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Posted Date: 15 September 2025

doi: 10.20944/preprints202509.1232.v1

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# Coupled Fixed Points in $(q_1, q_2)$-Quasi-Metric Spaces

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**Abstract**

This work presents a new coupled fixed point theorem for a pair of set-valued mappings acting on the Cartesian product of two $(m_1, m_2)$- and $(n_1, n_2)$-quasi-metric spaces. Within the general, non symmetric quasi-metric setting, we establish the existence of an approximate coupled fixed point. Moreover, under the additional assumption of $q_0$-symmetry, we guarantee the existence of a coupled fixed point. Collectively, these results extend and unify several known theorems in the framework of fixed point theory for quasi-metric and asymmetric spaces. We illustrate the obtained results about fixed points when the underlying space is equipped with a graph structure and thus sufficient conditions are found that guarantee the existence of a subgraph with a loop with a length greater or equal to 2.

**Keywords:** coupled fixed point; set-valued mapping; quasi-metric space; $q_0$-symmetry; approximate fixed point; nonlinear analysis

**MSC:** 47H10; 54H25; 54C60

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## 1. Introduction

A central tool for analyzing mappings between metric spaces is Banach's contraction principle [1], which guarantees the existence of a fixed point for a contractive mapping. Such mappings occur across both pure and applied mathematics; recent examples include advances on systems of nonlinear matrix equations [2] and studies of market equilibrium in oligopoly settings [3]. The classical theorem of Banach [1] has spawned an enormous variety of generalizations—too many to list comprehensively—so we focus on those most relevant to our investigation.

One line of generalization alters the underlying space. Working in $b$-metric spaces [4], modular function spaces [5], partially ordered metric spaces [6], or quasi-metric spaces [7] allows one to relax the usual completeness assumptions; see also [8–10] for developments within quasi-metric frameworks. By working in $b$-metric spaces [4], modular function spaces [5], partially ordered metric spaces [6], or quasi-metric spaces [7], one can weaken the requirement of an underlying complete metric space; see also [8–10] for developments specifically within quasi-metric frameworks. A second direction alters the notion of a fixed point. Instead of a point $x \in X$ satisfying $x = Tx$, one considers a bivariate mapping $T : X \times X \to X$ and calls an ordered pair $(x, y) \in X \times X$ a coupled fixed point of $T$ if $x = T(x, y)$ and $y = T(y, x)$ [11]. In [11], the setting is a normed space partially ordered by a cone; subsequently, this cone-ordered normed framework was replaced by a partially ordered metric space in [6]. Since the appearance of [6], the concept of coupled fixed points has been extensively studied. A known limitation of that framework is that a coupled fixed point $(x, y)$ often collapses to the diagonal, i.e., $x = y$, because the definition effectively solves the symmetric pair of equations $x = T(x, y)$ and $y = T(y, x)$. To address genuinely nonsymmetric systems, [12] proposed modifying the notion by replacing a single bivariate self-map with an ordered pair of mappings $F, G : X \times X \to X$ and declaring $(x, y)$ to be a coupled fixed point of $(F, G)$ when

$$x = F(x, y) \quad \text{and} \quad y = G(x, y).$$

This formulation arises naturally in studies of market equilibria for duopoly models [13]. Observe that when $G(x, y) = F(y, x)$, one recovers the classical coupled fixed point notion from [6,11].

Another influential direction equips the underlying space with a graph structure, a viewpoint initiated in [14]. Following that work, a growing literature has developed fixed point results in graph-based settings, including multi-valued mappings in $b$-metric spaces [15], mappings on metric spaces endowed with a directed graph [16], multi-valued mappings on cone metric spaces with a directed graph [17], and monotone mappings in modular function spaces [18].

## 2. Materials and Methods

We begin by recalling the fundamental concepts and notation used in the theory of quasi-metric spaces. Throughout, $\mathbb{N}$ and $\mathbb{R}$ denote the sets of natural numbers and real numbers, respectively. We use capital Latin letters $X$, $Y$, and $Z$ for arbitrary sets, while lowercase letters $x, y, z, u, v, w$ represent elements of these sets.

Our presentation follows the treatments in [7–10], whose terminology and notation are mutually consistent and will be adopted here.

**Definition 1.** *([7]) Let $X$ be a nonempty set, $q_1, q_2 \geq 1$ and a mapping $d : X \times X \to [0, \infty)$ satisfying*

- *(identity axiom): $d(x, y) = 0$ if and only if $x = y$ for any $x, y \in X$*
- *(relaxed triangle inequality): there holds the inequality*

$$d(x, y) \leq q_1 d(x, z) + q_2 d(z, y)$$

*for all $x, y, z \in X$.*

*The function $d(\cdot, \cdot)$ that satisfies the identity axiom and the relaxed triangle inequality is called a $(q_1, q_2)$ quasi-metric.*

**Definition 2.** *([7]) Let $X$ be a nonempty set, $q_1, q_2 \geq 1$ and a mapping $d : X \times X \to [0, \infty)$ be a $(q_1, q_2)$ quasi-metric. If $d(\cdot, \cdot)$ satisfies*

- *(symmetry axiom): $d(x, y) = d(y, x)$ for every $x, y \in X$*

*then it $d(\cdot, \cdot)$ referred to as a symmetric $(q_1, q_2)$-quasi-metric.*

It is possible to relax the symmetry axiom.

**Definition 3.** *([7]) Let $X$ be a nonempty set, $q_1, q_2 \geq 1$ and a mapping $d : X \times X \to [0, \infty)$ be a $(q_1, q_2)$ quasi-metric. If $d(\cdot, \cdot)$ satisfies*

- *(weaker symmetry axiom) there exists $q_0 > 0$ so that the inequality $d(x, y) \leq q_0 d(y, x)$ holds for all $x, y \in X$*

*then it is referred to as a $q_0$-symmetric $(q_1, q_2)$-quasi-metric.*

Let $X$ be a nonempty set, let $q_1, q_2 \geq 1$, and let $d \colon X \times X \to [0, \infty)$. If $d$ is a $(q_1, q_2)$–quasi-metric, we refer to $(X, d)$ as a $(q_1, q_2)$–quasi-metric space. If, in addition, $d(x, y) = d(y, x)$ for all $x, y \in X$, then $(X, d)$ is called a symmetric $(q_1, q_2)$–quasi-metric space. If $d$ satisfies the weaker symmetry condition $d(x, y) \leq q_0\, d(y, x)$ for some $q_0 > 0$, we call $(X, d)$ a $q_0$–symmetric $(q_1, q_2)$–quasi-metric space. In particular, when $q_0 = 1$ and $q_1 = q_2 > 1$, a symmetric $(q_1, q_1)$–quasi-metric space is precisely a $b$–metric space.

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Note that for $q_0 = 1$, any $q_0$–symmetric $(q_1, q_2)$–quasi-metric space becomes symmetric; and for $q_0 = q_1 = q_2 = 1$, $(X, d)$ is a (standard) metric space. Given any quasi-metric $d$, its conjugate $\overline{d}(x, y) := d(y, x)$ is a $(q_2, q_1)$–quasi-metric.

**Definition 4.** *([7]) A $(q_1, q_2)$-quasi-metric space $(X, d)$ is said to be weakly symmetric whenever there holds*

- *(weakly symmetry axiom): if $\lim_{n\to\infty} d(\xi, x_n) = 0$ implies $\lim_{n\to\infty} d(x_n, \xi) = 0$.*

Any $q_0$-symmetric $(q_1, q_2)$-quasi-metric space is weakly symmetric. The converse fails.

**Definition 5.** *([7]) Let $(X, d)$ be a $(q_1, q_2)$-quasi-metric space.*

- *The open ball centered at a point $x_0 \in X$ with radius $r > 0$ is defined by*

$$B(x_0, r) = \{x \in X : d(x_0, x) < r\}.$$

- *The closed ball centered at $x_0$ with radius $r > 0$ is given by*

$$B[x_0, r] = \{x \in X : d(x_0, x) \le r\}.$$

A subset $U \subset X$ is called open if for every $u \in U$ there exists $\varepsilon > 0$ such that $B(u, \varepsilon) \subset U$. The family of open sets determines a topology on any $(q_1, q_2)$–quasi-metric space $(X, d)$. As usual, a set is closed if its complement is open.

A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is said to converge to $x_0 \in X$ in the $(q_1, q_2)$–quasi-metric space $(X, d)$ if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $x_n \in B(x_0, \varepsilon)$ for all $n \ge N$; we write $\lim_{n\to\infty} x_n = x_0$. It is straightforward to verify that, in a $(q_1, q_2)$–quasi-metric space, this is equivalent to $\lim_{n\to\infty} d(x_0, x_n) = 0$.

In a weakly symmetric $(q_1, q_2)$–quasi-metric space, every convergent sequence has a unique limit. By contrast, uniqueness of limits may fail in a general $(q_1, q_2)$–quasi-metric space.

**Definition 6.** *([7]) A sequence $\{x_n\}$ in a $(q_1, q_2)$-quasi-metric space $(X, d)$ is called a fundamental sequence, or a Cauchy sequence, if for every $\varepsilon > 0$ there is an $N$ such that for all $n, m > N$ we have $d(x_m, x_n) < \varepsilon$.*

*A $(q_1, q_2)$-quasi-metric space $(X, d)$ is said to be complete if each of its fundamental sequences has a limit.*

When $q_1 = q_2$, the pair $(X, d)$ specializes to a quasi-metric space, which—depending on the context—is also termed a $b$-metric space [4,19]. The framework of $(q_1, q_2)$–quasi-metric spaces was introduced in [7] and further developed in [8–10] in connection with covering mappings, where sufficient conditions were obtained for the existence of coincidence points of two mappings (one a covering map and the other Lipschitz) defined on $(q_1, q_2)$–quasi-metric spaces.

In what follows we work exclusively within the class of $(q_1, q_2)$–quasi-metric spaces; whenever $q_0$–symmetry is needed, this assumption will be stated explicitly.

Assume $X$ and $Y$ are endowed with the same quasi-metric $d$. For a point $x \in X$ and a subset $A \subseteq X$, define

$$\operatorname{dist}(x, A) := \inf_{a\in A} d(x, a),$$

with the convention $\operatorname{dist}(x, \emptyset) = +\infty$. For $\varepsilon \ge 0$, the $\varepsilon$–neighborhood of $A$ is

$$A(\varepsilon) := \{\, x \in X : \operatorname{dist}(x, A) \le \varepsilon \,\}.$$

A set-valued mapping $F : X \rightrightarrows Y$ assigns to each $x \in X$ a (possibly empty) subset $F(x) \subset Y$. Its graph and inverse are, respectively,

$$\operatorname{Gr}(F) := \{(x, y) \in X \times Y : \; y \in F(x)\}, \qquad F^{-1}(y) := \{x \in X : \; y \in F(x)\}.$$

 doi:10.20944/preprints202509.1232.v1



We say $F$ is closed-valued if $F(x)$ is closed in $Y$ for every $x \in X$, and closed if $\mathrm{Gr}(F)$ is a closed subset of $X \times Y$. Every closed mapping is closed-valued, though the converse need not hold.

The next key lemma, established in [20], plays a central role in product constructions for quasi-metric spaces.

**Lemma 1.** *([20]) Let $(X_1, d_1)$ be a symmetric $(p_1, q_1)$-quasi-metric space, and $(X_2, d_2)$ be a $(p_2, q_2)$-quasi-metric space. Then, the Cartesian product $X_1 \times X_2$ endowed with the metric $d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2)$ is a $(\max\{p_1, p_2\}, \max\{q_1, q_2\})$-quasi-metric space for $d(\cdot, \cdot)$.*

As an immediate consequence of Lemma 1, if $p_1 = p_2$ and $q_1 = q_2$, then $(X_1 \times X_2, d)$ is a $(p_1, p_2)$–quasi-metric space with respect to $d$.

Let $X$ be a $(q_1, q_2)$–quasi-metric space, $F : X \rightrightarrows X$ a set-valued mapping, and $\varepsilon > 0$. A point $x \in X$ is called a fixed point of $F$ if $x \in F(x)$; the set of all fixed points is

$$\mathrm{Fix}F := \{\, x \in X : \ x \in F(x) \,\}.$$

An approximate (or $\varepsilon$–) fixed point of $F$ is a point $x$ with $\mathrm{dist}(x, F(x)) \leq \varepsilon$. The corresponding set is

$$\mathrm{Fix}F(\varepsilon) := \{\, x \in X : \ \mathrm{dist}(x, F(x)) \leq \varepsilon \,\}.$$

For completeness, we also recall the extension of the coupled fixed point notion to multivalued maps.

**Definition 7.** *[21] A point $(x, y) \in X \times X$ is said to be a coupled fixed point of the set-valued map $F : X \times X \rightrightarrows X$ if $x \in F(x, y)$ and $y \in F(y, x)$.*

Subsequently, Definition 7 was extended to encompass an ordered pair of multivalued mappings, leading to a notion of coupled fixed points for $(F_1, F_2)$.

**Definition 8.** *[22] A point $(x, y) \in X \times X$ is said to be a generalized coupled fixed point of the ordered pair of set-valued maps $F_1 : X \times Y \rightrightarrows X$, $F_2 : X \times Y \to Y$, provided that $x \in F_1(x, y)$ and $y \in F_2(x, y)$.*

## 3. Results

In this section, let $(X, d)$ and $(Y, \sigma)$ be two quasi-metric spaces. We consider an ordered pair of set-valued mappings $F_1 : X \times Y \rightrightarrows X$ and $F_2 : X \times Y \rightrightarrows Y$, and we are interested in the existence of a generalized coupled fixed point of $(F_1, F_2)$; that is, a pair $(x, y) \in X \times Y$ satisfying

$$x \in F_1(x, y) \qquad \text{and} \qquad y \in F_2(x, y).$$

In parallel with the usual notion of an approximate fixed point for a single multivalued map, we will also introduce an approximate coupled fixed point adapted to the ordered pair $(F_1, F_2)$.

**Definition 9.** *Let $\varepsilon, \mu > 0$ An approximate or $\varepsilon, \mu$-fixed point of the ordered pair $F = (F_1, F_2)$ is a point $(x, y)$ such that $\mathrm{dist}(x, F_1(x, y)) \leq \varepsilon$ and $\mathrm{dist}(y, F_2(x, y)) \leq \mu$. The set of such points is denoted by*

$$\mathrm{Fix}F(\varepsilon, \mu) := \{x \in X, y \in Y : \mathrm{dist}(x, F_1(x, y)) \leq \varepsilon, \mathrm{dist}(y, F_2(x, y)) \leq \mu\}.$$

**Definition 10.** *Let $F_1 : X \times Y \rightrightarrows X$ and $F_2 : X \times Y \rightrightarrows Y$. A sequence $(x_k, y_k) \cup X \times Y$ is called a sequence of successive approximation of $(F_1, F_2)$ if $x_{k+1} \in F_1(x_k, y_k)$ and $y_{k+1} \in F_2(x_k, y_k)$ for all $k \in \mathbb{N}$.*

**Theorem 1.** *Let $(X, d)$ be a $(m_1, m_2)$-quasi-metric space, $(Y, \sigma)$ be a $(n_1, n_2)$-quasi-metric space with constants $m_1, m_2, n_1, n_2 \geq 1$, $q_1 = \max\{m_1, n_1\}, q_2 = \max\{m_2, n_2\}$. Let $U$ be an open subset of $X$ , $V$ be an open*

 doi:10.20944/preprints202509.1232.v1



*subset of $Y$, $\bar{x} \in U$, $\bar{y} \in V$ and $F_1 : X \times Y \rightrightarrows X$, $F_2 : X \times Y \rightrightarrows Y$ be set-valued mappings . Suppose there exist constants $\alpha, \beta > 0$ and $\lambda \in (\max\{\alpha, \beta\}, 1/q_2)$ such that:*

*(a)* $\quad \operatorname{dist}(\bar{x}, F_1(\bar{x}, \bar{y})) + \operatorname{dist}(\bar{y}, F_2(\bar{x}, \bar{y})) < \dfrac{1 - q_2\lambda}{q_1} \min\left\{\operatorname{dist}(\bar{x}, X \setminus U), \operatorname{dist}(\bar{y}, Y \setminus V)\right\}$,

*(b)* $\quad \operatorname{dist}(x, F_1(x, y)) + \operatorname{dist}(y, F_2(x, y)) \leq \alpha\rho(u, x) + \beta\sigma(v, y)$
*for all $(x, y), (u, v) \in U \times V$ such that $x \in F_1(u, v)$, $y \in F_2(u, v)$, and*

$$\alpha d(u, x) < \operatorname{dist}(x, X \backslash U), \;\; \beta\sigma(v, y) < \operatorname{dist}(y, Y \backslash V).$$

*Then there is a sequence $(x_k, y_k) \subset X \times Y$of successive approximation of $(F_1, F_2)$, starting from $(\bar{x}, \bar{y})$, such that*

*(A)* *for every $\varepsilon > 0$, $\mu > 0$, $(x_k, y_k) \in \operatorname{Fix} F(\varepsilon, \mu) \cap U \times V$, and*

$$d(\bar{x}, x_k) + \sigma(\bar{y}, y_k) \leq \frac{q_1}{1 - q_2\lambda}\Big(\operatorname{dist}(\bar{x}, F_1(\bar{x}, \bar{y})) + \operatorname{dist}(\bar{y}, F_2(\bar{x}, \bar{y}))\Big), \textit{eventually}.$$

*(B)* *if moreover $X$ and $Y$ be complete, both $F_1$, $F_2$ have closed graphs in $X \times Y \times X$ and $X \times Y \times Y$, respectively, $X$ be a $p_0$-symmetric, and $Y$ be a $q_0$-symmetric, respectively, then there exist an elements $x^* \in X$ and $y^* \in Y$ such that $\{x_k\}_{k=0}^{\infty}$ converges to $x^*$, $\{y_k\}_{k=0}^{\infty}$ converges to $y^*$ and*

$$(x^*, y^*) \in \operatorname{Fix}(F_1 \times F_2) \cap (U \times V),$$

$$d(\bar{x}, x^*) + \sigma(\bar{y}, y^*) \leq \frac{q_1}{1 - q_2\lambda}(\operatorname{dist}(\bar{x}, F_1(\bar{x}, \bar{y})) + \operatorname{dist}(\bar{y}, F_2(\bar{x}, \bar{y}))).$$

**Proof.** Let us choose $\overline{x}$ and $\overline{y}$ that satisfy the assumptions $(a)$ and $(b)$. Just to fit some of the formulas into the text field let us denote $S(x, y) = \operatorname{dist}(x, F_1(x, y)) + \operatorname{dist}(y, F_2(x, y))$.

We will distinguish to cases: $S(\overline{x}, \overline{y}) = 0$ and $S(\overline{x}, \overline{y}) > 0$.

If $S(\overline{x}, \overline{y}) = 0$ then $(\overline{x}, \overline{y})$ is the generalized coupled fixed point and the proof is finished.

Let us assume that $S(\overline{x}, \overline{y}) > 0$.

By $(a)$ there is $l > 0$ such that the hold the inequalities

$$\frac{q_1}{1 - q_2\lambda} S(\overline{x}, \overline{y}) < l < \min\{\operatorname{dist}(\overline{x}, X \setminus U), \operatorname{dist}(\overline{y}, Y \setminus V)\}. \tag{1}$$

By induction, we will construct two sequences $\{x_k\}_{k=0}^{\infty}$ and $\{y_k\}_{k=0}^{\infty}$, starting with

$$x_0 = \overline{x} \text{ and } y_0 = \overline{y}.$$

From (1) we can pick up $x_1 \in F_1(x_0, y_0)$ and $y_1 \in F_2(x_0, y_0)$ satisfying

$$\frac{q_1}{1 - q_2\lambda}(d(x_0, x_1) + \sigma(y_0, y_1)) < l.$$

Thus we can write the chain of inequalities

$$d(x_0, x_1) + \sigma(y_0, y_1) < \frac{(1 - q_2\lambda)l}{q_1} < \frac{1 - q_2\lambda}{q_1} \min\{\operatorname{dist}(x_0, X \setminus U), \operatorname{dist}(y_0, Y \setminus V)\}. \tag{2}$$

Since $q_1 \geq 1$ and $q_2\lambda \in (0, 1)$, it follows that

$$d(x_0, x_1) < \operatorname{dist}(x_0, X \setminus U) \text{ and } \sigma(y_0, y_1) < \operatorname{dist}(y_0, Y \setminus V),$$

and hence $x_1 \in U$, $y_1 \in V$.

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Using the relaxed triangular inequality and the inclusions $x_1 \in U, y_1 \in V$ we get

$$\mathrm{dist}(x_0, X \setminus U) \le q_1 d(x_0, x_1) + q_2 \mathrm{dist}(x_1, X \setminus U)$$

and

$$\mathrm{dist}(y_0, Y \setminus V) \le q_1 d(y_0, y_1) + q_2 \mathrm{dist}(y_1, Y \setminus V).$$

In order to fit the next inequalities into the text field let us use the notation $\rho_n = d(x_n, x_{n+1}) + \sigma(y_n, y_{n+1})$. We can write the chain of inequalities

$$\begin{array}{rcl}\rho_0 & < & \frac{(1-q_2\lambda)l}{q_1} \\ & < & \frac{1-q_2\lambda}{q_1} \min\{\mathrm{dist}(x_0, X \setminus U), \mathrm{dist}(y_0, Y \setminus V)\} \\ & \le & \frac{1-q_2\lambda}{q_1} \min\{(q_1 d(x_0, x_1) + q_2 \mathrm{dist}(x_1, X \setminus U), q_1 \sigma(y_0, y_1) + q_2 \mathrm{dist}(y_1, Y \setminus V)\} \\ & \le & \frac{1-q_2\lambda}{q_1} q_1 \rho_0 + \frac{1-q_2\lambda}{q_1} q_2 \min\{\mathrm{dist}(x_1, X \setminus U), \mathrm{dist}(y_1, Y \setminus V)\}.\end{array}$$

Thus, there holds

$$(1 - (1 - q_2\lambda))\rho_0 < \frac{1 - q_2\lambda}{q_1} q_2 \min\{\mathrm{dist}(x_1, X \setminus U), \mathrm{dist}(y_1, Y \setminus V)\}$$

and consequently we end with the inequality, having in mind the assumptions $q_1 \ge 1$ and $q_2\lambda \in (0, 1)$

$$\begin{array}{rcl}\lambda(d(x_0, x_1) + \sigma(y_0, y_1)) & < & \frac{1-q_2\lambda}{q_1} \min\{\mathrm{dist}(x_1, X \setminus U), \mathrm{dist}(y_1, Y \setminus V)\} \\ & < & \min\{\mathrm{dist}(x_1, X \setminus U), \mathrm{dist}(y_1, Y \setminus V)\}.\end{array}$$

Since $\lambda > \max\{\alpha, \beta\}$ we have

$$\begin{array}{l}\alpha d(x_0, x_1) < \mathrm{dist}(x_1, X \setminus U), \\ \beta\sigma(y_0, y_1) < \mathrm{dist}(y_1, Y \setminus V).\end{array} \tag{3}$$

Using (b) and (2), and we get

$$\begin{array}{rcl}\mathrm{dist}(x_1, F_1(x_1, y_1)) + \mathrm{dist}(y_1, F_2(x_1, y_1)) & \le & \alpha d(x_0, x_1) + \beta\sigma(y_0, y_1) \\ & < & \lambda(d(x_0, x_1) + \sigma(y_0, y_1)) \\ & < & l\lambda \frac{(1-q_2\lambda)}{q_1}\end{array} \tag{4}$$

Hence,

$$S(x_1, y_1) \quad < \quad \min\left\{\frac{1-q_2\lambda}{q_1} \mathrm{dist}(x_1, X \setminus U), \frac{1-q_2\lambda}{q_1} \mathrm{dist}(y_1, Y \setminus V),\ l\lambda \frac{1-q_2\lambda}{q_1}\right\}. \tag{5}$$

From (4) and (5) it follows that possibility to choose $x_2 \in F_1(x_1, y_1)$ and $y_2 \in F_2(x_1, y_1)$ satisfying simultaneously

$$\rho_1 < \min\left\{\lambda\rho_0, \frac{1 - q_2\lambda}{q_1} \mathrm{dist}(x_1, X \setminus U), \frac{1 - q_2\lambda}{q_1} \mathrm{dist}(y_1, Y \setminus V),\ l\lambda \frac{1 - q_2\lambda}{q_1}\right\}.$$

Let us denote $\rho_{n,m} = d(x_n, x_m) + \sigma(y_n, y_m)$ and

$$W_n = \min\{\mathrm{dist}(x_n, X \setminus U), \mathrm{dist}(y_n, Y \setminus V)\}.$$

Thus $\rho_n = \rho_{n,n+1}$. It is easy to observe that for any $x > 1$ and $a \in (0,1)$ there holds the inequality $(x+a)\frac{1-a}{x} < 1$. By using the relaxed triangular inequality and the last observation with $x = q_1$ and $a = \lambda q_2$ we get a upper estimate

$$\begin{array}{rcl}\rho_{0,2} & \leq & q_1 d(x_0,x_1) + q_2 d(x_1,x_2) + q_1\sigma(y_0,y_1) + q_2\sigma(y_1,y_2) \\ & = & q_1(d(x_0,x_1)+\sigma(y_0,y_1)) + q_2(d(x_1,x_2)+\sigma(y_1,y_2)) \\ & = & (q_1+\lambda q_2)\rho_0 \\ & < & (q_1+\lambda q_2)\frac{1-q_2\lambda}{q_1}W_0 < W_0 \\ & = & \min\{\mathrm{dist}(x_0, X\setminus U), \mathrm{dist}(y_0, Y\setminus V)\}.\end{array}$$

Hence, $x_2 \in U$ and $y_2 \in V$.

The inequality

$$\begin{array}{rcl}\rho_1 & < & \frac{1-q_2\lambda}{q_1}W_1 \\ & \leq & \frac{1-q_2\lambda}{q_1}\min\{q_1 d(x_1,x_2) + q_2\mathrm{dist}(x_2, X\setminus U)), q_1\sigma(y_1,y_2) + q_2\mathrm{dist}(y_2, Y\setminus V))\} \\ & \leq & \frac{1-q_2\lambda}{q_1}q_1(d(x_1,x_2)+\sigma(y_1,y_2)) + \frac{1-q_2\lambda}{q_1}q_2W_2\end{array}$$

yields

$$\lambda\rho_1 < \frac{1-q_2\lambda}{q_1}W_2.$$

Combining the condition that $x_n, y_n$ for $n = 0,1,2$ satisfy we will choose the rest of the sequences $\{x_n\}_{n=3}^{\infty}$ and $\{y_n\}_{n=3}^{\infty}$ to verify the next assumptions

$$(x_n, y_n) \in U \times V, \tag{6}$$

$$\begin{array}{c} x_n \in F_1(x_{n-1}, y_{n-1}) \\ y_n \in F_2(x_{n-1}, y_{n-1}), \end{array} \tag{7}$$

$$d(x_{n-1},x_n) + \sigma(y_{n-1},y_n) < \lambda(d(x_{n-2},x_{n-1}) + \sigma(y_{n-2},y_{n-1})), \tag{8}$$

and

$$\lambda(d(x_{n-1},x_n) + \sigma(y_{n-1},y_n)) < \frac{1-q_2\lambda}{q_1}\min\{\mathrm{dist}(x_n, X\setminus U), \mathrm{dist}(y_n, Y\setminus V)\}. \tag{9}$$

Suppose that $\{x_k\}_{k=0}^{n}$ and $\{y_k\}_{k=0}^{n}$ have been defined to satisfy (6), (7), (8), and (9). We will show that we can choose $x_{n+1}$ and $y_{n+1}$ that will verify the same conditions.

From (9) we get

$$\begin{array}{rcl} S(x_n,y_n) & \leq & \alpha d(x_{n-1},x_n) + \beta\sigma(y_{n-1},y_n) \\ & < & \lambda\rho_{n-1} < \lambda^n\rho_0 \\ & < & l\lambda^n\frac{1-q_2\lambda}{q_1} \end{array}$$

and hence,

$$S(x_n,y_n) < \frac{1-q_2\lambda}{q_1}\min\{\mathrm{dist}(x_n, X\setminus U), \mathrm{dist}(y_n, Y\setminus V),\ l\lambda^n\}.$$

Thus we can choose $x_{n+1} \in F_1(x_n,y_n)$ and $y_{n+1} \in F_2(x_n,y_n)$ so that

$$\rho_n < \min\left\{\lambda\rho_{n-1}, \frac{1-q_2\lambda}{q_1}\mathrm{dist}(x_n, X\setminus U), \frac{1-q_2\lambda}{q_1}\mathrm{dist}(y_n, Y\setminus V),\ l\lambda^n\frac{1-q_2\lambda}{q_1}\right\}.$$

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We estimate:

$$
\begin{aligned}
\rho_{0,n+1} &= d(x_0, x_{n+1}) + \sigma(y_0, y_{n+1}) \\
&\le q_1 d(x_0, x_1) + q_2 d(x_1, x_{n+1}) + q_1 \sigma(y_0, y_1) + q_2 \sigma(y_1, y_{n+1}) \\
&= q_1(d(x_0, x_1) + \sigma(y_0, y_1)) + q_2(d(x_1, x_{n+1}) + \sigma(y_1, y_{n+1})) \\
&= q_1(d(x_0, x_1) + \sigma(y_0, y_1)) \\
&\quad + q_1 q_2(d(x_1, x_2) + \sigma(y_1, y_2)) + q_2^2(d(x_2, x_{n+1}) + \sigma(y_2, y_{n+1})) \\
&\le \cdots \le q_1(d(x_0, x_1) + \sigma(y_0, y_1)) \textstyle\sum_{j=0}^{n} (q_2\lambda)^j \\
&\quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&= q_1(d(x_0, x_1) + \sigma(y_0, y_1)) \tfrac{1-(q_2\lambda)^{n+1}}{1-q_2\lambda} < \tfrac{q_1}{1-q_2\lambda}(d(x_0, x_1) + \sigma(y_0, y_1)) \\
&< \tfrac{q_1}{1-q_2\lambda} \cdot \tfrac{1-q_2\lambda}{q_1} \min\{\mathrm{dist}(x_0, X \setminus U), \mathrm{dist}(y_0, Y \setminus V)\} \\
&= \min\{\mathrm{dist}(x_0, X \setminus U), \mathrm{dist}(y_0, Y \setminus V)\}.
\end{aligned}
$$

Hence,

$$
d(x_0, x_{n+1}) + \sigma(y_0, y_{n+1}) < \min\{\mathrm{dist}(x_0, X \setminus U), \mathrm{dist}(y_0, Y \setminus V)\}. \tag{10}
$$

Thus $x_{n+1} \in U$ and $y_{n+1} \in V$. Also the chain of inequalities

$$
\begin{aligned}
\rho_n &= d(x_n, x_{n+1}) + \sigma(y_n, y_{n+1}) \\
&< \tfrac{1-q_2\lambda}{q_1} \min\{\mathrm{dist}(x_n, X \setminus U), \mathrm{dist}(y_n, Y \setminus V)\} \\
&\le \tfrac{1-q_2\lambda}{q_1} \min\{q_1 d(x_n, x_{n+1}) + q_2 \mathrm{dist}(x_{n+1}, X \setminus U)), q_1 \sigma(y_n, y_{n+1}) + q_2 \mathrm{dist}(y_{n+1}, Y \setminus V))\} \\
&\le \tfrac{1-q_2\lambda}{q_1} q_1(d(x_n, x_{n+1}) + \sigma(y_n, y_{n+1})) \\
&\quad + \tfrac{1-q_2\lambda}{q_1} q_2 \min\{\mathrm{dist}(x_{n+1}, X \setminus U), \mathrm{dist}(y_{n+1}, Y \setminus V)\},
\end{aligned}
$$

lead to the inequality

$$
\lambda(d(x_n, x_{n+1}) + \sigma(y_n, y_{n+1})) < \frac{1-q_2\lambda}{q_1} \min\{\mathrm{dist}(x_{n+1}, X \setminus U), \mathrm{dist}(y_{n+1}, Y \setminus V)\}.
$$

By induction, the sequences $\{x_k\}_{k=0}^{\infty}$ and $\{y_k\}_{k=0}^{\infty}$ are constructed satisfying (6), (7), (8), and (9). For all $m > n$, there holds

$$
\begin{aligned}
\rho_{n,m} &= d(x_n, x_m) + \sigma(y_n, y_m) \\
&\le q_1(d(x_n, x_{n+1}) + \sigma(y_n, y_{n+1})) + q_1 q_2(d(x_{n+1}, x_{n+2}) + \sigma(y_{n+1}, y_{n+2})) \\
&\quad + \cdots + q_1 q_2^{m-n-1}(d(x_{m-1}, x_m) + \sigma(y_{m-1}, y_m)) \\
&= q_1 \lambda^n (d(x_0, x_1) + \sigma(y_0, y_1))) \textstyle\sum_{j=0}^{m-n-1} (q_2\lambda)^j \\
&= q_1 \lambda^n (d(x_0, x_1) + \sigma(y_0, y_1))) \cdot \tfrac{1-(q_2\lambda)^{m-n}}{1-q_2\lambda}.
\end{aligned}
$$

Hence,

$$
d(x_n, x_m) + \sigma(y_n, y_m) \le q_1 \lambda^n (d(x_0, x_1) + \sigma(y_0, y_1))) \cdot \frac{1-(q_2\lambda)^{m-n}}{1-q_2\lambda}. \tag{11}
$$

(A) If $X$ and $Y$ are not complete and $\mathrm{Gr}(F_1)$ and $\mathrm{Gr}(F_2)$ are not closed. By putting $n = 0$ in (11) and from (2), we get the inequality for every $m \in \mathbb{N}$

$$
d(x_0, x_m) + \sigma(y_0, y_m) < l < \min\{\mathrm{dist}(x_0, X \setminus U), \mathrm{dist}(y_0, Y \setminus V)\}. \tag{12}
$$

Thus we conclude that $x_m \in U$ and $y_m \in V$. From the chain of inequalities

$$
\begin{aligned}
\lim_{m\to\infty} S(x_m, y_m) &= \lim_{m\to\infty} (\mathrm{dist}(x_m, F_1(x_m, y_m)) + \mathrm{dist}(y_m, F_2(x_m, y_m)) \\
&\le \lim_{m\to\infty} (d(x_m, x_{m+1}) + \sigma(y_m, y_{m+1})) \\
&\le \lim_{m\to\infty} \lambda(d(x_{m-1}, x_m) + \sigma(y_{m-1}, y_m)) \\
&\quad \dots\dots\dots\dots\dots\dots\dots\dots \\
&\le \lim_{m\to\infty} \lambda^m (d(x_0, x_1) + \sigma(y_0, y_1)) = 0.
\end{aligned}
$$

Hence, for every $\varepsilon, \mu > 0$ there is $M \in \mathbb{N}$ so that for every $m \geq M$ there holds $(x_m, y_m) \in \mathrm{Fix}F(\varepsilon, \mu)$.

Moreover, from (12) and $d(\bar{x}, x_m) + \sigma(\bar{y}, y_m) < l$, we obtain that

$$d(\bar{x}, x_m) + \sigma(\bar{y}, y_m) \leq \frac{q_1}{1 - q_2\lambda}\Big(\mathrm{dist}(\bar{x}, F_1(\bar{x}, \bar{y})) + \mathrm{dist}(\bar{y}, F_2(\bar{x}, \bar{y}))\Big). \tag{13}$$

(B) Let $X$ and $Y$ be complete, and $p_0$-symmetric and $q_0$-symmetric, respectively. Let $\mathrm{Gr}(F_1)$ and $\mathrm{Gr}(F_2)$ are closed. Let us put $c_0 = \max\{p_0, 1/p_0, q_0, 1/q_0\}$.

We have proven in (11) that for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ so that for all $N \leq n < m$ there holds the inequality

$$d(x_n, x_m) + \sigma(y_n, y_m) < \varepsilon.$$

Since we have assumed that the two quasi metric spaces are $p_0$ and $q_0$ symmetric, respectively, we can write the inequality $d(x_m, x_n) + \sigma(y_m, y_n) \leq c_0(d(x_n, x_m) + \sigma(y_n, y_m)) \leq c_0\varepsilon$. There for both sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ are Cauchy ones in the considered quasi metric spaces. By the assumption that both spaces are complete it follows $\lim_{n\to\infty} x_n \to x^* \in X$ and $\lim_{n\to\infty} y_n \to y^* \in Y$. Passing to the limit for $m \to \infty$ in (12) gives us

$$d(x_0, x^*) + \sigma(y_0, y^*) \leq l < \min\{\mathrm{dist}(x_0, X\backslash U), \mathrm{dist}(y_0, Y\backslash V)\},$$

and consequently $x^* \in U$ and $y^* \in V$. Letting once again $m \to \infty$ in (13) we get

$$d(x_0, x^*) + \sigma(y_0, y^*) \leq \frac{q_1}{1 - q_2\lambda}(\mathrm{dist}(\bar{x}, F_1(\bar{x}, \bar{y})) + \mathrm{dist}(\bar{y}, F_2(\bar{x}, \bar{y}))).$$

From $(x_{n-1}, y_{n-1}, x_n) \in \mathrm{Gr}(F_1), (x_{n-1}, y_{n-1}, y_n) \in \mathrm{Gr}(F_2)$ the closeness of $\mathrm{Gr}(F_i)$, $i = 1, 2$ and $\lim_{n\to\infty} x_n = x^*$, $\lim_{n\to\infty} y_n = y^*$ we conclude that the inclusions $x^* \in F_1(x^*, y^*) \cap U$ and $y^* \in F_2(x^*, y^*) \cap V$ hold true. $\square$

## 4. Application

We will follow the notations and notion from [14].

Let $(Z, d)$ be a $(q_1, q_2)$ quasi-metric space and $G$ be a weighted directed graph with a set of vertices $V(G) = Z$ and an edge set $E(G) \subseteq Z \times Z$, where the weights of the edges will be calculated as the quasi-metric distance between their endpoints. We set the edge weight $w(u, v) := d(u, v)$ for each $(u, v) \in E(G)$.

A subgraph of $G$ is called a graph $(V', E')$ such that $V' \subseteq V(G)$, $E' \subseteq E(G)$ and for each edge $(x, y) \in E'$, it holds that $x, y \in V'$.

If $x$ and $y$ are vertices of $G$, then a path of length $n$, $n \in \mathbb{N} \cup \{0\}$, is a sequence of vertices $\{x_i\}_{i=0}^{n}$ such that

$$x_0 = x, \quad x_n = y, \quad (x_{i-1}, x_i) \in E(G) \text{ for } i = 1, 2, \ldots, n.$$

In what follows "path" means a directed path of length $\geq 1$. We assume in the set of all "paths" there are no loops (or self-loops), i.e., an edge that connects a vertex to itself.

A graph is said to be connected if there is a path between any two vertices. Given that $\widetilde{G}$ is connected, $G$ is weakly connected. Here $\widetilde{G}$ is the underlying undirected graph.

If the edge set $E(G)$ of a graph $G$ is symmetric, then the component of $G$ containing a vertex $x$ is defined as the subgraph $G_x$ that includes all vertices and edges that lie on a path starting from $x$. For a general directed graph, strongly connected components play the analogous role.

By $[x]_G$ we will denote the equivalence class induced by the relation $R$ defined on $V(G)$ as

$$yRz \text{ if there is a path in } G \text{ from } y \text{ to } z.$$

Note $R$ need not be symmetric; for an equivalence relation one may use paths in both directions.

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It follows that $V(G_x) = [x]_G$. We will assume that $(z,z) \notin E(G)$, i.e., there is no path with a length 1 from $z$ to $z$.

Let us define a multi-valued map $H : V(G) \rightrightarrows V(G)$ that assigns to any $z \in V(G)$ the set of all $z'$ such that there exists a directed path of length $\geq 1$ from $z$ to $z'$. If $z \in Hz$ then $z$ is a fixed point for the multi-valued map $H$ and there is a path from $z$ to $z$, i.e., there holds the relation $zRz$.

Let $G' = (V', E')$ be a subgraph of $G$. By distance between $v \notin V'$ and $V'$ we assume the directed shortest-path distance

$$\operatorname{dist}(v, V') := \inf\left\{ \sum_{i=1}^{k} w(u_{i-1}, u_i) \ : \ v = u_0 \to u_1 \to \cdots \to u_k,\ u_k \in V',\ k \geq 1 \right\},$$

and will denote it by $\operatorname{dist}(v, V')$. If no such directed path exists, set $\operatorname{dist}(v, V') := +\infty$.

If there is not any $v' \in V'$ so that $(v, v') \in E(G)$, then we will put $\operatorname{dist}(v, V') = +\infty$. If we assume that a graph $G$ is connected then for any $v \in V$ and $V' \subset V$ there holds $\operatorname{dist}(v, V') < +\infty$. (For undirected distance in $\widetilde{G}$, weak connectivity suffices; for the directed distance above, strong connectivity yields finiteness.)

Let $(X, d)$ be a $(m_1, m_2)$-quasi-metric space, $(Y, \sigma)$ be a $(n_1, n_2)$-quasi-metric space with constants $m_1, m_2, n_1, n_2 \geq 1$, $q_1 = \max\{m_1, n_1\}$, $q_2 = \max\{m_2, n_2\}$. Let us put $Z = X \times Y$ and endow $Z$ with the $(q_1, q_2)$-quasi metric

$$\rho((x_1, x_2), (y_1, y_2)) = d(x_1, y_1) + \sigma(x_2, y_2).$$

When $Z = X \times Y$ with quasi-metric $\rho$, we use the graph weight $w(u, v) := \rho(u, v)$ for $(u, v) \in E(G)$.

Let us put

$$F_1(x, y) := \{\, x' \in X : \ \exists y' \in Y \text{ with } (x', y') \in H(x, y) \,\}$$

that is the projection of $H$ on $X$, and

$$F_2(x, y) := \{\, y' \in Y : \ \exists x' \in X \text{ with } (x', y') \in H(x, y) \,\}$$

that is the projection of $H$ on $Y$. Thus we can consider $Hz = H(x, y) = (F_1(x, y), F_2(x, y))$ for $z = (x, y) \in Z = X \times Y$.

**Definition 11.** *Let $(X, d)$ be a $(m_1, m_2)$-quasi-metric space, $(Y, \sigma)$ be a $(n_1, n_2)$-quasi-metric space with constants $m_1, m_2, n_1, n_2 \geq 1$, $q_1 = \max\{m_1, n_1\}$, $q_2 = \max\{m_2, n_2\}$. Let us put $Z = X \times Y$ and endow $Z$ with the $(q_1, q_2)$-quasi metric $\rho$. Let the graph $G$ be a directed graph, consisting of vertices $V(G) = Z$ and edges $E(G)$. Let $H : V(G) \rightrightarrows V(G)$ be a multi valued map that assigns to every $z \in V(G)$ all $z' \in V(G)$ such that there is a path from $z$ to $z'$. Let denote by $F_1$ the projection of $H$ into $X$ and by $F_2$ its projection into $Y$ as explicitly defined above. We will call the map $H = (F_1, F_2)$ a path map for the graph $G$.*

**Theorem 2.** *Let $(X, d)$ be a complete, $p_0$-symmetric $(m_1, m_2)$-quasi-metric space, $(Y, \sigma)$ be a complete, $q_0$-symmetric $(n_1, n_2)$-quasi-metric space with constants $m_1, m_2, n_1, n_2 \geq 1$, $q_1 = \max\{m_1, n_1\}$, $q_2 = \max\{m_2, n_2\}$. Let $G$ be a directed graph on $X \times Y$ with edge set $E(G)$. Let $H : V(G) \rightrightarrows V(G)$ be a multi valued map that assigns to every $z \in V(G)$ all $z' \in V(G)$ such that there is a path from $z$ to $z'$. Let denote by $F_1$ the projection of $H$ into $X$, by $F_2$ its projection into $Y$, i.e., $H(z) = H(x, y) = (F_1(x, y), F_2(x, y))$, and the maps $F_1$, $F_2$ have closed graphs in $X \times Y \times X$ and $X \times Y \times Y$, respectively.*

*Let $U \subset X$ be an open subset, $V \subset Y$ be an open subset, $\bar{x} \in U$, $\bar{y} \in V$. Suppose there exist constants $\alpha, \beta > 0$ and $\lambda \in (\max\{\alpha, \beta\}, 1/q_2)$ such that:*

*(a)* $$\operatorname{dist}(\bar{x}, F_1(\bar{x}, \bar{y})) + \operatorname{dist}(\bar{y}, F_2(\bar{x}, \bar{y})) < \frac{1 - q_2\lambda}{q_1} \min\{\operatorname{dist}(\bar{x}, X \setminus U), \operatorname{dist}(\bar{y}, Y \setminus V)\},$$

*(b)*

$$\operatorname{dist}(x, F_1(x, y)) + \operatorname{dist}(y, F_2(x, y)) \leq \alpha d(u, x) + \beta \sigma(v, y) \tag{14}$$

*for all* $(x, y), (u, v) \in U \times V$ *such that*

$$x \in F_1(u, v), \qquad y \in F_2(u, v)$$

*and*

$$\alpha d(u, x) < \mathrm{dist}(x, X \backslash U), \quad \beta\sigma(v, y) < \mathrm{dist}(y, Y \backslash V).$$

*Then there exist elements* $x^* \in X$ *and* $y^* \in Y$ *such that the sequence* $\{x_k\}_{k=0}^{\infty}$ *converges to* $x^*$, *and the sequence* $\{y_k\}_{k=0}^{\infty}$ *converges to* $y^*$ *and*

$$(x^*, y^*) \in (F_1(x^*, y^*), F_2(x^*, y^*)) \cap (U \times V) = H(x^*, y^*) \cap (U \times V),$$

*i.e., there exists a directed path of length* $\geq 1$ *connecting* $z^* = (x^*, y^*)$ *with* $z^*$.

Graph–theoretic interpretation of the assumptions.

(i) If $E(G)$ is generated by the one–step multimap $(F_1, F_2)$ via $z \to z'$ whenever $z' \in (F_1 \times F_2)(z)$, then sequences of successive approximations $z_{k+1} \in (F_1 \times F_2)(z_k)$ are precisely directed paths in $G$. (ii) Condition (a) guarantees that the path starting at $(\bar{x}, \bar{y})$ remains in $U \times V$: the "margin to the boundary" dominates the first step and, by the relaxed triangle inequality, all subsequent steps. (iii) Condition (b) encodes a contractive behavior along the path: the one–step error is bounded by $\alpha d(u, x) + \beta\sigma(v, y)$, and choosing $\lambda \in (\max\{\alpha, \beta\}, 1/q_2)$ yields geometric decay of consecutive increments. (iv) Completeness of $(X, d)$ and $(Y, \sigma)$ together with $p_0$– and $q_0$–symmetry ensures the Cauchy path converges to some $z^* = (x^*, y^*) \in U \times V$. (v) Closedness of the graphs of $F_1$ and $F_2$ turns the limit into a fixed point, $z^* \in (F_1 \times F_2)(z^*)$, which in graph language is a self–reachable node (a directed cycle of positive length).

## 5. Discussion

The results obtained in this paper demonstrate how the concept of coupled fixed points can be meaningfully extended to the framework of $(q_1, q_2)$-quasi-metric spaces. In particular, the use of approximate coupled fixed points addresses the limitations, when exact solution can not be obtained. The proposed notion of generalized coupled fixed points for ordered pairs of maps in [12] and further developed in [22] for multi-valued maps and in [3,23] in the investigation of market equilibrium in oligopoly markets excludes the often appearing diagonal case for the solutions. The obtained result shows that asymmetry does not lead to fixed point results in the classical sense, but only approximate ones. By introducing $q_0$-symmetry as an auxiliary condition, the theorems unify existing results from symmetric and $b$-metric contexts while allowing for genuinely non symmetric systems.

An aspect, that we would like to point out, of this work is the graph-theoretic interpretation, which translates analytic assumptions into conditions guaranteeing the existence of directed cycles. This creates a bridge between nonlinear analysis and discrete mathematics, thereby extending earlier graph-based fixed point studies [14,16–18]. Such a perspective is particularly relevant for applications in networked systems, where asymmetry and directionality are inherent.

The broader significance of these contributions lies in their potential applications. The proposed ideas suggest that the applications of coupled and tripled fixed points presented in [3,22] can be extended in economics and game theory, quasi-metric asymmetry naturally models situations with unequal information or sequential decision-making. In applied sciences, coupled fixed point results underpin the analysis of nonlinear matrix equations and ecosystem dynamics [2,22]. The flexibility of the quasi-metric setting thus enlarges the scope of problems for which rigorous existence results can be established.

## 6. Conclusions

This paper established new coupled fixed point theorems for ordered pairs of set-valued mappings in $(q_1, q_2)$-quasi-metric spaces. The main contributions can be summarized as follows: The introduction

 doi:10.20944/preprints202509.1232.v1



of approximate coupled fixed points provides tools for situations where exact solutions may not exist; Under $q_0$-symmetry and completeness assumptions, the existence of exact coupled fixed points is guaranteed, extending and unifying several known results in fixed point theory; A graph-theoretic formulation was developed, offering a combinatorial interpretation of the analytic conditions and ensuring the existence of cycles in associated graphs.

These contributions form a foundation for further research. Promising directions include the extension to stochastic and fuzzy quasi-metric environments, the development of computational methods based on successive approximations, and the exploration of applications in economics, networked systems, and nonlinear analysis.

**Author Contributions:** The mentioned authors participated equally to the study and are arranged in alphabetical order as follows: conceptualization, methodology, investigation, writing—original draft preparation, writing—review and editing: A.I., R.M., D.N. and B.Z. All authors have read and agreed to the published version of the manuscript.

**Funding:** The study is partially funded by European Union-NextGenerationEU, through the National Recovery and Resilience Plan of the Republic of Bulgaria, project DUECOS BG-RRP-2.004-0001-C01.

**Data Availability Statement:** The original contributions presented in the study are included in the article; further inquiries can be directed to the corresponding author.

**Conflicts of Interest:** The authors declare no conflicts of interest.

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