related to the Riemann zeta function", *Proc. Nat. Acad. Sci.* 41 pg. 312-314, 1955.
5. Buescu, J., Paixão A. C., (2007) "Eigenvalue distribution of Mercer-like kernels", *Math. Nachr.* 280, No. 9–10, pg. 984 – 995, 2007.
6. Chang, C.H., Ha, C.W. (1999) "On eigenvalues of differentiable positive definite kernels", *Integr. Equ. Oper. Theory* 33 pg. 1-7, 1999.
7. ClayMath, Ins. (2024) "<https://www.claymath.org/millennium/riemann-hypothesis/>", 2024
8. Furdui, O., "Fractional Part Integrals", in *Limits, Series, and Fractional Part Integrals. Problem Books in Mathematics*, Springer NY, 2013.
9. Mercer, J., (1909) "Functions of positive and negative type and their connection with the theory of integral equations", *Philosophical Transactions of the Royal Society A* 209, 1909
10. Rudin, W., "Real and Complex Analysis", McGraw-Hill International, Third Edition 1987
11. Adam, D., (2024) "On the Method for Proving the RH Using the Alcantara-Bode Equivalence", doi: 10.20944/preprints202411.1062.v1.
12. Adam, D. (1994) "Mesh Independence of Galerkin Approach by Preconditioning", *Preconditioned Iterative Methods - Johns Hopkins Libraries, Lausanne, Switzerland; [Langhorne, Pa.] Gordon and Breach*, 1994.
13. Broughan, K., (2017) "Equivalents of the Riemann Hypothesis Volume One: Arithmetic Equivalents", *Cambridge University Press* ISBN: 978-1-107-19704-6
14. Cislo, J., Wolf, M., "Criteria equivalent to the Riemann Hypothesis", <https://doi.org/10.1063/1.3043867> arXiv, 2008
15. Conrey, J.B., Farmer W.D., (2019) "Equivalences to the Riemann Hypothesis", <https://www.scribd.com/document/416320708/Equivalences-to-the-Riemann-Hypothesis>
16. AIMATH, <https://www.aimath.org/WWN/rh/>

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