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A remark on quasi-automorphisms and deformable structures in quasi-set theory and its account to the logical foundations of quantum theory

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Abstract

Quantum theory is the land of indiscernible things, of things that in some situations cannot be put apart. This contrasts in much with standard mathematics and classical logic, which were elaborated to deal with *individuals*, discernible things. Many thinkers, among them the mathematician Yuri Manin, have proposed that a more general theory of 'sets' than standard set theories (where sets are collections of discernible elements) should be elaborated to cope with quantum physics. Quasi-set theory was proposed with such an aim. In this paper, we consider some aspects of this theory and discuss the way we can use such a theory as a framework for constructing deformable (not rigid) structures where indiscernible things can be considered but with one additional and fundamental detail: these structures cannot be extended to rigid structures as it happens in the standard frameworks. Really, the way of dealing with indiscernible (or indistinguishable) elements within a 'standard' framework such as a 'classical' set theory (like the ZFC system) requires the confinement of the discussion to deformable (non-rigid) structures. But in ZFC every structure can be extended to a rigid structure (encompassing only the trivial automorphism – the identity function), so that in the extended structure (or in the whole universe) one can realize that the

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supposed indiscernible objects are not indiscernible at all. Quasi-set theory is such that the universe of quasi-sets is deformable and in such a theory one can construct deformable structures which cannot be extended to rigid ones. The aim of this paper is to put these things in a clear way.

Keywords: logical foundations of quantum theory, indistinguishability, indiscernibility, quantum objects, deformable structures, quasi automorphisms.

"In the microworld, we need uniformity of the strongest kind: complete indistinguishability".

Frank Wilczek and Betsy Devine (1987)

1 Introduction

This is a paper dealing with the logical foundations of quantum theory. There are several aspects to be considered in such an investigation, one of them being the attempt to *represent* absolutely indistinguishable objects, as typical in quantum theory, that is, using Wilczek and Devine's words in the motto, *complete* indistinguishability. By this term we understand that if two things are 'absolutely', or 'completely' indistinguishable, there is no way to discern them, not even in principle or, as Dalla Chiara has put it, not even *in mente Dei* (Dalla Chiara, 1985). In mathematical terms, we should not be able to discern things neither by a property nor by a relation. We shall see soon that if we use a standard mathematical framework such as a standard set theory such as ZF, then the distinction *is always* possible. Really, the problem with the use of a standard framework, such as a standard set theory, is that we can consider indiscernible elements only within a deformable (not rigid) structure, one admitting nontrivial automorphisms.[1] But this is a bogus solution, for it can be shown that *every*

[1]The identity function is the trivial automorphism of any structure. If it is its only automorphism, the structure is *rigid*. A simple example of a deformable structure is that of the additive group of the integers, $\mathcal{Z} = \langle \mathbb{Z}, + \rangle$, which beyond the identity function, has $h(x) = -x$ as an automorphism. So, *inside* the structure, 2 and -2 are indiscernible, although we can see *outside the structure*, say by extending it to $\mathcal{Z}' = \langle \mathbb{Z}, +, < \rangle$, that they differ.

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structure built in a theory such as the ZF system can be extended to a rigid structure, where the apparently indistinguishable elements are shown to be discernible. Furthermore, the whole universe of sets of a theory such as ZF is rigid (Jech, 2003, p.66), that is, standard set theories are theories of individuals.[2]

A typical example of entities of this kind would be quantum objects, which in determined situations cannot be put apart by any device provided by the theory. A sample involves Bose-Einstein Condensates, which can be formed by milliards of 'completely' indistinguishable atoms or particles behaving as they were one, forming a *big molecule* (Ketterle, 2007); despite being many, there is no way to discern them. In some situations, things are a little bit different but the results are similar. Fermions, as is well known, cannot share the same quantum state. If we have more than one, they differ, generally with respect to their spins taken in a certain direction. But, despite the existence of such a 'difference', we cannot say which is which, that is, if one of them (suppose being two) has spin UP in a certain direction, which forces the other to have spin DOWN in the same direction, there is no way to say *which* has spin UP. In this sense, they are also indiscernible in a strong sense (other examples will be mentioned below).

Emphasizing the main problem in using standard mathematics (grounded on classical logic) for describing these entities is that anything a can always (even if only in principle) be discerned from any other entity b. The proof is simple; take the unitary set $\{a\}$ and define the *identity of* a as $I_a(x) := x \in \{a\}$. Done! Only a has this 'property', so there will be a difference to any other b, that is, to any other entity not obeying I_a. This is typical of standard mathematics, and it is expected to be so since the theory was designed for coping with individuals.[3] Of course, we can mimic indiscernibles in such frameworks by confining them to deformable (not-rigid) structures, that is, structures having nontrivial automorphisms. But this is a trick since one can prove that in set theories such as the ZFC system, every structure can be extended to a rigid structure (da Costa and Rodrigues, 2007), one that encompasses only the trivial automorphism and, in this extended structure, we realize that the supposed indiscernible enti-

[2]The Axiom of Choice is not used in these discussions, so we could refer to either ZF or ZFC.

[3]Some philosophers don't accept that the distinctive properties involve the identity sign or individual constants; others prefer to speak of 'legitimate' properties, and I_a would not be one of them. I think that these restrictions are unfounded; if the underlying logic is classical, as in the case of these philosophers, there is no escape: I_a can always be defined for any a and this should not be dismissed; for a discussion, see Krause (2023a).

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ties are, in fact, individuals.

But quantum physics brings questionings to such a view, and according to some, such as Yuri Manin, we "should consider the possibilities of developing a totally new language to speak about infinity"; of course, he is speaking of set theory, and his motivations are collections of quantum objects. Quasi-set theory was elaborated to be such a new theory of the infinite, answering Manin's problem (Manin, 1976).[4]

The main objective of this paper is to show that in the quasi-set theory $\mathfrak{Q}$ we can construct structures which cannot be extended to rigid structures; in reality, the whole universe of qsets is not rigid, contrary to the standard universe of sets of the Zermelo-Fraenkel set theory (Jech, 2003, p.66). In order to do so following the standards, we need to speak about automorphisms. Thus I was pushed by Eliza Wajch, whom I thank, in defining automorphism in $\mathfrak{Q}$ in order to do the next step. I hope this is achieved here.

2 The core notions

We shall be working in the quasi-set theory $\mathfrak{Q}$, so we suppose a previous knowledge of it, but we shall provide a minimum in between the text; for references, see (French and Krause, 2006). The theory comprises two kinds of *atoms*, entities that are not qsets but which can be elements of qsets, the M-atoms, which mimic the behaviour of the atoms in ZFA (the Zermelo-Fraenkel set theory with atoms (Suppes, 1972)), and the m-atoms, which are posed to represent quantum entities. To these ones, the standard notion of identity (given by classical logic) does not apply (the original idea that identity is senseless for quantum entities was advanced by Schrödinger (1998), but see French and Krause (2006); we approach this by saying that expressions like '$x = y$' (so as their negations) are not well-formed if either x or y represents an m-atom. An *extensional identity* is defined for M-atoms and qsets (these are the entities that are not atoms).[5]

Let $\mathfrak{A} = \langle D, R_i \rangle$ $(i \in I)$ be a q-structure (a structure in $\mathfrak{Q}$) where (to consider the most significative case) D is a quasi-set (qset) of indiscernible m-atoms and the R_i are q-relations on D. We shall consider only binary q-

[4]Presently, $\mathfrak{Q}$ it was extended to a theory involving *quasi-classes* by Eliza Wajch, who also corrected some loopholes; see Wajch (2023).

[5]The definition goes as follows, where $M(x)$ says that x is an M-atom and $Q(x)$ that x is a qset: $x =_E y := \big(M(x) \wedge M(y) \to \forall z(Q(z) \to (x \in z \leftrightarrow y \in z))\big) \vee \big(Q(x) \wedge Q(y) \to \forall z(z \in x \leftrightarrow z \in y)\big)$. It results that this relation has all the properties of standard identity in ZF.

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relations to illustrate. We recall that indiscernible m-atoms can are linked by the indistinguishability relation $\equiv$, which has the properties of an equivalence relation but doesn't obey full substitutivity, as remarked below. Even so, a qset with indistinguishable m-atoms can have a cardinal, a quasi-cardinal, which stands for the quantity of elements it has.[6]

A quasi-relation, or q-relation between quasi-sets A and B is a qset of 'ordered pairs' $\langle a, b \rangle$, with $a \in A$ and $b \in B$, as usual. The notion of 'ordered pair' needs to be relativised to indiscernibility – see below; the pair $\langle a, b \rangle$ is the qset of the indiscernible from a and from b. A q-relation is a q-function if being $\langle a, b \rangle \in R$ and $\langle c, d \rangle \in R$, if $a \equiv c$, then $b \equiv d$; furthermore, no element of A is out of some pair. We can define also q-injections and q-surjections, so also q-bijections, which cope with our intuitions regarding standard functions (for details, please see French and Krause (2006)).

Let $h : D \to D$ be a q-bijective q-function such that, for any $x, y \in D$ and for any relation $R \in R_i$, $R(x, y) \leftrightarrow R(h(x), h(y))$. Such a q-function is a candidate to be a q-automorphism of the structure. The most interesting case seems to be that where $h(x) \equiv x$ for every $x \in D$. Since the identity relation doesn't hold for m-atoms, then, of course, h is not the identity function, which cannot be defined on such a qset.

Suppose that $R(x, y)$ holds for $x, y \in D$. We shall see that $R(h(x), h(y))$ holds as well. In fact, suppose it fails, that is, $\neg R(h(x), h(y))$ is the case (let us name this expression '$\star$'). Remembering that $\langle x, y \rangle$ means $[[x], [x, y]]$ (we are supposing that the elements of these qsets belong to D), and that if $x' \equiv x$ and $y' \equiv y$, it results that $[[x], [x, y]] \in R$ entails $[[x'], [x', y']] \in R$ as well. Consequently, due to the definition of h, '$\star$' cannot be the case.

This argumentation requires explanation since $\mathfrak{Q}$ is a theory that defies our intuitions. Firstly, notice that m-atoms don't have proper names,[7] that is, the language cannot be extended to a language with a term referring to a *specific* m-atom. When we say 'Peter' do some m-atom, it stands for *whatever* indiscernible from Peter, that is, it makes reference to *something of the kind 'Peter'*, and not a specific object: in a certain sense, all Peters are indiscernible and when some of them is making something, it could

[6] As remarked in Krause (2023b); Wajch (2023), the existence of a quasi-cardinal does not imply that the elements of the qset are discernible, contrarily to what is said for instance in Jantzen (2011). Beyond m-atoms, the theory encompasses the M-atoms, which work like the *Urelemente* of ZFA, the Zermelo-Fraenkel set theory with atoms, in particular obeying the standard theory of identity.

[7] We use the notions of proper names and variables in the sense of Church (1956). For an argument questioning the use of proper names in the quantum domain, see Dalla Chiara and Toraldo di Francia (1993) and (French and Krause, 2006, Chap.5).

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be any 'other' of them who were doing that, and the results would be the same (invariance by permutations). Thus, proper names, or individual constants in a formal language, here act as individual variables, designating, or having associated to it, a range of possible values, so denoting *kinds* of objects. Thus, if $x \in y$, this means that *some* object *of the kind* x belongs to some qset of the kind y. We notice, and insist on noticing that this way of seeing is different to what a classical logician could expect. So, we beg the reader not to judge us from the point of view of classical logic and standard semantics.

The (apparently) only situation where the substitution by indiscernibles doesn't hold is with respect to the membership relation. Really, let R be membership. Then if $x \in y$ and $x' \equiv x$, nothing in the theory grants that $x' \in y$. For instance, let y be a strong singleton of x, namely, the qset $[\![x]\!]$ whose only element (recall that the quasi-cardinality of this qset is one) belongs to D. It should be emphasised that despite such a strong singleton having just one element, we cannot identify it since for doing that we need identity. Thus, even if $x' \equiv x$, the theory doesn't grant that *any* indistinguishable from x will belong to *any* strong singleton of x; these strong singletons are *indiscernible*, not identical, that is, $[\![x]\!] \equiv [\![x']\!]$, but not $[\![x]\!] = [\![x']\!]$. Consequently, the equivalence relation '$\equiv$' is distinct from identity.

One could argue against this conclusion by suggesting, say, that we can define an identity for m-atoms, for instance as follows: let x and y be indiscernible m-atoms and let $[\![x]\!]$ and $[\![y]\!]$ strong singletons of these elements. Then we can pose

$$x =^* y := [\![x]\!] =_E [\![y]\!]. \tag{1}$$

Recall from the short review above that the extensional equality ('$=_E$') applies also to qsets (see the footnote 5) when they have 'the same elements'. So, since the q-cardinals of both $[\![x]\!]$ and $[\![y]\!]$ are one, we may lead of concluding that their elements are the same and hence the defined identity would have the properties of standard identity. But this is a mistake. In the object language, that is, in the language of $\mathfrak{Q}$, we cannot say that the elements of these strong singletons are 'the same' since this requires identity and then we would be committing a clear *petitio principii* in assuming that we wish to define. We need to proceed formally: if $x =^* y$, we can say that they are (say) $*$-*identical*, but never that they are 'the same'. Furthermore, $=^*$ has not all the properties of standard identity; in particular, it fails substitution. Let us explain.

The Weak Extensionality Axiom (WEA) of $\mathfrak{Q}$ says that qsets with 'the same quantity' (expressed by means of q-cardinals) of 'elements of the

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same kind' (indiscernible among them) are indistinguishable by their way. For instance, the reader may think of two water molecules, so we could write '$H_2O \equiv H_2O$' once they have the same quantity of elements of the same species. The axiom captures things of this kind. But the defined 'identity' ($=^*$) is not equivalent to the identity of classical logic, which must obey substitutivity for *every* formula α, that is, we must have $x = y \leftrightarrow (\alpha(x) \leftrightarrow \alpha(y))$ (Mendelson, 1997, p.95). Notwithstanding, in $\mathfrak{Q}$, let α be the formula $x \in y$ and let x' be such that $x' =^* x$ according to the above definition. Then, we should have $x \in y \to x' \in y$, but this does not hold.

To see why, we can reason as follows. Suppose $qc([x, x']) =_E 2$ with x and x' denoting indistinguishable m-atoms; insisting, the q-cardinal being 2 suggests that we are in the presence of *two* indistinguishable m-atoms, two entangled bosons for instance. If $y =_E [\![x]\!]$ and if $x \in y$, even if $x =^* x'$, there is no sense in saying that x' belongs to y as well, for in this case, the quasi-cardinal of y would be greater than one, and it is not by the definition of the strong singletons. Furthermore, if both x and x' belong to y and the q-cardinal of y is one, they should be *the same element*, and this is not something that the language of $\mathfrak{Q}$ enables us to do. Really, we should reason as follows: in saying that $x \in y$, with $y =_E [\![x]\!]$, we are stating that *some* indistinguishable from x belongs to y, but there is no way the theory grants that such an element is this or that m-atom, since for asserting this we would need identity. But one thing is certain: *if an element x belongs to y, then no 'other' element can belong to it*. The meaning of the word 'other' is captured not by identity but by considering the quasi-cardinality of $[x, x']$ being 2. The same would happen with whatever relation you propose to cope with identity of m-atoms. We make a further remark that if $x' \equiv x$, then the most the theory proves is that $[\![x']\!] \equiv [\![x]\!]$, that is, the strong singletons are indiscernible, as resulting from the Weak Extensionality Axiom.

An analogy might be useful here. Suppose a neutral Sodium atom, whose electronic decomposition is $1s^2\ 2s^2\ 2p^6\ 3s^1$. Take the outer shell as a strong singleton of an electron we shall call 'x'. Of course, the q-cardinal of such a qset is one, but the problem is that although it has just one electron as its element, we cannot state which is it in the sense that we can say that Barak Obama was a US president (that is, a 'specific' person; electrons have not such specificities). So, if we consider two electrons of this atom and call them x and x', then, as above, the q-cardinal of $[x, x']$ is two and although they have different quantum numbers (since x is the only electron of the outer shell), we cannot say more than 'there is one electron in the outer shell and we named it 'x' '. But this does not grant any identification but of course x' *does not belong* to the outer shell, which

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is occupied by just one electron. The individual variables, when referring to m-atoms, act as *indefinite descriptions*.

A further remark in order here to clarify a little bit more. In saying that $x \in [\![x]\!]$, since the quasi-cardinal of the strong singleton is one, we are saying (in $\mathfrak{Q}$) that *some* indiscernible from x is in the strong singleton, but not that the object denoted by x is there, since we cannot attribute to x a specific denotation.

So, the above considerations concerning the q-automorphisms of course presuppose that no relation in R_i is membership, but this is expected in building mathematical structures since membership is already embedded in the underlying logic.

3 Another physical example

Let us consider another 'physical' example. Suppose R is a relation, or some collection of relations which give the main properties of a Helium neutral atom, He. It has two electrons, and by a chemical process, we can ionize the atom by realizing one of its electrons, let us call it e; notice that this is not a proper name, but just a parameter to make reference to the electron. Then we get a cation He^{+}. By another chemical process, we can make the cation to absorb an electron e' in order to get a neutral atom again. Questions: (1) are the two electrons, the realized one and the absorbed one, the same electron? (2) are the two neutral He atoms, the first before the ionization process and the second one, after absorbing e' the same atom? Of course these questions don't have answers, and in fact they do not make any physical sense due to the indiscernibility of the electrons and of the He atoms. It is a physical fact that the two He atoms have all the same properties, so as the two electrons. Thus, we may conclude that the supposed relation R (described by quantum physics) does preserve the substitution by indiscernibles. In other words, operations that resemble automorphisms seem to be a physical reality.

The main conclusion is that no q-automorphism over a qset of indiscernible m-atoms is the identity function. Consequently, due to the possibility of defining a q-automorphism of the structure $\mathfrak{A}$, we can conclude that it is not a rigid structure. This conclusion can be extended to the whole universe of quasi-sets (for instance as defined in Wajch (2023)); since it encompasses m-atoms at its bottom, the hierarchy of qsets turns out to be not rigid as well.

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4 Conclusions

What is the importance of this discussion? The answer goes this way. We are searching for a suitable mathematical framework for considering 'completely' indiscernible elements. In a standard set theory such as the ZFC system (other systems encompassing the standard theory of identity could be supposed as well), we can consider, or better, *mimic*, indiscernibles only within deformable (nonrigid) structures. These are those structures that admit other automorphisms than the identity function. But the problem is that this is a trick since the $\mathfrak{A}$-indiscernible elements, where $\mathfrak{A}$ is a structure, are indiscernible only *within* $\mathfrak{A}$. This is the case, for instance, with permutation models in ZFA (the Zermelo-Fraenkel system with Atoms). The atoms cannot be discerned only within the model, but it holds for ZFA, so as for ZFC, that *every structure can be extended to a rigid structure*, that is, one encompassing only the identity function as its automorphism. By 'extended' we mean another structure got by adding to the original one more relations and/or operations, as for instance the unitary sets of the elements of the domain.[8] Since the whole set-theoretical universe is rigid, we can see that every object represented in such a framework is an individual, something presenting identity in the sense that it can be discerned from any other individual, even if only in principle.

If we are looking for a framework where indiscernibility is to be attributed 'right from the start', that is, as a primitive notion and not as a 'mock' one, then we need a different framework. Quasi-set theory enables us to construct deformable structures which cannot be extended to rigid ones. So, it seems to be a nice place to accommodate indiscernible things.

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[8] If the domain is infinite, we can use an infinitary language and, if the Axiom of Choice holds, just add a well-ordering over the domain to the structure.

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