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Article

A Comparison Index for Interval Costs Linear Programming Models

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Abstract: Interval linear programming (ILP) reserves many interesting considerations when it is applied to real problems that are not easily describable by a robust uncertainty modeling. In this paper we apply the comparison index for interval ordering based on the generalized Hukuhara difference to compare solutions in the ILP and we discuss the robustness of the introduced methodology.

Keywords: interval linear programming; generalized Hukuhara difference; interval analysis

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1. Introduction

In the mathematical theory of linear programming (LP) models, established by Dantzig [1], the coefficients of the objective function, of the constraint matrix and of the right-hand sides of the constraints are assumed to be constant when uncertainty does not affect the description of a phenomenon. However, uncertainty is unavoidable when a real phenomenon is the object of interest and one of the possibilities if to replace constant coefficients with intervals of possible values (on the basis of seminal Moore's book [2]); on this way, the area of interest becomes ILP (interval linear programming) allowing the modelling of several possible scenarios and the description of the best and worst cases (as it is explained in [3] also in the case of the inverse problem).

In this paper we compare the objective solutions in the ILP by adopting the comparison index introduced in [4] and [5] that is based on the generalized Hukuhara difference for intervals; in the two mentioned papers we show that the index summarizes the order relations proposed and analyzed by Ishibuchi and Tanaka in [6] where they consider the coefficients in mathematical programming problems as intervals and introduce five order relations for ranking two intervals. We provide evidence of the superiority of the comparison index in terms of errors and of worst case loss in many optimization models. The same index has also been tested in the research of the average rate of return for investment appraisal in uncertain conditions (see [7]).

A rich literature is devoted to ILP, here we summarize some crucial contributions. Optimization problems in which the coefficients of the objective function and the constraints are interval numbers have been investigated in a seminal paper by Tong in [8]: the interval of the solution is deduced by taking the maximum value range and minimum value range inequalities as constraint conditions. Sengupta and Pal, in [9], studied the same problem and proposed the concept of the acceptability index; see also reference [10], an extended presentation of many contributions around the main theme.

A unified scenario for optimal solutions is introduced in [11] where necessary and sufficient criteria for testing a class of optimality are developed according to the Karush-Kuhn-Tucker conditions.

Hladik in [12] presents conditions for necessary efficiency in interval multiobjective linear programming and in [13] he studies robustness of optimal solutions in terms of their capability to stay optimal when perturbations occur; Alolyan investigates interval constraints in [14].

In [15] an ensemble framework for assessing solutions of interval programming problems is developed when interval dominance rules are defined and their correlations are described via exclusion, inclusion and equivalence.

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The optimal solution set of ILP is deduce as the intersection of regions arising from best and worst scenarios in [16]; in [17], a review of some methods for solving ILP (Interval Linear Programming) models with inequality constraints is presented; in [18] some existing methods for solving interval linear programming problems are described when the model is transformed into two submodels and improvements about them are studied. In [19] an algorithm useful for large-scale problems is introduced and it is based on the construction of an interval linear equations system that is the union of linear equations coming from the binding constraint indices of the optimal solution.

In [20], a nonlinear interval programming problem is studied when coefficients are uncertain and the key methodology adopted to solve it is to convert the interval single-objective problem into a two-objective problem, which considers both of the average value and the robustness of the design.

An extended overview of the different approaches reported in the literature to deal with uncertainty in multiple objective linear models through interval programming can be found in [21] and the research of efficient solutions is focused in [22] with evidence on some practical financial aspects.

A new method for solving fully fuzzy linear programming problems with inequality constraints and parameterized fuzzy numbers, by means of solving multiobjective linear programming problems, is presented in [23]. Previously, Arana-Jiménez in [24] proposes an algorithm, that does not use ranking functions, to find the fuzzy optimal (nondominated) solutions of fully fuzzy linear programming problems with inequality constraints, with triangular fuzzy numbers and not necessarily symmetric, via solving a multiobjective linear problem with crisp numbers.

A unique optimal solution for linear programming problem is obtained in [25] through a lexicographic ranking-based solution methodology.

Finally, a general study for studying interval optimization problems is developed in [26].

The paper is organized as follows: after the introduction, in the second section we recall the main properties about the way to compare interval numbers. Section three is devoted to the introduction of linear programming solutions when costs are modelled through interval numbers. Numerical examples and sensitivity analysis are collected in section four and section five closes the paper.

2. Interval Numbers Comparison

An order relation for the ranking of interval numbers has been introduced in [4] and then it has been detailed in [5]. Basically we need the following notation:

an interval $A = [a^-, a^+]$ with $a^- \leq a^+$ has a midpoint-radius representation $A = (\widehat{a}; \overline{a})$ that is defined by the following values

$$\widehat{a} = \frac{a^+ + a^-}{2}, \overline{a} = \frac{a^+ - a^-}{2} \tag{1}$$

where $\overline{a} \geq 0$ and such that:

$$a^- = \widehat{a} - \overline{a}, a^+ = \widehat{a} + \overline{a}. \tag{2}$$

Canonical operations are defined in both notations:

$$A + B = \left[a^- + b^-, a^+ + b^+\right] = (\widehat{a} + \widehat{b}; \overline{a} + \overline{b}) \tag{3}$$

$$A - B = \left[a^- - b^-, a^+ - b^+\right] = (\widehat{a} - \widehat{b}; \overline{a} - \overline{b}) \tag{4}$$

and if λ is a scalar then:

$$\lambda A = (\lambda \widehat{a}; |\lambda| \overline{a}). \tag{5}$$

We need also the generalized Hukuhara difference (detailed in [27]) that is defined as

$$A \ominus_{gH} B = (\widehat{a} - \widehat{b}; \left|\overline{a} - \overline{b}\right|) \tag{6}$$

in order to define the index for interval numbers comparison:

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Definition 1. *Given two distinct intervals $A \neq B$, the gH-comparison index (of order 2) is defined as*

$$CI(A,B) = \frac{\widehat{A \ominus_{gH} B}}{\|A \ominus_{gH} B\|} \tag{7}$$

where $A \ominus_{gH} B$ is the gH-difference,

$$\forall A, B, \widehat{A \ominus_{gH} B} = \frac{(A \ominus_{gH} B)^+ + (A \ominus_{gH} B)^-}{2} \tag{8}$$

is the midpoint value and

$$\|A \ominus_{gH} B\|_2 = \left(\left|\widehat{A \ominus_{gH} B}\right|^2 + \overline{A \ominus_{gH} B}^2 \right) \tag{9}$$

is the Hausdorff distance.

A comparison index ratio can be defined when $\widehat{a} \neq \widehat{b}$ as:

$$\gamma_{A,B} = \frac{\overline{a} - \overline{b}}{\widehat{a} - \widehat{b}} \tag{10}$$

and the relationship between the two indexes is:

$$CI(A,B) = \frac{1}{\pm\sqrt{1 + (\gamma_{A,B})^2}} \tag{11}$$

where the sign $+$ holds when $\widehat{a} > \widehat{b}$ and $-$ when $\widehat{a} < \widehat{b}$.

The index $\gamma_{A,B}$ satisfies some properties as the invariance of scale and the invariance to interval translation.

In [5] we also show how it is possible to extend the definition of a comparison index to fuzzy intervals.

When comparing overlapping intervals $A = (\widehat{a}, \overline{a})$ and $B = (\widehat{b}, \overline{b})$, if the choice is based on the midpoint values $\widehat{a}$ and $\widehat{b}$ then two kinds of risk arise and they can be focused as follows:

Definition 2. *A type I risk is defined to be the possible worst-case loss when, comparing A and B, we choose A but there exist elements in B which are better than all elements of A.*

Definition 3. *A type II risk is defined to be the possible worst-case loss when, comparing A and B, we choose B but there exist elements in A which are worst than all elements of B.*

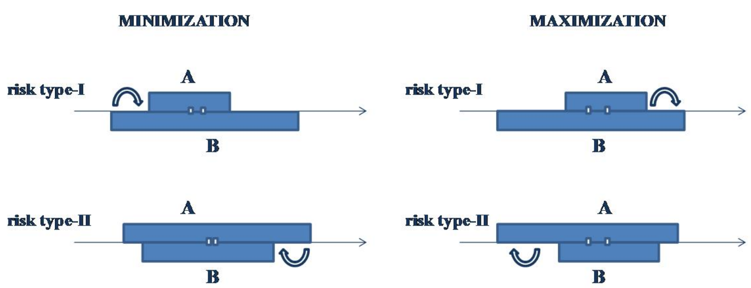

Figure 1. The type-I and type-II risks may occur in minimization and maximization problems.

The defined value $\gamma_{A,B}$ enables the definition of a "risk" measure that is detailed in [5] and that we recall as follows:

in a case of a minimization problem, we have a preliminary preference for A against B if $\widehat{a} < \widehat{b}$ (for the moment $\widehat{a} \neq \widehat{b}$) because the difference $\widehat{b} - \widehat{a} > 0$ represents the mid-point gain if we choose A.

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On the other hand, taking into account the uncertainties given by $\bar{a}$ and $\bar{b}$, two possible bad situations may arise
(i) $b^- < a^-$, i.e. $\widehat{b} - \widehat{a} < \bar{b} - \bar{a}$, and this means that there exist values $b \in B$ such that $b < a$ for all $a \in A$;
(ii) $a^+ > b^+$, i.e. $\bar{b} - \bar{a} < \widehat{a} - \widehat{b}$, and this means that there exist values $a \in A$ such that $a > b$ for all $b \in B$.
In case (i) the positive quantity $\frac{a^- - b^-}{\widehat{b} - \widehat{a}} = \gamma_{A,B} - 1$ measures the possible regret, relative to the midpoint gain;
in case (ii) a measure of the possible regret is given by the positive quantity $\frac{a^+ - b^+}{\widehat{b} - \widehat{a}} = -\gamma_{A,B} - 1$.
In terms of the the comparison index ratio $\gamma_{A,B}$, then, the relative regret measure is positive if $\gamma_{A,B} > 1$ or if $\gamma_{A,B} < -1$ and the regret measures increases if $\gamma_{A,B}$ increases far from the threshold 1 (case (i)) or decreases far from the threshold -1 (case (ii)).

In the presence of worst case loss for the two types of risk, the value $\gamma_{A,B}$ can be limited by two fixed values $\gamma_m < 0$ and $\gamma_M > 0$ such that $\gamma_m \leq \gamma_{A,B} \leq \gamma_M$ and a new order relation can be introduced:

Definition 4. *Given two intervals $A = [a^-, a^+] = (\widehat{a}; \bar{a})$ and $B = [b^-, b^+] = (\widehat{b}; \bar{b})$ and $\gamma_m < 0$, $\gamma_M > 0$ we define the following (strict) order relation, denoted $<_{\gamma_m, \gamma_M}$,*

$$A <_{\gamma_m,\gamma_M} B \Longleftrightarrow \begin{cases} \widehat{a} < \widehat{b} \\ \gamma_m \leq \gamma_{A,B} \leq \gamma_M \end{cases} \Longleftrightarrow \begin{cases} \widehat{a} < \widehat{b} \\ \gamma_M \left(\widehat{a} - \widehat{b}\right) \leq \bar{a} - \bar{b} \\ \gamma_m \left(\widehat{a} - \widehat{b}\right) \geq \bar{a} - \bar{b} \end{cases} \tag{12}$$

It is immediate to see that the relation $<_{\gamma_m,\gamma_M}$ with $\gamma_m < 0$, $\gamma_M > 0$ is antisymmetric and transitive; furthermore, there are specific values of γ_m and γ_M which make the order relation (12) equivalent to the most cited order relations in literature known with the acronyms LU, LC, UC, CW_M, CW_m as it is shown in the following definition.

Definition 5. $A \leq_{LU} B \Leftrightarrow -1 \leq \gamma_{A,B} \leq +1 \Leftrightarrow a^- \leq b^-$ *and* $a^+ \leq b^+$

Let A and B be two intervals with $\widehat{a} < \widehat{b}$ and $\gamma_m \leq \gamma_{A,B} \leq \gamma_M$ then it holds that
1) $A <_{LU} B \Longleftrightarrow \gamma_{A,B} \in [-1, 1] \Rightarrow \gamma_m = -1, \gamma_M = +1$
2) $A <_{CW_M} B \Longleftrightarrow \gamma_{A,B} \leq 0 \Rightarrow \gamma_m = -\infty, \gamma_M = 0$
3) $A <_{CW_m} B \Longleftrightarrow \gamma_{A,B} \geq 0 \Rightarrow \gamma_m = 0, \gamma_M = +\infty$
4) $A <_{LC} B \Longleftrightarrow \gamma_{A,B} \leq 1 \Rightarrow \gamma_m = -\infty, \gamma_M = +1$
5) $A <_{UC} B \Longleftrightarrow \gamma_{A,B} \geq -1 \Rightarrow \gamma_m = -1, \gamma_M = +\infty$.

By varying the two parameters $\gamma_m \leq 0$, $\gamma_M \geq 0$, we obtain a continuum of strict (partial) order relations for interval. The set of real intervals $\mathbb{I}$ with the order relation $\leq_{\gamma_m,\gamma_M}$ defined by $A \leq_{\gamma_m,\gamma_M} B \iff (A <_{\gamma_m,\gamma_M} B$ or $A = B)$ with $\gamma_m \leq 0, \gamma_M \geq 0$ is a complete lattice $(\mathbb{I}, \leq_{\gamma_m,\gamma_M})$

Consequently, the decision making midpoint representation can be defined as follows:

Definition 6. $A \leq_{\gamma_m,\gamma_M} X \Leftrightarrow \begin{cases} \widehat{a} \leq \widehat{x} \\ \bar{x} \leq \widehat{a} + \gamma_M(\widehat{x} - \widehat{a}) \quad \textit{with } \widehat{x} \in R, \bar{x} \in R^+ \cup \{0\} \\ \bar{x} \geqslant \widehat{a} + \gamma_m(\widehat{x} - \widehat{a}) \end{cases}$

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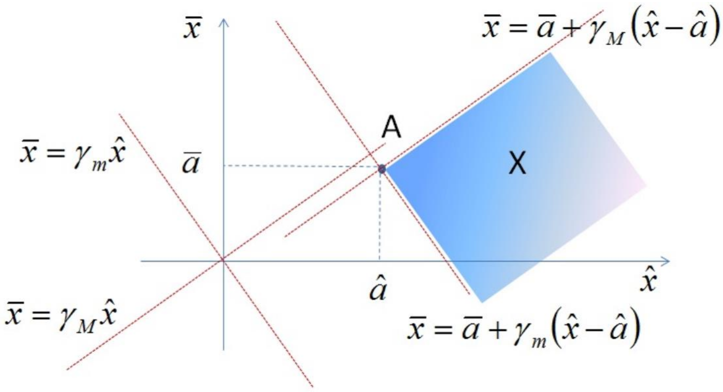

Figure 2. Visualization of the set of intervals $\mathbb{D}_{\gamma_m,\gamma_M}(A)$

For a given interval $A = (\widehat{a}; \overline{a})$, consider the set of intervals

$$\begin{aligned}\mathbb{D}_{\gamma_m,\gamma_M}(A) &= \{X \in \mathbb{I} | A \leq_{\gamma_m,\gamma_M} X\} \\ &= \{(\widehat{x}; \overline{x}) | \widehat{a} < \widehat{x} \text{ and } \gamma_m \leq \gamma_{A,B} \leq \gamma_M\} \cup \{(\widehat{a}; \overline{a})\}.\end{aligned}$$

Proposition 7. *For any real $\gamma_m < 0$ and $\gamma_M > 0$ (the slopes of the tangent lines) and any intervals $A, B \in \mathbb{I}$, we have*

1. *$A \leq_{\gamma_m,\gamma_M} B$ if and only if $\mathbb{D}_{\gamma_m,\gamma_M}(B) \subseteq \mathbb{D}_{\gamma_m,\gamma_M}(A)$, and*
2. *$A = B$ if and only if $\mathbb{D}_{\gamma_m,\gamma_M}(A) = \mathbb{D}_{\gamma_m,\gamma_M}(B)$.*

Given a family $\mathbb{A} = \{A_i | i \in \mathcal{I}\}$ of intervals (for any finite or infinite index set $\mathcal{I}$) the infimum and the supremum operators with respect to partial order $\leq_{\gamma_m,\gamma_M}$, respectively $C = \inf\{A \in \mathbb{A}\}$ and $D = \sup\{A \in \mathbb{A}\}$, are defined by the two intervals (in mid-point notation) $C = (\widehat{c}; \overline{c})$ and $D = (\widehat{d}; \overline{d})$

$$\begin{aligned}\widehat{c} &= \frac{\gamma_M c' - \gamma_m c''}{\gamma_M - \gamma_m}, \ \overline{c} = \frac{\gamma_M \gamma_m (c' - c'')}{\gamma_M - \gamma_m} \geq 0 \\ \widehat{d} &= \frac{\gamma_M d' - \gamma_m d''}{\gamma_M - \gamma_m}, \ \overline{d} = \frac{\gamma_M \gamma_m (d' - d'')}{\gamma_M - \gamma_m} \geq 0\end{aligned} \tag{13}$$

where $c' \leq c''$ are

$$\begin{aligned}c' &= \inf\{\widehat{a} - \frac{\overline{a}}{\gamma_M} | A \in \mathbb{A}\} \\ c'' &= \inf\{\widehat{a} - \frac{\overline{a}}{\gamma_m} | A \in \mathbb{A}\}\end{aligned} \tag{14}$$

and $d' \leq d''$ are

$$\begin{aligned}d' &= \sup\{\widehat{a} - \frac{\overline{a}}{\gamma_M} | A \in \mathbb{A}\} \\ d'' &= \sup\{\widehat{a} - \frac{\overline{a}}{\gamma_m} | A \in \mathbb{A}\}.\end{aligned} \tag{15}$$

Remark 8. *To focus on the interest for an interval ordering index, we mention that the acceptability index for inequality $A < B$, introduced by Sengupta and Pal (see [9,10]) and defined by (assuming $\overline{a} + \overline{b} > 0$)*

$$Acc(A < B) = \frac{\widehat{b} - \widehat{a}}{\overline{a} + \overline{b}}, \tag{16}$$

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is successfully used to convert an interval inequality $Ax \le B$, with $x \ge 0$, into a "crisp equivalent" form as follows

$$Ax <_\alpha B \Longleftrightarrow \begin{cases} a^+x \le b^+ \\ Acc(B < Ax) \ge \alpha \end{cases} \tag{17}$$

where $\alpha \in]0,1]$ is an assumed fixed (optimistic) threshold; substituting the expression for $Acc(B < Ax)$ we obtain

$$Ax <_\alpha B \Longleftrightarrow \begin{cases} \hat{a}x + \bar{a}x \le \hat{b} + \bar{b} \\ \hat{a}x - \alpha\bar{a}x \ge \hat{b} + \alpha\bar{b} \end{cases}. \tag{18}$$

This set of inequalities, being $\alpha > 0$, implies that $\hat{a}x \ge \hat{b} + \alpha\bar{b} + \alpha\bar{a}x > \hat{b}$ and does not imply a control on the possible worse case losses. In fact we can see that $Ax <_{\gamma_m,\gamma_M} B$ is not equivalent to $Ax <_\alpha B$ in the sense that the one can not be transformed into the other.

The comparison index can be applied when a variable interval of the form Ax is compared with a fixed interval B;.two possible worst case losses may occur and they are related to the value of γ. Supposing $x \ge 0$, it follows that the value of γ for the inequality $Ax < B$ is:

$$\gamma_{Ax,B} = \frac{\bar{a}x - \bar{b}}{\hat{a}x - \hat{b}}. \tag{19}$$

In order to control the extent of the possible worst case loss for the two types of risk, we can require that the value $\gamma_{Ax,B}$ be controlled for the type I risk and/or for the type II risk. To do this, we fix two values $\gamma_m < 0$ and $\gamma_M > 0$ and we require that valid values of x satisfy $\hat{a}x < \hat{b}$ and $\gamma_m \le \gamma_{Ax,B} \le \gamma_M$.

The two types of risk are eliminated when $\gamma_m \in [-1, 0]$ and $\gamma_M \in [0, 1]$. The values $1 - \gamma_M$ and $1 + \gamma_m$, if negative, give the relative worst case loss with respect to $\hat{a}x - \hat{b}$.

In terms of (12), we can write:

$$Ax <_{\gamma_m,\gamma_M} B \Longleftrightarrow \begin{cases} \hat{a}x < \hat{b} \\ \gamma_M\hat{a}x - \bar{a}x \le \gamma_M\hat{b} - \bar{b} \\ \gamma_m\hat{a}x - \bar{a}x \ge \gamma_m\hat{b} - \bar{b} \end{cases}. \tag{20}$$

If we are minimizing and we do not accept a risk of type II, we may require that $1 + \gamma_{Ax,B} \ge 0$ (we eventually accept only a risk of type I) and we choose $\gamma_m = -1$, $\gamma_M = BIG > 0$; a risk of type II represents the possibility that we realize values in Ax that are greater than all values in B. Similarly, if we do not accept a risk of type I, then we choose $\gamma_M = 1$, $\gamma_m = -BIG < 0$; a risk of type I represents the possibility that we realize values in B that are less than all values in Ax. If $\gamma_m = -1$ and $\gamma_M = 1$ no risk of the two types is accepted. It follows that the use of the acceptability index does not avoid the two types of risk that are controlled using the order relationship (12).

3. Interval Costs in Linear Programming

We apply the order relation $\le_{\gamma_m,\gamma_M}$ to linear programming with interval costs (ILP problem) that has the following general form:

$$\min z(\underline{x}) = [z^-(x), z^+(x)] = (\hat{z}(x); \bar{z}(x)) \tag{21}$$

$$\hat{z}(x) = \sum_{j=1}^{n} \hat{c}_j x_j \qquad \bar{z}(x) = \sum_{j=1}^{n} \bar{c}_j x_j \tag{22}$$

$$s.t.\ (Ax)_j \le b_i \quad i = 1,..,m \quad A = [a_{i,j}] \in \mathbb{R}^{m\times n} \quad x_j \ge 0\ j = 1,..,n$$

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where A is a matrix with m rows and n columns, b is m-vector of right-hand side terms, $c_j = (\widehat{c}_j; \overline{c}_j)$, $j = 1, 2, ..., n$ are n intervals representing the coefficients of the linear objective function $z(x)$ to be minimized that is represented by an interval

$$z(x) = \left[z^-(x), z^+(x)\right] = (\widehat{z}(x); \overline{z}(x)) \tag{23}$$

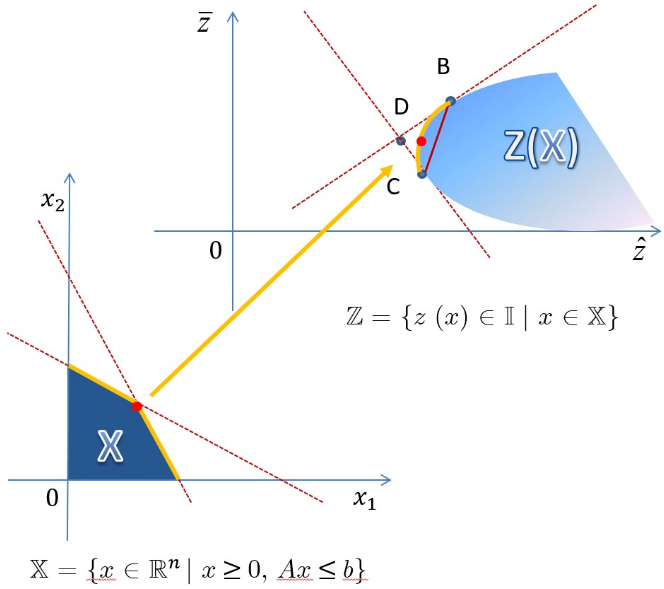

Figure 3. Visualization of the process from the feasible convex set to the feasible objective intervals

where $\widehat{z}(x)$ and $\overline{z}(x)$ are given by

$$\widehat{z}(x) = \sum_{j=1}^{n} \widehat{c}_j x_j \tag{24}$$
$$\overline{z}(x) = \sum_{j=1}^{n} \overline{c}_j x_j$$

An optimal solution x^* is computed with the corresponding objective value: $\widehat{z}_{\min} = \sum_{j=1}^{n} \widehat{c}_j x_j^*$ where x is in the feasible convex set:

$$\mathbb{X} = \{x \in \mathbb{R}^n \mid x \geq 0,\ Ax \leq b\}. \tag{25}$$

and the set of feasible objective intervals is again a convex set:

$$\mathbb{Z} = \{z(x) \in \mathbb{I} \mid x \in \mathbb{X}\}$$

Remark 9. *It is well known that $\mathbb{X}$ is a convex polygon in $\mathbb{R}^n$ and this implies that $\mathbb{Z}$ is a convex polygon in the space $\mathbb{I}$ of intervals.*

Definition 10. *If $x^{'}, x^{''} \in \mathbb{X}$ are two feasible solutions and $z^{'} = (\widehat{z}'; \overline{z}'), z^{''} = (\widehat{z}''; \overline{z}'')$ are the corresponding objective interval values then $x^{'}$ dominates $x^{''}$ (or in other words $z^{'}$ dominates $z^{''}$) if and only if $z^{'} <_{\gamma_m, \gamma_M} z^{''}$.*

The search of a solution $x \in \mathbb{X}$ requires the following method: among all the feasible objective values, the not dominated values, with respect to the interval relation order $<_{\gamma_m, \gamma_M}$(with fixed $\gamma_m < 0$, $\gamma_M > 0$) have to be selected . In particular, given two feasible solutions x' and x'' with the corresponding objective intervals given by: $z' = (\hat{z}', \bar{z}')$ and $z'' = (\hat{z}'', \bar{z}'')$ the problem is to chose the best interval between them. Three possible situations may occur:

Given two feasible solutions x' and x'' with the corresponding objective intervals given by: $z' = (\hat{z}', \bar{z}')$ and $z'' = (\hat{z}'', \bar{z}'')$, the best interval between them corresponds to three cases:

- x' is better than (dominates) x'' or equivalently $z' \leq_{\gamma_m,\gamma_M} z''$;
- x'' is better than (dominates) x' or equivalently $z'' \leq_{\gamma_m,\gamma_M} z'$;
- x' and x'' are not comparable because of the incomparability between z' and z''.

Consequently, a solution x is called efficient if there is no feasible solution x' such that x' dominates x. It follows that, as it is shown in Figure 2, the tangent lines to $\mathbb{Z}$ determine the ideal objective interval that has values between the tangent points called $P = (\hat{p}, \bar{p})$ and $Q = (\hat{q}, \bar{q})$.

The main interpretation of the expression $z(x') \leq_{\gamma_m,\gamma_M} z(x'')$ is that:

- x' dominates x'' in the variable space X
- $z(x')$ dominates $z(x'')$ in the space of interval object

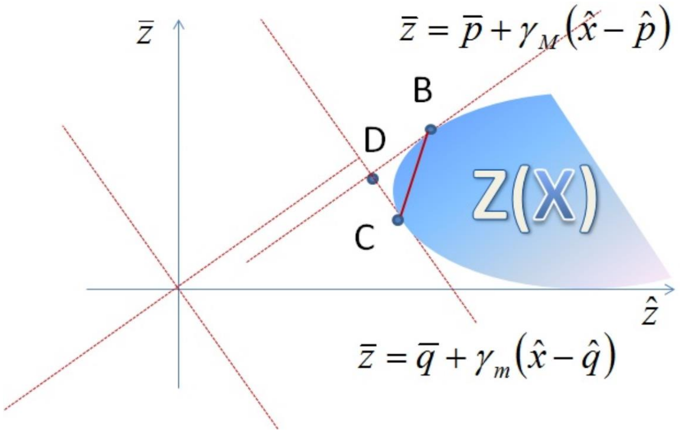

Figure 4. Representation of tangent lines to $\mathbb{Z}$

The DBC triangle construction provides two important information :

- the efficient boundary of the LP problem, in the target value space $\mathbb{Z}(\mathbb{X})$ $[z(x_1, x_2, ..., x_n) \in \mathbb{Z}(\mathbb{X})]$, is inside the triangle DBC;
- value D identifies the ideal objective solution, it is unique in $\mathbb{Z}$, but not necessarily in $\mathbb{X}$ and this implies
  $z_D = GLB(\mathbb{Z}(\mathbb{X}))$ Greatest Lower Bounded respect to $\leq_{\gamma_m,\gamma_M}$
  if $D \in \mathbb{Z}(\mathbb{X})$ then D is unique and is the ideal solution

The efficient frontier is determined by two "tangent" lines or better "support lines":
1) $\bar{z} = \bar{p} + \gamma_M(\hat{z} - \hat{p})$ where P is feasible,
$\hat{p} = \hat{z}(x_P)$ and $\bar{p} = \bar{z}(x_P)$ with $x_P \in \mathbb{X}$ such that $\bar{p} - \gamma_M\hat{p}$ is the largest.
2) $\bar{z} = \bar{q} + \gamma_m(\hat{z} - \hat{q})$ where Q is feasible,
$\hat{q} = \hat{z}(x_Q)$ and $\bar{q} = \bar{z}(x_Q)$ with $x_Q \in \mathbb{X}$ such that $\bar{q} - \gamma_m\hat{q}$ is the smallest.
The intersection between tangent lines is the ideal solution $z_{id} = (\hat{z}_{id}, \bar{z}_{id})$:

$$\begin{cases} \bar{z} = \bar{p} + \gamma_M(\hat{z} - \hat{p}) \\ \bar{z} = \bar{q} + \gamma_m(\hat{z} - \hat{q}) \end{cases} \begin{cases} \hat{z}_{id} = \frac{\bar{q} - \bar{p} + \gamma_M\hat{p} - \gamma_m\hat{q}}{\gamma_M - \gamma_m} \\ \bar{z}_{id} = \frac{\gamma_m(\gamma_M\hat{p} - \bar{p}) - \gamma_M(\gamma_m\hat{q} - \bar{q})}{\gamma_M - \gamma_m} \end{cases} \quad (26)$$

It follows that:

- if $z_{id} \in \mathbb{Z}$ (it is feasible) that it can be considered the optimal solution;

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- if $z_{id} \notin \mathbb{Z}$ (it is not feasible) then a goal programming technique is applied in order to find the feasible solution $z^* = (\widehat{z}^*, \overline{z}^*)$ with the smallest distance to z_{id}, i.e. the optimal feasible solution that solves the following optimization problem:

$$\min \left(\sum_{j=1}^{n} \widehat{c}_j x_j - \widehat{z}_{id} \right)^2 + \left(\sum_{j=1}^{n} \overline{c}_j x_j - \overline{z}_{id} \right)^2 \tag{27}$$

$$\begin{aligned} s.t.\ Ax_j &\leq b_i \ \ i = 1,..,m \ \ A = [a_{i,j}] \in \mathbb{R}^{m \times n} \\ x_j &\geq 0 \ j = 1,..,n \end{aligned}$$

A solution x is efficient if there is no feasible solution x' such that x' dominates x.

Tangent lines (support lines) to $\mathbb{Z}$ determine the ideal objective interval that has values between the tangent points called $P = (\widehat{p}, \overline{p})$ and $Q = (\widehat{q}, \overline{q})$.

The interval valued LP optimal solution is

- if $z_{id} \in \mathbb{Z}$ (it is feasible) that it is the unique optimal solution;
- if $z_{id} \notin \mathbb{Z}$ (it is not feasible) then a goal programming technique is applied in order to find the feasible solution $z^* = (\widehat{z}^*, \overline{z}^*)$ with the smallest distance to z_{id}, i.e. the optimal feasible solution that solves the following optimization problem:

$$\min \left(\sum_{j=1}^{n} \widehat{c}_j x_j - \widehat{z}_{id} \right)^2 + \left(\sum_{j=1}^{n} \overline{c}_j x_j - \overline{z}_{id} \right)^2 \tag{28}$$

$$\begin{aligned} s.t.\ (Ax)_j &\leq b_i \ \ i = 1,..,m \ \ A = [a_{i,j}] \in \mathbb{R}^{m \times n} \\ x_j &\geq 0 \ j = 1,..,n \end{aligned}$$

4. Numerical Examples and Sensitivity Analysis

We now apply the mentioned results to some linear programming problems with interval costs and we extend preliminary results shown in [28]. An exahustive scenario of properties concerning calculus of interval-valued functions can be found in [29,30].

Example 1. The first example is specified with the following data:

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 2 \\ 2 & 4 \end{bmatrix}; b = \begin{bmatrix} 40 \\ 132 \\ 140 \end{bmatrix}$$

$$\widehat{c} = \begin{bmatrix} -250 \\ -220 \end{bmatrix}; \overline{c} = \begin{bmatrix} 50 \\ 0 \end{bmatrix}$$

Figure 4 shows the two red tangent lines and the third red line connecting the tangent points. The area described by blue lines is the feasible set, D is z_{id} in (??) that does not belong to the feasible set and it requires the mentioned goal programming technique to identify the minimum value $P0$ as z^*.

The sensitivity analysis for the optimal solution is carried out through a scatter plot where values of the two variables x_1 and x_2 and their frequencies are shown in order to observe the robustness of the solution. The vertical and horizontal histograms show the distribution of the values of x_1 and x_2 around the optimal solution.

Example 2. The second example comes from [31] where the optimization problem is described as follows ($x_j \geq 0 \ j = 1, 2$):

$$\max[-20, 50]x_1 + [0, 10]x_2$$

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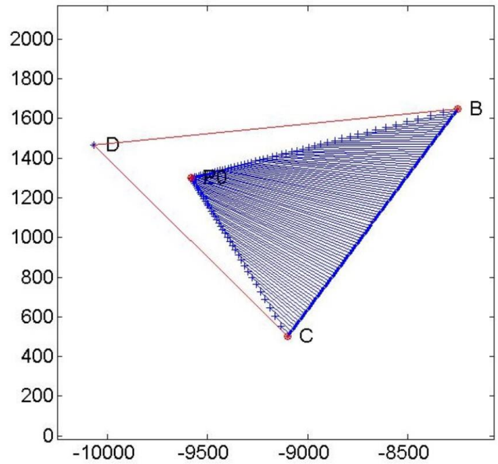

Figure 5. Ideal and optimal solution of example 1 are shown

$$s.t. \begin{bmatrix} 10 & 60 \\ 10 & 20 \\ 10 & 10 \\ 30 & 10 \\ 40 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x2 \end{bmatrix} \leq \begin{bmatrix} 1080 \\ 400 \\ 240 \\ 420 \\ 520 \end{bmatrix}$$

We have solved the problem in the minimization form so that the midpoint values of the objective function are negative. Figure 2 shows the minimum value $P0$ obtained with the slope of tangent lines specified as follows: $\gamma_m = 0$ and $\gamma_M = 5$. The point $P1 = (-90; 90)$ is the minimum obtained in [31] and it is the same a the one obtained by our method (as the nearest interval to the ideal solution D. corresponding to the chosen values of γ_m and γ_M. The second solution suggested by Chanas and Kuchta is $(-210; 390)$, obtained by our method with the partial order generated by $\gamma_m = -2$ and $\gamma_M = 1.5$ and represented in Figure 3.

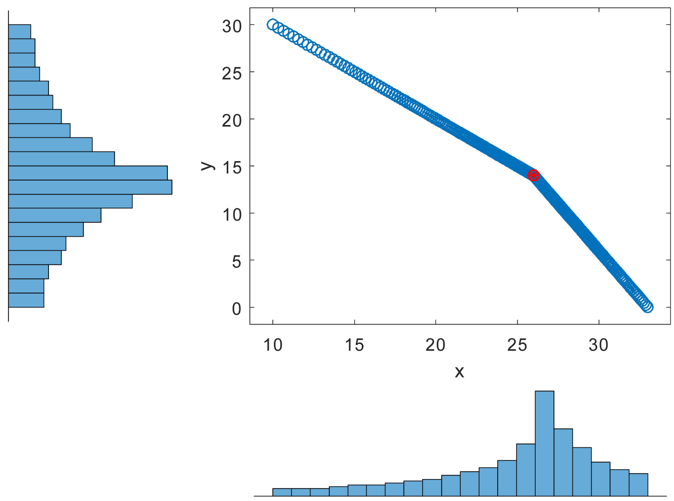

Figure 6. Sensitivity analysis for example 1

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Example 3. The third numerical example is a problem called ITPMPF (Interval-valued Transportation Problem with Multiple Penalty Factors) that comes from chapter 7 of the book [10] and it is defined as:

$$\min Z^1 = \sum_{i=1}^{3}\sum_{j=1}^{4} c_{ij}^1 x_{ij} \tag{29}$$

$$\min Z^2 = \sum_{i=1}^{3}\sum_{j=1}^{4} c_{ij}^2 x_{ij} \tag{30}$$

$$s.t. \sum_{j=1}^{4} x_{1j} = [7,9], \sum_{j=1}^{4} x_{2j} = [17,21], \sum_{j=1}^{4} x_{3j} = [16,18]$$

$$\sum_{i=1}^{3} x_{i1} = [10,12], \sum_{i=1}^{3} x_{i2} = [2,4] \tag{31}$$

$$\sum_{i=1}^{3} x_{i3} = [13,15], \sum_{i=1}^{3} x_{i4} = [15,17] \tag{32}$$

$$x_j \geq 0 \;\; i = 1,2,3 \;\; j = 1,2,3,4$$

where

$$C^1 = \left(c_{ij}^1\right) = \begin{matrix} [1,2] & [1,3] & [5,9] & [4,8] \\ [1,2] & [7,10] & [2,6] & [3,5] \\ [7,9] & [7,11] & [3,5] & [5,7] \end{matrix} \tag{33}$$

$$C^2 = \left(c_{ij}^2\right) = \begin{matrix} [3,5] & [2,6] & [2,4] & [1,5] \\ [4,6] & [7,9] & [7,10] & [9,11] \\ [4,8] & [1,3] & [3,6] & [1,2] \end{matrix}$$

We chose $\gamma_m = -0.25$ and $\gamma_M = 0.25$. In figure 3, $P0 = (167.75; 47.0625)$ is the minimum obtained with our methodology, $P1 = (176.5; 47)$ and $P2 = (191; 42)$ are the minima obtained with the reduction into a standard LPP structure as in [10] and $P3 = (178.875; 48.985)$ is the solution of the same problem obtained in [32]. We remark that the solutions $P1$, $P2$ and $P3$ are not on the efficient frontier of the feasible objective values.

The figure shows that $P0$ is the closest to the ideal minimum D, stressing the goodness of our methodology.

A further example is the same as in papers [15,18]; they consider a numerical interval linear programming example as follows:

$$\max \; f = [3,3.5]x_1 - [1,1.2]x_2$$

$$s.t. \quad \begin{matrix} [1,1.1]x_1 + [1.6,1.8]x_2 \leq [11.6,12] \\ [3,4]x_1 - [2,3]x_2 \leq [5,7] \\ x_1, x_2 \geq 0 \end{matrix}$$

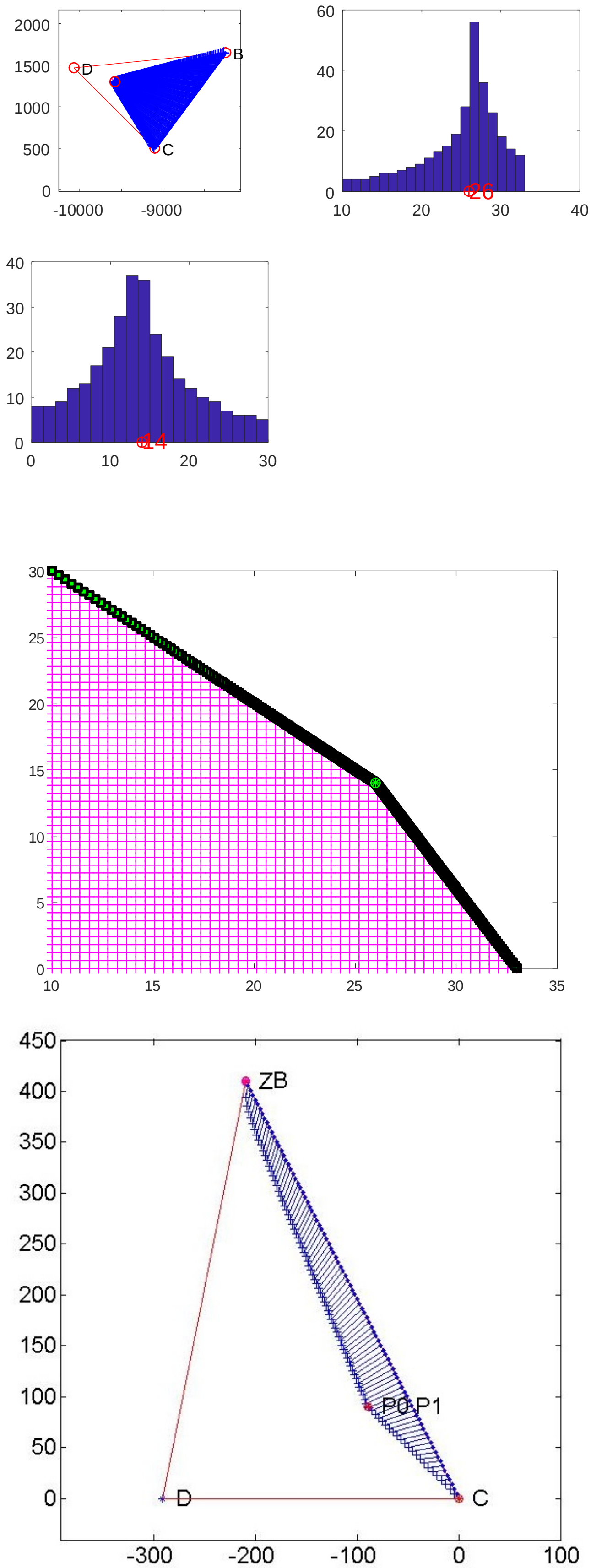

Figure 7. Numerical example n.2 (Chanas and Kuchta) with $\gamma_m = 0$, $\gamma_M = 5.0$

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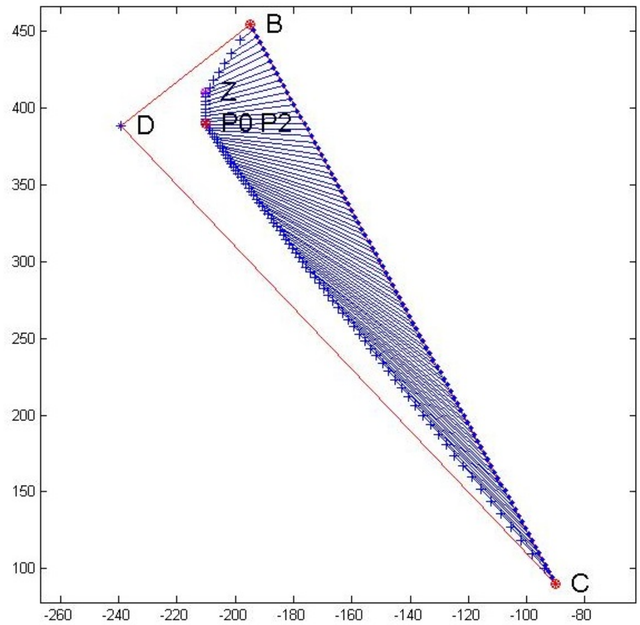

Figure 8. Numerical example n.2 (Chanas and Kuchta) with $\gamma_m = -2$, $\gamma_M = 1.5$

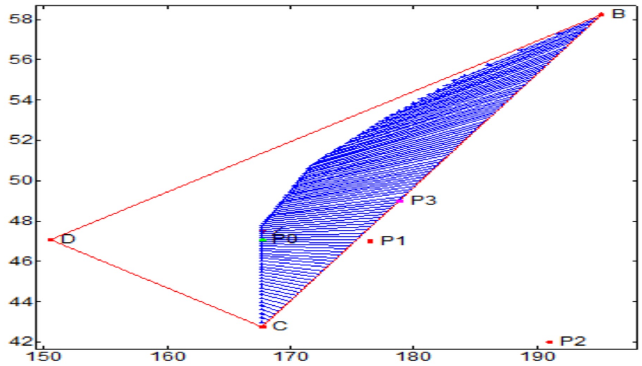

Figure 9. Numerical example 3. (Sengupta and Pal)

5. Further Directions and Conclusions

The application of the comparison index in interval linear programming is the main topic of the present paper. The preliminary results seem to encourage some additional research due to the numerical efficiency and robustness of the suggested methodology and to the large number of possible applications in many areas.

When intervals are not enough to model a more complex uncertainty, then coefficients can be replaced by fuzzy numbers. Fuzzy linear programming (FLP) was born in 1970 with the seminal work on Decision Theory by Bellman and Zadeh ([33]) but FLP problems were formally born in 1974 when Tanaka et al. ([34]) and Zimmermann ([35]) published their works modelling the set of constraints in LP as fuzzy sets. Ramík extensively worked on the topic and in [36] he introduces a class of fuzzy optimization problems with objective function depending on fuzzy parameters; in [37] it is underlined that FLP can tackle highly complex situations in an elegant and efficient way.

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