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Functional Ghobber-Jaming Uncertainty Principle

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Abstract: Let $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ and $(\{g_k\}_{k=1}^n, \{\omega_k\}_{k=1}^n)$ be two p-orthonormal bases for a finite dimensional Banach space $\mathcal{X}$. Let $M, N \subseteq \{1, \dots, n\}$ be such that

$$o(M)^{\frac{1}{q}} o(N)^{\frac{1}{p}} < \frac{1}{\max\limits_{1\leq j,k\leq n} |g_k(\tau_j)|},$$

where q is the conjugate index of p. Then for all $x \in \mathcal{X}$, we show that

$$\|x\| \leq \left(1 + \frac{1}{1 - o(M)^{\frac{1}{q}} o(N)^{\frac{1}{p}} \max\limits_{1\leq j,k\leq n} |g_k(\tau_j)|}\right) \left[\left(\sum_{j\in M^c} |f_j(x)|^p\right)^{\frac{1}{p}} + \left(\sum_{k\in N^c} |g_k(x)|^p\right)^{\frac{1}{p}}\right].$$

We call Inequality (1) as **Functional Ghobber-Jaming Uncertainty Principle**. Inequality (??) improves the uncertainty principle obtained by Ghobber and Jaming *[Linear Algebra Appl., 2011]*.

Keywords: uncertainty principle; orthonormal basis; Hilbert space; Banach space

MSC: 42C15; 46B03; 46B04

1. Introduction

Let $d \in \mathbb{N}$ and $\widehat{\cdot}: \mathcal{L}^2(\mathbb{R}^d) \to \mathcal{L}^2(\mathbb{R}^d)$ be the unitary Fourier transform obtained by extending uniquely the bounded linear operator

$$\widehat{\cdot}: \mathcal{L}^1(\mathbb{R}^d) \cap \mathcal{L}^2(\mathbb{R}^d) \ni f \mapsto \widehat{f} \in C_0(\mathbb{R}^d); \quad \widehat{f}: \mathbb{R}^d \ni \xi \mapsto \widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \xi \rangle}\, dx \in \mathbb{C}.$$

In 2007, Jaming [1] extended the uncertainty principle obtained by Nazarov for $\mathbb{R}$ in 1993 [2] (cf. [3]). In the following theorem, Lebesgue measure on $\mathbb{R}^d$ is denoted by m. Mean width of a measurable subset E of $\mathbb{R}^d$ having finite measure is denoted by $w(E)$.

Theorem 1 ([1,2]). *(**Nazarov-Jaming Uncertainty Principle**) For each $d \in \mathbb{N}$, there exists a universal constant C_d (depends upon d) satisfying the following: If $E, F \subseteq \mathbb{R}^d$ are measurable subsets having finite measure, then for all $f \in \mathcal{L}^2(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} |f(x)|^2\, dx \leq C_d e^{C_d \min\{m(E)m(F), m(E)^{\frac{1}{d}} w(F), m(F)^{\frac{1}{d}} w(E)\}} \left[\int_{E^c} |f(x)|^2\, dx + \int_{F^c} |\widehat{f}(\xi)|^2\, d\xi\right]. \quad (1)$$

In particular, if f is supported on E and $\widehat{f}$ is supported on F, then $f = 0$.

Theorem 1 and the milestone paper [4] of Donoho and Stark which derived finite dimensional uncertainty principles, motivated Ghobber and Jaming [5] to ask what is the exact finite dimensional analogue of Theorem 1? Ghobber and Jaming were able to derive the following beautiful theorem. Given a subset $M \subseteq \{1, \dots, n\}$, the number of elements in M is denoted by $o(M)$.

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Theorem 2 ([5]). *(**Ghobber-Jaming Uncertainty Principle**) Let $\{\tau_j\}_{j=1}^n$ and $\{\omega_j\}_{j=1}^n$ be orthonormal bases for the Hilbert space $\mathbb{C}^n$. If $M, N \subseteq \{1, \dots, n\}$ are such that*

$$o(M)o(N) < \frac{1}{\max\limits_{1\leq j,k\leq n} |\langle \tau_j, \omega_k\rangle|^2}, \tag{2}$$

then for all $h \in \mathbb{C}^n$,

$$\|h\| \leq \left(1 + \frac{1}{1 - \sqrt{o(M)o(N)}\max\limits_{1\leq j,k\leq n} |\langle \tau_j, \omega_k\rangle|}\right)\left[\left(\sum_{j\in M^c} |\langle h, \tau_j\rangle|^2\right)^{\frac{1}{2}} + \left(\sum_{k\in N^c} |\langle h, \omega_k\rangle|^2\right)^{\frac{1}{2}}\right].$$

In particular, if h is supported on M in the expansion using basis $\{\tau_j\}_{j=1}^n$ and h is supported on N in the expansion using basis $\{\omega_j\}_{j=1}^n$, then $h = 0$.

It is reasonable to ask whether there is a Banach space version of Ghobber-Jaming Uncertainty Principle, which when restricted to Hilbert space, reduces to Theorem 2? We are going to answer this question in the paper.

2. Functional Ghobber-Jaming Uncertainty Principle

In the paper, $\mathbb{K}$ denotes $\mathbb{C}$ or $\mathbb{R}$ and $\mathcal{X}$ denotes a finite dimensional Banach space over $\mathbb{K}$. Identity operator on $\mathcal{X}$ is denoted by $I_\mathcal{X}$. Dual of $\mathcal{X}$ is denoted by $\mathcal{X}^*$. Whenever $1 < p < \infty$, q denotes conjugate index of p. For $d \in \mathbb{N}$, the standard finite dimensional Banach space $\mathbb{K}^d$ over $\mathbb{K}$ equipped with standard $\|\cdot\|_p$ norm is denoted by $\ell^p([d])$. Canonical basis for $\mathbb{K}^d$ is denoted by $\{\delta_j\}_{j=1}^d$ and $\{\zeta_j\}_{j=1}^d$ be the coordinate functionals associated with $\{\delta_j\}_{j=1}^d$. Motivated from the properties of orthonormal bases for Hilbert spaces, we set the following notion of p-orthonormal bases which is also motivated from the notion of p-approximate Schauder frames [6] and p-unconditional Schauder frames [7].

Definition 1. *Let $\mathcal{X}$ be a finite dimensional Banach space over $\mathbb{K}$. Let $\{\tau_j\}_{j=1}^n$ be a basis for $\mathcal{X}$ and let $\{f_j\}_{j=1}^n$ be the coordinate functionals associated with $\{\tau_j\}_{j=1}^n$. The pair $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ is said to be a **p-orthonormal basis** $(1 < p < \infty)$ for $\mathcal{X}$ if the following conditions hold.*

(i) $\|f_j\| = \|\tau_j\| = 1$ for all $1 \leq j \leq n$.
(ii) For every $(a_j)_{j=1}^n \in \mathbb{K}^n$,

$$\left\|\sum_{j=1}^n a_j\tau_j\right\| = \left(\sum_{j=1}^n |a_j|^p\right)^{\frac{1}{p}}.$$

Given a p-orthonormal basis $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$, we easily see from Definition 1 that

$$\|x\| = \left\|\sum_{j=1}^n f_j(x)\tau_j\right\| = \left(\sum_{j=1}^n |f_j(x)|^p\right)^{\frac{1}{p}}, \quad \forall x \in \mathcal{X}.$$

Example 1. *The pair $(\{\zeta_j\}_{j=1}^d, \{\delta_j\}_{j=1}^d)$ is a p-orthonormal basis for $\ell^p([d])$.*

Like orthonormal bases for Hilbert spaces, the following theorem characterizes all p-orthonormal bases.

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Theorem 3. *Let $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ be a p-orthonormal basis for $\mathcal{X}$. Then a pair $(\{g_j\}_{j=1}^n, \{\omega_j\}_{j=1}^n)$ is a p-orthonormal basis for $\mathcal{X}$ if and only if there is an invertible linear isometry $V : \mathcal{X} \to \mathcal{X}$ such that*

$$g_j = f_jV^{-1},\ \omega_j = V\tau_j, \quad \forall 1 \leq j \leq n.$$

Proof. $(\Rightarrow)$ Define $V : \mathcal{X} \ni x \mapsto \sum_{j=1}^n f_j(x)\omega_j \in \mathcal{X}$. Since $\{\omega_j\}_{j=1}^n$ is a basis for $\mathcal{X}$, V is invertible with inverse $V^{-1} : \mathcal{X} \ni x \mapsto \sum_{j=1}^n g_j(x)\tau_j \in \mathcal{X}$. For $x \in \mathcal{X}$,

$$\|Vx\| = \left\|\sum_{j=1}^n f_j(x)\omega_j\right\| = \left(\sum_{j=1}^n |f_j(x)|^p\right)^{\frac{1}{p}} = \left\|\sum_{j=1}^n f_j(x)\tau_j\right\| = \|x\|.$$

Therefore V is isometry. Note that we clearly have $\omega_j = V\tau_j, \forall 1 \leq j \leq n$. Now let $1 \leq j \leq n$. Then

$$f_j(V^{-1}x) = f_j\left(\sum_{k=1}^n g_k(x)\tau_k\right) = \sum_{k=1}^n g_k(x)f_j(\tau_k) = g_j(x), \quad \forall x \in \mathcal{X}.$$

$(\Leftarrow)$ Since V is invertible, $\{\omega_j\}_{j=1}^n$ is a basis for $\mathcal{X}$. Now we see that $g_j(\omega_k) = f_j(V^{-1}V\tau_k) = f_j(\tau_k) = \delta_{j,k}$ for all $1 \leq j,k \leq n$. Therefore $\{g_j\}_{j=1}^n$ is the coordinate functionals associated with $\{\omega_j\}_{j=1}^n$. Since V is an isometry, we have $\|\omega_j\| = 1$ for all $1 \leq j \leq n$. Since V is also invertible, we have

$$\begin{aligned}\|g_j\| &= \sup_{x\in\mathcal{X},\|x\|\leq 1} |g_j(x)| = \sup_{x\in\mathcal{X},\|x\|\leq 1} |f_j(V^{-1}x)| = \sup_{Vy\in\mathcal{X},\|Vy\|\leq 1} |f_j(y)| \\ &= \sup_{Vy\in\mathcal{X},\|y\|\leq 1} |f_j(y)| = \|f_j\| = 1, \quad \forall 1 \leq j \leq n.\end{aligned}$$

Finally, for every $(a_j)_{j=1}^n \in \mathbb{K}^n$,

$$\left\|\sum_{j=1}^n a_j\omega_j\right\| = \left\|\sum_{j=1}^n a_jV\tau_j\right\| = \left\|V\left(\sum_{j=1}^n a_j\tau_j\right)\right\| = \left\|\sum_{j=1}^n a_j\tau_j\right\| = \left(\sum_{j=1}^n |a_j|^p\right)^{\frac{1}{p}}.$$

□

In the next result we show that Example 1 is prototypical as long as we consider p-orthonormal bases.

Theorem 4. *If $\mathcal{X}$ has a p-orthonormal basis $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$, then $\mathcal{X}$ is isometrically isomorphic to $\ell^p([n])$.*

Proof. Define $V : \mathcal{X} \ni x \mapsto \sum_{j=1}^n f_j(x)\delta_j \in \ell^p([n])$. By doing a similar calculation as in the direct part in the proof of Theorem 3, we see that V is an invertible isometry. □

Now we derive main result of this paper.

Theorem 5. *(**Functional Ghobber-Jaming Uncertainty Principle**) Let $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ and $(\{g_k\}_{k=1}^n, \{\omega_k\}_{k=1}^n)$ be p-orthonormal bases for $\mathcal{X}$. If $M, N \subseteq \{1, \dots, n\}$ are such that*

$$o(M)^{\frac{1}{q}}o(N)^{\frac{1}{p}} < \frac{1}{\max\limits_{1\leq j,k\leq n} |g_k(\tau_j)|},$$

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then for all $x \in \mathcal{X}$,

$$\|x\| \leq \left(1+\frac{1}{1-o(M)^{\frac{1}{q}}o(N)^{\frac{1}{p}}\max\limits_{1\leq j,k\leq n}|g_k(\tau_j)|}\right)\left[\left(\sum_{j\in M^c}|f_j(x)|^p\right)^{\frac{1}{p}}+\left(\sum_{k\in N^c}|g_k(x)|^p\right)^{\frac{1}{p}}\right]. \quad (3)$$

In particular, if x is supported on M in the expansion using basis $\{\tau_j\}_{j=1}^n$ and x is supported on N in the expansion using basis $\{\omega_k\}_{k=1}^n$, then $x = 0$.

Proof. Given $S \subseteq \{1,\ldots,n\}$, define

$$P_S x := \sum_{j\in S} f_j(x)\tau_j, \quad \forall x \in \mathcal{X}, \quad \|x\|_{S,f} := \left(\sum_{j\in S}|f_j(x)|^p\right)^{\frac{1}{p}}, \quad \|x\|_{S,g} := \left(\sum_{j\in S}|g_j(x)|^p\right)^{\frac{1}{p}}.$$

Also define $V : \mathcal{X} \ni x \mapsto \sum_{k=1}^n g_k(x)\tau_k \in \mathcal{X}$. Then V is an invertible isometry. Using V we make following important calculations:

$$\|P_S x\| = \left\|\sum_{j\in S} f_j(x)\tau_j\right\| = \left(\sum_{j\in S}|f_j(x)|^p\right)^{\frac{1}{p}} = \|x\|_{S,f}, \quad \forall x \in \mathcal{X}$$

and

$$\begin{aligned}\|P_S V x\| &= \left\|\sum_{j\in S} f_j(Vx)\tau_j\right\| = \left\|\sum_{j\in S} f_j\left(\sum_{k=1}^n g_k(x)\tau_k\right)\tau_j\right\| = \left\|\sum_{j\in S}\sum_{k=1}^n g_k(x)f_j(\tau_k)\tau_j\right\| \\ &= \left\|\sum_{j\in S} g_j(x)\tau_j\right\| = \left(\sum_{j\in S}|g_j(x)|^p\right)^{\frac{1}{p}} = \|x\|_{S,g}, \quad \forall x \in \mathcal{X}.\end{aligned}$$

Now let $y \in \mathcal{X}$ be such that $\{j \in \{1,\ldots,n\} : f_j(y) \neq 0\} \subseteq M$. Then $\|P_N V y\| = \|P_N V P_M y\| \leq \|P_N V P_M\|\|y\|$ and

$$\|y\|_{N^c,g} = \|P_{N^c} V y\| = \|Vy - P_N V y\| \geq \|Vy\| - \|P_N V y\| = \|y\| - \|P_N V y\| \geq \|y\| - \|P_N V P_M\|\|y\|.$$

Therefore

$$\|y\|_{N^c,g} \geq (1 - \|P_N V P_M\|)\|y\|. \quad (4)$$

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Let $x \in \mathcal{X}$. Note that $P_M x$ satisfies $\{j \in \{1,\dots,n\} : f_j(P_M x) \neq 0\} \subseteq M$. Now using (4) we get

$$\begin{aligned}
\|x\| &= \|P_M x + P_{M^c}x\| \leq \|P_M x\| + \|P_{M^c}x\| \leq \frac{1}{1-\|P_N V P_M\|}\|P_M x\|_{N^c,g} + \|P_{M^c}x\| \\
&= \frac{1}{1-\|P_N V P_M\|}\|P_{N^c} V P_M x\| + \|P_{M^c}x\| = \frac{1}{1-\|P_N V P_M\|}\|P_{N^c} V (x - P_{M^c}x)\| + \|P_{M^c}x\| \\
&\leq \frac{1}{1-\|P_N V P_M\|}\|P_{N^c} V x\| + \frac{1}{1-\|P_N V P_M\|}\|P_{N^c} V P_{M^c}x\| + \|P_{M^c}x\| \\
&\leq \frac{1}{1-\|P_N V P_M\|}\|P_{N^c} V x\| + \frac{1}{1-\|P_N V P_M\|}\|P_{M^c}x\| + \|P_{M^c}x\| \\
&= \frac{1}{1-\|P_N V P_M\|}\|P_{N^c} V x\| + \left(1 + \frac{1}{1-\|P_N V P_M\|}\right)\|P_{M^c}x\| \\
&\leq \|P_{N^c} V x\| + \frac{1}{1-\|P_N V P_M\|}\|P_{N^c} V x\| + \left(1 + \frac{1}{1-\|P_N V P_M\|}\right)\|P_{M^c}x\| \\
&= \left(1 + \frac{1}{1-\|P_N V P_M\|}\right)[\|P_{N^c} V x\| + \|P_{M^c}x\|] = \left(1 + \frac{1}{1-\|P_N V P_M\|}\right)[\|x\|_{N^c,g} + \|P_{M^c}x\|] \\
&= \left(1 + \frac{1}{1-\|P_N V P_M\|}\right)\left[\left(\sum_{j\in M^c}|f_j(x)|^p\right)^{\frac{1}{p}} + \left(\sum_{k\in N^c}|g_k(x)|^p\right)^{\frac{1}{p}}\right].
\end{aligned}$$

For $x \in \mathcal{X}$, we now find

$$\begin{aligned}
\|P_N V P_M x\|^p &= \left\|\sum_{k\in N} f_k(V P_M x)\tau_k\right\|^p = \left(\sum_{k\in N}|f_k(V P_M x)|^p\right)^{\frac{1}{p}} = \sum_{k\in N}\left|(f_k V)\left(\sum_{j\in M} f_j(x)\tau_j\right)\right|^p \\
&= \sum_{k\in N}\left|\sum_{j\in M} f_j(x) f_k(V\tau_j)\right|^p = \sum_{k\in N}\left|\sum_{j\in M} f_j(x) f_k\left(\sum_{r=1}^n g_r(\tau_j)\tau_r\right)\right|^p = \sum_{k\in N}\left|\sum_{j\in M} f_j(x)\sum_{r=1}^n g_r(\tau_j) f_k(\tau_r)\right|^p \\
&= \sum_{k\in N}\left|\sum_{j\in M} f_j(x) g_k(\tau_j)\right|^p \leq \sum_{k\in N}\left(\sum_{j\in M}|f_j(x) g_k(\tau_j)|\right)^p \leq \left(\max_{1\leq j,k\leq n}|g_k(\tau_j)|\right)^p \sum_{k\in N}\left(\sum_{j\in M}|f_j(x)|\right)^p \\
&= \left(\max_{1\leq j,k\leq n}|g_k(\tau_j)|\right)^p o(N)\left(\sum_{j\in M}|f_j(x)|\right)^p \leq \left(\max_{1\leq j,k\leq n}|g_k(\tau_j)|\right)^p o(N)\left(\sum_{j\in M}|f_j(x)|^p\right)^{\frac{p}{p}}\left(\sum_{j\in M}1^q\right)^{\frac{p}{q}} \\
&\leq \left(\max_{1\leq j,k\leq n}|g_k(\tau_j)|\right)^p o(N)\left(\sum_{j=1}^n|f_j(x)|^p\right)^{\frac{p}{p}}\left(\sum_{j\in M}1^q\right)^{\frac{p}{q}} = \left(\max_{1\leq j,k\leq n}|g_k(\tau_j)|\right)^p o(N)\|x\|^p o(M)^{\frac{p}{q}}.
\end{aligned}$$

Therefore

$$\|P_N V P_M\| \leq \max_{1\leq j,k\leq n}|g_k(\tau_j)| o(N)^{\frac{1}{p}} o(M)^{\frac{1}{q}}$$

which gives the theorem. □

Corollary 1. *Theorem 2 follows from Theorem 5.*

Proof. Let $\{\tau_j\}_{j=1}^n$, $\{\omega_j\}_{j=1}^n$ be two orthonormal bases for a finite dimensional Hilbert space $\mathcal{H}$. Define

$$f_j : \mathcal{H} \ni h \mapsto \langle h, \tau_j\rangle \in \mathbb{K}; \quad g_j : \mathcal{H} \ni h \mapsto \langle h, \omega_j\rangle \in \mathbb{K}, \quad \forall 1 \leq j \leq n.$$

Then $p = q = 2$ and $|f_j(\omega_k)| = |\langle\omega_k, \tau_j\rangle|$ for all $1 \leq j,k \leq n$. □

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By interchanging p-orthonormal bases in Theorem 5 we get the following theorem.

Theorem 6. *(Functional Ghobber-Jaming Uncertainty Principle) Let* $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ *and* $(\{g_k\}_{k=1}^n, \{\omega_k\}_{k=1}^n)$ *be p-orthonormal bases for* $\mathcal{X}$. *If* $M, N \subseteq \{1, \dots, n\}$ *are such that*

$$o(M)^{\frac{1}{q}} o(N)^{\frac{1}{p}} < \frac{1}{\max\limits_{1\leq j,k\leq n} |f_j(\omega_k)|},$$

then for all $x \in \mathcal{X}$,

$$\|x\| \leq \left(1 + \frac{1}{1 - o(M)^{\frac{1}{q}} o(N)^{\frac{1}{p}} \max\limits_{1\leq j,k\leq n} |f_j(\omega_k)|}\right) \left[\left(\sum_{k\in M^c} |g_k(x)|^p\right)^{\frac{1}{p}} + \left(\sum_{j\in N^c} |f_j(x)|^p\right)^{\frac{1}{p}}\right].$$

In particular, if x *is supported on* M *in the expansion using basis* $\{\omega_k\}_{k=1}^n$ *and* x *is supported on* N *in the expansion using basis* $\{\tau_j\}_{j=1}^n$, *then* $x = 0$.

Observe that the constant

$$C_d e^{C_d \min\{m(E)m(F), m(E)^{\frac{1}{d}} w(F), m(F)^{\frac{1}{d}} w(E)\}}$$

in Inequality (1) is depending upon subsets E, F and not on the entire domain $\mathbb{R}$ of functions $f, \widehat{f}$. Thus it is natural to ask whether there is a constant sharper in Inequality (3) depending upon subsets M, N and not on $\{1, \dots, n\}$. A careful observation in the proof of Theorem 5 gives following result.

Theorem 7. *Let* $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ *and* $(\{g_k\}_{k=1}^n, \{\omega_k\}_{k=1}^n)$ *be p-orthonormal bases for* $\mathcal{X}$. *If* $M, N \subseteq \{1, \dots, n\}$ *are such that*

$$o(M)^{\frac{1}{q}} o(N)^{\frac{1}{p}} < \frac{1}{\max\limits_{j\in M,k\in N} |g_k(\tau_j)|},$$

then for all $x \in \mathcal{X}$,

$$\|x\| \leq \left(1 + \frac{1}{1 - o(M)^{\frac{1}{q}} o(N)^{\frac{1}{p}} \max\limits_{j\in M,k\in N} |g_k(\tau_j)|}\right) \left[\left(\sum_{j\in M^c} |f_j(x)|^p\right)^{\frac{1}{p}} + \left(\sum_{k\in N^c} |g_k(x)|^p\right)^{\frac{1}{p}}\right].$$

Similarly we have the following result from Theorem 6.

Theorem 8. *Let* $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ *and* $(\{g_k\}_{k=1}^n, \{\omega_k\}_{k=1}^n)$ *be p-orthonormal bases for* $\mathcal{X}$. *If* $M, N \subseteq \{1, \dots, n\}$ *are such that*

$$o(M)^{\frac{1}{q}} o(N)^{\frac{1}{p}} < \frac{1}{\max\limits_{j\in N,k\in M} |f_j(\omega_k)|},$$

then for all $x \in \mathcal{X}$,

$$\|x\| \leq \left(1 + \frac{1}{1 - o(M)^{\frac{1}{q}} o(N)^{\frac{1}{p}} \max\limits_{j\in N,k\in M} |f_j(\omega_k)|}\right) \left[\left(\sum_{k\in M^c} |g_k(x)|^p\right)^{\frac{1}{p}} + \left(\sum_{j\in N^c} |f_j(x)|^p\right)^{\frac{1}{p}}\right].$$

Theorem 5 brings the following question.

Question 9. *Given p and a Banach space $\mathcal{X}$ of dimension n, for which subsets $M, N \subseteq \{1, \dots, n\}$ and pairs of p-orthonormal bases $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$, $(\{g_k\}_{k=1}^n, \{\omega_k\}_{k=1}^n)$ for $\mathcal{X}$, we have equality in Inequality (3)?*

It is clear that we used $1 < p < \infty$ in the proof of Theorem 5. However, Definition 1 can easily be extended to include cases $p = 1$ and $p = \infty$. This therefore leads to the following question.

Question 10. *Whether there are Functional Ghobber-Jaming Uncertainty Principle (versions of Theorem 5) for 1-orthonormal bases and ∞-orthonormal bases?*

We end by mentioning that Donoho-Stark-Elad-Bruckstein-Ricaud-Torrésani Uncertainty Principle for finite dimensional Banach spaces is derived in [8] (actually, in [8] the functional uncertainty principle was derived for p-Schauder frames which is general than p-orthonormal bases. Thus it is worth to derive Theorem 5 or a variation of it for p-Schauder frames, which we are unable).

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