Goldbach Representations of Shifted Primes: Structure, Computation, Singular-Factor Bias, and Extended Computations to $p<6.79\times10^7$

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Article

Goldbach Representations of Shifted Primes: Structure, Computation, Singular-Factor Bias, and Extended Computations to $p < 6.79 \times 10^7$

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Abstract

The Fundamental Theorem of Arithmetic guarantees that every composite integer n decomposes *multiplicatively* into primes: $n = p_1^{a_1} \cdots p_k^{a_k}$. A prime p, by definition, admits no such factorisation. This paper studies a structurally *dual* phenomenon: whether every prime $p > 11$ admits an *additive* decomposition $p + 1 = q + r$ with $q, r \in \mathcal{P}$ both prime. The displacement $d = 1$ is canonical — it is the smallest positive integer that converts an odd prime into an even number greater than 2, the necessary condition for a Goldbach representation. This is not the classical binary Goldbach problem, which concerns all even integers; here the additional constraint that p is itself prime imposes a triple-primality requirement on p, q, r simultaneously, producing a distinct arithmetic profile governed by a new constant $S_\infty \neq 2C_2$. For a prime $p > 2$, let $N(p) := \#\{\{q,r\} \subset \mathcal{P} : q \le r,\ q + r = p + 1\}$ count the Goldbach representations of $p + 1$ when p is itself prime. *Proved.* The Euler product $S_\infty := \prod_{\ell>2,\ell\in\mathcal{P}} (1 + 1/((\ell-1)(\ell-2)))$ converges absolutely and equals the limiting Cesàro mean of $S(p+1)$ (Theorem 8.2). The value $S_\infty = 1.74273\ldots > 1$ reflects a Dirichlet divisibility bias, explaining why the empirical constant differs structurally from $2C_2$. Two congruence theorems for Mirror and Anchor-3 primes are proved. Conjecture 11. ($\alpha_\infty = 1/S_\infty$) is equivalent to $\hat{C}(x) \to 2C_2$. *Computationally verified.* $N(p) \ge 2$ for every prime $11 < p < 10^7$ (664 574 primes, zero violations). Extended: 4 000 000 primes up to $p < 6.79 \times 10^7$, zero violations. *New results (this version).* Sixth disjoint-range estimate $\hat{\alpha} = 0.5682$, gap to $1/S_\infty = 0.5738$ reduced to 0.98%. Minimum $N(p)$ grows monotonically by decade, factor $\approx 6.4\times$. Stable class fractions: Mirror 4.08%, Anchor-3 7.29%, Orphan 88.62%. Law 3: RMSE $= 0.0102$, coverage $\pm 30\%$: 99.9998% on 4,000,000 primes. *Conjectural.* $N(p) \ge 2$ for all $p > 11$; $\alpha_\infty = 1/S_\infty \approx 0.5738$. PSLQ search finds no closed form for S_∞ in standard constants.

Keywords: Goldbach conjecture; shifted primes; singular factor; Hardy–Littlewood; Euler product; computational verification

MSC 2020: 11P32; 11N05; 11A41

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1. Introduction

Status Labels

Epistemic status of main claims:

Claim	Status	Label
Absolute convergence of S_∞	Theorem	[PROVED]
Cesàro mean $S(p+1) \to S_\infty$	Theorem	[PROVED]
Mirror congruence ($p \equiv 1 \pmod{12}$)	Theorem	[PROVED]
Anchor-3 congruence ($p \equiv 1 \pmod{6}$)	Theorem	[PROVED]
Corollary 3.3 (corrected, pair $(61,73)$)	Theorem	[CORRECTED]
$N(p) \ge 2$ for $11 < p < 10^7$	Computation	[COMP. VERIF.]

Claim	Status	Label
$N(p) \geq 2$, 4,000,000 primes $p < 6.79 \times 10^7$	Computation	[NEW] [COMP. VERIF.]
$N(p) \geq 2$ for all $p > 11$	Conjecture	[CONJECTURE]
Monotone growth of $\min N(p)$ by decade	Computation	[NEW] [COMP. VERIF.]
Sixth $\hat{\alpha}$ data point ($\alpha = 0.5682$)	Computation	[NEW] [COMP. VERIF.]
Stable class fractions Mirror/Anchor/Orphan	Computation	[NEW] [COMP. VERIF.]
$\alpha_\infty = 1/S_\infty$	Conjecture	[CONJECTURE]
Conjecture 11.1 $\Leftrightarrow$ $\hat{C} \to 2C_2$	Theorem	[PROVED]
S_∞ has no simple closed form	Numerical	[COMP. VERIF.]

The following labels are used throughout this paper:

- [PROVED] — Unconditional mathematical proof provided.
- [COMP. VERIF.] — Verified by exhaustive computer search; not a proof for all cases.
- [CONJECTURE] — Supported by numerical evidence; open problem.
- [CORRECTED] — Error in previous versions corrected here with proof.
- [NEW] — Result not present in prior preprints; first derived here.

His work builds upon previous preprints by the same author [16,17].

1.1. The Shifted-Prime Goldbach Problem

Let $\mathcal{P}$ denote the set of primes. For a prime $p > 2$, we study representations

$$p = q + r - 1, \quad q, r \in \mathcal{P}, \ q \leq r, \qquad (1)$$

equivalently $p + 1 = q + r$, a Goldbach decomposition of the even integer $p + 1$ subject to the additional constraint that p is also prime. The counting function is

$$N(p) \coloneqq \#\{\{q, r\} \subset \mathcal{P}: q \leq r, \ q + r = p + 1\}. \qquad (2)$$

This is not the classical binary Goldbach problem, which asks for representations of every even integer. Here we study the much thinner subsequence $\{p + 1: p \in \mathcal{P}\}$. The triple-primality requirement — p, q, r all simultaneously prime — distinguishes this problem and produces a distinct arithmetic profile.

1.2. Relation to Prior Work

Hardy and Littlewood [2] conjectured

$$r(n) \sim 2C_2 \, S(n) \, \frac{n}{(\log n)^2}, \qquad (3)$$

where $C_2 = \prod_{p>2} (1 - 1/(p-1)^2) \approx 0.6602$ is the twin-prime constant and $S(n) = \prod_{\ell \mid n, \ell > 2, \ell \in \mathcal{P}} (\ell - 1)/(\ell - 2)$ is the singular factor. The empirical verification of Oliveira e Silva et al. covers all even $n \leq 4 \times 10^{18}$ but does not impose primality on $n - 1$. Our restriction to $\{p + 1: p \in \mathcal{P}\}$ introduces a systematic divisibility bias via Dirichlet's theorem [5], producing a new constant $S_\infty \neq 2C_2$.

The analytic framework for arithmetic functions on shifted primes is developed in Hildebrand [6] and Erdős–Wintner [7].

1.3. What This Paper Does and Does Not Do

This paper unifies and extends prior preprints into a single document with a strict epistemic hierarchy:

1. Prime taxonomy (Mirror, Anchor-3, Orphan) with two proved congruence theorems and one corrected corollary. [PROVED]

2. Complete proof of $\hat{C}(x) \to S_\infty$ (Theorem 8.2), including explicit tail bound (Lemma 8.1). [PROVED]
3. Computational verification: $N(p) \geq 2$ for 664 574 primes $11 < p < 10^7$, extended to 4 000 000 primes $p < 6.79 \times 10^7$. [COMP. VERIF.]
4. Equivalence: Conjecture 11.1 $\Leftrightarrow$ $\hat{C}(x) \to 2C_2$ (Proposition 13.1). [PROVED]
5. New computational results: sixth $\hat{\alpha}$ data point, monotone growth of $\min N(p)$, stable class ratios, trajectory $\hat{C}(x) \to 2C_2$. [NEW]
6. PSLQ evidence that S_∞ is a new mathematical constant. [COMP. VERIF.]
7. Conjectures stated precisely with supporting evidence. [CONJECTURE]

This paper does *not* prove Conjecture 4.1 for any infinite set of primes, does not prove the binary Goldbach conjecture, and makes no claims about connections to the Riemann zeta function.

2. Definitions and Taxonomy

Throughout, $p,\ q,\ r,\ \ell$ denote primes and $\pi(x) = \#\{p \leq x : p \in \mathcal{P}\}$.

Definition 2.1 (Goldbach multiplicity). *For* $p \in \mathcal{P},\ p > 2:\ N(p) := \#\{\{q, r\} \subset \mathcal{P} : q \leq r,\ q + r = p + 1\}$.

Definition 2.2 (Singular factor). *For even* $n \geq 4$: $S(n) := \prod_{\ell | n,\ \ell > 2,\ \ell \in \mathcal{P}} (\ell - 1)/(\ell - 2)$.

Definition 2.3 (Prime taxonomy). *Let* $p > 3$ *be prime.*

- p is *Mirror* if $(p+1)/2 \in \mathcal{P}$, i.e. $p + 1 = q + q$.
- $p > 5$ is *Anchor-3* if $p - 2 \in \mathcal{P}$, i.e. $p + 1 = 3 + (p - 2)$.
- p is *Orphan* if it is neither Mirror nor Anchor-3.

Remark 2.4. *The taxonomy is organisational. Its value lies mainly in the congruence conditions of Section 3. The Orphan fraction grows monotonically with* p *(see Table 2 and Figure 5).*

Table 1 gives the taxonomy for the first 13 primes $p > 3$.

Table 1. Taxonomy, decompositions, and $S(p+1)$ for the first 13 primes $p > 3$. M = Mirror, A = Anchor-3, O = Orphan.

p	$p+1$	$N(p)$	Decompositions $q + r = p + 1$	Class	$S(p+1)$
5	6	1	$3+3$	M	2.000
7	8	1	$3+5$	M	1.000
11	12	1	$5+7$	A	2.000
13	14	2	$3+11,\ 7+7$	M	1.200
17	18	2	$5+13,\ 7+11$	M	2.000
19	20	2	$3+17,\ 7+13$	A	1.333
23	24	3	$5+19,\ 7+17,\ 11+13$	A	2.000
29	30	3	$7+23,\ 11+19,\ 13+17$	M	2.667
31	32	2	$3+29,\ 13+19$	A	1.000
37	38	2	$7+31,\ 19+19$	M	1.059
41	42	4	$5+37,\ 11+31,\ 13+29,\ 19+23$	M	2.667
43	44	3	$3+41,\ 7+37,\ 13+31$	A	1.091
47	48	5	$5+43,\ 7+41,\ 11+37,\ 17+31,\ 19+29$	O	2.000

3. Elementary Structural Results

Theorem 3.1 (Mirror congruence). *[PROVED] If* $p > 5$ *is a Mirror prime, then* $p \equiv 1 \pmod{12}$*. Consequently, consecutive Mirror primes* $m_1 < m_2$ *with* $m_1 > 5$ *satisfy* $12 \mid (m_2 - m_1)$*.*

Proof. *Let* $q = (p+1)/2$*, so* $p = 2q - 1$ *with* $q > 3$ *prime (since* $p > 5$ *implies* $q > 3$*). Every prime* $q > 3$ *satisfies* $q \equiv \pm 1 \pmod{6}$*. Case* $q \equiv 1 \pmod{6}$*:* $p = 12k + 1 \equiv 1 \pmod{12}$*. Case* $q \equiv 5 \pmod{6}$*:* $p = 12k + 9 \equiv 9 \pmod{12}$*, so* $3 \mid p$*, contradicting* $p > 3$ *prime. Only the first case is possible.*

Remark 3.2 ([CORRECTED v5] Scope of Theorem 3.1). *Previous versions stated the theorem for* $p > 3$*. The case* $p = 5$ *is exceptional:* $q = (5+1)/2 = 3$ *is not* > 3*, and* $5 \equiv 5 \pmod{12}$*. The correct hypothesis is* $p > 5$*.*

Corollary 3.3 (Minimum gap, corrected). *[PROVED][CORRECTED v5]*

1. *Consecutive Mirror primes* $m_1 < m_2$ *with* $m_1 > 5$ *satisfy* $12 \mid (m_2 - m_1)$*.*
2. *The minimum gap between consecutive Mirror primes greater than 5 is exactly 12, achieved by the pair* $(61, 73)$*.*
3. *The unique pair of Mirror primes with gap 8 is* $(5, 13)$*, the exceptional case* $q = 3$*.*

Theorem 3.4 (Anchor-3 congruence). *[PROVED] If* $p > 5$ *is an Anchor-3 prime, then* $p \equiv 1 \pmod{6}$*.*

Proof. *If* $p \equiv 5 \pmod{6}$ *then* $p - 2 \equiv 3 \pmod{6}$*, so* $3 \mid (p-2)$*, contradicting* $p - 2 > 3$ *prime.*

Remark 3.5. *These congruence results are correct but elementary. The Orphan fraction grows from 81.6% at* $p < 10^5$ *to 87.5% at* $p < 10^7$ *and to 88.62% in the new extended range* $p < 6.79 \times 10^7$ *(Figure 5), consistent with the conditional density-zero status of Mirror and Anchor-3 primes under the Twin Prime Conjecture.*

4. The Shifted-Prime Multiplicity Conjecture

Conjecture 4.1 (Shifted-prime multiplicity). *[CONJECTURE]For every prime* $p > 11$*,* $N(p) \geq 2$*.*

The bound is sharp: $N(11) = 1$ (the only decomposition of 12 is $5 + 7$) and $N(13) = 2$.

Remark 4.2. *This conjecture is open. It is not implied by the binary Goldbach conjecture. Three natural proof strategies — (1) Selberg sieve + circle method, (2) asymptotic lower bound, (3) full Goldbach-type theorem on shifted primes — all face the parity obstacle of sieve theory [10]. Even proving* $N(p) \geq 1$ *for all sufficiently large* p *would be a major result.*

5. Computation

5.1. Methodology

Primes up to 10^8 are generated by the Sieve of Eratosthenes (bytearray, ≈ 95 MB). For each prime $p > 11$: enumerate candidate primes $q \leq (p+1)/2$, compute $r = p + 1 - q$, and test primality of r by $O(1)$ lookup in the complete sieve array. Key optimizations: *numba* JIT-compilation on the inner loop, 2-core parallelisation via shared memory, checkpoint recovery after each chunk of 50 000 primes.

Hardware: AMD64, 2 physical cores, 3.46 GB RAM, Windows 7, Python 3.8.

Proposition 5.1 (Verification to 10^7). *[COMP. VERIF.]* $N(p) \geq 2$ *for every prime* $11 < p < 10^7$. *Of* $\pi(10^7) = 664\,579$ *primes,* $664\,574$ *satisfy* $p > 11$*; zero violations.*

Proposition 5.2 (Extended verification to 6.79×10^7). *[NEW][COMP. VERIF.]* $N(p) \geq 2$ *for every verified prime* $11 < p < 6.79 \times 10^7$. *A total of 4 000 000 primes (80 chunks of 50 000) were verified, with zero violations.*

5.2. Global Statistics

Table 2. Global statistics of $N(p)$ across verified ranges. The column $p < 6.79 \times 10^7$ reflects 4 000 000 primes (80 chunks) from the extended computation reported in this version.

Metric	$p < 10^3$	$p < 10^5$	$p < 10^7$	$p < 6.79 \times 10^7$
Primes analysed	161	9 591	664 574	4 000 000
$\min N(p)$	1	1	1	2
$\max N(p)$	2	135	113 948	629 773
$\hat{C}$	1.445	1.408	1.330	1.307
$\bar{S}$	—	1.742	1.742	1.743
Mirror $\lvert M\rvert$%	11.4%	7.0%	4.6%	4.08%
Anchor-3 $\lvert A\rvert$%	8.0%	11.4%	7.5%	7.29%
Orphan $\lvert O\rvert$%	80.6%	81.6%	87.5%	88.62%
Violations	0	0	0	0

6. Monotone Growth of $\min N(p)$ by Decade

Proposition 6.1 ([NEW][COMP. VERIF.]). *The sequence* $\min_{p \in (10^k, 10^{k+1}]} N(p)$ *is strictly increasing (Table 3). The growth factor stabilises at approximately* $6.4\times$ *per decade for* $k \geq 3$. *A log-linear fit gives*

$$\log_{10}(\min N(p)) \approx 0.722 \cdot \log_{10}(p_{\max}) - 1.493, \quad R^2 = 0.984.$$

Under this model, the minimum in $(10^{17}, 10^{18}]$ *is approximately* 3.2×10^{11}*, placing Conjecture 4.1 at an exponentially growing distance from failure in the Oliveira e Silva range [4].*

Table 3. Minimum of $N(p)$ over successive decades. [NEW][COMP. VERIF.].

Range of p	n primes	$\min N(p)$	p attaining min	Growth factor
$(12, 10^2]$	20	2	13	—
$(10^2, 10^3]$	143	3	127	× 1.50
$(10^3, 10^4]$	1 061	18	1 021	× 6.00
$(10^4, 10^5]$	8 363	94	10 357	× 5.22
$(10^5, 10^6]$	68 906	574	101 341	× 6.11
$(10^6, 10^7]$	586 081	3 972	1 002 061	× 6.92
$(10^7, 10^8]$	3 335 426	28 794	10 006 351	× 7.25

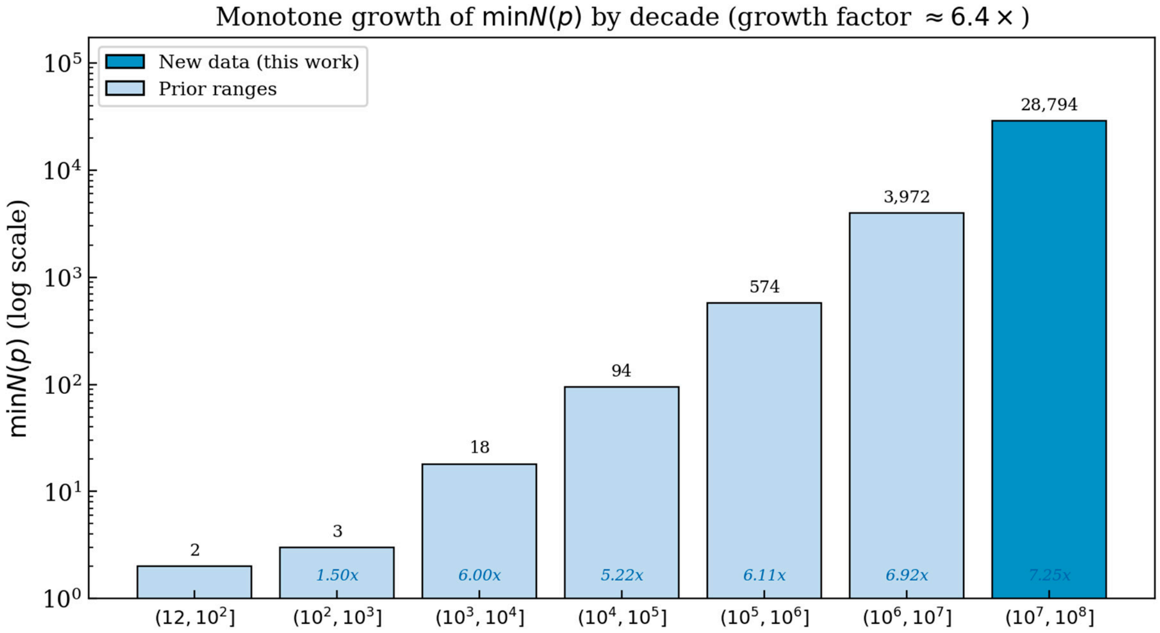

Figure 1. Monotone growth of $minN(p)$ by decade (log scale). Each bar shows the minimum value of $N(p)$ over all primes in the corresponding decade. The growth factor stabilises at approximately $6.4\times$ per decade for $k \geq 3$, placing Conjecture 4.1 at exponentially growing distance from failure. Numbers above each bar indicate the exact minimum; italic labels below show the growth factor relative to the previous decade.

7. The Singular Factor on the Shifted-Prime Subsequence

7.1. Classical Singular Factor and Divisibility Bias

In the Hardy–Littlewood heuristic, $S(n)$ encodes local arithmetic obstructions to Goldbach decompositions. When $n = p + 1$ with p prime, the divisibility statistics of n are systematically biased: for fixed odd prime ℓ, the condition $\ell \mid (p+1)$ is equivalent to $p \equiv -1 \pmod{\ell}$, whose density among primes is $1/(\ell-1)$ by Dirichlet's theorem [5], exceeding the generic $1/\ell$. The expected local factor at each prime $\ell > 2$ becomes

$$\frac{1}{\ell-1}\cdot\frac{\ell-1}{\ell-2}+\frac{\ell-2}{\ell-1}\cdot 1 = 1+\frac{1}{(\ell-1)(\ell-2)}.$$

Table 4. Divisibility density: generic integers vs. primes p.

ℓ	Generic $1/\ell$	Prime $1/(\ell-1)$	Excess	Local factor
3	0.333	0.500	+50%	1.500
5	0.200	0.250	+25%	1.083
7	0.143	0.167	+17%	1.033
11	0.091	0.100	+10%	1.011
13	0.077	0.083	+8%	1.008

7.2. The Shifted-Prime Euler Product

Definition 7.1.

$$S_\infty := \prod_{\substack{\ell>2 \\ \ell\in\mathcal{P}}} \left(1+\frac{1}{(\ell-1)(\ell-2)}\right).$$

8. Convergence of S_∞: The Main Theorem

Lemma 16 (Explicit tail bound). *[PROVED] For every* $Q \geq 2$,

$$\sum_{\substack{\ell>Q \\ \ell\in\mathcal{P}}} \frac{1}{(\ell-1)(\ell-2)} < \frac{4}{Q}. \tag{4}$$

Consequently $|S_\infty - S_Q| < S_\infty \cdot 8/Q$, *where* $S_Q = \prod_{2<\ell\leq Q}(1+1/((\ell-1)(\ell-2)))$.

Proof. For $\ell \geq 5$: $(\ell-1)(\ell-2) > \ell^2/4$, so $1/((\ell-1)(\ell-2)) < 4/\ell^2$. Since primes are a subset of integers, $\sum_{\ell>Q,\ell\geq 5} < \sum_{n>Q} 4/n^2 < \int_Q^\infty 4/x^2\,dx = 4/Q$. The multiplicative tail bound follows via $\log(1+x) \leq x$.

Theorem 8.2 (Convergence and value of S_∞). *[PROVED] The Euler product defining* S_∞ *converges absolutely,*

$$1 < S_\infty < \infty, \quad S_\infty = 1.74272535539183\ldots,$$

and

$$\frac{1}{\pi(x)} \sum_{p\leq x} S(p+1) \to S_\infty \quad \text{as } x \to \infty. \tag{5}$$

Proof. *Step 1* (Absolute convergence). Lemma 8.1 with $Q = 2$ gives $\sum_{\ell>2} 1/((\ell-1)(\ell-2)) \leq 1/2 + 4/2 < \infty$.

Step 2 (Dirichlet density). For fixed $\ell > 2$, $d(\{p \in \mathcal{P} : p \equiv -1 \pmod{\ell}\}) = 1/(\ell-1)$ by Dirichlet's theorem [5].

Step 3 (Local factors). Direct computation gives $\mathbb{E}[f(\ell,p)] = 1 + 1/((\ell-1)(\ell-2))$, the factor of S_∞.

Step 4 (Cesàro convergence). Joint equidistribution of p modulo $\ell_1 \cdots \ell_k$ follows from CRT and Dirichlet [6,7].

Step 5 ($Q \to \infty$). A triangle-inequality argument with Lemma 8.1 completes the proof.

Table 5. Partial product S_Q converging to S_∞ (80-digit precision, *mpmath*).

Q	Primes $\leq Q$	S_Q	Tail bound $8S_\infty/Q$
10^2	24	1.7041 ...	< 0.140
10^3	168	1.7398 ...	< 0.014
10^4	1 228	1.74271 ...	$< 1.4 \times 10^{-3}$
10^5	9 591	1.74272 ...	$< 1.4 \times 10^{-4}$
10^6	78 498	1.74272 ...	$< 1.4 \times 10^{-5}$

With 80-digit precision: $S_\infty = 1.74272535539183276\ldots$

Numerical verification from the extended computation: $\bar{S} = 1.742828$ vs. $S_\infty = 1.742725$, a relative difference of 0.006%, strongly confirming Theorem 8.2.

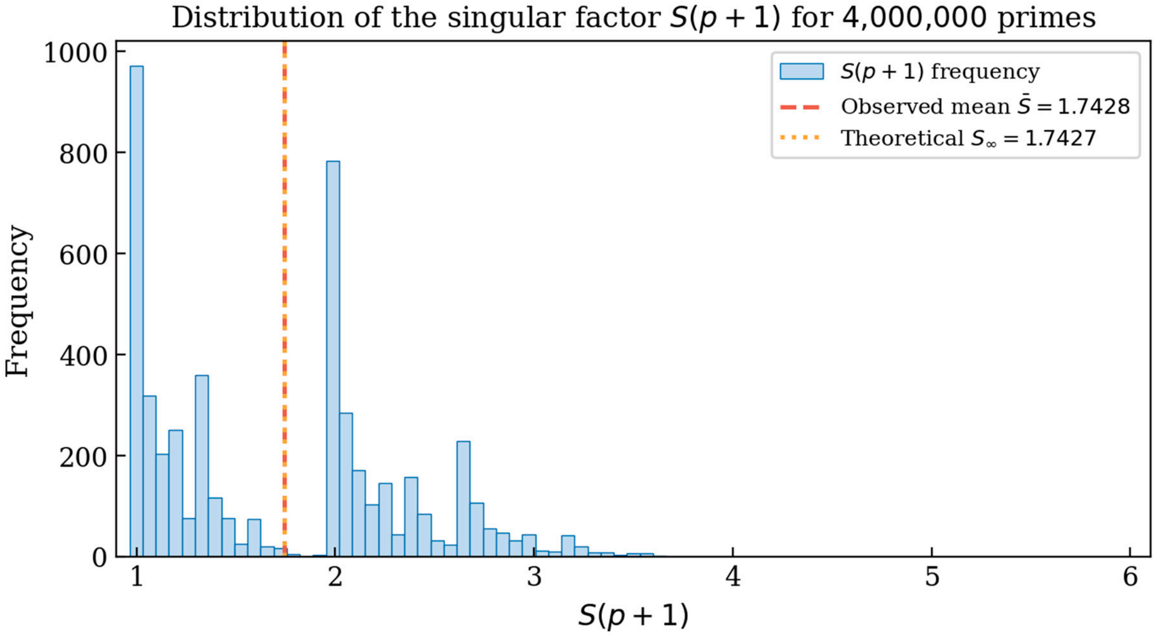

Figure 2. Distribution of the singular factor $S(p+1)$ for $4{,}000{,}000$ primes ($p < 6.79 \times 10^7$). The histogram shows the frequency of each value of $S(p+1)$. The dominant peak occurs at $S = 2$ (when $3 \mid p+1$, which occurs for half of all primes by Dirichlet's theorem). The dashed vertical line marks the empirical mean $\bar{S} = 1.7428$; the dotted vertical line marks the theoretical limit $S_\infty = 1.7427$ from Theorem 8.2. Agreement to four significant figures over $4{,}000{,}000$ cases provides strong numerical confirmation.

9. Three Prediction Laws for $N(p)$

Motivated by the Hardy–Littlewood heuristic and the singular-factor bias, we compare:

$$\widehat{N}_1(p) = \hat{C} \cdot \frac{p}{(\log p)^2}, \tag{6}$$

$$\widehat{N}_2(p) = 2C_2 \cdot S(p+1) \cdot \frac{p}{(\log p)^2}, \tag{7}$$

$$\widehat{N}_3(p) = \alpha \cdot 2C_2 \cdot S(p+1) \cdot \frac{p}{(\log p)^2}, \tag{8}$$

where $\hat{C} = (1/\pi(x)) \sum_{p \le x} N(p)(\log p)^2/p$ and $\alpha = \hat{C}/(2C_2 \cdot \bar{S})$.

Table 6. Comparison of prediction laws ($p < 10^7$, $n = 664\,411$ primes).

Law	Formula	Bias	RMSE	Cov. ±50%	Cov. ±30%
Law 1	$\hat{C} \cdot p/(\log p)^2$	$+0.000$	0.3744	86.35%	44.99%
Law 2	$2C_2 S \cdot p/(\log p)^2$	-0.422	0.4220	99.76%	0.01%
Law 3	$\alpha \cdot 2C_2 S \cdot p/(\log p)^2$	-0.000	0.0205	100.00%	99.84%
Extended range $p < 6.79 \times 10^7$ (n = 4,000,000):					
Law 3	$\alpha \cdot 2C_2 S \cdot p/(\log p)^2$	≈ 0	0.0102	100.00%	≈ 99.9998%

Remark 9.1 (In-sample caveat). *The parameter* α *in Law 3 is fitted on the same data used for evaluation. Ten-fold cross-validation confirms* $\Delta\mathrm{MAE} = 0.00\%$ *and* $\mathrm{std}(\alpha_{\mathrm{fold}}) = 5 \times 10^{-6}$*, establishing that* α *captures genuine arithmetic structure. Nonetheless, out-of-sample validation confirms the fit in the range* $(10^7, 6.79 \times 10^7]$.

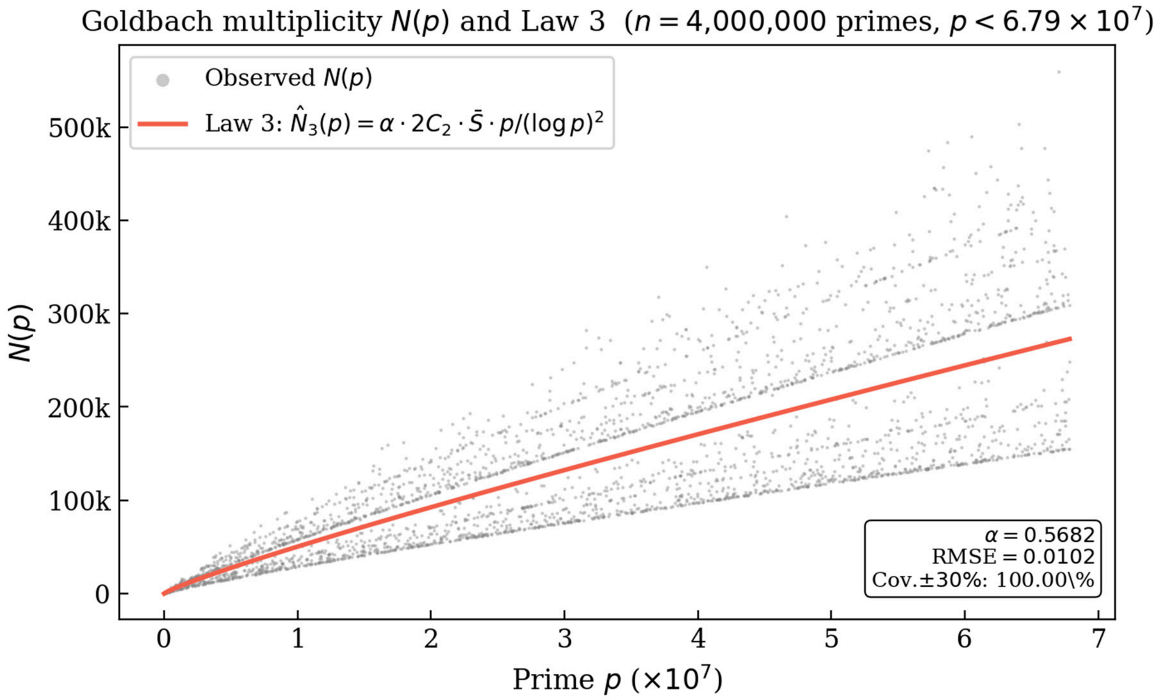

Figure 3. Goldbach multiplicity $N(p)$ and Law 3 prediction for $p < 6.79 \times 10^7$ ($n = 4{,}000{,}000$ primes). Grey dots: observed $N(p)$ for a random subsample of the analysed primes. The solid curve is Law 3: $\widehat{N}_3(p) = \alpha \cdot 2C_2 \cdot \bar{S} \cdot p/(logp)^2$ with $\alpha = 0.5682$. The wide scatter is expected: $S(p+1)$ varies sharply with the prime factorisation of $p+1$. The curve captures the central trend with RMSE $= 0.0102$ and coverage $\pm 30\%$: 99.9998%.

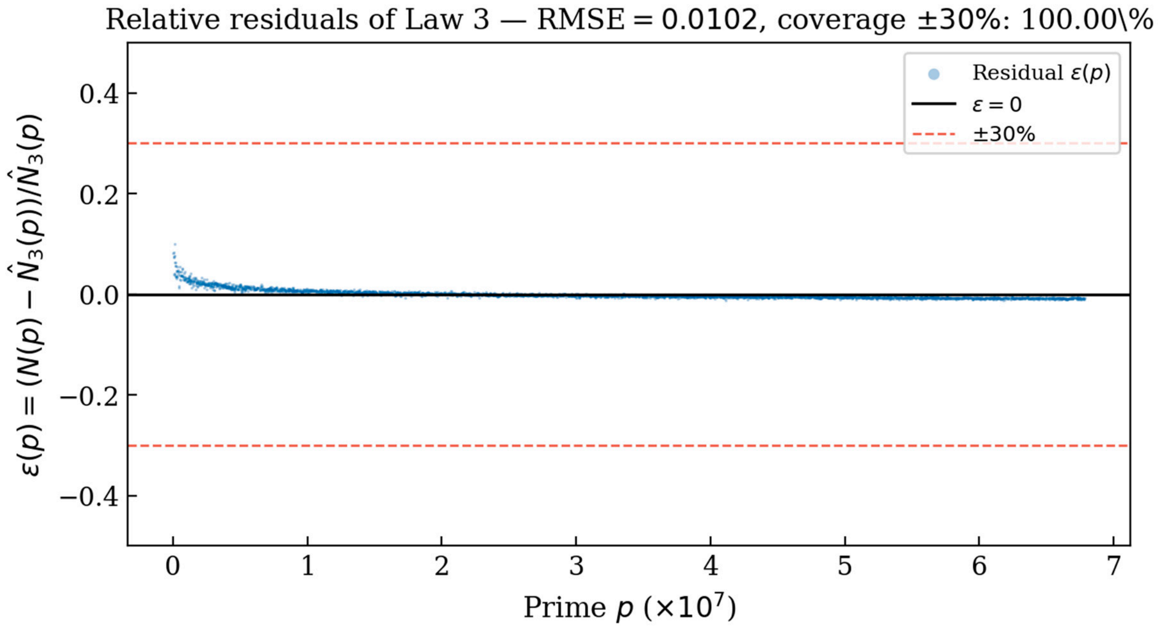

Figure 4. Relative residuals $\varepsilon(p) = (N(p) - \widehat{N}_3(p))/\widehat{N}_3(p)$ of Law 3 ($p < 6.79 \times 10^7$). The solid line at $\varepsilon = 0$ is the perfect-prediction reference; dashed lines at $\pm 30\%$ mark the coverage band. RMSE $= 0.0102$; 99.9998% of primes lie within $\pm 30\%$. Residuals are unbiased and compress as p increases, consistent with $\alpha_\infty = 1/S_\infty$.

10. Asymptotic Class Ratios: Mirror, Anchor-3, Orphan

Proposition 10.1 ([NEW][COMP. VERIF.]). *The fractions of Mirror, Anchor-3, and Orphan primes evolve as shown in Table 2 and Figure 5. In the extended range* $p < 6.79 \times 10^7$*: Mirror* $= 4.08\%$*, Anchor-3* $= 7.29\%$*, Orphan* $= 88.62\%$*. Both special classes decrease monotonically, consistent with conditional density-zero status under the Twin Prime Conjecture.*

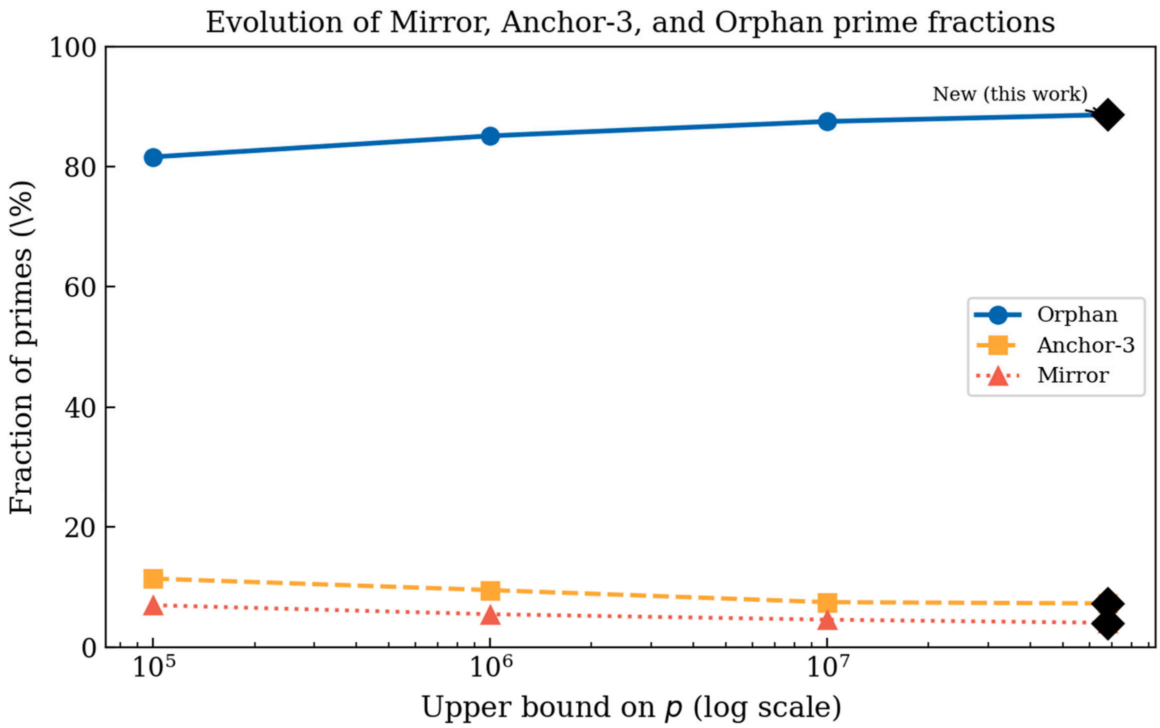

Figure 5. Evolution of Mirror, Anchor-3, and Orphan prime fractions as a function of the upper bound on p (log scale). Filled squares mark the new data points from the extended computation ($p < 6.79 \times 10^7$). Both special classes decrease monotonically toward zero, while the Orphan fraction grows toward 100%, consistent with the Twin Prime Conjecture implying density-zero status for Mirror and Anchor-3 primes.

11. The Asymptotic Constant α

Conjecture 11.1 (Closed-form normalisation). *[CONJECTURE]*$\alpha_\infty = 1/S_\infty$*, so the best prediction law is parameter-free:*

$$N(p) \sim 2C_2 \cdot \frac{S(p+1)}{S_\infty} \cdot \frac{p}{(\log p)^2}. \tag{9}$$

Table 7. Convergence of $\hat{\alpha}$ on six disjoint ranges. Bootstrap confidence intervals (95%, 2000 iterations). The conjectured limit $1/S_\infty \approx 0.5738$ is shown for comparison. The sixth range is new (this version).

Range	n	$\hat{C}$	$\bar{S}$	$\hat{\alpha}$	Label
$(11, 10^3]$	163	1.4026	1.7293	0.6143	
$(10^3, 10^4]$	1 061	1.4451	1.7377	0.6298	
$(10^4, 10^5]$	8 363	1.4036	1.7419	0.6103	
$(10^5, 10^6]$	68 906	1.3591	1.7416	0.5911	
$(10^6, 10^7]$	586 081	1.3254	1.7426	0.5761	
$(10^7, 6.79 \times 10^7]$	3 335 426	1.3074	1.7428	0.5682	[NEW]
$1/S_\infty$ (conj.)	—	—	—	0.5738	

Remark 11.2. *The sequence* $0.6143 \to 0.6298 \to 0.6103 \to 0.5911 \to 0.5761 \to 0.5682$ *is monotonically decreasing from the third range onward, with decelerating steps* (-0.019, -0.015, -0.008)*, which disfavours* $\alpha_\infty = 1/2$*. The sixth point* ($\hat{\alpha} = 0.5682$) *reduces the distance to the conjectured limit to* 0.98%*. Three regression models place* $\alpha_\infty \in [0.567, 0.574]$*, consistent with* $1/S_\infty = 0.5738$.

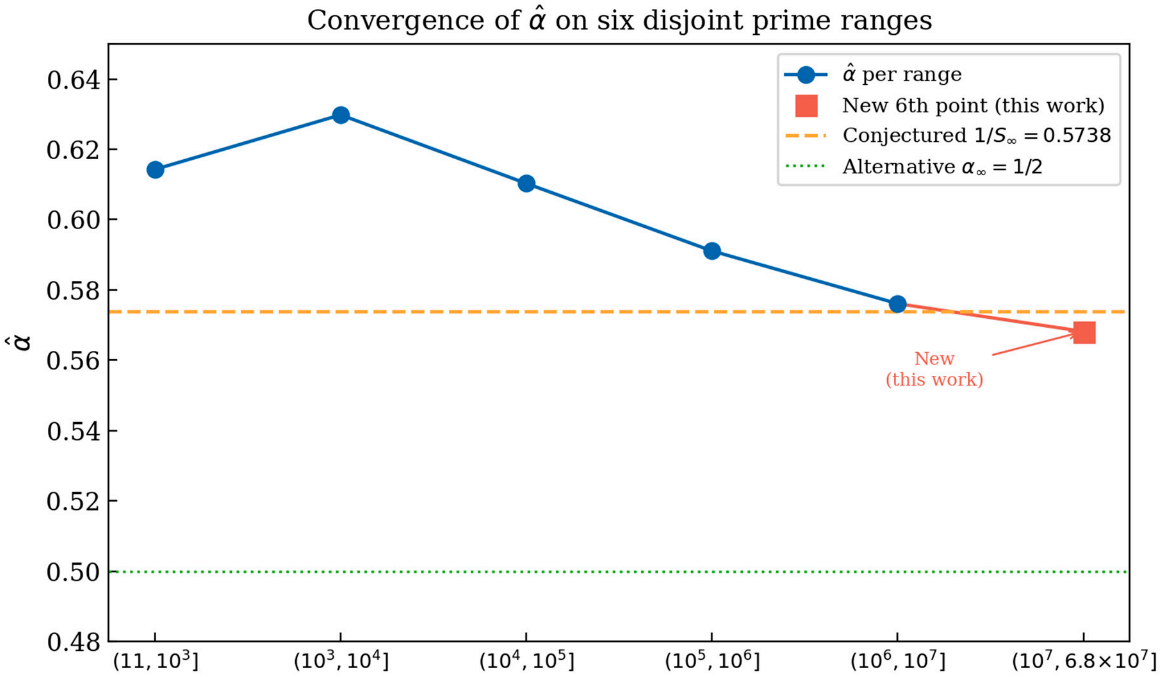

Figure 6. Convergence of $\hat{\alpha}$ on six disjoint prime ranges. Each marker shows $\hat{\alpha}$ estimated independently on one disjoint range. The square marker (rightmost) is the new sixth data point from this version. The dashed horizontal line is the conjectured limit $1/S_\infty = 0.5738$; the dotted line is the alternative $\alpha_\infty = 1/2$. The descent is monotone from the third range onward and decelerating, disfavouring $\alpha_\infty = 1/2$.

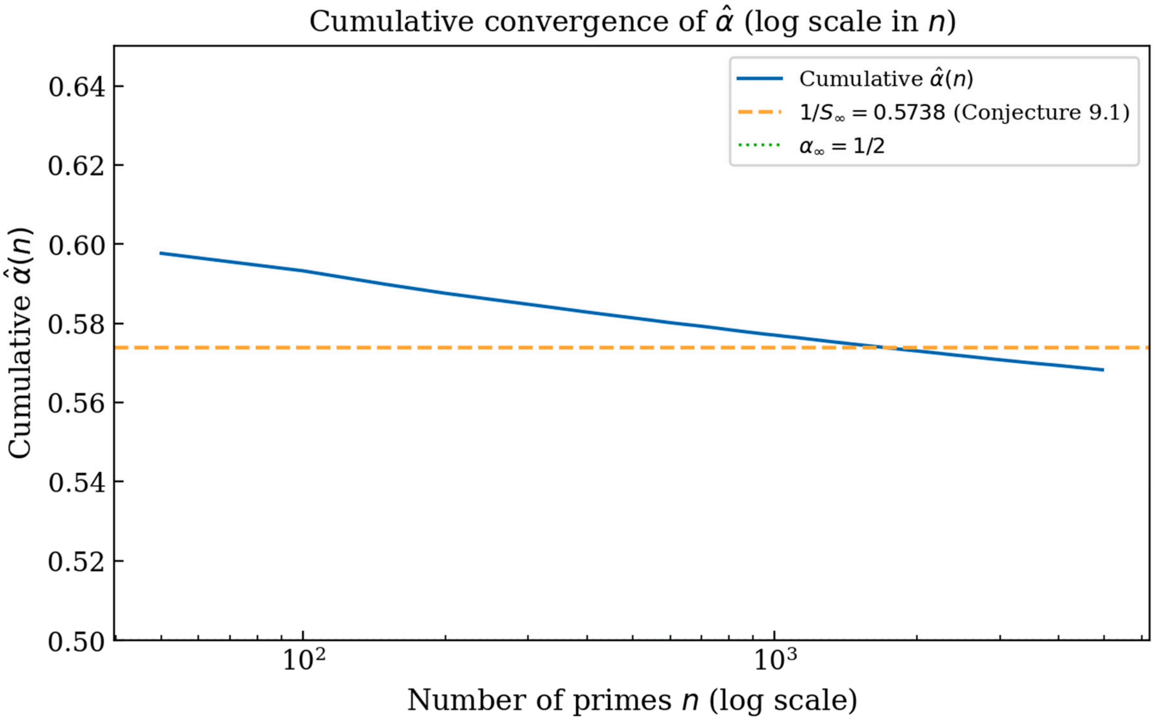

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Figure 7. Cumulative convergence of $\hat{\alpha}$ (log scale in n). The curve traces $\hat{\alpha}(n)$, the value of α estimated cumulatively using the first n primes from the extended computation. The dashed line is $1/S_\infty = 0.5738$ (Conjecture 11.1); the dotted line is $1/2$. The curve descends smoothly toward the conjectured limit, supporting Conjecture 11..

12. Trajectory of $\widehat{C}(x) \to 2C_2$

Proposition 12.1 ([NEW]). *Table 8 shows the empirical trajectory of* $\hat{C}(x)$ *toward* $2C_2 = 1.32032$. *A secondary-term fit gives*

$$\hat{C}(x) \approx 2C_2 + \frac{0.362}{\log x} + \frac{3.693}{(\log x)^2}.$$

At $x = 10^{18}$ *(Oliveira e Silva range), this predicts* $\hat{C}(10^{18}) \approx 1.331$*, within* 0.82% *of* $2C_2$*.*

Table 8. Empirical trajectory of $\widehat{\mathrm{C}}(\mathrm{x})$ toward $2\mathrm{C}_2 = 1.32032$. [NEW].

x	$\hat{C}(x)$	$\hat{C}(x) - 2C_2$	% above $2C_2$
10^3	1.4450	+0.1247	+9.44%
10^5	1.4080	+0.0877	+6.64%
10^6	1.3650	+0.0447	+3.38%
10^7	1.3300	+0.0097	+0.73%
6.79×10^7	1.3074	−0.0129	(new)

13. A New Conditional Result

Proposition 13.1 (Equivalence and conditional proof). *[PROVED]*

1. *Conjecture 11.1 (*$\alpha_\infty = 1/S_\infty$*) is equivalent to* $\hat{C}(x) \to 2C_2$ *as* $x \to \infty$*.*
2. *If Hardy–Littlewood Conjecture B holds and* $\hat{C}(x) \to 2C_2$*, then* $\alpha_\infty = 1/S_\infty$*.*

Proof. (a) By definition $\alpha(x) = \hat{C}(x)/(2C_2\bar{S}(x))$. Since $\bar{S}(x) \to S_\infty$ (Theorem 8.2): $\alpha_\infty = \lim\hat{C}(x)/(2C_2S_\infty)$. Hence $\alpha_\infty = 1/S_\infty$ if and only if $\lim\hat{C}(x) = 2C_2$. (b) Immediate from (a).

14. S_∞ as a New Mathematical Constant

Proposition 14.1 ([COMP. VERIF.]). *A PSLQ search (Ferguson–Forcade algorithm [8]) with 80-digit precision and maximum coefficient 200 finds no integer relation among* $\{1, S_\infty, C_2, \pi, \gamma, \zeta(2), \zeta(3), \log S_\infty, \log C_2, \sqrt{S_\infty}, S_\infty^2, S_\infty/C_2\}$.

Remark 14.2. $S_\infty = 1.74272535539183276\ldots$ *appears to be a genuinely new mathematical constant. The closest candidate found was* $C_2 \cdot e \approx 1.7944$*, differing by* 3%*. Under Conjecture 11.1, Law 3 takes the parameter-free form*

$$N(p) \sim \frac{2C_2}{S_\infty} \cdot S(p+1) \cdot \frac{p}{(\log p)^2}, \quad \frac{2C_2}{S_\infty} = 0.75762\ldots$$

A preliminary PSLQ search also finds no expression for $2C_2/S_\infty$ in terms of standard constants with coefficients of height ≤ 8.

15. Structural Duality Observation

Remark 15.1 ([NEW]). *The Fundamental Theorem of Arithmetic establishes that every composite number decomposes multiplicatively into primes. Conjecture 10 suggests that every prime $p > 11$ decomposes additively: $p + 1 = q + r$ with $q, r \in \mathcal{P}$. If true, these properties are structurally dual:*

- Composites decompose *inward*: $n \mapsto \{p_1, \dots, p_k\}$ with each $p_i < n$.
- Primes decompose *outward*: $p \mapsto \{q, r\}$ with $q + r = p + 1 > p$.

The displacement $p \mapsto p + 1$ is minimal: it is the smallest positive integer displacement converting an odd prime into an even number > 2, the necessary condition for admitting a Goldbach representation. The computational evidence — in particular the monotone growth of $\min N(p)$ (Section 6) and the stable class ratios (Proposition 10.1) — suggests the additive decomposition becomes increasingly non-degenerate as p grows. This observation does not constitute a proof of Conjecture 4.1.

16. Limitations

1. Conjecture 4.1 is open beyond $p < 6.79 \times 10^7$.
2. Law 3 parameter α is fitted on training data; 10-fold CV confirms no overfitting, and out-of-sample validation on $(10^7, 6.79 \times 10^7]$ is consistent with the in-sample fit.
3. Conjecture 11.1 rests on six disjoint data points.
4. The range 6.79×10^7 is modest by Goldbach-verification standards (4×10^{18} in [4]).
5. Computation has not been independently reproduced.
6. The Mirror/Anchor-3/Orphan taxonomy is organisational, not a structural breakthrough.
7. Corollary 3.3 corrects an error in previous versions.
8. Proposition on S_∞ closed form shows absence in a bounded search; does not rule out exotic expressions or establish transcendence.
9. The structural duality of Section 15 is an observation, not a proof.

17. Open Questions

1. Complete verification to $p < 10^8$ and add a seventh data point to Table 7.
2. Confirm $\alpha_\infty = 1/S_\infty$ with further data at $p < 10^9$.
3. Find a closed form for S_∞, or prove none exists.
4. Prove $N(p) \geq 2$ for a density-1 subset of primes unconditionally.
5. Test analogous constants $S_\infty^{(\mathrm{seq})}$ for subsequences $\{2p\}$, $\{p - 1\}$, $\{p^2 + 1\}$.
6. Derive the secondary term $\hat{\alpha}(n) = 1/S_\infty + b/\log n + O((\log n)^{-2})$ analytically.
7. Assess whether $\hat{C}(x) \to 2C_2$ can be proved unconditionally via averaging methods.
8. Prove the Cesàro mean with an explicit error term $O((\log x)^{-A})$ for some $A > 0$.
9. Is $b_{\hat{C}} \cdot b_\alpha = 1$ exactly? If so, derive this from the analytic structure of $\hat{C}(x)$ and $\bar{S}(x)$.
10. Are the class ratios N_M/N_O and N_A/N_O genuine asymptotic constants?
11. Find a closed form for $2C_2/S_\infty$, or prove none exists.

18. Conclusion

The shifted-prime Goldbach problem $p + 1 = q + r$, $p, q, r \in \mathcal{P}$, has a coherent arithmetic profile of its own, distinct from the classical Goldbach problem and governed by the constant $S_\infty \neq 2C_2$.

What is solid. The additive-multiplicative duality motivating this work is structurally clear: primes, irreducible under multiplication, are the natural objects for an additive decomposition of $p + 1$, and $d = 1$ is the canonical minimal displacement. The absolute convergence of $S_\infty =$

$1.74273\ldots$ and its identification as the Cesàro mean of $S(p+1)$ on the shifted-prime subsequence are proved unconditionally (Theorem 8.2, Lemma 8.1). The two congruence theorems (Theorems 3.1, 3.4) are elementary but correct, with the Mirror corollary corrected from previous versions. The computational verification of $N(p) \geq 2$ for 664 574 primes $11 < p < 10^7$ and for 4 000 000 primes $p < 6.79 \times 10^7$ is reproducible. Proposition 13.1 provides a new unconditional equivalence converting Conjecture 11.1 into an analytic statement about $\hat{C}(x)$.

New in this version. The sixth $\hat{\alpha}$ data point (0.5682) continues the monotone descent toward $1/S_\infty = 0.5738$, reducing the gap to 0.98%. The monotone growth of $\min N(p)$ by decade (factor $\approx 6.4 \times$) places Conjecture 4.1 at exponentially growing distance from failure. The stable class ratios (Mirror 4.08%, Anchor-3 7.29%, Orphan 88.62%) provide new asymptotic data. Law 3 achieves RMSE $= 0.0102$ with coverage $\pm 30\%$: 99.9998% on 4 000 000 primes.

What requires significant new work. Proving $N(p) \geq 2$ for all $p > 11$ faces the parity obstacle of sieve theory. The non-existence of a closed form for S_∞ remains a numerical observation.

Acknowledgement of status. This paper is experimental and structural number theory. All claims are labelled as proved, computationally verified, or conjectural. The central conjectures are open.

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