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# The Distribution and Quantiles of the Sample Autocovariance and Other Functions of Sample Moments from a Stationary Process

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**Abstract**

We give expansions for the distribution, density and quantiles of any smooth function of the sample cross-moments of a stationary process. We do this by showing that the sample cross-moments are *standard estimates*. The 8 examples include the sample autocovariance and the sample autocorrelation.

**Keywords:** Edgeworth-Cornish-Fisher expansions; autocovariance; autocorrelation; stationary process

**MSC:** Classification 62E20

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## 1. Introduction and Summary

In Withers and Nadarajah (2012), we gave the extended Edgeworth-Cornish-Fisher expansions, *the eECF expansions*, for smooth functions of the sample cross-moments of a linear process. Here we extend these results to a stationary process.

Suppose that $\hat{\theta}$ is a *standard estimate* of an unknown parameter $\theta \in R^q$ of a statistical model, based on a sample of size $n$. That is, its cross cumulants can be expanded in the form (3.3), or if $q = 1$, in the form (2.1). Then eECF expansions are available for its distribution that improve upon the Central Limit Theorem. In order to be self-contained, Section 2 summarises these expansions to $O(n^{-3/2})$, and Section 3 gives the leading cumulant coefficients for functions of an unbiased standard estimate. In Section 4 we show that the sample moments of a stationary process are standard estimates. (In fact their cumulant expansions have exactly 2 terms!) So by Withers (1982, 2024), smooth functions of them are also standard estimates, and so have the eECF expansions of Sections 2 and 3.

Theorem 4.1 gives 2 choices for the cumulant coefficients needed for these expansions. The examples of Section 5 include the distributions of the sample mean, the sample autocovariance and the sample autocorrelation. Section 6 shows how to extend these results to a multivariate stationary process.

This nonparametric approach removes the need for modelling time series parametrically as a moving average or autoregressive or ARIMA process. One weakness of such approaches is that a wrong model will have the wrong asymptotic variance, so that inference will not be even asymptotically correct. Another is the difficulty of estimating their parameters. A vast collection of software has been developed for economists and others for this purpose, all now unnecessary if one moves to our non-parametric method.

**Note 1.1.** *Withers and Nadarajah (2010c, 2012) considered the general stationary linear process*

$$X_i = \sum_{j=0}^{\infty} \rho_j e_{i-j}$$

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*where* $\{\rho_j\}$ *are constants, and* $\{e_i\}$ *are independent and identically distributed random variables from a distribution on* $R$ *with finite cumulants* $\{\tau_j\}$. *We showed that its cross-cumulants are*

$$\kappa_{i_1\cdots i_r} = \alpha(i_1 i_2 \cdots i_r)\,\tau_r, \textit{ where} \tag{1.1}$$
$$\alpha(i_1 i_2 \cdots i_r) = \alpha(I_1 I_2 \cdots I_r) = \sum_{j=0}^{\infty} \rho_{j+I_1}\rho_{j+I_2}\cdots\rho_{j+I_r},$$

*for* $I_k$ *of (4.3), and that* $\alpha(i_1 i_2 \cdots i_r)$ *is finite for processes like ARMA processes where* $\rho_j \to 0$ *exponentially as* $n \to \infty$. *See there for a simulation study. For* $\{X_j\}$ *an autoregressive process, see p158 of Withers and Nadarajah (2012).*

## 2. eECF Expansions for Standard Estimates

This section summarises results to be found in Withers (1984, 2025a).

**Univariate estimates.** Suppose that $\hat{\theta}$ is *a standard estimate* of an unknown $\theta \in R$ with respect to $n$, typically the sample size. That is, $E\,\hat{\theta} \to \theta$ as $n \to \infty$, and its cumulants can be expanded as

$$\kappa_r(\hat{\theta}) \approx \sum_{j=r-1}^{\infty} n^{-j} a_{rj} \text{ for } r \geq 1, \tag{2.1}$$

where the *cumulant coefficients* $a_{rj}$ may depend on $n$ but are bounded as $n \to \infty$, and $a_{21}$ is bounded away from 0. Here and below $\approx$ indicates an asymptotic expansion that need not converge. So (2.1) holds in the sense that

$$\kappa_r(\hat{\theta}) = \sum_{j=r-1}^{I-1} a_{rj}\, n^{-j} + O(n^{-I}) \text{ for } I \geq r \geq 1,$$

where $y_n = O(x_n)$ means that $y_n/x_n$ is bounded in $n$. For non-lattice estimates, the distribution and quantiles of

$$Y_n = (n/a_{21})^{1/2}(\hat{\theta} - \theta) \tag{2.2}$$

have asymptotic expansions in powers of $n^{-1/2}$ of the form

$$P_n(x) = Prob(Y_n \leq x) \approx \Phi(x) - \phi(x) \sum_{r=1}^{\infty} n^{-r/2} h_r(x), \tag{2.3}$$
$$p_n(x) = dP_n(x)/dx \approx \phi(x)\,[1 + \sum_{r=1}^{\infty} n^{-r/2} \bar{h}_r(x)], \tag{2.4}$$
$$\Phi^{-1}(P_n(x)) \approx x - \sum_{r=1}^{\infty} n^{-r/2} f_r(x),\ P_n^{-1}(\Phi(x)) \approx x + \sum_{r=1}^{\infty} n^{-r/2} g_r(x), \tag{2.5}$$

where $\Phi(x) = Prob(N \leq x)$, $N \sim \mathcal{N}(0,1)$ is a unit normal random variable with density $\phi(x) = (2\pi)^{-1/2} e^{-x^2/2}$, and $h_r(x), \bar{h}_r(x), f_r(x), g_r(x)$ are polynomials in $x$ and $\{A_{ri}\}$ where $A_{ri} = a_{ri}/a_{21}^{r/2}$. The expansions (2.3)–(2.5) are given in Withers (1984) to $O(n^{-5/2})$ and Withers and Nadarajah (2010a) to $O(n^{-3})$, starting

$$h_1(x) = f_1(x) = g_1(x) = A_{11} + A_{32}H_2/6,\ \bar{h}_1(x) = A_{11}H_1 + A_{32}H_3/6, \tag{2.6}$$
$$h_2(x) = (A_{11}^2 + A_{22})H_1/2 + (A_{11}A_{32} + A_{43}/4)H_3/6 + A_{32}^2 H_5/72,$$
$$f_2(x) = (A_{22}/2 - A_{11}A_{32}/3)H_1 + A_{43}H_3/24 - A_{32}^2(4x^3 - 7x)/36,$$
$$g_2(x) = A_{22}H_1/2 + A_{43}H_3/24 - A_{32}^2(2x^3 - 5x)/36,$$
$$\bar{h}_2(x) = (A_{11}^2 + A_{22})H_2/2 + (A_{11}A_{32} + A_{43}/4)H_4/6 + A_{32}^2 H_6/72, \tag{2.7}$$

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where for $k \geq 0$, $H_k$ is the $k$th Hermite polynomial,

$$
\begin{aligned}
H_k = H_k(x) &= \phi(x)^{-1}(-d/dx)^k \phi(x) \\
&= E\,(x + iN)^k \text{ for } i = \sqrt{-1},\ N \sim \mathcal{N}(0,1). \qquad (2.8) \\
\text{So, } H_{k+1} &= (x - d/dx)H_k,\ (d/dx)H_{k+1} = (k+1)H_k, \\
H_0 = 1,\ H_1 &= x,\ H_2 = x^2 - 1,\ H_3 = x^3 - 3x,\ H_4 = x^4 - 6x^2 + 3, \ldots
\end{aligned}
$$

So (2.3)– (2.8) give the eECF expansions to $O(n^{-3/2})$ for the distribution and quantiles of $Y_n$ of (2.2) in terms of $a_{21}, a_{11}, a_{32}, a_{22}, a_{43}$, while . $a_{21}, a_{11}, a_{32}$ alone only give the eECF expansions to $O(n^{-1})$, and $a_{21}$ alone gives the eECF expansions to $O(n^{-1/2})$.

**Note 2.1.** *Cornish and Fisher (1937) and Cornish and Fisher (1960) gave (2.5) for $r \leq 6$ for the special case when $a_{ri} = 0$ for $i \neq r-1$. This was extended to standard estimates in Withers (1984). For (2.8), see Withers (2000).*

We now reserve $i_1, i_2, \ldots$ for any sequence from $1, 2, \ldots, p$. To avoid double subscripts, we introduce the bar notation.

*The multivariate Hermite polynomial*, $\bar{H}^{1-k} = \bar{H}^{1-k}(x, V)$, is

$$\bar{H}^{1-k}(x, V) = \phi_V(x)^{-1}(-\bar{\partial}_1) \ldots (-\bar{\partial}_k)\phi_V(x), \text{ where } \partial_i = \partial/\partial x_i \text{ and } \bar{\partial}_k = \partial_{i_k}.$$

By Withers (2020), for $i = \sqrt{-1}$,

$$
\begin{aligned}
&\bar{H}^{1-k}(x, V) = E\,\Pi_{j=1}^{k}(\bar{y}_j + i\bar{Y}_j) \text{ where } \bar{y}_j = y_{i_j},\ y = V^{-1}x, \\
&\bar{Y}_j = Y_{i_j},\ Y \sim \mathcal{N}_p(0, V^{-1}). \\
&\text{So, } H^1 = y_1,\ \bar{H}^1 = \bar{y}_1,\ H^{12} = y_1 y_2 - V^{12},\ \bar{H}^{12} = \bar{y}_1\bar{y}_2 - \bar{V}^{12}, \\
&H^{1-3} = y_1 y_2 y_3 - \sum^{3} V^{12} y_3, \text{ where } \sum^{3} V^{12} y_3 = V^{12}y_3 + V^{13}y_2 + V^{23}y_1,
\end{aligned}
$$

$V^{i_1 i_2}$ is the $(i_1, i_2)$ element of $V^{-1}$, and $\bar{V}^{j_1 j_2}$ is the $(i_{j_1}, i_{j_2})$ element of $V^{-1}$. For their dual form see Withers and Nadarajah (2014). Their integrated form, $\bar{H}_*^{1-k} = \bar{H}_*^{1-k}(x, V)$, is

$$\bar{H}_*^{1-k} = (-\bar{\partial}_1) \ldots (-\bar{\partial}_k)\ \Phi_V(x) = \int_{-\infty}^{x} \bar{H}^{1-k}\phi_V(x)dx. \qquad (2.9)$$

**Multivariate estimates.** Suppose that $\hat{w}$ is a *standard estimate* of $w \in R^p$ with respect to $n$. That is, $E\,\hat{w} \to w$ as $n \to \infty$, and for $r \geq 1$, $1 \leq i_1, \ \ldots, \ i_r \leq p$, the $r$th order cumulants of $\hat{w}$ can be expanded as

$$\kappa(\hat{w}_{i_1}, \ldots, \hat{w}_{i_r}) \approx \sum_{j=r-1}^{\infty} \bar{k}_j^{1-r}\ n^{-j} \text{ where } \bar{k}_j^{1-r} = k_j^{i_1 \ldots i_r}, \qquad (2.10)$$

and the *cumulant coefficients* $\bar{k}_j^{1-r} = k_j^{i_1 \ldots i_r}$ may depend on $n$ but are bounded as $n \to \infty$. So $\bar{k}_0^1 = \bar{w}_1 = w_{i_1}$. Then

$$X_n = n^{1/2}(\hat{w} - w) \xrightarrow{\mathcal{L}} \mathcal{N}_p(0, V) \text{ where } V = (k_1^{i_1 i_2}),\ p \times p, \qquad (2.11)$$

with density and distribution

$$\phi_V(x) = (2\pi)^{-q/2}(det(V))^{-1/2}\exp(-x'V^{-1}x/2),\ \Phi_V(x) = \int_{-\infty}^{x} \phi_V(x)dx.$$

$V$ may depend on $n$, but we assume that $det(V)$ is bounded away from 0. By Withers and Nadarajah (2010b) or Withers (2024), the distribution and density of $X_n = n^{1/2}(\hat{w} - w)$ can be expanded as

$$Prob.(X_n \leq x) \approx \sum_{r=0}^{\infty} n^{-r/2} P_r(x),\ p_{X_n}(x) \approx \sum_{r=0}^{\infty} n^{-r/2} p_r(x),\ x \in R^p, \tag{2.12}$$

$$\text{where } P_0(x) = \Phi_V(x),\ p_0(x) = \phi_V(x), \text{ and for } r \geq 1, \tag{2.13}$$

$$P_r(x) = \sum_{k=1}^{3r} [\bar{P}_r^{1-k}\ \bar{H}_*^{1-k}(x, V):\ k - r \text{ even}], \tag{2.14}$$

$$p_r(x)/\phi_V(x) = \sum_{k=1}^{3r} [\bar{P}_r^{1-k} \bar{H}^{1-k}(x, V):\ k - r \text{ even}] = \tilde{p}_r(x) \text{ say}. \tag{2.15}$$

(2.14) and (2.15) use *the tensor summation convention* of implicitly summing $i_1, \ldots, i_r$ over their range $1, \ldots, p$ and $\bar{P}_r^{1-k}$ are *the Edgeworth coefficients*. These are polynomials in the cumulant coefficients $\bar{k}_j^{1-r}$ of (2.10). See Withers (2025a) for their definition. The Edgeworth coefficients needed for Edgeworth expansions to $O(n^{-3/2})$ are

$$\bar{P}_1^1 = \bar{k}_1^1,\ \bar{P}_1^{1-3} = \bar{k}_2^{1-3}/6,\ \bar{P}_2^{12} = \bar{k}_1^1 \bar{k}_1^2 + \bar{k}_2^{12}/2,$$
$$\bar{P}_2^{1-4} = \bar{k}_3^{1-4}/24 + \mathcal{S}\bar{k}_1^1 \bar{k}_2^{2-4}/6 + \mathcal{S}\bar{k}_1^4 \bar{k}_2^{1-3}/6,\ \bar{P}_2^{1-6} = \mathcal{S}\bar{k}_2^{1-3} \bar{k}_2^{4-6}/36,$$

where $\mathcal{S}$ is the operator that symmetrizes $\bar{f}^{1-k} = f^{i_1 \ldots, i_k}$. So,

$$P_1(x) = \bar{k}_1^1\ \bar{H}_*^1(x, V) + \bar{k}_2^{1-3}\ \bar{H}_*^{1-3}(x, V)/6,$$
$$\tilde{p}_1(x) = \bar{k}_1^1\ \bar{H}^1(x, V) + \bar{k}_2^{1-3}\ \bar{H}^{1-3}(x, V)/6,$$
$$P_2(x) = \sum_{k=2,4,6} \bar{P}_2^{1-k}\ (-\bar{\partial}_1) \ldots (-\bar{\partial}_k)\ \Phi_V(x),\ \tilde{p}_2(x) = \sum_{k=2,4,6} \bar{P}_2^{1-k}\ \bar{H}^{1-k}(x, V).$$

For $\bar{P}_3^{1-k}$ see Withers (2025a). These give the Edgeworth expansions for the distribution of $X_n = n^{1/2}(\hat{w} - w)$ to $O(n^{-2})$. See Withers (2025a) for more details.

eECF expansions for parametric and nonparametric standard estimates were first given in Withers (1984), and extended to functions of them in Withers (1982, 1983).

In Section 4 we show that for $\hat{\theta}$ and $\hat{w}$ sample cross-moments, only the first 2 terms in (2.1) and (2.10) are non-zero for $r \geq 2$.

## 3. Cumulant Coefficients for Functions of Unbiased Standard Estimates

This section summarises the results needed in Section 5 from Withers (1982, 2024).

Suppose that $\hat{w} \in R^p$ is a standard estimate satisfying (2.10), and that

$$\theta = t(w) : R^p \to R^q \tag{3.1}$$

is a smooth function with finite derivatives

$$t_{\cdot a_1 a_2 \cdots} = \partial_{a_1} \partial_{a_2} \cdots t(w) \text{ where } \partial_a = \partial/\partial w_a. \tag{3.2}$$

For $b = 1, \cdots, q$, let $\theta^b$ be the $b$th component of $\theta$. Then by Withers (1982, 2024), $\hat{\theta} = t(\hat{w}) \in R$ is a standard estimate of $\theta = t(w)$. So its $r$th order cross-cumulants are of magnitude $n^{1-r}$ with an expansion of the form

$$\kappa(\hat{\theta}^{b_1}, \cdots, \hat{\theta}^{b_r}) \approx \sum_{j=r-1}^{\infty} n^{-j} K_{nj}^{b_1 \cdots b_r} \tag{3.3}$$

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where $b_1, \cdots b_r$ lie in $1, \cdots, q$, $K^b_{n0} = \theta^b$, and the cumulant coefficients for $\hat{\theta}$, $K^{b_1 \cdots b_r}_{nj}$, are given in terms of the cumulant coefficients for $\hat{w}$, $\{\bar{k}^{1-r}_j = k^{a_1 \ldots a_r}_j\}$ of (2.10) and the derivatives $t_{\cdot a_1 a_2 \cdots}$, by the appendix to Withers (1982), and in more detail in Theorem 2 of Withers (2024), where we now assume that $E\,\hat{w} = w$. So,

$$Z_n = n^{1/2}(\hat{\theta} - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}_q(0, \tilde{V}) \text{ as } n \to \infty \text{ where } \tilde{V} = (K^{b_1 b_2}_{n1}), \tag{3.4}$$

$$K^{b_1 b_2}_{n1} = t^{b_1}_{\cdot a_1} k^{a_1 a_2}_1 t^{b_2}_{\cdot a_2},\ K^{b_1}_{n1} = t^{b_1}_{\cdot a_1 a_2} k^{a_1 a_2}_1 / 2, \tag{3.5}$$

where we use the tensor summation convention of implicit summation of the repeated pairs, in this case $a_1, a_2$, over their range $1, \cdots, p$. That is,

$$K^{b_1 b_2}_{n1} = \sum_{a_1, a_2 = 1}^{p} t^{b_1}_{\cdot a_1} k^{a_1 a_2}_1 t^{b_2}_{\cdot a_2},\ K^{b_1}_{n1} = \sum_{a_1, a_2 = 1}^{p} t^{b_1}_{\cdot a_1 a_2} k^{a_1 a_2}_1 / 2.$$

Similarly,

$$K^{b_1 b_2 b_3}_{n2} = t^{b_1}_{\cdot a_1} t^{b_2}_{\cdot a_2} t^{b_3}_{\cdot a_3} k^{a_1 a_2 a_3}_2 + \sum_{b_1 b_2 b_3}^{3} s^{b_1}_{a_1} t^{b_2}_{\cdot a_1 a_3} s^{b_3}_{a_3} \text{ where } s^{b_1}_{a_1} = k^{a_1 a_2}_1 t^{b_1}_{\cdot a_2}, \tag{3.6}$$

$$\begin{aligned} K^{b_1 b_2}_{n2} &= t^{b_1}_{\cdot a_1} t^{b_2}_{\cdot a_2} k^{a_1 a_2}_2 + \sum_{b_1 b_2}^{2} [t^{b_1}_{\cdot a_1 a_2} t^{b_2}_{\cdot a_3} k^{a_1 a_2 a_3}_2 + t^{b_1}_{\cdot a_1 a_2 a_3} s^{b_2}_{a_3} k^{a_1 a_2}_1]/2 \\ &+ v^{b_1 a_2}_{a_3} v^{b_2 a_3}_{a_2}/2 \text{ where } v^{b_1 a_2}_{a_3} = t^{b_1}_{\cdot a_1 a_3} k^{a_1 a_2}_1, \end{aligned} \tag{3.7}$$

$$\begin{aligned} K^{b_1 \ldots b_4}_{n3} &= t^{b_1}_{\cdot a_1} \ldots t^{b_4}_{\cdot a_4} k^{a_1 \ldots a_4}_3 + k^{a_1 a_2 a_3}_2 \sum^{12} t^{b_2}_{\cdot a_2} t^{b_3}_{\cdot a_3} u^{b_1 b_4}_{a_1} + \sum^{4} t^{b_1}_{\cdot a_1 a_3 a_5} s^{b_2}_{a_1} s^{b_3}_{a_3} s^{b_4}_{a_5} \\ &+ \sum^{12} u^{b_1 b_3}_{a_1} k^{a_1 a_2}_1 u^{b_2 b_4}_{a_2} \text{ where } u^{b_1 b_4}_{a_1} = t^{b_1}_{\cdot a_1 a_4} s^{b_4}_{a_4}, \end{aligned} \tag{3.8}$$

These are the coefficients needed for $P_r(x), p_r(x)$ of (2.12) for $r = 1, 2$ when $X_n$ of (2.11) is replaced by $Z_n$ of (3.4).

**The case** $q = 1$. By (2.1), we can replace $K^{b_1 \ldots b_r}_{nj}$ by $a_{rj}$ where by (3.3)–(3.6),

$$a_{10} = \theta,\ a_{21} = t_{\cdot a_1} k^{a_1 a_2}_1 t_{\cdot a_2} = t_{\cdot a_1} s_{a_1} \text{ for } s_{a_1} = k^{a_1 a_2}_1 t_{\cdot a_2},, \tag{3.9}$$

$$a_{11} = t_{\cdot a_1 a_2} k^{a_1 a_2}_1 / 2,\ a_{32} = t_{\cdot a_1} t_{\cdot a_2} t_{\cdot a_3} k^{a_1 a_2 a_3}_2 + 3 s_{a_1} t_{\cdot a_1 a_3} s_{a_3}, \tag{3.10}$$

$$a_{22} = t_{\cdot a_1} k^{a_1 a_2}_2 t_{\cdot a_3} + t_{\cdot a_1 a_2} k^{a_1 a_2 a_3}_2 t_{\cdot a_3} + s_{a_3} t_{\cdot a_1 a_2 a_3} k^{a_1 a_2}_1 + v^{a_2}_{a_3} v^{a_3}_{a_2}, \tag{3.11}$$

$$\begin{aligned} a_{43} &= t_{\cdot a_1} \ldots t_{\cdot a_4} k^{a_1 \ldots a_4}_3 + 12 u_{a_1} k^{a_1 a_2 a_3}_2 t_{\cdot a_2} t_{\cdot a_3} + 4 t_{\cdot a_1 a_3 a_5} s_{a_1} s_{a_3} s_{a_5} \\ &+ 12 u_{a_1} k^{a_1 a_2}_1 u_{a_2} \text{ for } v^{a_2}_{a_3} = t_{\cdot a_1 a_3} k^{a_1 a_2}_1,\ u_{a_1} = t_{\cdot a_1 a_2} s_{a_2}. \end{aligned} \tag{3.12}$$

This gives the $a_{ri}$ needed for $\{h_r, \bar{h}_r, f_r, g_r,\ r = 1, 2\}$ of Section 2. So if $a_{21}$ is bounded away from 0 as $n \to \infty$, then $Y_n = (n/a_{21})^{1/2}(\hat{\theta} - \theta)$ has the eECF expansions (2.3)–(2.5).

$$\text{For any } f_{12}, \text{ set } \sum_{12}^{2} f_{12} = f_{12} + f_{21}, \text{ and } \sum_{123}^{3} f_{12} = f_{12} + f_{23} + f_{31}. \tag{3.13}$$

If $p = 1$, then $s_1 = t_{\cdot 1} k^{11}_1,\ u_1 = t_{\cdot 11} s_1 = t_{\cdot 1} t_{\cdot 11} k^{11}_1,\ v_{11} = t_{\cdot 11} k^{11}_1,$

$$\begin{aligned} a_{21} &= t^2_{\cdot 1} k^{11}_1 = t_{\cdot 1} s_1,\ a_{11} = t_{\cdot 11} k^{11}_1 / 2,\ a_{32} = t^3_{\cdot 1} k^{111}_2 + 3 s^2_1 t_{\cdot 11}, \\ a_{22} &= t^2_{\cdot 1} k^{11}_2 + t_{\cdot 1} t_{\cdot 11} k^{111}_2 + s_1 t_{\cdot 111} k^{11}_1 + (v^1_1)^2 / 2, \\ a_{43} &= t^4_{\cdot 1} k^{1111}_3 + 12 k^{111}_2 t^2_{\cdot 1} t_{\cdot 11} s_1 + 4 t_{\cdot 111} s^3_1 + 12 k^{11}_1 u^2_1. \end{aligned} \tag{3.14}$$

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If $p = 2$, as in Examples 5.3 and 5.4 below, then $s_j = \sum_{i=1}^2 k_1^{ji} t_{.i}$, $u_k = \sum_{j=1}^2 t_{.kj} s_j$, and (3.9)– (3.12) can be written using (3.13) as

$$a_{21} = \sum_{i=1}^2 t_{.i}^2 k_1^{ii} + 2t_{.1}t_{.2}k_1^{12},\ a_{11} = \sum_{i=1}^2 t_{.ii}k_1^{ii}/2 + t_{.12}k_1^{12}, \tag{3.15}$$

$$a_{32} = \sum_{i=1}^2 t_{.i}^3 k_2^{iii} + 3\sum_{12}^2 t_{.1}^2 t_{.2} k_2^{112} + 3\sum_{j=1}^2 s_j^2 t_{.jj} + 6s_1 s_2 t_{.12}, \tag{3.16}$$

$$a_{22} = t_{.i} k_2^{ij} t_{.j} + \sum_{i=1}^2 t_{.i}(t_{.ii}k_2^{iii} + 2t_{.12}k_2^{12i}) + \sum_{12}^2 t_{.11}k_2^{112} t_{.2} + \sum_{i=1}^2 s_i t_{.iii} k_1^{ii}$$

$$+ \sum_{12}^2 s_1(2t_{.112}k_1^{12} + t_{.122}k_1^{22}) + v_j^i v_i^j/2 = \sum_{k=1}^6 a_{22k} \text{ say}, \tag{3.17}$$

$$a_{43} = \sum_{k=1}^4 a_{43k} \text{ where } a_{431} = \sum_{i=1}^2 t_{.i}^4 k_3^{1111} + 4\sum_{12}^2 t_{.1}^3 t_{.2} k_3^{1112} + 6t_{.1}^2 t_{.2}^2 k_3^{1122}, \tag{3.18}$$

$$a_{432} = 12u_{a_1} t_{.a_2} t_{.a_3} k_2^{a_1 a_2 a_3},\ a_{433}/4 = \sum_{i=1}^2 s_i^3 t_{.iii} + 3s_1^2 s_2 t_{.112} + 3s_1 s_2^2 t_{.122},$$

$$a_{434}/12 = \sum_{i=1}^2 u_i^2 k_1^{ii} + 2u_1 u_2 k_1^{12}.$$

If $p = 3$, as in Examples 5.5 and 5.6 below, then $s_j = \sum_{i=1}^3 k_1^{ji} t_{.i}$, and (3.9) and (3.10) can be written using (3.13), as

$$a_{21} = \sum_{i=1}^3 t_{.i}^2 k_1^{ii} + 2\sum_{123}^3 t_{.1}t_{.2}k_1^{12},\ a_{11} = \sum_{i=1}^3 t_{.ii}k_1^{ii}/2 + \sum_{123}^3 t_{.12}k_1^{12}, \tag{3.19}$$

$$a_{32} = \sum_{i=1}^3 t_{.i}^3 k_2^{iii} + 3\sum_{123}^3 t_{.1}^2 t_{.2} k_2^{112} + 6t_{.1}t_{.2}t_{.3}k_2^{123} + 3\sum_{j=1}^3 s_j^2 t_{.jj} + 2\sum_{123}^3 s_1 s_2 t_{.12}.$$

## 4. The Cumulants of the Sample Cross-Moments

This section uses the notation of Section 2 of Withers (2025b). Let $\cdots, X_{-1}, X_0, X_1, \cdots$ be any real stationary process with finite mean, cross-moments, and cross-cumulants,

$$\mu = E\ X_0,\ M_{i_1 \cdots i_r} = E\ X_{i_1} \cdots X_{i_r},\ \mu_{i_1 \cdots i_r} = E\ (X_{i_1} - \mu) \cdots (X_{i_r} - \mu), \tag{4.1}$$

$$k(i_1, \cdots, i_r) = \kappa(X_{i_1}, \cdots, X_{i_r}). \text{ So, } \mu_{0^r} = \mu_r(X_0),\ k(0^r) = \kappa_r(X_0), \tag{4.2}$$

where $0^r$ denotes a string of $r$ zeros. For relationships between them see Section 3.30 of Stuart and Ord (1987). Multivariate relations can be written down from their univariate versions. For example for $M_r$, $\mu_r$ and $\kappa_r$ the $r$th central moment and cumulant of $X \in R$,

$$M_2 = \kappa_2 + \kappa_1^2 \Rightarrow M_{11} = \kappa_{11} + \kappa_{10}\kappa_{01},$$

$$\kappa_4 = \mu_4 - 3\mu_2^2 \Rightarrow k(1234) = \mu_{1234} - \mu_{12}\mu_{34} - \mu_{13}\mu_{24} - \mu_{14}\mu_{23}.$$

Given a sequence of integers $i_1, \cdots, i_r$, set

$$i_0 = \min_{k=1}^r i_k,\ I_k = i_k - i_0 \geq 0,\ I_0 = I_0(i_1 \ldots i_r) = \max_{k=1}^r I_k = \max_{k=1}^r i_k - i_0. \tag{4.3}$$

$$\text{So, } M_{i_1 \cdots i_r} = M_{I_1 \cdots I_r},\ \mu_{i_1 \cdots i_r} = \mu_{I_1 \cdots I_r},\ \kappa_{i_1 \cdots i_r} = \kappa_{I_1 \cdots I_r}. \tag{4.4}$$

These are not changed by permuting subscripts. Also at least one $I_k$ is zero. For $a \geq 0$ and $k(.)$ of (4.2), the $a$th autocovariance and autocorrelation are

$$\text{covar}(X_0, X_a) = k(0,a), \text{ and } \text{corr}(X_0, X_a) = k(0,a)/k(0^2). \tag{4.5}$$
$$\text{Also, } k(0,T) = k(0,|T|), \text{ and if } T_1 < T_2 < 0 \text{ or } T_1 < 0 < T_2, \text{ then}$$
$$k(0,T_1,T_2) = k(0,T_2 - T_1, -T_1) = k(0,|T_2 - T_1|,|T_1|).$$

Now suppose that we observe only $X_1, \cdots, X_n$. For $I_1, \ldots, I_r$ and $n > I_0 = I_0(i_1 \ldots i_r)$ of (4.3), define the *sample non-central cross-moment*

$$\hat{M}_{i_1 \cdots i_r} = \hat{M}_{I_1 \cdots I_r} = N^{-1} \sum_{t=1}^{N} X_{t+I_1} \cdots X_{t+I_r} \text{ where } N = n - I_0. \tag{4.6}$$

This is an unbiased estimate of $M_{i_1 \cdots i_r}$ of (4.1). For example $\mu = E\ X_0$ and $M_{0a} = E\ X_0 X_a$ have unbiased estimates

$$\hat{\mu} = \bar{X} = \hat{M}_0 = n^{-1} \sum_{j=1}^{n} X_j, \tag{4.7}$$

$$\hat{M}_{0a} = N^{-1} \sum_{j=1}^{N} X_j X_{j+a} \text{ for } N = n - a > 0, \tag{4.8}$$

and $a \geq 0$ an integer. We can write (4.6) as

$$\hat{M}_\pi = N^{-1} \sum_{t=1}^{N} X_t(\pi) \text{ for } N = n - I_0(\pi) > 0, \text{ that is, for } n > I_0(\pi).$$

Given $p \geq 1$ and sequences of integers $\pi_1, \cdots, \pi_p$, for $j = 1, \ldots, p$, set

$$w_j = M_{\pi_j},\ \hat{w}_j = \hat{M}_{\pi_j},\ w = (w_1, \cdots, w_p),\ \hat{w} = (\hat{w}_1, \cdots, \hat{w}_p),\ N_j = n - I_0(\pi_j). \tag{4.9}$$

We shall see that (2.10) holds with $\bar{k}_j^{1-r} = 0$ for $j > r \geq 2$. So $\hat{w}$ is a standard estimate.

$$\kappa(\hat{w}_1, \cdots, \hat{w}_r) = (N_1 \ldots N_r)^{-1} \sum_{t_1=1}^{N_1} \cdots \sum_{t_r=1}^{N_r} K\begin{pmatrix} \pi_1 \ldots \pi_r \\ t_1 \ldots t_r \end{pmatrix} \tag{4.10}$$

$$\text{where } K\begin{pmatrix} \pi_1 \ldots \pi_r \\ t_1 \ldots t_r \end{pmatrix} = \kappa(X_{t_1}(\pi_1), \ldots, X_{t_r}(\pi_r)), \tag{4.11}$$
$$\text{and } X_t(i_1 \ldots i_r) = X_{t+I_1} \cdots X_{t+I_r}.$$

**Example 4.1.** *If $\pi_1 = (0)$ and $\pi_2 = (0,a)$, then $I_0(\pi_1) = 0$, $I_0(\pi_2) = a,\ N_1 = n,\ N_2 = n - a$,*

$$X_t(\pi_1) = X_t,\ X_t(\pi_2) = X_t X_{t+a},\ K\begin{pmatrix} \pi_1 \pi_2 \\ t_1 t_2 \end{pmatrix} = \kappa(X_{t_1}, X_{t_2} X_{t_2+a}),$$

$$\kappa_2(\hat{w}_1) = n^{-2} \sum_{t_1,t_2=1}^{n} \kappa(X_{t_1}, X_{t_2}),\ \kappa_2(\hat{w}_2) = N_2^{-2} \sum_{t_1,t_2=1}^{N_2} \kappa(X_{t_1} X_{t_1+a}, X_{t_2} X_{t_2+a}),$$

$$\kappa(\hat{w}_1, \hat{w}_2) = (nN_2)^{-1} \sum_{t_1=1}^{n} \sum_{t_2=1}^{N_2} \kappa(X_{t_1}, X_{t_2} X_{t_2+a}).$$

$K\binom{\pi_1 \ldots \pi_r}{t_1 \ldots t_r}$ can be written in terms of the cross-cumulants of $\{X_j\}$ using p254–265 of McCullagh (1987) if $L = L_1 + \cdots + L_r \leq 8$ where $L_i$ the length of the sequence $\pi_i$. See his p58 and the appendix below for some examples. So this covers $\kappa_4(\hat{M}_{i_1 i_2})$, which has $L = 8$, but not $\kappa_3(\hat{M}_{i_1 i_2 i_3})$, which has

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$L = 9$. So at present we can obtain the eECF expansions for $\hat{M}_{i_1 i_2}$ to $O(n^{-3/2})$, but the eECF expansions for $\hat{M}_{i_1 i_2 i_3}$ only to $O(n^{-1})$.

Under mild conditions, $n^{r-1}\kappa(\hat{w}_{i_1}, \cdots, \hat{w}_{i_r})$ is bounded as $n$ increases, as the examples show.

**Theorem 4.1.** *Take $r \geq 2$. Let $K(t_1, \ldots, t_r)$ be any symmetric function of integers $t_1, \ldots, t_r$ such that*

$$\text{for any integer, } K(t_1, \ldots, t_r) \equiv K(t_1 - T, \ldots, t_r - T).$$

*Take $n \geq 1$ and integers $a_1 < n, \ldots, a_r < n$. Then for $N_j = n - a_j$,*

$$\sum_{t_1=1}^{N_1} \cdots \sum_{t_r=1}^{N_r} K(t_1, \cdots, t_r) = \sum_{-N_1 < T_j < N_j,\ j=2,\cdots,r} D_{r\mathbf{NT}}\, K(0, T_2, \cdots, T_r), \tag{4.12}$$

$$\text{where } D_{r\mathbf{NT}} = \min(N_1, N_2 - T_2, \cdots, N_r - T_r) + m_{r\mathbf{T}} \tag{4.13}$$

$$= n - \delta_{r\mathbf{T}} = N_1 - \delta'_{r\mathbf{T}},\ m_{r\mathbf{T}} = \min(0, T_2, \cdots, T_r), \tag{4.14}$$

$$\delta_{r\mathbf{T}} = \max(a_1, T_2 + a_2, \cdots, T_r + a_r) - m_{r\mathbf{T}}, \tag{4.15}$$

$$\delta'_{r\mathbf{T}} = \delta_{r\mathbf{T}} - a_1 = \max(0, T_2 + a_2 - a_1, \cdots, T_r + a_r - a_1) - m_{r\mathbf{T}}. \tag{4.16}$$

$$\text{So LHS (4.12)} = n\, a^K_{r,r-1} + a^K_{rr} = N_1\, a^K_{r,r-1} + a'^K_{rr} \text{ where} \tag{4.17}$$

$$(a^K_{r,r-1}, a^K_{rr}, a'^K_{rr}) = \sum_{-N_1 < T_j < N_j,\ j=2,\cdots,r} (1, -\delta_{r\mathbf{T}}, -\delta'_{r\mathbf{T}})\, K(0, T_2, \cdots, T_r) \tag{4.18}$$

$$\to (a_{r,r-1}, a_{rr}, a'_{rr}) = \sum_{|T_j| < \infty,\ j=2,\cdots,r} (1, -\delta_{r\mathbf{T}}, -\delta'_{r\mathbf{T}})\, K(0, T_2, \cdots, T_r) \tag{4.19}$$

*as $n \to \infty$, when finite, and*

$$a^K_{rr} = a'^K_{rr} - a_1\, a^K_{r,r-1},\ a_{rr} = a'_{rr} - a_1\, a'_{r,r-1}, \tag{4.20}$$

$$a^K_{21} = \sum_{-N_1 < T < N_2} K(0, T) \to a_{21} = \sum_{T=-\infty}^{\infty} K(0, T), \tag{4.21}$$

$$a^K_{32} = \sum_{-N_1 < T_j < N_j,\ j=2,3} K(0, T_2, T_3) \to a_{32} = \sum_{T_2, T_3 = -\infty}^{\infty} K(0, T_2, T_3), \tag{4.22}$$

$$a'^K_{22} = -\sum_{-N_1 < T < N_2} \delta'_{2T}\, K(0, T) \to a'_{22} = -\sum_{T=-\infty}^{\infty} \delta'_{2T}\, K(0, T), \tag{4.23}$$

$$\text{where } \delta'_{2T} = \max(0, T + a_2 - a_1) - \min(0, T),$$

$$a^K_{43} = \sum_{-N_1 < T_j < N_j,\ j=2,3,4} K(0, T_2, T_3, T_4) \to a_{43} = \sum_{T_2, T_3, T_4 = -\infty}^{\infty} K(0, T_2, T_3, T_4). \tag{4.24}$$

*Now suppose that $a_i \equiv a$. Then $N_i \equiv N = n - a$,*

$$\delta'_{r\mathbf{T}} = \max(0, T_2, \cdots, T_r) - \min(0, T_2, \cdots, T_r),\ \delta'_{2\mathbf{T}} = |T_2|, \tag{4.25}$$

$$\delta'_{3\mathbf{T}} = T_3\, I(0 \leq T_2 < T_3) + (T_3 - T_2)\, I(T_2 \leq 0 < T_3) - T_2\, I(T_2 < T_3 \leq 0),$$

$$(a^K_{r,r-1}, a^K_{rr}, a'^K_{rr}) = \sum_{|T_j| < N,\ j=2,\cdots,r} (1, -\delta_{r\mathbf{T}}, -\delta'_{r\mathbf{T}})\, K(0, T_2, \cdots, T_r). \tag{4.26}$$

PROOF Transform from $t_i$ to $T_i = t_i - t_1$ for $i = 2, \cdots, r$. So (4.12) holds with

$$D_{r\mathbf{NT}} = \sum_{t_1=1}^{N_1} I(-T_j < t_1 \leq N_j - T_j,\ j = 2, \ldots, r)$$

$$= \min(N_1, N_2 - T_2, \cdots, N_r - T_r) - \max(0, -T_2, \ldots, -T_r). \ \square$$

For $i = r-1, r$ this will give us two choices for $(a_{r,r-1}, a_{rr})$ needed for (2.1)–(2.5), namely $(a^K_{r,r-1}, a^K_{rr})$ of (4.18) and its limit, $(a_{r,r-1}, a_{rr})$ of (4.19). When $a_i \equiv a$, we have another 2 choices for $a_{rr}$, namely $a'^K_{rr}$ of (4.18) and its limit, $a'_{rr}$ of (4.19). These limits can be used when the differences, for example $a'^K_{rr} - a'_{rr}$, are exponentially small in $n$, that is, they have magnitude $O(e^{-n\lambda})$ for some $\lambda > 0$. When $a_i \equiv a$, $\delta'_{r\mathbf{T}}$ of (4.25) is simpler than $\delta_{r\mathbf{T}} = a + \delta'_{r\mathbf{T}}$, so $a'^K_{rr}$ is simpler than $a^K_{rr}$, and the expansions of Section 2 may be simpler if $n$ is replaced by $N_1$ as in Example 5.2.

**Corollary 4.1.** *Take $a_j = I_0(\pi_j)$, $N_j = n - a_j$, $w_j, \hat{w}_j$ of (4.9), and $D_{r\mathbf{NT}}$, $\delta_{r\mathbf{T}}$, $\delta'_{r\mathbf{T}}$ of (4.13)–(4.16). Then for K of (4.11),*

$$\kappa(\hat{w}_1, \cdots, \hat{w}_r) = \sum_{j=r-1}^{r} n^{-j} k_j^{1\ldots r} \text{ where } k_j^{1\ldots r} = n^r (N_1 \ldots N_r)^{-1} a^K_{rj}, \tag{4.27}$$

$$(a^K_{r,r-1}, a^K_{rr}) = \sum_{-N_1 < T_j < N_j,\ j=2,\cdots,r} (1, -\delta_{r\mathbf{T}})\ K\begin{pmatrix} \pi_1 \pi_2 \ldots \pi_r \\ 0\ T_2 \ldots T_r \end{pmatrix}. \tag{4.28}$$

*So $D_{r\mathbf{NT}} = n - \delta_{r\mathbf{T}} = N_{i_1} - \delta'^*_{r\mathbf{T}}$, and (4.17)–(4.24) hold with $K(0, T_2, \ldots, T_r)$ replaced by $K\binom{\pi_{i_1} \pi_{i_2} \ldots \pi_{i_r}}{0\ T_2 \ldots T_r}$.*

PROOF By (4.10), (4.12), and (4.18), for $k_j^{1\ldots r}$ of (4.27), $LHS(4.10) =$

$$(N_1 \ldots N_r)^{-1}\ RHS(4.12) = (N_1 \ldots N_r)^{-1}\ (n\ a^K_{r,r-1} + a^K_{rr}) = \sum_{j=r-1}^{r} n^{-j} k_j^{1\ldots r}. \quad \square$$

For its univariate version, we have the option of expanding in $N^{-1/2}$ rather than in $n^{-1/2}$ as in Section 2.

**Corollary 4.2.** *Given a sequence of integers $\pi$, set*

$$N = n - I_0(\pi),\ \pi^r = (\pi, \ldots, \pi).$$

$$\text{Then } \kappa_r(\hat{M}_\pi) = N^{1-r} a^K_{r,r-1} + N^{-r} a'^K_{rr} \tag{4.29}$$

$$\text{where } (a^K_{r,r-1}, a'^K_{rr}) = \sum_{|T_j| < N,\ j=1,\ldots,r} (1, -\delta'_{r\mathbf{T}})\ K\begin{pmatrix} \pi\pi \ldots \pi \\ 0\ T_2 \ldots T_r \end{pmatrix}. \tag{4.30}$$

*for $a^K_{rj}$ of (4.28). So the eECF expansions for $Y_n$ of (2.2), that is, (2.3)–(2.5), hold with $(n, Y_n, a_{r,r-1}, a_{rr})$ replaced by $(N, Y_N, a^K_{r,r-1}, a'^K_{rr})$,*

$$\text{where } Y_N = (N/a^K_{21})^{1/2} (\hat{M}_\pi - M_\pi).$$

## 5. Examples

**Example 5.1.** *The eECF expansions to $O(n^{-3/2})$ for the sample mean, $\hat{\mu}$ of (4.7).*

*Take $p = 1$, $\pi = (0)$ so that $\hat{M}_0 = \hat{\mu} = \bar{X}$. For $r \geq 2$ and $K(.) = k(.)$ of (4.2),*

$$\begin{aligned} \kappa_r(\bar{X}) &= n^{1-r} a^k_{r,r-1} + n^{-r} a^k_{rr} \text{ of (4.18) with } N_1 = n \\ &= n^{1-r} a_{r,r-1} + n^{-r} \tilde{a}_{rr} + O(e^{-n\lambda}) \text{ of (4.19) where } \lambda > 0, \end{aligned} \tag{5.1}$$

*under mild conditions. (2.1)–(2.5) hold for*

$$\hat{w} = \bar{X} \text{ with } a_{10} = \mu,\ a_{1j} = 0 \text{ for } j \geq 1.$$

*So (2.3)–(2.7) hold for $Y_n = (n/a_{21})^{1/2} (\bar{X} - \mu)$ where for $r \geq 2$,*

$$(a_{r,r-1}, a_{rr}) = (a^k_{r,r-1}, a^k_{rr}) \text{ of (4.18) or } (a_{r,r-1}, a_{rr}) \text{ of (4.19)},$$

*and* $a_{ri} = 0$ *for* $i > r$. *This example was the subject of Withers (2025b). See there for extensions to missing values and weighted means.*

**Example 5.2.** *The eECF expansions to* $O(n^{-3/2})$ *for the sample autocovariance* **assuming that** $\mu = 0$. *(This assumption is common in the literature on the grounds that the series can be adjusted by subtracting the estimated mean. But by Example 5.3, it gives the wrong variance if* $\mu \neq 0$*!) In this case,* $p = 1$ *and for* $a \geq 0$, $w = M_{0a} = E\, X_0 X_a$ *has unbiased estimate* $\hat{w} = \hat{M}_{0a}$ *of (4.8). So* $\pi = (0, a)$, $N = n - a$, *and by Corollary 4.2,*

$$\begin{aligned} \kappa_r(\hat{M}_{0a}) &= N^{1-r} a^K_{r,r-1} + N^{-r} a'^K_{rr} \\ &\approx N^{1-r} a_{r,r-1} + N^{-r} a'_{rr} \text{ of (4.18) and (4.19)}, \quad (5.2) \\ &\text{where } K(t_1, \ldots, t_r) = \kappa(X_{t_1}(\pi), \ldots, X_{t_r}(\pi)) \text{ for } X_t(\pi) = X_t X_{t+a}. \quad (5.3) \end{aligned}$$

*So (2.3)–(2.5) hold with* $(n, Y_n)$ *replaced by* $(N, Y_N)$ *where*

$$Y_N = (N/a_{21})^{1/2} (\hat{M}_{0a} - M_{0a}),$$

*for* $a_{ri} = a^K_{ri}$ *of (4.26), or its limit when the difference is exponentially small. Alternatively, by (4.27),*

$$\kappa_r(\hat{M}_{0a}) = n^{1-r} a^K_{r,r-1} + n^{-r} a^K_{rr} \approx n^{1-r} a_{r,r-1} + n^{-r} a_{rr} \quad (5.4)$$

*so that (2.3)–(2.5) hold for* $Y_n = (n/a_{21})^{1/2} (\hat{M}_{0a} - M_{0a})$. *We now give* $K(.)$ *needed for (4.21)–(4.24), that is,* $K(0, T_2, \ldots, T_r)$ *of (5.3) for* $2 \leq r \leq 4$, *in terms of the cross-cumulants of the process,* $k(.)$ *of (4.2). By (A2), since* $\mu = 0$, $A_2 = A_4 = 0$, $K(0, T)$ *needed by (4.21) and (4.23) for* $a_{21}$ *and* $a_{22}$, *is*

$$\begin{aligned} K(0, T) &= \kappa(X_0 X_a, X_T X_{T+a}) = A_1 + A_3 \quad (5.5) \\ &\text{where } A_1 = k(0, a, T, T+a), \ A_3 = k(0, T)^2 + k(0, T+a) k(a, T). \end{aligned}$$

$a_{32}$ *of (4.22) needs*

$$K(0, T_2, T_3) = \kappa(X_0 X_a, X_{T_2} X_{T_2+a}, X_{T_3} X_{T_3+a}) = \sum_{j=1}^{11} C_j \quad (5.6)$$

*is given by identifying* $(0, a, T_2, T_2 + a, T_3, T_3 + a)$ *with* $(1, \ldots, 6)$ *in (A13). So* $C_j = 0$ *for* $j = 2, 4, 7, 8, 9, 11$, $C_1 = k(0, a, T_2, T_2 + a, T_3, T_3 + a)$,

$$\begin{aligned} C_3 = \sum_{23}^{2} &[k(0, T_2) k(0, T_2, T_3, T_3 - a) + k(0, T_2 + a) k(0, T_2 - a, T_3, T_3 - a) \\ &+ k(0, T_2 - a) k(0, T_2 + a, T_3, T_3 + a) + k(0, T_2) k(0, T_2, T_3, T_3 + a) \\ &+ k(0, T_3 - T_2 + a) k(0, a, T_2 + a, T_3)] + k(0, T_3 - T_2)[k(0, a, T_2, T_3) \\ &+ k(0, -a, T_2, T_3)] \text{ by (A14)}. \end{aligned}$$

$$\begin{aligned} C_5 &= \sum_{23}^{2} [k(0, a, T_2) k(0, a, T_2 - T_3 + a) + k(0, a, T_2 + a) k(0, a, T_2 - T_3) \\ &+ k(0, T_2, T_2 + a) k(0, T_3, T_3 - a)] \text{ by (A15)}. \\ C_6 &= k(0, T_2, T_3)^2 + k(0, T_2 + a, T_3 + a) k(0, T_2 - a, T_3 - a) \\ &+ \sum_{23}^{2} k(0, T_2, T_3 + a) k(0, T_2, T_3 - a) \text{ by (A16)}. \end{aligned}$$

$$\begin{aligned} &\text{By (A17)}, \ C_{10} = k(T_2, T_3) \sum_{23}^{2} k(0, T_2)\, [k(a, T_3) + k(0, T_3 + a)] \\ &+ k(0, T_2) k(0, T_3) \sum_{23}^{2} k(T_2, T_3 + a) + \sum_{23}^{2} k(0, T_2 + a) k(0, T_3 - a) k(T_2, T_3 + a). \end{aligned}$$

*This completes* $K(0,T_2,T_3)$ *needed for* $a_{32} = a^K_{32}$ *of (4.22) or its limit. Lastly* $a_{43}$ *of (4.24) needs*

$$K(0,T_2,T_3,T_4) = \kappa(X_0X_a, X_{T_2}X_{T_2+a}, X_{T_3}X_{T_3+a}, X_{T_4}X_{T_4+a}). \tag{5.7}$$

*This is given by identifying* $(0,a,T_2,T_2+a,T_3,T_3+a,T_4,T_4+a)$ *with* $(1,\ldots,8)$ *in (A23), starting with* $D_1 = \kappa(X_0,X_a,X_{T_2},X_{T_2+a},X_{T_3},X_{T_3+a},X_{T_4},X_{T_4+a})$.

**Example 5.3.** *The eECF expansions to* $O(n^{-3/2})$ *for the sample autocovariance* **without** *assuming that* $\mu = 0$.

*In this case we take* $p = 2$ *in (3.1),* $a \geq 0$,

$$w_1 = \mu,\ w_2 = M_{0a},\ \theta = t(w) = \kappa(X_0,X_a) = w_2 - w_1^2.$$

*The non-zero derivatives are* $t_{.1} = -2\mu,\ t_{.2} = 1,\ t_{.11} = -2$. *For* $i = r-1$ *and* $r$, $k_i^{1^r} = a^k_{ri}$ *of Example 5.1, and* $k_i^{2^r} = a^K_{ri}$ *of Example 5.2. So,*

$$\begin{aligned}&(k_1^{11}, k_2^{11}, k_2^{111}, k_3^{1111}) = (a^k_{21}, a^k_{22}, a^k_{32}, a^k_{43}) \text{ of (4.18)},\\ &\text{and } (k_1^{22}, k_2^{22}, k_2^{222}, k_3^{2222}) = (a^K_{21}, a^K_{22}, a^K_{32}, a^K_{43}) \text{ of (5.4)–(5.7)}.\end{aligned}$$

*By (3.15) and (3.16),*

$$a_{21} = 4\mu^2 k_1^{11} - 4\mu k_1^{12} + k_1^{22},\ a_{11} = -k_1^{11}, \tag{5.8}$$

$$a_{32} = -8\mu^3 k_2^{111} + 12\mu^2 k_2^{112} - 6\mu k_2^{122} + k_2^{222} - 2s_1^2,\ s_j = -2\mu k_1^{1j} + k_1^{2j}. \tag{5.9}$$

*By (4.27) with* $r = 2,\ N_1 = n,\ N_2 = N = n-a$,

$$\begin{aligned}&\kappa(\hat{\mu}, \hat{M}_{0a}) = (nN)^{-1}(na^K_{21} + a^K_{22}) = \sum_{j=1}^{2} n^{-j} k_j^{12} \text{ where}\\ &k_j^{12} = (n/N)a^K_{2j},\ a^K_{21} = \sum_{T=1-n}^{N-1} K(0,T),\ a^K_{22} = -\sum_{T=1-n}^{N-1} \delta_{2T}\, K(0,T), \qquad (5.10)\\ &\delta_{2T} = \max(0,T+a) - \min(0,T), \text{ and}\\ &K(0,T) = \kappa(X_0, X_T X_{T+a}) = k(0,T,T+a) + \mu\,[k(0,T+a) + k(0,T)], \qquad (5.11)\end{aligned}$$

*by (A1). This gives* $a_{21}$ *of (5.8). By (3.15),* $a_{11} = -k_1^{11}$.

$a_{32}$ *of (5.9) needs* $k_2^{112}$ *and* $k_2^{122}$.

*By (4.27) with* $r = 3, N_1 = N_2 = n, N_3 = N = n-a$,

$$\begin{aligned}&\kappa(\hat{\mu}, \hat{\mu}, \hat{M}_{0a}) = (n^2N)^{-1}(na^{K_1}_{32} + a^{K_1}_{33}) = \sum_{j=2}^{3} n^{-j} k_j^{112} \text{ where}\\ &k_j^{112} = (n/N)\, a^{K_1}_{3j},\ \delta_{3\mathbf{T}} = \max(0,T_2,T_3+a) - m_{3\mathbf{T}}, \text{ and by (A3)}, \qquad (5.12)\\ &K_1(0,T_2,T_3) = \kappa(X_0, X_{T_2}, X_{T_3}X_{T_3+a}) = k(0,T_2,T_3,T_3+a) + \mu\, k(0,T_2,T_3+a)\\ &+ \mu\, k(0,T_2,T_3) + k(0,T_3)k(T_2,T_3+a) + k(0,T_3+a)k(T_2,T_3).\end{aligned}$$

*By (4.27) with* $r = 3,\ N_1 = n,\ N_2 = N_3 = N = n-a$,

$$\begin{aligned}&\kappa(\hat{\mu}, \hat{M}_{0a}, \hat{M}_{0a}) = (nN^2)^{-1}(na^{K_2}_{32} + a^{K_2}_{33}) = \sum_{j=2}^{3} n^{-j} k_j^{122} \text{ where}\\ &k_j^{122} = (n/N)\, a^{K_2}_{3j}, \text{ and } \delta_{3\mathbf{T}} = \max(a,T_2,T_3) - \min(0,T_2,T_3), \qquad (5.13)\end{aligned}$$

$$\text{and by (A6), } K_2(0,T_2,T_3)=\kappa(X_0,X_{T_2}X_{T_2+a},X_{T_3}X_{T_3+a})=\sum_{j=1}^{6}B_j \text{ where}$$

$$B_1=k(T_2,T_2+a,T_3,T_3+a,0),\ B_2/\mu=k(0,T_2,T_2+a,T_3)$$
$$+k(0,T_2,T_2+a,T_3+a)+k(0,T_2,T_3,T_3+a)+k(0,T_2+a,T_3,T_3+a),$$
$$B_3=k(T_2,T_2+a,T_3)k(0,T_3+a)+k(T_2,T_2+a,T_3+a)k(0,T_3)$$
$$+k(T_2,T_3,T_3+a)k(0,T_2+a)+k(T_2+a,T_3,T_3+a)k(0,T_2),$$
$$B_4+B_5=k(0,T_2,T_3)(k(T_2,T_3)+\mu^2)+k(0,T_2,T_3+a)(k(T_2+a,T_3)+\mu^2)$$
$$+k(0,T_2+a,T_3)(k(T_2,T_3+a)+\mu^2)+k(0,T_2+a,T_3+a)(k(T_2,T_3)+\mu^2),$$

$$B_6/\mu=k(T_2,T_3)[k(0,T_2+a)+k(0,T_3+a)]+k(T_2,T_3+a)[k(0,T_2+a)$$
$$+k(0,T_3)]+k(0,T_2)[k(T_2+a,T_3)+k(T_2,T_3)]+k(T_2+a,T_3)k(0,T_3+a)$$
$$+k(T_2,T_3)k(0,T_3).$$

*This completes $a_{32}$ of (5.9), giving $h_1,\bar{h}_1,f_1,g_1$ of (2.3)–(2.5).*

*Next we give $a_{22}$. By (3.17),*

$$a_{22}=\sum(a_{22j}:\ j=1,2,3,6) \text{ where}$$
$$a_{221}=4\mu^2k_2^{11}-4\mu k_2^{12}+k_2^{22},\ a_{222}=4\mu k_2^{11},\ a_{223}=-2k_2^{112},\ a_{226}=2(k_1^{11})^2.$$

*Finally we give $a_{43}$. By (3.18), $a_{43}$ is the sum of*

$$a_{431}=16\mu^4k_3^{1111}-32\mu^3k_3^{1112}+24\mu^2k_3^{1122}-8\mu k_3^{1222}+k_3^{2222},$$
$$a_{432}=-24s_2\ (4\mu^2k_2^{111}-4\mu k_3^{112}+k_2^{122},\ s_j=-2\mu k_1^{j1}+k_1^{j2},\ u_1=-2s_2,\ u_2=0,$$
$$a_{433}=0,\ a_{434}=12u_1^2k_1^{11}=4s_2^2k_1^{11}.$$

*So we need $k_3^{ijkl}$. Set*

$$K_{31}=\kappa(\hat{\mu},\hat{\mu},\hat{\mu},\hat{M}_{0a}),\ K_{22}=\kappa(\hat{\mu},\hat{\mu},\hat{M}_{0a},\hat{M}_{0a}),\ K_{13}=\kappa(\hat{\mu},\hat{M}_{0a},\hat{M}_{0a},\hat{M}_{0a}).$$

*By (4.27) with $r=3, N_1=N_2=N_3=n,\ N_4=N=n-a$,*

$$K_{31}=n^{-3}N^{-1}(na_{43}^K+a_{44}^K)=\sum_{j=3}^{4}n^{-j}k_j^{1112} \text{ where } k_j^{1112}=(n/N)a_{4j}^K,$$
$$\delta_{4\mathbf{T}}=\max(0,T_2,T_3,T_4+a)-m_{4\mathbf{T}} \text{ for } m_{4\mathbf{T}} \text{ of (4.14). By (A10),}$$
$$K(0,T_2,T_3,T_4)=\kappa(X_0,X_{T_2},X_{T_3},X_{T_4}X_{T_4+a})=\sum_{j=1}^{3}E_j \text{ where}$$

$$E_1=k(0,T_2,T_3,T_4,T_4+a),\ E_2/\mu=k(0,T_2,T_3,T_4+a)+k(0,T_2,T_3,T_4),$$
$$E_3=k(0,T_4)\,k(T_2,T_3,T_4+a)+k(T_2,T_4)\,k(0,T_3,T_4+a)+k(0,a)\,k(0,T_2,T_3)$$
$$+k(0,T_4+a)\,k(T_2,T_3,T_4)+k(T_2,T_4+a)\,k(0,T_3,T_4)+k(T_3,T_4+a)\,k(0,T_2,T_4).$$

*By (4.27) with $r=3, N_1=N_2=n,\ N_3=N_4=N=n-a$,*

$$K_{22}=(nN)^{-2}(na_{43}^K+a_{44}^K)=\sum_{j=3}^{4}n^{-j}k_j^{1122} \text{ where } k_j^{1112}=(n/N)^2a_{4j}^K,$$
$$\delta_{4\mathbf{T}}=\max(0,T_2,T_3+a,T_4+a)-m_{4\mathbf{T}},$$

*and by (A20),*

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$$K(0, T_2, T_3, T_4) = \sum_{j=1}^{10} F_j \text{ where for example } F_1 = k(0, T_2, T_3, T_3 + a, T_4, T_4 + a),$$
$$F_2/\mu = \sum_{34}^{2} [k(0, T_2, T_3 + a, T_4, T_4 + a) + k(0, T_2, T_3, T_4, T_4 + a)].$$

*By (4.27) with $r = 3, N_1 = n,\ N_2 = N_3 = N_4 = N = n - a$,*

$$K_{13} = (nN^3)^{-1}(na^K_{43} + a^K_{44}) = \sum_{j=3}^{4} n^{-j} k^{1222}_j \text{ where } k^{1122}_j = (n/N)^3 a^K_{4j},$$
$$\delta_{4\mathbf{T}} = \max(0, T_2 + a, T_3 + a, T_4 + a) - m_{4\mathbf{T}},$$

*and by (A21),*

$$K(0, T_2, T_3, T_4) = \sum_{j=1}^{9} G_j \text{ where for example } G_1 = k(0, T_2, T_3, T_3 + a, T_4, T_4 + a),$$
$$G_2/\mu = \sum_{234}^{3} [k(0, T_2)\, k(T_2 + a, T_3, T_3 + a, T_4, T_4 + a)$$
$$+ k(0, T_2 + a)\, k(T_2, T_3, T_3 + a, T_4, T_4 + a)].$$

*This completes $a_{43}$, giving $h_2, \bar{h}_2, f_3, g_3$ of (2.3)–(2.5).*

**Example 5.4.** *The eECF expansions to $O(n^{-1})$ for the ath sample autocorrelation* **assuming that** $\mu = 0$.

*Take $w_1 = M_{00},\ w_2 = M_{0a}$ at $a_1 = 0,\ a_2 = a$. Then $\theta = t(w) = w_2/w_1$ is the ath autocorrelation and $\hat{\theta} = \hat{M}_{0a}/\hat{M}_{00}$ is the ath sample autocorrelation. So $\hat{M}_{00} = \overline{X^2}$, $p = 2$, $a_{21}, a_{11}, a_{32}$ are given by (3.15) and (3.16), and*

$$t_{.1} = -w_2/w_1^2,\ t_{.2} = 1/w_1,\ t_{.11} = 2w_2/w_1^3,\ t_{.12} = -1/w_1^2,\ t_{.22} = 0.$$
$$\text{So, } a_{21} = (t_1^2 k_1^{11} - 2t k_1^{12} + k_1^{22}) w_1^{-2},\ a_{11} = (\theta k_1^{11} - k_1^{12}), w_1^{-2},$$
$$a_{32} = (-\theta^3 k_2^{111} + k_2^{222} + 3t^2 k_2^{112} - 3t k_2^{122} + 6w_2 s_1^2 - 6w_1 s_1 s_2) w_1^{-3},$$
$$s_1 = (-\theta k_1^{11} + k_1^{12}) w_1^{-1},\ s_2 = (-\theta k_1^{21} + k_1^{22}) w_1^{-1}.$$

*These need $k_1^{ij}$ and $k_2^{ijk}$. $k_i^{2^r}$ is $a^K_{ri}$ of Example 5.2. So,*

$$k_1^{22} = \sum_{|T|<N} K(0, T) \text{ of (5.5)},\ k_2^{222} = \sum_{|T_i|<N, i=2,3} K(0, T_2, T_3) \text{ of (5.6)}.$$

*$k_i^{1^r}$ is $k_i^{2^r}$ at $a = 0$. So,*

$$k_1^{11} = \sum_{|T|<N} K(0, T) \text{ where } K(0, T) = k(0, 0, T, T) + 2k(0, T)^2,$$
$$k_2^{111} = \sum_{|T_i|<N, i=2,3} K(0, T_2, T_3) \text{ where } K(0, T_2, T_3) = \sum_{j=1,3,5,6,10} C_j,$$
$$C_1 = k(0, 0, T_2, T_2, T_3, T_3),$$
$$C_3 = 4 \sum_{23}^{2} [k(0, T_2) k(0, T_2, T_3, T_3) + k(0, T_3 - T_2) k(0, 0, T_2, T_3)].$$
$$C_5 = \sum_{23}^{2} [2k(0, 0, T_2) k(0, 0, T_2 - T_3) + k(0, T_2, T_2) k(0, T_3, T_3)].$$
$$C_6 = k(0, T_2, T_3)^2 + 3k(0, T_2, T_3)^2.\ C_{10} = 8k(T_2, T_3)\, k(0, T_2)\, k(0, T_3).$$

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*This completes* $K(0,T_2,T_3)$ *needed for* $a_{32} = a^K_{32}$ *or its limit. We now give* $k^{12}_1, k^{112}_2, k^{122}_2$. *By (4.27) with* $r = 2, N_1 = n, N_2 = N = n - a$,

$$\kappa(\hat{M}_{00}, \hat{M}_{0a}) = (nN)^{-1}(na^K_{21} + a^K_{22}) = \sum_{j=1}^{2} n^{-j} k^{12}_j \text{ where}$$

$$k^{12}_j = (n/N)a^K_{2j},\ K(0,T) = \sum_{j=1}^{4} A_j \text{ by (A2) where}$$

$$A_1 = k(0,0,T,T+a),\ A_2 = A_4 = 0,\ A_3 = 2k(0,T)k(0,T+a)$$

*By (4.27) with* $r = 3, N_1 = N_2 = n,\ N_3 = N = n - a$,

$$\kappa(\hat{M}_{00}, \hat{M}_{00}, \hat{M}_{0a}) = (n^2N)^{-1}(na^K_{32} + a^K_{33}) = \sum_{j=2}^{3} n^{-j} k^{112}_j \text{ where } k^{112}_j = (n/N)\ a^K_{3j},$$

$$\text{and } K(0,T_2,T_3) = \kappa(X^2_0, X^2_{T_2}, X_{T_3}X_{T_3+a}) = \sum_{j=1}^{11} C_j$$

*of Example 5.5. By (4.27) with* $r = 3,\ N_1 = n,\ N_2 = N_3 = N = n - a$,

$$\kappa(\hat{M}_{00}, \hat{M}_{0a}, \hat{M}_{0a}) = (nN^2)^{-1}(na^K_{32} + a^K_{33}) = \sum_{j=2}^{3} n^{-j} k^{122}_j \text{ where } k^{122}_j = (n/N)\ a^K_{3j},$$

$$\text{and } K(0,T_2,T_3) = \kappa(X^2_0, X_{T_2}X_{T_2+a}, X_{T_3}X_{T_3+a}) = \sum_{j=1}^{11} C_j \text{ of (A13)}$$

*with* $(123456)$ *identified with* $(0,0,T_2,T_2+a,T_3,T_3+a)$. *Similarly* $a_{22}, a_{43}$ *needed for the eECF expansions to* $O(n^{-3/2})$, *can be obtained from (3.17) and (3.18).*

**Example 5.5.** *The eECF expansions to* $O(n^{-1})$ *for the ath sample autocorrelation* **without** *assuming that* $\mu = 0$. *Take*

$$\begin{aligned} &w_1 = \mu,\ w_2 = M_{00},\ w_3 = M_{0a}, \\ &\hat{w}_1 = \bar{X},\ \hat{w}_2 = \overline{X^2},\ \hat{w}_3 = \hat{M}_{0a} \text{ of (4.8)}, \\ &D = var(X_0) = w_2 - w^2_1,\ E = covar(X_0, X_a) = w_3 - w^2_1, \\ &\theta = t(w) = covar(X_0, X_a)/var(X_0) = E/D. \end{aligned} \tag{5.14}$$

*So* $p = 3$ *and* $a_{21}, a_{11}, a_{32}$ *are given by (3.19) with*

$$\begin{aligned} &t_{.1} = 2\mu(\theta - 1)/D,\ t_{.2} = D^{-1},\ t_{.3} = -\theta D^{-1},\ t_{.11} = 2(\theta-1)w_3D^{-2}, \\ &t_{.12} = 2w_1D^{-2},\ t_{.13} = 2w_1(1-2\theta)D^{-2},\ \ t_{.22} = 0,\ t_{.23} = -D^{-2},\ t_{.33} = 2\theta D^{-2}. \end{aligned}$$

$k^{11}_1$ *and* $k^{111}_2$ *are* $a^k_{21}$ *and* $a^k_{32}$ *of (5.1).* $k^{33}_1$ *and* $k^{333}_2$ *are* $a^k_{21}$ *and* $a^k_{32}$ *of (5.2).* $k^{22}_1$ *and* $k^{22}_1$ *are just* $k^{33}_1$ *and* $k^{333}_2$ *with* $a = 0$. $k^{13}_1$ *is* $k^{12}_1$ *of (5.10),* $k^{12}_1$ *is* $k^{13}_1$ *with* $a = 0$. $k^{113}_2$ *is* $k^{112}_2$ *of (5.12).* $k^{112}_2$ *is* $k^{113}_2$ *with* $a = 0$. $k^{133}_2$ *is* $k^{122}_2$ *of (5.13).* $k^{122}_2$ *is* $k^{133}_2$ *with* $a = 0$. $k^{23}_1$ *is* $k^{0a}_1$ *of Example 4.6 of Withers and Nadarajah (2012). (This corrects the derivatives of t given in Example 4.5 there.)* $a_{32}$ *also needs* $k^{123}_2, k^{223}_2, k^{233}_2$. *Set* $N = n - a$. *By (4.27) with* $r = 3, N_1 = N_2 = n, N_3 = N = n - a$,

$$\kappa(\hat{w}_1, \hat{w}_2, \hat{w}_3) = (n^2N)^{-1}(na^K_{32} + a^K_{33}) = \sum_{j=2}^{3} n^{-j} k^{112}_j \text{ where}$$

$$k^{123}_j = (n/N)\ a^K_{3j},\ \text{and } K(0,T_2,T_3) = \kappa(X_0, X^2_{T_2}, X_{T_3}X_{T_3+a}).$$

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*Identifying* $(12, 34, 5)$ *of (A6) with* $(T_2, T_2, T_3, T_3 + a, 0)$ *gives*

$$
\begin{aligned}
&K(0, T_2, T_3) = \sum_{j=1}^{6} B_j \textit{ where } B_1 = k(0, T_2, T_2, T_3, T_3 + a),\\
&B_2 = \mu\,[k(0, T_2, T_2, T_3) + k(0, T_2, T_2, T_3 + a) + 2k(0, T_2, T_3, T_3 + a)],\\
&B_3 = k(T_2, T_2, T_3)k(0, T_3 + a) + k(T_2, T_2, T_3 + a)k(0, T_3) + 2k(T_2, T_3, T_3 + a)k(0, T_2),\\
&B_4 = 2k(T_2, T_3)k(0, T_2, T_3 + a) + 2k(T_2, T_3 + a)k(0, T_2, T_3),\\
&B_5 = 2\mu^2\,[k(0, 0, T_3 + a) + k(T_2, T_3, T_3 + a)],\\
&B_6 = 2\mu\, k(0, T_2)[k(T_2, T_3) + k(T_2, T_3 + a)] + \mu\, k(0, T_3)[k(0, 0) + k(T_2, T_3 + a)]\\
&\ + \mu\, k(0, T_3 + a)[k(0, 0) + k(T_2, T_3)].
\end{aligned}
$$

*This gives* $k_2^{123}$*. Next we give* $k_2^{223}$*. By (4.27) with* $r = 3,\ N_1 = N_2 = n,\ N_3 = N = n - a,$

$$
\begin{aligned}
&\kappa(\hat{w}_2, \hat{w}_2, \hat{w}_3) = (n^2N)^{-1}(na^K_{32} + a^K_{33}) = \sum_{j=2}^{3} n^{-j} k_j^{223} \textit{ where } k_j^{223} = (n/N)\, a^K_{3j},\\
&\textit{and } K(0, T_2, T_3) = \kappa(X_0^2, X_{T_2}^2, X_{T_3}X_{T_3+a}).
\end{aligned}
$$

*Identifying* $(12, 34, 56)$ *of (A13) with* $(0, 0, T_2, T_2, T_3, T_3 + a)$ *gives*

$$
\begin{aligned}
&K(0, T_2, T_3) = \kappa(X_0^2, X_{T_2}^2, X_{T_3}X_{T_3+a}) = \sum_{j=1}^{11} C_j,\ C_1 = k(0, 0, T_2, T_2, T_3, T_3 + a),\\
&C_2 = \mu\,[k(0, 0, T_2, T_2, T_3) + k(0, 0, T_2, T_2, T_3 + a) + 2k(0, 0, T_2, T_3, T_3 + a)\\
&\ + 2k(0, T_2, T_2, T_3, T_3 + a)],\\
&C_3 = 2k(0, T_2)k(0, T_2, T_3, T_3 + a) + 2k(0, T_3)k(0, T_2, T_2, T_3 + a)\\
&\ + 2k(0, T_3 + a)k(0, T_2, T_2, T_3) + 2k(0, T_2)k(0, T_2, T_3, T_3 + a)\\
&\ + 2k(T_2, T_3)k(0, 0, T_2, T_3 + a) + 2k(T_2, T_3 + a)k(0, 0, T_2, T_3),
\end{aligned}
$$

$$
\begin{aligned}
&C_4 = \mu^2\,[2k(0, T_2, T_3, T_3 + a) + k(0, T_2, T_2, T_3 + a) + k(0, T_2, T_2, T_3)\\
&\ + k(0, 0, T_2, T_3 + a) + k(0, 0, T_2, T_3),\\
&C_5 = 2\sum_{23}^{2} k(0, 0, T_2)k(T_2, T_3, T_3 + a) + k(0, 0, T_3)k(T_2, T_2, T_3 + a)\\
&\ + k(0, 0, T_3 + a)k(T_2, T_2, T_3),\\
&C_6 = 4k(0, T_2, T_3)k(0, T_2, T_3 + a),\\
&C_7 = \mu\,[2k(T_2, T_3)k(0, 0, T_2) + 2k(0, T_3)k(0, T_2, T_2) + 2k(T_2, T_3 + a)k(0, 0, T_2)\\
&\ + k(0, T_3 + a)(2k(0, T_2, T_2) + k(0, T_2, T_3)) + k(T_3, T_3 + a)k(0, 0, T_2)\\
&\ + k(T_2, T_3 + a)(k(0, 0, T_3) + k(0, T_2, T_3))].\\
&C_8/4\mu = k(0, T_2, T_3)\,[k(0, T_2) + k(0, T_3 + a) + k(T_2, T_3 + a)]\\
&\ + k(0, T_2, T_3 + a)\,[k(0, T_2) + k(0, T_3) + k(T_2, T_3)],\\
&C_9 = 4\mu^3\,[k(0, T_2, T_3) + k(0, T_2, T_3 + a)],\\
&C_{10} = 4k(0, T_2)\,[k(0, T_3)k(T_2, T_3 + a) + k(0, T_3 + a)k(T_2, T_3)],\\
&C_{11}/4\mu^2 = k(0, T_2)\,[k(0, T_3) + k(0, T_3 + a) + k(T_2, T_3) + k(T_2, T_3 + a)]\\
&\ + k(0, T_3)k(T_2, T_3 + a) + k(0, T_3 + a)k(T_2, T_3).
\end{aligned}
$$

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*Finally* $a_{32}$ *needs* $k_2^{233}$. *By* (4.27) *with* $r = 3,\ N_1 = n,\ N_2 = N_3 = N = n - a,$

$$\kappa(\hat{w}_2, \hat{w}_3, \hat{w}_3) = (nN^2)^{-1}(na_{32}^K + a_{33}^K) = \sum_{j=2}^{3} n^{-j} k_j^{233} \text{ where } k_j^{233} = (n/N)^2\, a_{3j}^K,$$
$$\text{and } K(0, T_2, T_3) = \kappa(X_0^2, X_{T_2}X_{T_2+a}, X_{T_3}X_{T_3+a}).$$

*Identifying* $(12, 34, 56)$ *of* (A13) *with* $(0, 0, T_2, T_2 + a, T_3, T_3 + a)$ *gives*

$$K(0, T_2, T_3) = \kappa(X_0^2, X_{T_2}X_{T_2+a}, X_{T_3}X_{T_3+a}) = \sum_{j=1}^{11} C_j,\ C_1 = k(0, 0, T_2, T_2 + a, T_3, T_3 + a),$$
$$C_2 = \mu\, [k(0, 0, T_2, T_2 + a, T_3) + k(0, 0, T_2, T_2 + a, T_3 + a) + k(0, 0, T_2, T_3, T_3 + a)$$
$$+ k(0, 0, T_2 + a, T_3, T_3 + a) + 2k(0, T_2, T_2 + a, T_3, T_3 + a)],$$
$$C_3 = 2k(0, T_2)k(0, T_2 + a, T_3, T_3 + a) + 2k(0, T_2 + a)k(0, T_2, T_3, T_3 + a)$$
$$+ 2k(0, T_3)k(0, T_2, T_2 + a, T_3 + a) + 2k(0, T_3 + a)k(0, T_2, T_2 + a, T_3)$$
$$+ k(T_2, T_3)k(0, 0, T_2 + a, T_3 + a) + k(T_2, T_3 + a)k(0, 0, T_2 + a, T_3)$$
$$+ k(T_2 + a, T_3)k(0, 0, T_2, T_3 + a) + k(T_2, T_3)k(0, 0, T_2, T_3),$$

$$C_4/\mu^2 = 2k(0, T_2 + a, T_3, T_3 + a) + 2k(0, T_2, T_3, T_3 + a)$$
$$+ 2k(0, T_2, T_2 + a, T_3 + a) + 2k(0, T_2, T_2 + a, T_3) + k(0, 0, T_2 + a, T_3)$$
$$+ k(0, 0, T_2 + a, T_3) + k(0, 0, T_2, T_3 + a) + k(0, 0, T_2, T_3),$$

*and similarly for* $C_5, \ldots, C_{11}$.

**Example 5.6.** *Fix* $a_1 \geq 0$ *and* $a_2 \geq 0$. *For* $i = 1, 2$, *set* $N_i = n - a_i > 0$. *We give* $a_{21}$ *needed for the eECF expansions for* $\hat{\theta} = t(\hat{w})$ *when*

$$\theta = t(w) = \kappa(X_0X_{a_1}, X_0X_{a_2}) = w_3 - w_1w_2,$$
$$w_1 = M_{0a_1},\ w_2 = M_{0a_2},\ w_3 = M_{00a_1a_2} = E\ X_0^2X_{a_1}X_{a_2}.$$
$$\text{So, } \hat{w}_3 = N_3^{-1}\sum_{j=1}^{N_3} X_j^2X_{j+a_1}X_{j+a_2}, \text{ where } N_3 = \min(N_1, N_2).$$

*So* $p = 3$, *and* $a_{21}, a_{11}, a_{32}$ *are given by* (3.19) *with non-zero derivatives*

$$t_{\cdot 1} = -w_2,\ t_{\cdot 2} = -w_1,\ t_{\cdot 3} = 1,\ t_{\cdot 12} = -1.$$

*The eECF expansions for* $\hat{\theta} = t(\hat{w})$ *to* $O(n^{-1/2})$ *is given by* $a_{21}$. *By* (3.19), $a_{21}$ *needs* $k_1^{ij}$. *For* $i = 1, 2$, $(k_1^{ii}, k_2^{iii}) = (a_{21}^K, a_{32}^K)$ *of* (5.4) *with* $a = a_i$. *By* (4.27) *with* $r = 2$,

$$\kappa(\hat{w}_1, \hat{w}_2) = (N_1N_2)^{-1}(n\, a_{21}^K + a_{22}^K) = \sum_{j=1}^{2} n^{-j} k_j^{12} \text{ where } k_j^{12} = (n^2/N_1N_2)a_{2j}^K,$$
$$K(0, T) = \kappa(X_0X_{a_1}, X_TX_{T+a_2}) = \sum_{j=1}^{4} A_j \text{ by (A2) where}$$
$$A_1 = k(0, a_1, T, T + a_2),\ A_2/\mu = k(a_1, T, T + a_2)\ + k(0, T, T + a_2)$$
$$+ k(0, a_1, T + a_2) + k(0, a_1, T),$$
$$A_3 = k(0, T)k(a_1, T + a_2) + k(0, T + a_2)k(a_1, T),$$
$$A_4/\mu^2 = k(a_1, T + a_2) + k(a_1, T) + k(0, T + a_2) + k(0, T).$$

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*By (4.27), for $i = 1, 2$,*

$$\kappa(\hat{w}_i, \hat{w}_3) = (N_i N_3)^{-1}(n\, a_{21}^K + a_{22}^K) = \sum_{j=1}^{2} n^{-j} k_j^{i3} \text{ where } k_j^{i3} = (n^2/N_i N_3) a_{2j}^K,$$

$$K(0,T) = \kappa(X_0 X_{a_i}, X_T^2 X_{T+a_1} X_{T+a_2}) = \sum_{j=1}^{19} A_j \text{ by (A11) where for example}$$

$$A_1 = k(0, a_i, T, T, T+a_1, T+a_2),$$
$$A_2/\mu = k(0, T, T, T+a_1, T+a_2) + k(a_i, T, T, T+a_1, T+a_2),$$
$$A_3/\mu = k(0, a_i, T, T, T+a_1) + k(0, a_i, T, T, T+a_2) + 2k(0, a_i, T, T+a_1, T+a_2).$$

*By (4.27),*

$$\kappa_2(\hat{w}_3) = N_3^{-2}(n\, a_{21}^K + a_{22}^K) = \sum_{j=1}^{2} n^{-j} k_j^{33} \text{ where } k_j^{33} = (n/N_3)^2 a_{2j}^K,$$

$$\text{and by (A22), } K(0,T) = \kappa(X_0^2 X_{a_1} X_{a_2}, X_T^2 X_{T+a_1} X_{T+a_2}) = \sum_{j=1}^{16} H_j$$

$$\text{where for example } H_1 = k(0, 0, a_1 a_2, T, T, T+a_1, T+a_2).$$

*This gives $k_1^{33}$. $a_{21}$ is now given by (3.19). However $a_{32}$ needs the joint cumulants of length $L = 9$, which are not given in McCullagh (1987).*

**Example 5.7.** *Fix $a_1, a_2$. Let us find $a_{21}$ for $w = M_{0a_1a_2}$, $\hat{w} = \hat{M}_{0a_1a_2}$ of (4.6):*

$$\hat{w} = N^{-1} \sum_{t=1}^{N} X_{t+I_1} \cdots X_{t+I_r} \text{ where } N = n - I_0, \tag{5.15}$$

*and $I_0 = \max(0, a_1, a_2) - \min(0, a_1, a_2)$. By Corollary 4.2, we can take*

$$a_{21}^K = (n/N)^2 \sum \{K^*(0,T) : |T| < N\} \text{ where}$$
$$K^*(0,T) = \kappa(X_0 X_{a_1} X_{a_2}, X_T X_{T+a_1} X_{T+a_2}) = \sum_{i=1}^{15} A_i$$

*is given by identifying $0, a_1, a_2, T, T+a_1, T+a_2$ with $1, \ldots, 6$ in (A12). But it has 1+6+6.4+9.5+18.6=178 terms, so software is needed.*

**Example 5.8.** *Take $p = 5$,*

$$w_1 = \mu, w_2 = M_{0a_1}, w_3 = M_{0a_2}, w_4 = M_{a_1a_2}, w_5 = M_{0a_1a_2},$$
$$\theta = t(w) = \kappa(X_0, X_{a_1}, X_{a_2}) = w_5 - w_1(w_2 + w_3 + w_4) + 2w_1^3.$$
$$\text{So, } t_{\cdot 1} = 6w_1^2 - w_2 - w_3 - w_4,\ t_{\cdot 2} = t_{\cdot 3} = t_{\cdot 4} = -w_1,\ t_{\cdot 5} = 1.$$

*$a_{21}$ is given by (3.9) in terms of $k_1^{a_1a_2}$, $a_1, a_2 = 1, \ldots, 5$. By Example 5.1, $k_1^{11} = \sum\{k(0,T) : |T| < n\}$. For $a \geq 0$, set $k_1^{22}(a) = a_{21}$ of (5.8). Then for $i = 2, 3$, $k_1^{ii} = k_1^{22}(|a_i|)$, $k_1^{44} = k_1^{22}(|a_1 - a_2|)$. $k_1^{55} = a_{21}$ of Example 5.8. Set $k_1^{12}(a) = k_1^{12} = (n/N)\sum\{K(0,T) : -n < T < N\}$, for $N = n - |a|$ and $K(0,T)$ of (5.11). Then $k_1^{12} = k_1^{12}(|a_1|)$, $k_1^{13} = k_1^{12}(|a_2|)$, $k_1^{14} = k_1^{12}(|a_1 - a_2|)$. By (4.27), for $N$ of (5.15), $k_1^{15} = (n/N)\sum\{K(0,T) : -n < T < N\}$, where now $K(0,T) = \kappa(X_0, X_T X_{T+a_1} X_{T+a_2}) = M_{0,T,T+a_1,T+a_2} - \mu M_{0a_1a_2}$.*

*$k_1^{23} = k_1^{12}$ of Example 5.6. $k_1^{24}$ needs $\kappa(\hat{M}_{0a_1}, \hat{M}_{0a_2})$, a special case of*

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$$\kappa(\hat{M}_{0a_1}, \hat{M}_{a_2a_3}) = \sum_{j=1}^{2} n^{-j} k_j^{24}, \text{ where} \tag{5.16}$$

$$k_j^{24} = (n^2/N_2N_4) a_{2j}^K,\ N_2 = n - |a_1|,\ N_4 = n - |a_2 - a_3|,$$

$$K(0,T) = \kappa(X_0X_{a_1}, X_{T+a_2}X_{T+a_3}) = \sum_{j=1}^{4} A_j \tag{5.17}$$

*by (A2) is given by identifying* $(0, a_1, T + a_2, T + a_3)$ *with* $(1, 2, 3, 4)$. *By (4.27),*

$$\kappa(\hat{\mu}, \hat{M}_{0a_1a_1}) = (n/N) \sum_{-n<T<N} K^*(0,T) \text{ where} \tag{5.18}$$

$$N = n - \max(0, a_1, a_2) + \min(0, a_1, a_2),\ K^*(0,T) = \kappa(X_0, X_TX_{T+a_1}X_{T+a_2}),$$

*given by identifying* $0, T, T + a_1, T + a_2$ *with* $1, 2, 3, 4$ *in (A4).*

*Next we obtain* $k_1^{34}$, $k_1^{35}$, $k_1^{45}$. $k_1^{34}$ *is a special case of (5.16).* $k_1^{35}$ *is covered by* $k_1^{25}$ *of (5.18).* $k_1^{45}$ *is a special case of the lead coefficient in* $\kappa(\hat{M}_{a_1a_2}, \hat{M}_{a_3a_4a_5})$. *In this case, for* $N_1 = \max(a_1, a_2) - \min(a_1, a_2)$, $N_2 = \max(a_3, a_4, a_5) - \min(a_3, a_4, a_5)$,

$$k_1^{45} = (n^2/N_1N_2) \sum_{-N_1<T<N_2} K^*(0,T) \text{ where } K^*(0,T) = \kappa(X_{a_1}X_{a_2}, X_{T+a_3}X_{T+a_4}X_{T+a_5}),$$

*is given by identifying* $a_1, a_2, T + a_3, T + a_4, T + a_5$ *with* $12345$ *in (A5). This completes the terms needed for* $a_{21}$.

Note how Example 5.5 built on Example 5.1, Example 5.6 built on Example 5.5, and Example 5.9 built on Example 5.7. So the above results form a catalogue upon which other examples can build.
Example 5.1 gave $a_{21} = k_1^{11}, a_{32} = k_2^{111}, a_{22} = k_2^{11}, a_{43} = k_3^{1111}$ for $\hat{\mu}$.
Example 5.3 gave $a_{21} = k_1^{11}, a_{32} = k_2^{111}$ for $\hat{M}_{0a}$.
Example 5.5 gave $k_1^{ij}, k_2^{ijk}$ for $\hat{w}_1 = \bar{X},\ \hat{w}_2 = \overline{X^2},\ \hat{w}_3 = \hat{M}_{0a}$.
Example 5.6 gave $k_1^{ij}$ for $\hat{w}_1 = \hat{M}_{0a_1},\ \hat{w}_2 = \hat{M}_{0a_2},\ \hat{w}_3 = \hat{M}_{00a_1a_2}$.
Example 5.7 gave $a_{21} = k_1^{11}$ for $\hat{w}_1 = \hat{M}_{0a_1a_2}$.
Example 5.8 gave $a_{21} = k_1^{11}$ for $\kappa(X_0, X_{a_1}, X_{a_2})$.

## 6. Multivariate Stationary Processes

Suppose that $\cdots, X_{-1}, X_0, X_1, \cdots$ lie in $R^p$ and are stationary with finite moments. For $j = 1, \cdots, p$, denote the $j$th component of $X_i$ by $X_i^j$ and the cross-moments,and the cross-cumulants, by

$$\mu = E\ X_0,\ \mu^j = E\ X_0^j,\ M_{i_1\cdots i_r}^{j_1\cdots j_r} = E\ X_{i_1}^{j_1} \cdots X_{i_r}^{j_r},$$

$$\mu_{i_1\cdots i_r}^{j_1\cdots j_r} = E\ (X_{i_1}^{j_1} - \mu^{j_1}) \cdots (X_{i_r}^{j_r} - \mu^{j_r}),\ k\begin{pmatrix} j_1 \cdots j_r \\ i_1, \cdots, i_r \end{pmatrix} = \kappa(X_{i_1}^{j_1}, \cdots, X_{i_r}^{j_r}).$$

Given a sequence of integers $i_1, \cdots, i_r$, define $i_0, I_k$ as in (4.3). (4.4) becomes

$$M_{i_1\cdots i_r}^{j_1\cdots j_r} = M_{I_1\cdots I_r}^{j_1\cdots j_r},\ \mu_{i_1\cdots i_r}^{j_1\cdots j_r} = \mu_{I_1\cdots I_r}^{j_1\cdots j_r},\ k\begin{pmatrix} j_1 \cdots j_r \\ i_1 \cdots i_r \end{pmatrix} = k\begin{pmatrix} j_1 \cdots j_r \\ I_1 \cdots I_r \end{pmatrix}.$$

However in general $k\binom{j_1j_2}{0,I} \neq k\binom{j_1j_2}{0,-I}$. An unbiased estimate of $M\binom{j_1\cdots j_r}{i_1\cdots i_r}$ is

$$\hat{M}_{i_1\cdots i_r}^{j_1\cdots j_r} = N^{-1} \sum_{t=1}^{N} X_{t+I_1}^{j_1} \cdots X_{t+I_r}^{j_r} \text{ for } N = n - I_0 > 0.$$

We can abbreviate 2 sequences of integers of the same length $r$, as $\pi = (i_1 \cdots i_r),\ \tau = (j_1 \cdots j_r)$. Given $P \geq 1$ and finite sequences of integers $\pi_1, \cdots, \pi_P$ and $\tau_1, \cdots, \tau_P$ not depending on $n$ with $\tau_i$ of the same length as $\pi_i$,

$$\text{set } w_a = M_{\pi_a}^{\tau_a},\ \hat{w}_a = \hat{M}_{\pi_a}^{\tau_a},\ w = (w_1, \cdots, w_p),\ \hat{w} = (\hat{w}_1, \cdots, \hat{w}_p). \tag{6.1}$$

$$\text{Then } \kappa(\hat{w}_{a_1}, \cdots, \hat{w}_{a_r}) = \sum_{j=r-1}^{r} n^{-j} k_j^{a_1 \cdots a_r} \text{ for } a_1, \cdots, a_r \in \{1, \cdots, p\}.$$

Let $\theta = t(w) : R^p \to R^q$ be a function with finite derivatives (3.2) at $w = E\, X_0$. Then $\hat{\theta} = t(\hat{w})$ satisfies (3.3), and $Y_n = n^{1/2}(\hat{w} - w)$ and $Z_n = n^{1/2}(\hat{\theta} - \theta)$ have multivariate Edgeworth expansions. The results of Section 5 apply with $K(t_1 \ldots t_r)$ replaced by its extension $K(\substack{j_1 \cdots j_r \\ t_1 \cdots t_r})$.

## 7. Discussion

For the mean of a random sample, the coefficients of the Edgeworth expansions are functions of the cumulants. However when the observations are dependent, the usual unbiased estimate of $var(X_0)$ fails, so we use empirical estimates. The role of the empirical distribution is eclipsed by having to deal with the cross-cumulants.

Winterbottom (1979, 1980, 1984) showed that Cornish-Fisher expansions gave substantial improvements to the Central Limit Theorem, even for lattice estimates like the binomial and the Poisson. Phillips (1987) gave an extension to nonstationary processes. Bose (1988) obtained an Edgeworth correction by bootstrap in autoregressions. Loh (1996) gave an Edgeworth expansion for U-statistics with dependent observations. Kitamura (1997) used an Edgeworth expansion to study empirical likelihood methods for dependent processes. Lieberman et al. (2001) gave an Edgeworth expansion for the sample autocorrelation function under long range dependence. For a review of a paper by Taniguchi and Kakizawa on Edgeworth and saddlepoint expansions for time series, see Lieberman (2002). Andrews and Lieberman (2005) gave an Edgeworth expansion for maximum likelihood estimator for stationary long-memory Gaussian time series. Lieberman et al. (2003) gave expansions for the maximum likelihood estimator for a stationary Gaussian process.

However until now, there has been no non-parametric theory for the distribution of the general standard estimate based on a sample from a stationary process. Consequently, econometrics, the foremost user of time series, has been dominated by software contructing estimates for parametric autoregressive (AR), moving average (MA), ARMA, and integrated ARMA processes. But parametric models suffer from a severe drawback: if the model is wrong, the results will generally not even give 1st order (Central Limit Theorem) accuracy. This is where a non-parametric method is far superior. Software is still needed when the required $a_{ri}$ of (2.1) have a large numbers of terms.

**Future directions.**

1. These results can easily be extended to both weighted estimates, and estimates based on missing data, as done in Withers (2025b).

2. Lahiri (2003) gave $a_{21}$ and a Central Limit Theorem for weighted sums of a spatial process. It should not be too hard to extend this to give the other leading $a_{rj}$.

3. Software to implement the results of Section 3 for arbitrary $t(w)$ would also be very useful.

4. There is a need to extend the results of McCullagh (1987), and to spell out his succinct notation as done in the appendix below. As noted, his results cover $a_{43}$ for $\kappa_4(\hat{M}_{i_1 i_2})$ but not $a_{32}$ for $\kappa_3(\hat{M}_{i_1 i_2 i_3})$.

5. To extend these results to confidence intervals for a general function of cross-cumulants, say $t(w)$, would be a huge advance. The 1st step is to extend them to Edgeworth expansions for its Studentized form, $\hat{\theta} = (t(\hat{w}) - t(w))/\hat{a}_{21}^{1/2}$, where $\hat{a}_{21} = \sum\{\hat{K}(0,T) :\ |T| < l_n$ is a consistent estimate of $a_{21}$ of (4.21), and one can take $l_n = Kn^{1/2}$ for some $K > 0$. Lahiri (2010) gave Edgeworth expansions for the case $t(w) = \mu$, so that $K(0,T) = \kappa(X_0, X_T)$ and $\hat{K}(0,T) = \hat{M}_{0T}$ of Example 5.3. The expansions are a superposition of three distinct series, respectively, given by one in powers of $n^{-1/2}$, one in powers of $(n/l_n)^{-1/2}$ (resulting from the standard error of the studentizing factor), and one in powers of the bias of $\hat{a}_{21}$. But this is typicallly exponentially small, taking us to the choice $l_n = Kn^{1/2}$.

6. It may be useful to replace our empirical estimate of the cross-cumulant, by estimates with lower bias. While a jack-knife or bootstrap could be used, analytic results are more easily obtained by adjusting it with an estimate of its bias.

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## Appendix A. Some results of McCullagh (1987)

Here we spell out some results from p254–265 of McCullagh (1987). His results are not specified clearly for application. For example the 12 terms that we have written out in full for $C_3$ below, are merely denoted as $1235|46[4][3]$. For applications like ours, all such terms need to be made specific. McCullagh and Wilks (1988) gives the software used to obtain his results. It should be possible to extend these using Matlab in a way that they can be easily applied to examples like ours. We have not been able to access McCullagh and Wilks (1985). Set

$$\kappa^{1,2,\ldots,r} = \kappa(X_1, X_2, \ldots, X_r),\ \kappa^{12,345,\ldots} = \kappa(X_1X_2, X_3X_4X_5, \ldots),$$

and so on. McCullagh denotes these by $1|2|\ldots|r$ and $12|345|\ldots$ His formulas give the cross-cumulants like $\kappa^{12,345,\ldots}$ in terms of the raw cumulants $\kappa^{1,\ldots,r}$.

$$\text{By } 12|3 \text{ p254, } \kappa^{1,23} = \kappa^{1,2,3} + \kappa^{1,3}\kappa^2 + \kappa^{1,2}\kappa^3. \tag{A1}$$

$$\text{By } 12|34 \text{ p254, } \kappa^{12,34} = \kappa^{1,2,3,4} + \sum^4 \kappa^1\kappa^{2,3,4} + \sum^2 \kappa^{1,3}\kappa^{2,4} + \sum^4 \kappa^1\kappa^3\kappa^{2,4}$$

$$= 1234 + \sum^4 1.234 + \sum^2 13.24 + \sum^4 1.3.24 = \sum_{j=1}^4 A_j \text{ say.} \tag{A2}$$

$$\text{By } 12|3|4 \text{ p254, } \kappa^{12,3,4} = \kappa^{1,2,3,4} + \sum^2 (\kappa^2\kappa^{1,3,4} + \kappa^{1,2}\kappa^{2,4}). \tag{A3}$$

$$\text{By } 123|4 \text{ p254, } \kappa^{123,4} = \sum_{i=1}^4 A_i \text{ where } A_1 = \kappa^{1,2,3,4}, \tag{A4}$$

$$A_2 = \sum^3 \kappa^{1,2,4}\kappa^3,\ A_3 = \sum^3 \kappa^{1,2}\kappa^{3,4},\ A_4 = \sum^3 \kappa^{1,4}\kappa^2\kappa^3.$$

$$\text{By } 123|45 \text{ p255, } \kappa^{123,45} = \sum_{i=1}^{10} A_i \text{ where } A_1 = \kappa^{1,\ldots,5}, \tag{A5}$$

$$A_2 = \sum^2 \kappa^{1,2,3,4}\kappa^5,\ A_3 = \sum^3 \kappa^{1,2,4,5}\kappa^3,\ A_4 = \sum^6 \kappa^{1,2,4}\kappa^{3,5},\ A_5 = \sum^3 \kappa^{1,4,5}\kappa^{2,3},$$

$$A_6 = \sum^6 \kappa^{1,2,4}\kappa^3\kappa^5,\ A_7 = \sum^3 \kappa^{1,4,5}\kappa^2\kappa^3,\ A_8 = \sum^6 \kappa^{1,2}\kappa^{3,4}\kappa^5,$$

$$A_9 = \sum^6 \kappa^{1,4}\kappa^{2,5}\kappa^3,\ A_{10} = \sum^6 \kappa^{1,4}\kappa^2\kappa^3\kappa^5.$$

$$\text{By p255, } \kappa^{12,34,5} = 12345 + \sum_1^4 + \sum_2^4 + \sum_3^4 + \sum_4^4 + \sum_1^8 = \sum_{j=1}^6 B_j, \tag{A6}$$

where $B_1 = 12345 = \kappa^{1,2,3,4,5}$, and for $1235.4 = \kappa^{1,2,3,5}\kappa^4$,

$$B_2 = \sum_1^4 = \sum^4 1235.4 = 1235.4 + 1245.3 + 1345.2 + 2345.1,$$

$$B_3 = \sum_2^4 = \sum^4 123.45 = 123.45 + 124.35 + 134.25 + 234.15, \tag{A7}$$

$$B_4 = \sum_3^4 = \sum^4 135.24 = 13.245 + 14.235 + 23.145 + 24.135,$$

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$$B_5 = \sum_4^4 = \sum^4 135.2.4 = 1.3.245 + 1.4.235 + 2.3.145 + 2.4.135,$$

$$B_6 = \sum_1^8 = \sum^8 13.25.4 = 15(23.4 + 24.3) + 25(13.4 + 14.3) + 35(12.4 + 24.1)$$

$$+ 45(13.2 + 12.3). \tag{A8}$$

By p255, $\kappa^{1,2,3,45} = \sum_{j=1}^{3} E_j$ where $E_1 = 12345,\ E_2 = 4.1235 + 5.1234,$ (A9)

$$E_3 = 41.235 + 42.135 + 45.231 + 51.234 + 52.134 + 53.124. \tag{A10}$$

By 1234|56 p256, $\kappa^{1234,56} = \sum_{j=1}^{19} A_j$ where for example

$$A_1 = 123456,\ A_2 = 12345.6 + 12346.5,$$

$$A_3 = \sum^4 12356.4 = 1.23456 + 2.13456 + 3.12456 + 4.12356. \tag{A11}$$

By 123|456 p255, $\kappa^{123,456} = \sum_{i=1}^{15} A_i$ where $A_1 = \kappa^{1,\dots,6}$, (A12)

$$A_2 = \sum^6 \kappa^{1,2,3,4,5}\kappa^6,\ A_3 = \sum^6 \kappa^{1,2,3,4}\kappa^{5,6},\ A_4 = \sum^9 \kappa^{1,2,4,5}\kappa^{3,6},\ \text{ and so on.}$$

By p256, $\kappa^{12,34,56} = \sum_{j=1}^{11} C_j$, where (A13)

$$C_1 = 123456 = \kappa^{1,2,3,4,5,6},\ C_2 = \sum^6 12345.6,$$

$$C_3 = \sum^{12} 1235.46 = 13.2456 + 14.2356 + 15.2346 + 16.2345 + 23.1456$$

$$+ 24.1356 + 25.1356 + 26.1345 + 35.1246 + 36.1245 + 45.1236 + 46.1235, \tag{A14}$$

$$C_4 = \sum^{12} 1235.4.6 = 1.3.2456 + 1.4.2356 + 1.5.2346 + 1.6.2345 + 2.3.1456$$

$$+ 2.4.1356 + 2.5.1346 + 2.6.1345 + 3.5.1246 + 3.6.1245 + 4.5.1236 + 4.6.1235,$$

$$C_5 = \sum^6 123.456 = 123.456 + 124.356 + 125.346 + 126.345 + 341.256$$

$$+ 342.156, \tag{A15}$$

$$C_6 = \sum^4 135.246 = 135.246 + 136.245 + 145.236 + 146.235, \tag{A16}$$

$$C_7 = \sum^{24} 123.45.6 = 6.(54.123 + 53.124 + 52.134 + 51.234) + 5.(64.123$$
$$+ 63.124 + 62.134 + 61.234) + 4.(65.123 + 63.125 + 62.135 + 61.235)$$
$$+ 3.(46.125 + 45.126 + 42.156 + 41.256) + 2.(16.345 + 15.346 + 14.356 + 13.456),$$

$$C_8 = \sum^{24} 135.46.2 = 135.(2.46 + 4.26 + 6.24) + 136.(2.45 + 4.25 + 5.24)$$
$$+ 145.(2.36 + 3.26 + 6.23) + 146.(2.35 + 3.25 + 5.23) + 235.(1.46 + 4.16 + 6.14)$$
$$+ 236.(1.45 + 4.15 + 5.14) + 245.(1.36 + 3.16 + 6.13) + 246.(1.35 + 3.15 + 5.13),$$

$$C_9 = \sum^{8} 135.24.6 = 135.(24.6 + 2.46) + 136.(24.5 + 2.45) + 235.(14.6 + 1.46)$$
$$+ 236.(14.5 + 1.45),$$

$$C_{10} = \sum^{8} \kappa^{1,3}\kappa^{2,5}\kappa^{4,6} = \sum^{8} 13.25.46 \text{ say}, = 13(25.46 + 26.45)$$
$$+ 14(25.36 + 26.35) + 15(23.46 + 24.63) + 16(23.45 + 24.35). \tag{A17}$$

$$C_{11} = 13.(25.4.6 + 26.4.5) + 14.(25.3.6 + 26.3.5) + 15.(23.4.6 + 24.3.6)$$
$$+ 16.(23.4.5 + 24.3.5) + 31.(45.2.6 + 46.2.5) + 32.(45.1.6 + 46.1.5)$$
$$+ 35.(41.2.6 + 42.44.6) + 36.(41.2.5 + 42.1.5) + 51.(63.2.4 + 64.2.3)$$
$$+ 52.(63.1.4 + 64.1.3) + 53.(61.2.4 + 62.1.4) + 54.(61.2.3 + 62.1.3). \tag{A18}$$

$$\text{By p256, } \kappa^{1,2,34,56} = \sum_{j=1}^{10} F_j \text{ where for example } F_1 = 123456, \tag{A19}$$

$$F_2 = \sum_{1}^{4} = 3.12346 + 4.12356 + 5.12346 + 6.12345. \tag{A20}$$

$$\text{By p258, } \kappa^{1,23,45,67} = \sum_{j=1}^{9} G_j \text{ where for example } G_1 = 1234567, \tag{A21}$$

$$G_2 = \sum_{1}^{6} = 12.34567 + 13.24567 + 14.23567 + 15.23467 + 16.23457 + 17.23456.$$

$$\text{By p259, } \kappa^{1234,5678} = \sum_{j=1}^{16} H_j \text{ where for example} \tag{A22}$$

$$H_1 = 12345678, \; H_2 = \sum^{12} 123456.78.$$

$$\text{By p263, } \kappa^{12,34,56,78} = \sum_{j=1}^{12} D_j \text{ where } D_1 = 12345678, \tag{A23}$$

$$D_2 = \sum_{1}^{24} 123457.68, \; D_3 = \sum_{2}^{24} 12345.678, \; D_4 = \sum^{32} 13578.246,$$
$$D_5 = \sum_{3}^{24} 1235.4678, \; D_6 = \sum^{8} 1357.2468, \; D_7 = \sum_{1}^{72} 1235.47.68,$$
$$D_8 = \sum_{1}^{48} 1357.24.68, \; D_9 = \sum_{2}^{72} 123,457.68, \; D_{10} = \sum_{2}^{48} 134.567.28,$$
$$D_{11} = \sum^{96} 137.258.466, \; D_{12} = \sum_{3}^{48} 13.25.47.68.$$

## References

Andrews, D.W.K. and Lieberman, O. (2005) Valid Edgeworth expansions for the Whittle maximum likelihood estimator for stationary long-memory Gaussian time series. *Econometric Theory*, **21** (4), 710–734.

Bose, A. (1988) Edgeworth correction by bootstrap in autoregressions. *Annals of Statistics*, **16** (4), 1709–1722.

Cornish, E.A. and Fisher, R. A. (1937) *Moments and cumulants in the specification of distributions.* Rev. de l'Inst. Int. de Statist. **5**, 307-322. Reproduced in the collected papers of R.A. Fisher, **4**.

Fisher, R. A. and Cornish, E.A. (1960) The percentile points of distributions having known cumulants. *Technometrics*, **2**, 209–225.

Kitamura, Y. (1997) Empirical likelihood methods with weakly dependent processes. *Annals Statist.*, **25** (5), 2084–2102.

Lahiri,S N. (2003) Central limit theorems for weighted sums of a spatial process under a class of stochastic and fixed designs. *Sankhya: The Indian Journal of Statistics*, **65** (2), 356–388.

Lahiri,S N. (2010) Edgeworth expansions for Studentized statistics under weak dependence. *Annals Statist.*, **38** (1), 388–434.

Lieberman, O. (2002) Asymptotic theory of statistical inference for time series, by M. Taniguchi, Y. Kakizawa. Review. *Econometric Theory*,**18** (4) 993–999.

Lieberman, O. and Rousseau, J. and Zucker, D.M. (2001) Valid Edgeworth expansion for the sample autocorrelation function under long range dependence. *Econometric Theory*, **17** (1), 257–275.

Lieberman, O. and Rousseau, J. and Zucker, D.M. (2003) Valid asymptotic expansions for the maximum likelihood estimator of the parameter of a stationary, Gaussian, strongly dependent process. *Annals Statist.*, **31** (2), 586–612.

Loh, Wei-Liem (1996) An Edgeworth expansion for U-statistics with weakly dependent observations. *Statistica Sinica*, **6** (1), 171–186.

McCullagh, P. and Wilks, A.R. (1985) Extended tables of complementary set partitions. AT& T Bell Labs Tech. Memorandum 11214-850328-8.

McCullagh, P., (1987) *Tensor methods in statistics.* Chapman and Hall, London.

McCullagh, P. and Wilks, A.R. (1988) Complementary set partitions. *Proc. Royal Soc. London, Series A, Math. and Physical Sciences*, **415**, No. 1849, 347-362.

Phillips, P.C.B. (1987) Asymptotic expansions in nonstationary vector autoregressions. *Econometric Theory*, **3** (1), 45–68.

Stuart, A. and Ord, K. (1987). Kendall's advanced theory of statistics, **1**. 5th edition. Griffin, London.

Winterbottom, A. (1979) Cornish-Fisher expansions for confidence limits. *J.R. Statist. Soc.* **B 41,** 69–531.

Winterbottom, A. (1980) Asymptotic expansions to improve large sample confidence intervals for system reliability. *Biometrika*, **67**, 351–357.

Winterbottom, A. (1984) The interval estimation of system reliability component test data. *Operations Research*, **32,** 628–640.

Withers, C.S. (1982) The distribution and quantiles of a function of parameter estimates. *Ann. Inst. Statist. Math. A*, **34**, 55–68. A correction to $a_{22}$ is given in Appendix A of Withers (2024).

Withers, C.S. (1983) Expansions for the distribution and quantiles of a regular functional of the empirical distribution with applications to nonparametric confidence intervals. *Annals Statist.*, **11** (2), 577–587.

Withers, C.S. (1984) Asymptotic expansions for distributions and quantiles with power series cumulants. *Journal Royal Statist. Soc. B*, **46**, 389–396. Corrigendum (1986) **48**, 256.

Withers, C.S. (2024) 5th-Order multivariate Edgeworth expansions for parametric estimates. *Mathematics*, 12,905, Advances in Applied Probability and Statistical Inference. https://www.mdpi.com/2227-7390/12/6/905/pdf.

Withers, C.S. (2025a) Edgeworth coefficients for standard multivariate estimates. *Axioms*, 2025, Special Issue: New Perspectives in Mathematical Statistics, 2nd edition. https://www.mdpi.com/2075-1680/14/8/632 PDF Version: https://www.mdpi.com/2075-1680/14/8/632/pdf.

Withers, C.S. (2025b) The distribution and quantiles of the sample mean from a stationary process. *Axioms*, Special Issue: New Perspectives in Mathematical Statistics, https://www.mdpi.com/2075-1680/14/6/406.

Withers, C.S. and Nadarajah, S.N. (2010a) Expansions for log densities of asymptotically normal estimates, *Statistical Papers*, **51**, 247–257, Springer, DOI10.1007/s00362-008-0135-2.

Withers, C.S. and Nadarajah, S. (2010b) Tilted Edgeworth expansions for asymptotically normal vectors. *Ann. Institute Statist. Math.*, **62** (6), 1113–1142. DOI: 10.1007/s10463-008-0206-0.

doi:10.20944/preprints202509.2407.v1



Withers, C.S. and Nadarajah, S. (2010c) The joint cumulants of a linear process. *International Journal of Agric. and Statist. Sciences*, **6** (2), 343–347.

Withers, C.S. and Nadarajah, S. (2012) Cornish-Fisher expansions for sample autocovariances and other functions of sample moments of linear processes. *Brazilian Journ. Prob. and Statist.*, **26** (2), 149–166. DOI:10.1214/10-BJPS126.