Short Note

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Bramble for Submodular Partition Function

Takaaki Fujita *

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Bramble for Submodular Partition Function

Takaaki Fujita

Independence; t171d603@gunma-u.ac.jp

Abstract: In this compact and focused paper, we revisit the interplay between Bramble and Filter through the lens of a submodular partition function.

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1. Short Introduction

In this short paper, we revisit and contemplate the interplay between Bramble and Filter through the lens of a submodular partition function.

The submodular partition function, Bramble, and graph width parameter are well-known parameters that have received significant attention due to their importance, leading to numerous research studies (e.g., [1–28]). While the level of novelty may be limited, our objective is to make a modest contribution to the future research on graph width parameters.

2. Preliminaries

In this section, we present the essential definitions required for this paper. Throughout the paper, we utilize a finite set (referred to as the underlying set) X, a set of partitions P, and natural numbers i, k, and p.

Moreover, in this paper, we utilize the symbol α to denote a collection of subsets. It is worth noting that this notation is adopted from reference [22]. For instance, α represents a collection $A_1, ..., A_k$ of subsets of a finite set X. The collection α is considered a partition if the sets Ai are mutually disjoint and their union forms the set X. We introduce the following notation: if α represents the collection $A_1, ..., A_k$, and A is another subset, then $\alpha \cap A$ denotes the collection $A_1 \cap A, ..., A_k \cap A$. Similarly, we use $\alpha \setminus A$ as a related notation. Lastly, $[B_1, ..., B_p, \alpha]$ represents the collection obtained by inserting sets $B_1, ..., B_p$ into the collection α.

2.1. Submodular Partition Functions and Brambles

We will explain about submodular partition functions. The definition of a partition function and a submodular partition function of separations is provided below:

Definition 1 [21,22]. A partition function is a function that maps the set of all partitions to non-negative integers, satisfying the condition $\psi([\emptyset, \alpha]) = \psi(\alpha)$ for every partition α. In other words, inserting an empty set into a collection does not alter the value of the partition function. A partition function ψ is submodular if the following holds for every two partitions $[A, \alpha]$ and $[B, \beta]$:

$$\psi([A, \alpha]) + \psi([B, \beta]) \geq \psi([A \cup (X \setminus B), \alpha \cap B]) + \psi([B \cup (X \setminus A), \beta \cap A]) .$$

We will further assume that $\psi([X]) = 0$ since shifting all values of a submodular partition function by a constant does not break the property. $P_k[\psi]$ denote the set of partitions α of X such that $\psi(\alpha) \leq k$.

The submodular partition function possesses the following properties.

Lemma 2 [22]. Let ψ be a submodular partition function on X and $[A, \alpha]$ a partition. Then $\psi([A, \alpha]) \geq \psi([A, X \setminus A])$.

Lemma 3 [25]. Let ψ be a submodular partition function on X. Then $\psi([A, X \setminus A]) = \psi([X \setminus A, A])$.

Lemma 4 [25]. Let ψ be a submodular partition function on X. Then $\psi(\emptyset) = 0$.

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Inspired by reference [22], we define the concept of Bramble, which serves as a fundamental dual concept to width parameters such as Tree-width and branch-width, and tree-cut width [3,4,5,7,29–35].

Definition 5 [21]: Let ψ be a submodular partition function on a finite set X. A (non-principal) k-bramble, denoted as L, is a nonempty family of subsets of X satisfying the following conditions:

(B1) For any A and B belonging to L, their intersection $A \cap B$ is not empty.

(B2) For every $[A_1, \ldots, A_n] \in P_k[\psi]$, there exists A_i in L.

(B3) For all $e \in X$, if the partition $[\{e\}, X\backslash\{e\}]$ *belongs to* $P_k[\psi]$, then $\{e\} \notin L$,

In the case of a non-principal k-bramble, the following holds true.

Lemma 6: Let X be a finite set. A (non-principal) k-bramble satisfies following conditions:

(B4) If $A_1 \in F$, $A_1 \subseteq A_2$, $[A_2, X\backslash(A_2)] \in P_k[\psi]$, then $A_2 \in L$,

(B5) $\emptyset \notin$ L.

Proof of Lemma 6: We show that axiom (B4) holds. Suppose A_1 is a subset in L and $A_1 \subseteq A_2$. Additionally, we are given that $[A_2, X\backslash A_2]$ is in $P_k[\psi]$. According to the definition of a (non-principal) k-bramble (Definition 5), for every $[A_1, ..., A_n]$ in $P_k[\psi]$, there is some A_i in L (by Condition (B2)). Since $[A_2, X\backslash A_2]$ is in $P_k[\psi]$, and $A_1 \subseteq A_2$, we can conclude that A_2 must be in L.

We show that axiom (B5) holds. The condition (B1) of Definition 5 states that the intersection of any two subsets in L is not empty. If we assume that the empty set is in L, then the intersection of any subset in L with the empty set would also be empty, contradicting condition (B1). Thus, we conclude that the empty set is not in L.

This concludes the proof of Lemma 6.

Furthermore, it is known that the following holds true in the context of Bramble.

Lemma 7 [21]: Let L be a k-bramble corresponding to the partition function. For every A, B, C in L, the intersection $A \cap B \cap C$ is non-empty.

2.2. UltraFilter of partitions

The definition of Filter for submodular partition functions is below.

Definition 8 [25]: Let ψ be a submodular partition function on a finite set X. A $P_k[\psi]$-(non-principal) filter of partitions is a family F satisfying the following four axiom:

(F1) For all $e \in X$, if the partition $[\{e\}, X\backslash\{e\}]$ *belongs to* $P_k[\psi]$, then $\{e\} \notin F$,

(F2) If $A_1 \in F$, $A_1 \subseteq A_2$, $[A_2, X\backslash(A_2)] \in P_k[\psi]$, then $A_2 \in F$,

(F3) If $A_1, A_2, \ldots, A_i \in F$ for $i = 1, \ldots, p$, $[X\backslash A_1, \ldots, X\backslash A_p, X\backslash(X\backslash A_1 \cup \ldots \cup X\backslash A_p)] \in P_k[\psi]$, then $A_1 \cap \ldots \cap A_p \in F$,

(F4) $\emptyset \notin F$.

In this paper, we introduce an additional axiom (F5) for the Pk[ψ]-(non-principal) filter of partitions. We refer to this as a Pk[ψ]-(non-principal) ultrafilter.

(F5) If $[A_1, X\backslash A_1] \in P_k[\psi]$, *either* $A_1 \in F$ *or* $X \backslash A_1 \in F$

Lemma 9: Let L be a $P_k[\psi]$-(non-principal) ultrafilter corresponding to the partition function. For every A, B, C in L, the intersection $A \cap B \cap C$ is non-empty.

Proof of Lemma 9: Proof of Lemma 9 can be established similarly to Lemma 7.

3. Result: Filter of partitions and Bramble of partitions

The main result of this paper is presented below.

Theorem 10. Let ψ be a submodular partition function on a finite set X. T is a k-Bramble iff T is a $P_k[\psi]$-(non-principal) filter.

Proof of Theorem 10:

The proof of Theorem 10 proceeds in two parts.

Step 1: A k-Bramble is a $P_k[\psi]$-(non-principal) filter.

Let L be a k-Bramble. We need to show that L satisfies the conditions (F1) to (F5) of a $P_k[\psi]$- (non-principal) ultrafilter (Definition 8).

We show that axiom (F1) holds. This condition is precisely the same as condition (B3) of Definition 5.

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We show that axiom (F2) holds. By Lemma 6, condition (B4) implies condition (F2).

We show that axiom (F3) holds. Let's suppose, for contradiction, that the opposite is true: there exist $A_1, A_2, \ldots, A_p$ in L such that $A_1 \cap \ldots \cap A_p$ is not in L.

Define a partition $\beta = [X \setminus A_1, \ldots, X \setminus A_p, X \setminus (X \setminus A_1 \cup \ldots \cup X \setminus A_p)]$. Given that all Ai are in L, by our supposition, $\beta \in P_k[\psi]$.

Since L is a k-bramble, there must exist some B in L such that B is a subset of a part of β, i.e., $B \subseteq X \setminus A_i$ for some $i \in \{1, \ldots, p\}$, or $B \subseteq X \setminus (X \setminus A_1 \cup \ldots \cup X \setminus A_p)$.

Consider two cases:

Case 1: $B \subseteq X \setminus A_i$ for some i: In this case, we also have $A_i \subseteq X \setminus B$ due to the properties of set subtraction. By the definition of a k-bramble (Condition (B4)), if $A_1 \subseteq A_2$ and $[A_2, X \setminus (A_2)] \in P_k[\psi]$, then $A_2 \in L$. Here, $A_i \cap \ldots \cap A_p$ (which is a subset of A_i and thus a subset of $X \setminus B$) must be in L. This contradicts our original supposition that $A_1 \cap \ldots \cap A_p$ is not in L.

Case 2: $B \subseteq X \setminus (X \setminus A_1 \cup \ldots \cup X \setminus A_p)$: By definition, B is disjoint from each A_i. Hence, the intersection $A_1 \cap \ldots \cap A_p$ is empty, which contradicts the property of a bramble, that for any A and B in L, the intersection $A \cap B$ is not empty (Condition (B1) from the properties of a bramble).

We show that axiom (F4) holds. Condition (B5) of Lemma 6 corresponds directly to condition (F4) in the definition of $P_k[\psi]$-(non-principal) ultrafilter.

We show that axiom (F5) holds. If $[A_1, X \setminus A_1] \in P_k[\psi]$, then we know by the properties of a k-bramble that either $A_1 \in L$ or $X \setminus A_1 \in L$ (Condition (B2)).

Hence, all the conditions for L to be a $P_k[\psi]$-(non-principal) ultrafilter are satisfied.

Step 2: A $P_k[\psi]$-(non-principal) filter is a k-Bramble.

Now, suppose F is a $P_k[\psi]$-(non-principal) ultrafilter. We will show that F satisfies the properties of a k-Bramble.

We show that axiom (B1) holds. Condition (F3) of Definition 8 and Lemma 9 ensure the non-emptiness of the intersection of any subsets in F, hence satisfying Condition (B1) of Definition 5.

We show that axiom (B2) holds. If $[A_1, \ldots, A_n] \in P_k[\psi]$, we know from condition (F5) that there must exist some A_i in F, satisfying condition (B2).

We show that axiom (B3) holds. Condition (F1) of Definition 8 is precisely condition (B3) of Definition 5.

Therefore, all conditions for F to be a k-bramble are satisfied.

Hence, we conclude that a family of subsets of X is a k-Bramble if and only if it is a $P_k[\psi]$-(non-principal) ultrafilter. This completes the proof of Theorem 10.

4. Future tasks: Single Filter and Weak Filter

We will define a single filter as defined below and investigate its connection to graph width parameters as needed.

Definition 11: Let ψ be a submodular partition function on a finite set X. A $P_k[\psi]$-(non-principal) single filter of partitions is a family F satisfying the following four axiom:

(F1) For all $e \in X$, if the partition $[\{e\}, X \setminus \{e\}]$ *belongs to* $P_k[\psi]$, then $\{e\} \notin F$,

(F2) If $A_1 \in F$, $A_1 \subseteq A_2$, $[A_2, X \setminus (A_2)] \in P_k[\psi]$, then $A_2 \in F$,

(F3) If $X \setminus \{e_1\}, X \setminus \{e_2\}, \ldots, X \setminus \{e_i\} \in F$ for $i = 1, \ldots, p$, $[\{e_1\}, \ldots, \{e_p\}, X \setminus (\{e_1\} \cup \ldots \cup \{e_p\})] \in P_k[\psi]$, then $X \setminus \{e_1\} \cap \ldots \cap X \setminus \{e_p\} \in F$,

(F4) $\emptyset \notin F$.

And we will consider about Weak filter of submodular partition function. Weak filter is a concept used in the world of logic [36,37,38]. Definition of Weak filter of submodular partition function is below.

Definition 12: Let ψ be a submodular partition function on a finite set X. A $P_k[\psi]$-(non-principal) weak filter of partitions is a family F satisfying the following four axiom:

(F1) For all $e \in X$, if the partition $[\{e\}, X \setminus \{e\}]$ *belongs to* $P_k[\psi]$, then $\{e\} \notin F$,

(F2) If $A_1 \in F$, $A_1 \subseteq A_2$, $[A_2, X \setminus (A_2)] \in P_k[\psi]$, then $A_2 \in F$,

(F3) If $A_1, A_2, \ldots, A_i \in F$ for $i = 1, \ldots, p$, $[X \setminus A_1, \ldots, X \setminus A_p, X \setminus (X \setminus A_1 \cup \ldots \cup X \setminus A_p)] \in P_k[\psi]$, then $A_1 \cap \ldots \cap A_p \neq \emptyset$,

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(F4) $\emptyset \notin F$.

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