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What Is the Radius of Continuity in the Function Space $F(\mathbb{R}, \mathbb{R})$?

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Abstract

The standard ε–δ definition of continuity is inherently quantitative, yet the precise dependence of the admissible radius δ on the accuracy ε and the base point x_0 is rarely treated as an independent mathematical object. In this paper, we introduce the *radius of continuity* through two variants: the radius of pointwise continuity and the radius of uniform continuity, defined as explicit numerical invariants that capture the maximal symmetric neighborhood on which a real-valued function maintains a prescribed tolerance. We establish the fundamental structural properties of these radii, including their behavior under algebraic operations such as sums, products, and compositions, and demonstrate their inverse relationship to the classical modulus of continuity. Furthermore, we prove that the finiteness pattern of these radii characterizes constant versus non-constant functions. To illustrate the utility of this framework, we derive closed-form expressions for the pointwise radius of quadratic polynomials and the uniform radius of the normal probability density function. These examples highlight how the radius of continuity encodes geometric and probabilistic features, such as local curvature and global scale parameters. Ultimately, this perspective bridges the gap between real analysis and quantitative methods in metric geometry, offering a concrete measure of the stability of a function's continuity.

Keywords: continuity; foundations; elementary functions

MSC: 26A03; 26A06; 26A09; 26A15

> "When you can measure what you are speaking about, and express it in numbers, you know something about it."
> — Lord Kelvin (1824–1907)

1. Introduction

1.1. Real-Valued Functions and Continuity

We work throughout in the function space $F(\mathbb{R}, \mathbb{R})$ of all real-valued functions defined on the real line. Within this space, continuity remains one of the central regularity properties. In standard real analysis, a function $f : \mathbb{R} \to \mathbb{R}$ is said to be continuous at a point $x_0 \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$

$$|x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - f(x_0)| < \varepsilon. \tag{1}$$

This ε–δ definition, and its various equivalent formulations in terms of sequences or inverse images of open sets, provides the standard language for studying local behaviour of real-valued functions [1,2].

A stronger notion, uniform continuity on a set $E \subseteq \mathbb{R}$, requires that for each $\varepsilon > 0$ there is a single $\delta > 0$ that statement (1) works simultaneously for all points $x_0 \in E$. Uniform continuity is closely related to the classical *modulus of continuity* of a function, which quantifies the size of oscillations at a given spatial scale and plays an important role in analysis on metric spaces and in regularity theory [3].

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In the present paper we remain in this classical setting—real-valued functions on $\mathbb{R}$ equipped with the Euclidean metric d_E—but shift the emphasis from the qualitative statement "f is (uniformly) continuous" to a quantitative description of *how far* one can move in the domain while preserving a prescribed ε-accuracy in the range. Our goal is to refine the usual continuity framework by associating to each function a family of radii that encode the local and global "size" of its continuity.

1.2. Motivation

The ε–δ definition of continuity is inherently quantitative: given $\varepsilon > 0$ and a point x_0, any admissible $\delta > 0$ determines a neighbourhood on which f fluctuates by at most ε. However, in most textbooks and classroom proofs one only exhibits *some* convenient $\delta = \delta(\varepsilon, x_0)$; the collection of *all* admissible radii is rarely examined systematically [1,2]. This leaves several natural quantitative questions unanswered. For instance: for a fixed accuracy level ε, what is the *largest* radius around x_0 on which the ε-tolerance still holds? How does this maximal radius depend on x_0 and on the structure of f?

In earlier work, the author proposed a global classification of the function space $F(\mathbb{R}, \mathbb{R})$ into blocks according to the cardinalities of the sets of continuity and differentiability points [4]. That framework was at macroscale and qualitative: it distinguished functions by the *size* and *location* of their continuity sets but did not deal with it in microscale nor quantify the *strength* of continuity at individual points or uniformly on $\mathbb{R}$. In the follow-up work, the author surveys the $\epsilon - \delta$ definition for the classes of continuous functions at microscale level [5]. The present paper aims to complement that perspective by introducing and studying the *radius of continuity* at a point and its uniform counterpart. Seen in this light, the paper does not replace the classical ε–δ definition of continuity, but pushes it one step further: what Weierstrass's 1861 formulation makes existential, we make quantitative [6]. By assigning to each tolerance level an explicit pointwise or uniform radius, the present framework turns the hidden numerical geometry of continuity into an object of direct study.

More precisely, for a given function f, point x_0, and accuracy level $\varepsilon > 0$, we consider the set of all radii $\delta > 0$ for which the ε–δ condition holds, and define the *pointwise radius of continuity* $\delta_f(\varepsilon, x_0)$ as the supremum of this set. Analogously, we define the *uniform radius of continuity* $\delta_f^U(\varepsilon)$ by taking the largest radius that works simultaneously for all $x_0 \in \mathbb{R}$. These radii can be expressed in terms of pointwise and uniform oscillation functions, thereby connecting our viewpoint to classical moduli of continuity [3,7–9].

This radius-based approach has several advantages. It packages the infinitary quantifiers in the usual continuity definition into concrete numerical invariants; it allows for explicit closed-form computations in important families of functions (for example, quadratic polynomials or normal density functions); and it offers a way to compare the "stability" of different functions under perturbations of the input. At a more conceptual level, radii of continuity provide a bridge between textbook ε–δ arguments and quantitative notions that are commonly used in modern analysis and numerical applications, where explicit control over neighbourhood sizes is often required.

1.3. Organization of the Paper

The remainder of the paper is organized as follows. In Section 2 we introduce the pointwise and uniform radii of continuity, develop equivalent descriptions in terms of local and uniform oscillation, and establish basic structural properties. In particular, we characterize non-constant and constant functions in $F(\mathbb{R}, \mathbb{R})$ through the finiteness or infiniteness of their associated radii. We conclude the section with investigation of the relationship between two radii of continuity. Section 3 is devoted to explicit computations: we derive a closed-form formula for the pointwise radius of continuity of a quadratic polynomial and analyze the uniform radius of continuity for the normal probability density function, illustrating how the abstract theory translates into concrete expressions. Finally, Section 4 concludes the paper with a discussion on the main findings and outlines several directions for future work.

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2. The Radius of Continuity: Definitions & General Properties

2.1. Radius of Pointwise Continuity

In this section, we introduce, for each function f, point $x_0 \in \mathbb{R}$, and accuracy level $\varepsilon > 0$, the set $\Delta_f(\varepsilon, x_0)$ of all radii $\delta > 0$ for which the usual ε–δ condition holds at x_0, and we define the pointwise radius of continuity $\delta_f(\varepsilon, x_0)$ as the supremum of this set (Figure 1). This quantity admits an equivalent description in terms of the local oscillation of f [7,8] on the symmetric interval $(x_0 - \delta, x_0 + \delta)$, or, after a shift of variables, in terms of an h-dependent oscillation around the origin. We then collect a number of elementary but useful properties of δ_f, including an evenness property in the spatial variable, natural scaling relations, and inequalities that arise when f is globally Lipschitz. Further bounds describe how radii behave under algebraic operations: the radius for a sum or product of functions can be estimated in terms of the radii of the individual factors, and similar control holds for compositions under mild regularity conditions. These results show that the radius of continuity is compatible with the standard algebraic operations in $F(\mathbb{R}, \mathbb{R})$, and can therefore be propagated through many common constructions. Finally, we prove that the finiteness pattern of $\delta_f(\varepsilon, x_0)$ characterizes whether f is constant or not: a non-constant function necessarily has finite radius at some continuity point, whereas a function with infinite radius everywhere must be constant.

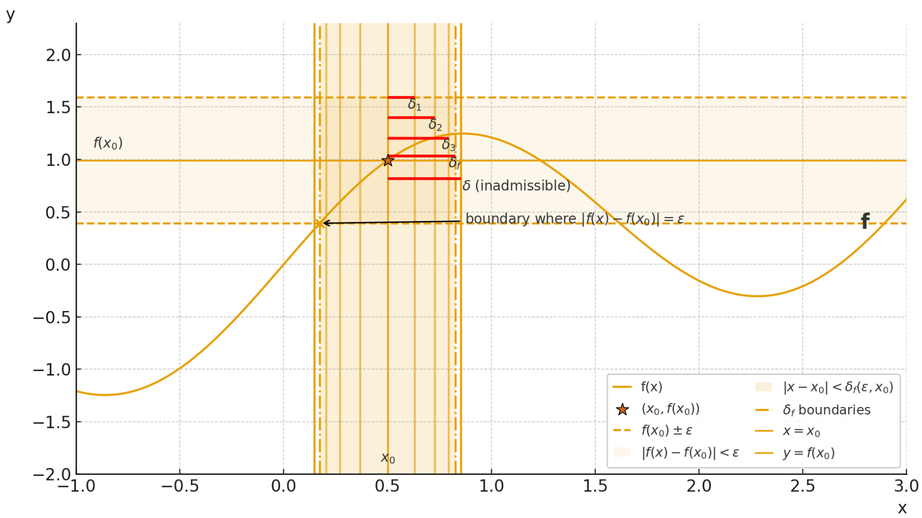

Figure 1. Geometric illustration of the pointwise radius of continuity $\delta_f(\varepsilon, x_0)$ for a continuous function f at a fixed point x_0: the horizontal band $f(x_0) \pm \varepsilon$ in the range is matched with symmetric intervals $|x - x_0| < \delta$ in the domain. Admissible radii correspond to intervals on which $|f(x) - f(x_0)| < \varepsilon$, while the maximal radius $\delta_f(\varepsilon, x_0)$ is attained at the first intersection of the graph of f with the ε-boundary.

Definition 1 (Radius of Pointwise Continuity). *Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function with a nonempty set of continuity points $C(f) \neq \varnothing$. For $x_0 \in \mathbb{R}$ and $\varepsilon > 0$ define the pointwise admissible continuity admissible radius as:*

$$\Delta_f(\varepsilon, x_0) := \{\delta > 0 : \forall x\, (|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon)\}, \quad (2)$$

$$\delta_f(\varepsilon, x_0) := \sup \Delta_f(\varepsilon, x_0). \quad (3)$$

Geometric Interpretation (Pointwise). Geometrically, the pointwise radius of continuity $\delta_f(\varepsilon, x_0)$ measures the largest symmetric horizontal interval centered at x_0 on which the graph of f remains trapped inside the vertical tolerance band $f(x_0) \pm \varepsilon$. In this sense, it quantifies how far one may move away from a fixed base point before the graph first touches or crosses the prescribed ε-boundary.

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Remark 0 (Shifted-h formulation of Δ_f and δ_f)**.** *Equivalently, in terms of the local oscillation*

$$\omega_f(x_0;\delta) := \sup_{|h|<\delta} |f(x_0+h) - f(x_0)|, \tag{4}$$

we have:

$$\Delta_f(\varepsilon, x_0) \;:=\; \{\delta > 0 : \; \omega_f(x_0;\delta) < \varepsilon\}, \tag{5}$$
$$\delta_f(\varepsilon, x_0) \;:=\; \sup\{\delta > 0 : \; \omega_f(x_0;\delta) < \varepsilon\}. \tag{6}$$

Proposition 1 (Properties of Radius of Pointwise Continuity)**.** *Let $f : \mathbb{R} \to \mathbb{R}$. Then, for $\varepsilon > 0$ and $x_0 \in \mathbb{R}$ the radious of pointwise continuity $\delta_f(\varepsilon, x_0)$ has the following properties:*

(A) Single–function properties.

- *(i)* ***Evenness.*** *If f is even, then $\delta_f(\varepsilon, x_0) = \delta_f(\varepsilon, -x_0)$.*
- *(ii)* ***Scaling.*** *If $c \neq 0$, then $\delta_{cf}(\varepsilon, x_0) = \delta_f(\varepsilon/|c|, x_0)$; if $c = 0$, then $\delta_{0\cdot f}(\varepsilon, x_0) = +\infty$.*
- *(iii)* ***Large–ε for bounded f.*** *If $|f| \leq M$ everywhere and $\varepsilon \geq 2M$, then $\delta_f(\varepsilon, x_0) = +\infty$ for all x_0 (the constant $2M$ is sharp in general).*
- *(iv)* ***Monotonicity in ε.*** *If $0 < \varepsilon_1 \leq \varepsilon_2$, then $\delta_f(\varepsilon_1, x_0) \leq \delta_f(\varepsilon_2, x_0)$.*
- *(v)* ***Input translation.*** *For $g(x) = f(x+a)$ one has $\delta_g(\varepsilon, x_0) = \delta_f(\varepsilon, x_0 + a)$.*
- *(vi)* ***Lipschitz lower bound.*** *If f is L-Lipschitz on $|x - x_0| < r$ [10], then $\delta_f(\varepsilon, x_0) \geq \min\{r, \varepsilon/L\}$; in particular, if f is globally L-Lipschitz, $\delta_f(\varepsilon, x_0) \geq \varepsilon/L$.*
- *(vii)* ***Continuity criterion.*** *f is continuous at x_0 iff $\delta_f(\varepsilon, x_0) > 0$ for every $\varepsilon > 0$.*

(B) Two–function properties.

- *(viii)* ***Composition.*** *If g is continuous at $f(x_0)$ with modulus η (i.e., $|u - v| < \eta(\varepsilon) \Rightarrow |g(u) - g(v)| < \varepsilon$), then*

$$\delta_{g\circ f}(\varepsilon, x_0) \;\geq\; \delta_f(\eta(\varepsilon), x_0).$$

- *(ix)* ***Sum (lower bound).*** *For f_1, f_2 and any $\alpha \in [0,1]$,*

$$\delta_{f_1+f_2}(\varepsilon, x_0) \;\geq\; \min\{\delta_{f_1}(\alpha\varepsilon, x_0),\; \delta_{f_2}((1-\alpha)\varepsilon, x_0)\}.$$

(There is no universal matching upper bound; e.g., $f_1 = g$, $f_2 = -g$.)

- *(x)* ***Linear combination (lower bound).*** *For $c_1, c_2 \in \mathbb{R}$ and any $\alpha \in [0,1]$,*

$$\delta_{c_1 f_1 + c_2 f_2}(\varepsilon, x_0) \;\geq\; \min\Big\{\delta_{f_1}\Big(\frac{\alpha\varepsilon}{|c_1|}, x_0\Big),\; \delta_{f_2}\Big(\frac{(1-\alpha)\varepsilon}{|c_2|}, x_0\Big)\Big\},$$

with the convention from (ii) when a coefficient is 0.

- *(xi)* ***Product (one convenient bound).*** *If $|f_1| \leq A$ on $|x - x_0| < r$, then for any $\varepsilon > 0$,*

$$\delta_{f_1 f_2}(\varepsilon, x_0) \;\geq\; \min\Big\{r,\; \delta_{f_2}\Big(\frac{\varepsilon}{2A}, x_0\Big),\; \delta_{f_1}\Big(\frac{\varepsilon}{2(|f_2(x_0)|+1)}, x_0\Big)\Big\}.$$

Proof. (i) $\omega_f(x_0;\delta) = \omega_f(-x_0;\delta)$ for even f (substitute $h \mapsto -h$). (ii) $\omega_{cf} = |c|\,\omega_f$; for $c = 0$, $\omega \equiv 0$. (iii) $|f(x) - f(x_0)| \leq |f(x)| + |f(x_0)| \leq 2M$. (iv) $\{\omega_f \leq \varepsilon_1\} \subseteq \{\omega_f \leq \varepsilon_2\}$. (v) $|f(x+a) - f(x_0+a)| = |g(x) - g(x_0)|$. (vi) On $|x - x_0| < r$, $|f(x) - f(x_0)| \leq L|x - x_0|$; take $\delta \leq \min\{r, \varepsilon/L\}$. (vii) Equivalent to the ε–δ definition. (viii) If $|f(x) - f(x_0)| < \eta(\varepsilon)$ then $|g(f(x)) - g(f(x_0))| < \varepsilon$; hence $\Delta_f(\eta(\varepsilon), x_0) \subseteq \Delta_{g\circ f}(\varepsilon, x_0)$. (ix) By triangle inequality and a split $\alpha\varepsilon, (1-\alpha)\varepsilon$; the counterexample $f_1 = g$, $f_2 = -g$ shows no universal upper bound. (x) Combine (ii) with (ix). (xi) Use $|f_1 f_2 - f_1(x_0) f_2(x_0)| \leq |f_1| \cdot |f_2 - f_2(x_0)| + |f_2(x_0)| \cdot |f_1 - f_1(x_0)|$, bound $|f_1| \leq A$ on $|x - x_0| < r$, and choose the two tolerances to sum to ε. □

Theorem 1 (Characterization of Radius of Pointwise Continuity). *Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function with a nonempty set of continuity points $C(f) \neq \varnothing$. For $x_0 \in \mathbb{R}$ and $\varepsilon > 0$, the following are equivalent:*

(i) *f is non-constant.*

(ii) *There is $x_0 \in C(f)$ there exists $\varepsilon > 0$ such that $\delta_f(\varepsilon, x_0) < \infty$.*

Proof. *(ii) ⇒ (i).* Suppose (ii) holds and, towards a contradiction, assume f is constant: $f(x) \equiv c_0$. Fix any $x_0 \in C(f)$ (indeed, $C(f) = \mathbb{R}$) and any $\varepsilon > 0$. Then $|f(x) - f(x_0)| = 0 < \varepsilon$ for all x, so $\Delta_f(\varepsilon, x_0) = (0, \infty)$ and thus $\delta_f(\varepsilon, x_0) = \infty$, contradicting (ii).

(i) ⇒ (ii). Assume f is non-constant. Fix $x_0 \in C(f) \neq \varnothing$. Then there are $a, b \in \mathbb{R}$ with $a \neq b$ and $f(a) \neq f(b)$. Set

$$d_a := |f(a) - f(x_0)|, \qquad d_b := |f(b) - f(x_0)|, \qquad \varepsilon_b := \tfrac{1}{2}(d_a + d_b).$$

Since $\max\{d_a, d_b\} \geq \varepsilon_b$, at least one of a or b (call it p) satisfies $|f(p) - f(x_0)| \geq \varepsilon_b$. Consequently, any $\delta > \max\{|a - x_0|, |b - x_0|\} \geq |p - x_0|$ forces the open interval $(x_0 - \delta, x_0 + \delta)$ to contain p, and hence to contain a point where $|f(x) - f(x_0)| \geq \varepsilon_b$. By the definition of δ_f we obtain the finite bound

$$\delta_f(\varepsilon_b, x_0) \;\leq\; \max\{|a - x_0|, |b - x_0|\} \;<\; \infty.$$

This supplies, for the given $x_0 \in C(f)$, an explicit $\varepsilon := \varepsilon_b > 0$ with $\delta_f(\varepsilon, x_0) < \infty$. Since $x_0 \in C(f)$ was arbitrary, (ii) follows. □

Corollary 1. *With $\delta_f(\varepsilon, x_0)$ defined as above, the following are equivalent:*

(i) *f is constant.*

(ii) *For every $x_0 \in C(f)$ and every $\varepsilon > 0$, we have $\delta_f(\varepsilon, x_0) = \infty$.*

2.2. Radius of Uniform Continuity

In this section we pass from the local picture at a single point to a global one by introducing, for each function f and accuracy level $\varepsilon > 0$, the set $\Delta_f^U(\varepsilon)$ of all radii $\delta > 0$ that make the ε–δ condition hold simultaneously for every $x_0 \in \mathbb{R}$. The *uniform radius of continuity* $\delta_f^U(\varepsilon)$ is then defined as the supremum of this set and measures, in a single number, how far one can move anywhere in the domain without exceeding the prescribed error ε (Figure 2). We show that δ_f^U admits a clean formulation in terms of the *uniform oscillation* of f [7,9], obtained by taking the supremum of $|f(x) - f(y)|$ over all pairs of points at distance less than δ. This viewpoint allows us to derive a collection of general properties of δ_f^U and to compare it directly with the pointwise radius $\delta_f(\varepsilon, x_0)$, for instance by bounding the former in terms of the infimum of the latter over all base points. In particular, whenever f is uniformly continuous, the uniform radius is strictly positive for each $\varepsilon > 0$, while for merely pointwise continuous functions $\delta_f^U(\varepsilon)$ may degenerate to zero even though each $\delta_f(\varepsilon, x_0)$ remains positive. Finally, we obtain a characterization analogous to the pointwise case: the pattern of finiteness or infiniteness of $\delta_f^U(\varepsilon)$ completely determines whether f is constant or non-constant, so that the uniform radius encodes global regularity information in a quantitative form.

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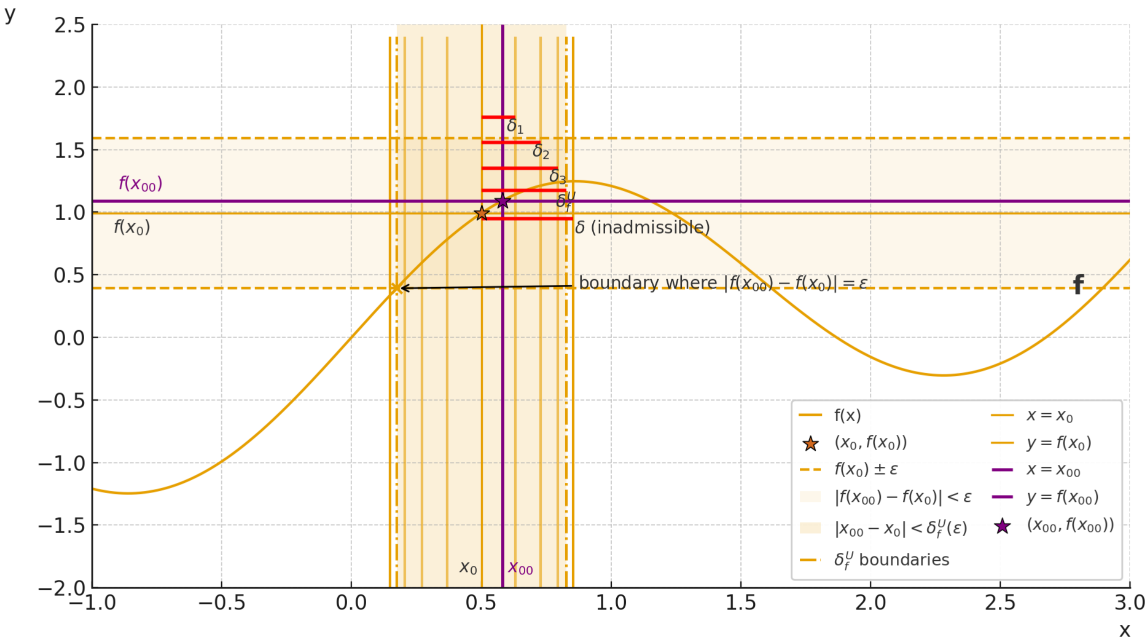

Figure 2. Geometric illustration of the uniform radius of continuity $\delta_f^U(\varepsilon)$: a single radius controls all pairs of points x_0, x_{00} with $|x_{00} - x_0| < \delta_f^U(\varepsilon)$, forcing their function values to lie in the same horizontal ε-band around $f(x_0)$. The highlighted pair $(x_0, f(x_0))$ and $(x_{00}, f(x_{00}))$ shows that, whenever $|x_{00} - x_0| < \delta_f^U(\varepsilon)$, the vertical lines through x_0 and x_{00} remain inside this band, illustrating that $\delta_f^U(\varepsilon)$ is chosen independently of the base point.

Definition 2 (Radius of Uniform Continuity)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function with a nonempty set of continuity points $C(f) \neq \emptyset$. For $\varepsilon > 0$ define the uniform admissible radius as:*

$$\Delta_f^U(\varepsilon) := \Big\{\delta > 0 : \ \forall x \in \mathbb{R} \forall y \in \mathbb{R} \ \big(|x - y| < \delta \ \Rightarrow \ |f(x) - f(y)| < \varepsilon\big)\Big\}, \tag{7}$$

$$\delta_f^U(\varepsilon) := \sup \Delta_f^U(\varepsilon). \tag{8}$$

Geometric Interpretation (Uniform). Geometrically, the uniform radius of continuity $\delta_f^U(\varepsilon)$ is the largest single horizontal scale that works simultaneously throughout the entire graph, regardless of the choice of base point. Thus it represents a global bottleneck radius: a universal distance below which every pair of points on the graph produces function values that still lie within the same vertical tolerance level ε.

Remark 1 (Shifted-h formulation of Δ_f^U and δ_f^U)**.** *For the uniform case, introduce the* uniform local oscillation

$$\omega_f^U(\delta) := \sup_{x \in \mathbb{R}} \sup_{|h| < \delta} |f(x + h) - f(x)|. \tag{9}$$

Then:

$$\Delta_f^U(\varepsilon) := \{\delta > 0 : \ \omega_f^U(\delta) < \varepsilon\}, \tag{10}$$

$$\delta_f^U(\varepsilon) := \sup\{\delta > 0 : \ \omega_f^U(\delta) < \varepsilon\}. \tag{11}$$

Proposition 2 (Properties of Radius of Uniform Continuity)**.** *Let $f : \mathbb{R} \to \mathbb{R}$. Then, for $\varepsilon > 0$ the radius of uniform continuity $\delta_f^U(\varepsilon)$ has the following properties:*
(A) Single–function properties.

(i) ***Scaling.*** *If $c \neq 0$, then $\delta_{cf}^U(\varepsilon) = \delta_f^U(\varepsilon/|c|)$; if $c = 0$, then $\delta_{0\cdot f}^U(\varepsilon) = +\infty$.*

(ii) ***Monotonicity in ε.*** *If $0 < \varepsilon_1 \leq \varepsilon_2$, then $\delta_f^U(\varepsilon_1) \leq \delta_f^U(\varepsilon_2)$.*

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(iii) ***Translation/reflection invariance.*** *For $g(x) = f(x+a)$ or $g(x) = f(-x)$, we have $\delta_g^U(\varepsilon) = \delta_f^U(\varepsilon)$.*

(iv) ***Large–ε for bounded f.*** *If $|f| \le M$ everywhere and $\varepsilon \ge 2M$, then $\delta_f^U(\varepsilon) = +\infty$ (sharp in general).*

(v) ***Lipschitz lower bound.*** *If f is globally L-Lipschitz, then $\delta_f^U(\varepsilon) \ge \varepsilon/L$.*

(vi) ***Uniform continuity criterion.*** *f is uniformly continuous on $\mathbb{R}$ iff $\delta_f^U(\varepsilon) > 0$ for every $\varepsilon > 0$.*

(vii) ***Link to pointwise radii.*** $\delta_f^U(\varepsilon) \le \inf_{x_0 \in \mathbb{R}} \delta_f(\varepsilon, x_0)$.

(B) Two–function properties.

(viii) ***Composition.*** *If g admits a global modulus η with $|u - v| < \eta(\varepsilon) \Rightarrow |g(u) - g(v)| < \varepsilon$, then*

$$\delta_{g \circ f}^U(\varepsilon) \;\ge\; \delta_f^U(\eta(\varepsilon)).$$

(ix) ***Sum (lower bound).*** *For any $\alpha \in [0,1]$,*

$$\delta_{f_1+f_2}^U(\varepsilon) \;\ge\; \min\left\{\delta_{f_1}^U(\alpha\varepsilon),\; \delta_{f_2}^U((1-\alpha)\varepsilon)\right\}.$$

(There is no universal matching upper bound; e.g., $f_1 = g$, $f_2 = -g$.)

(x) ***Linear combination (lower bound).*** *For $c_1, c_2 \in \mathbb{R}$ and any $\alpha \in [0,1]$,*

$$\delta_{c_1 f_1 + c_2 f_2}^U(\varepsilon) \;\ge\; \min\Big\{\delta_{f_1}^U\Big(\frac{\alpha\varepsilon}{|c_1|}\Big),\; \delta_{f_2}^U\Big(\frac{(1-\alpha)\varepsilon}{|c_2|}\Big)\Big\},$$

with the convention in (i) if a coefficient is 0.

(xi) ***Product (global bounds).*** *If $|f_1| \le A$ and $|f_2| \le B$ on $\mathbb{R}$, then for any $\varepsilon > 0$,*

$$\delta_{f_1 f_2}^U(\varepsilon) \;\ge\; \min\Big\{\;\delta_{f_2}^U\Big(\frac{\varepsilon}{2A}\Big),\; \delta_{f_1}^U\Big(\frac{\varepsilon}{2B}\Big)\Big\}.$$

Proof. *(i)* $\omega_{cf}^U(\delta) = |c|\,\omega_f^U(\delta)$. If $c = 0$, $\omega^U \equiv 0$. *(ii)* Inclusion of level sets $\{\omega_f^U \le \varepsilon_1\} \subseteq \{\omega_f^U \le \varepsilon_2\}$. *(iii)* Both translations and $x \mapsto -x$ leave ω^U unchanged: $|f(x+a+h) - f(x+a)|$ and $|f(-x-h) - f(-x)|$. *(iv)* For all x, y, $|f(x) - f(y)| \le |f(x)| + |f(y)| \le 2M$. *(v)* Global Lipschitz gives $|f(x) - f(y)| \le L|x-y|$; take $\delta \le \varepsilon/L$. *(vi)* This is exactly the ε–δ definition of uniform continuity. *(vii)* If $\delta \in \Delta_f^U(\varepsilon)$ then, fixing any x_0, $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$, hence $\delta \in \Delta_f(\varepsilon, x_0)$; take sups and then the infimum over x_0. *(viii)* If $|f(x) - f(y)| < \eta(\varepsilon)$ then $|g(f(x)) - g(f(y))| < \varepsilon$; thus $\Delta_f^U(\eta(\varepsilon)) \subseteq \Delta_{g \circ f}^U(\varepsilon)$. *(ix)* Triangle inequality with split $\alpha\varepsilon, (1-\alpha)\varepsilon$. *(x)* Apply (i) to $c_1 f_1$ and $c_2 f_2$ and then (ix). *(xi)* Using $|f_1 f_2(x) - f_1 f_2(y)| \le |f_1(x)|\,|f_2(x) - f_2(y)| + |f_2(y)|\,|f_1(x) - f_1(y)| \le A\Delta_2 + B\Delta_1$, pick $\Delta_2 < \varepsilon/(2A)$ and $\Delta_1 < \varepsilon/(2B)$ and invoke the uniform radii of f_2 and f_1, respectively. □

Theorem 2 (Characterization of Radius of Uniform Continuity). *Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function with a nonempty set of continuity points $C(f) \ne \varnothing$. For $\varepsilon > 0$ the following are equivalent:*

(i) *f is non-constant* $\iff$

(ii) *There exists $\varepsilon > 0$, $\delta_f^U(\varepsilon) < \infty$.*

Proof. If f is constant, then $|f(x) - f(y)| = 0 < \varepsilon$ for all x, y, so $\Delta_f^U(\varepsilon) = (0, \infty)$ and $\delta_f^U(\varepsilon) = \infty$.

Conversely, suppose f is non-constant, pick $a \ne b$ with $f(a) \ne f(b)$ and set $\varepsilon_0 := \frac{1}{2}|f(a) - f(b)| > 0$. Then for any $\delta > |a - b|$, the pair $(x, y) = (a, b)$ satisfies $|x - y| < \delta$ but $|f(x) - f(y)| \ge \varepsilon_0$, so $\delta \notin \Delta_f^U(\varepsilon_0)$. Hence

$$\delta_f^U(\varepsilon_0) \;\le\; |a - b| \;<\; \infty.$$

This mirrors the pointwise result with $\delta_f(\varepsilon, x_0)$ and shows the same finite-radius phenomenon in the uniform setting. □

Corollary 2. *With $\delta_f^U(\varepsilon)$ defined as above, the following are equivalent:*

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(i) *f is constant.*
(ii) *For every $\varepsilon > 0$, we have $\delta_f(\varepsilon) = \infty$.*

2.3. Connection Between the Pointwise and Uniform Radii of Continuity

In the previous two subsections we introduced the pointwise radius $\delta_f(\varepsilon, x_0)$ and the uniform radius $\delta_f^U(\varepsilon)$ as local and global quantitative versions of continuity. A natural question is how these two invariants are related. Since uniform continuity requires a single admissible radius to work simultaneously at every base point, one expects the uniform radius to be governed by the least favorable local radius. The next results make this intuition precise, first at the level of admissible-radius sets and then at the level of the corresponding suprema.

Lemma 1 (Downward Closedness of Admissible Radius Sets)**.** *Let $f : \mathbb{R} \to \mathbb{R}$, let $\varepsilon > 0$, and let $x_0 \in \mathbb{R}$. Then:*

$$\delta \in \Delta_f(\varepsilon, x_0),\ \ 0 < \eta \le \delta \quad \Longrightarrow \quad \eta \in \Delta_f(\varepsilon, x_0), \tag{12}$$

and

$$\delta \in \Delta_f^U(\varepsilon),\ \ 0 < \eta \le \delta \quad \Longrightarrow \quad \eta \in \Delta_f^U(\varepsilon). \tag{13}$$

In particular, both admissible-radius sets are downward closed.

Proof. Suppose $\delta \in \Delta_f(\varepsilon, x_0)$ and $0 < \eta \le \delta$. If $|x - x_0| < \eta$, then automatically $|x - x_0| < \delta$. Since δ is admissible, it follows that

$$|f(x) - f(x_0)| < \varepsilon.$$

Hence $\eta \in \Delta_f(\varepsilon, x_0)$, proving (12).

The proof of (13) is identical. Suppose $\delta \in \Delta_f^U(\varepsilon)$ and $0 < \eta \le \delta$. If $|x - y| < \eta$, then $|x - y| < \delta$, and therefore

$$|f(x) - f(y)| < \varepsilon.$$

Thus $\eta \in \Delta_f^U(\varepsilon)$. □

Lemma 2 (Uniform Admissible Radius as an Intersection)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$, and let $\varepsilon > 0$. Then*

$$\Delta_f^U(\varepsilon) \quad = \quad \bigcap_{x \in \mathbb{R}} \Delta_f(\varepsilon, x). \tag{14}$$

Proof. First, let $\delta \in \Delta_f^U(\varepsilon)$. Then by definition, for all $x, y \in \mathbb{R}$,

$$|x - y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon.$$

Fix any $x_0 \in \mathbb{R}$. Setting $y = x_0$, we obtain

$$|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \varepsilon \qquad (\forall x \in \mathbb{R}),$$

which shows that $\delta \in \Delta_f(\varepsilon, x_0)$. Since x_0 was arbitrary,

$$\delta \in \bigcap_{x \in \mathbb{R}} \Delta_f(\varepsilon, x).$$

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Therefore,

$$\Delta_f^U(\varepsilon) \quad \subseteq \quad \bigcap_{x\in\mathbb{R}} \Delta_f(\varepsilon, x). \tag{15}$$

Conversely, let

$$\delta \in \bigcap_{x\in\mathbb{R}} \Delta_f(\varepsilon, x).$$

Then for every $x_0 \in \mathbb{R}$,

$$|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \varepsilon \qquad (\forall x \in \mathbb{R}).$$

Now take arbitrary $x, y \in \mathbb{R}$ with $|x - y| < \delta$. Since $\delta \in \Delta_f(\varepsilon, y)$, choosing $x_0 = y$ gives

$$|x - y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon.$$

Hence $\delta \in \Delta_f^U(\varepsilon)$. Therefore,

$$\bigcap_{x\in\mathbb{R}} \Delta_f(\varepsilon, x) \quad \subseteq \quad \Delta_f^U(\varepsilon). \tag{16}$$

Combining (15) and (16), we obtain (14). □

Proposition 3 (Uniform Radius as the Infimum of the Pointwise Radii)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$, and let $\varepsilon > 0$. Then*

$$\delta_f^U(\varepsilon) \quad = \quad \inf_{x\in\mathbb{R}} \delta_f(\varepsilon, x). \tag{17}$$

In particular, the radius of uniform continuity is the global bottleneck among the pointwise continuity radii.

Proof. By Lemma 2,

$$\Delta_f^U(\varepsilon) \quad = \quad \bigcap_{x\in\mathbb{R}} \Delta_f(\varepsilon, x). \tag{18}$$

Hence, for every $x \in \mathbb{R}$,

$$\Delta_f^U(\varepsilon) \quad \subseteq \quad \Delta_f(\varepsilon, x). \tag{19}$$

Taking suprema in (19), we obtain

$$\delta_f^U(\varepsilon) \quad \leq \quad \delta_f(\varepsilon, x) \qquad (\forall x \in \mathbb{R}), \tag{20}$$

and therefore

$$\delta_f^U(\varepsilon) \quad \leq \quad \inf_{x\in\mathbb{R}} \delta_f(\varepsilon, x). \tag{21}$$

For the reverse inequality, set

$$\alpha \quad := \quad \inf_{x\in\mathbb{R}} \delta_f(\varepsilon, x). \tag{22}$$

Let $0 < \eta < \alpha$. Then for every $x \in \mathbb{R}$,

$$\eta \quad < \quad \delta_f(\varepsilon, x) = \sup \Delta_f(\varepsilon, x). \tag{23}$$

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Since f is continuous on $\mathbb{R}$, each set $\Delta_f(\varepsilon, x)$ is nonempty. Hence for every $x \in \mathbb{R}$ there exists $\delta_x \in \Delta_f(\varepsilon, x)$ such that

$$\eta < \delta_x.$$

By Lemma 1, the set $\Delta_f(\varepsilon, x)$ is downward closed, so from $\delta_x \in \Delta_f(\varepsilon, x)$ and $0 < \eta < \delta_x$ we conclude that

$$\eta \in \Delta_f(\varepsilon, x) \qquad (\forall x \in \mathbb{R}).$$

Therefore,

$$\eta \quad \in \quad \bigcap_{x \in \mathbb{R}} \Delta_f(\varepsilon, x) = \Delta_f^U(\varepsilon) \tag{24}$$

by Lemma 2. Since this holds for every $0 < \eta < \alpha$, we obtain

$$\delta_f^U(\varepsilon) = \sup \Delta_f^U(\varepsilon) \quad \geq \quad \alpha = \inf_{x \in \mathbb{R}} \delta_f(\varepsilon, x). \tag{25}$$

Combining (21) and (25), we conclude that

$$\delta_f^U(\varepsilon) \quad = \quad \inf_{x \in \mathbb{R}} \delta_f(\varepsilon, x). \tag{26}$$

This completes the proof. □

Corollary 3 (Applications)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$, and let $\varepsilon > 0$. Then the following consequences hold.*

(i) *If there exists $\varepsilon_0 > 0$ and a sequence $(x_n)_{n \geq 1} \subset \mathbb{R}$ such that*

$$\delta_f(\varepsilon_0, x_n) \quad \longrightarrow \quad 0, \tag{27}$$

then

$$\delta_f^U(\varepsilon_0) \quad = \quad 0, \tag{28}$$

and hence f is not uniformly continuous on $\mathbb{R}$.

(ii) *If for every $\varepsilon > 0$ there exists a constant $m(\varepsilon) > 0$ such that*

$$\delta_f(\varepsilon, x) \quad \geq \quad m(\varepsilon) \qquad \text{for all } x \in \mathbb{R}, \tag{29}$$

then

$$\delta_f^U(\varepsilon) \quad \geq \quad m(\varepsilon) \qquad \text{for all } \varepsilon > 0, \tag{30}$$

and therefore f is uniformly continuous on $\mathbb{R}$.

(iii) *If for some $\varepsilon > 0$ there exists $x_* \in \mathbb{R}$ such that*

$$\delta_f(\varepsilon, x_*) \quad = \quad \inf_{x \in \mathbb{R}} \delta_f(\varepsilon, x), \tag{31}$$

then

$$\delta_f^U(\varepsilon) \quad = \quad \delta_f(\varepsilon, x_*). \tag{32}$$

In particular, the point x_* is a geometric bottleneck at scale ε: the local behavior of f at x_* determines the global uniform radius.

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(iv) *Whenever the function $x \mapsto \delta_f(\varepsilon, x)$ is available in explicit form, the corresponding uniform radius may be computed by the one-dimensional optimization formula*

$$\delta_f^U(\varepsilon) \quad = \quad \inf_{x\in\mathbb{R}} \delta_f(\varepsilon, x). \tag{33}$$

Thus the global problem of finding a single admissible radius may be reduced to minimizing the pointwise radius over the base point.

Proof. Each assertion is an immediate consequence of the preceding proposition. Indeed, parts (i), (ii), and (iv) follow by taking the infimum of the family $\{\delta_f(\varepsilon, x) : x \in \mathbb{R}\}$, while part (iii) is obtained by evaluating that infimum at a point where it is attained. □

Example 4 (Application of part (i): detecting failure of uniform continuity). *Let $f(x) = e^x$. For fixed $\varepsilon > 0$ and base point $x_0 \in \mathbb{R}$, the admissible pointwise radius is determined by*

$$|e^x - e^{x_0}| < \varepsilon \qquad \text{whenever } |x - x_0| < \delta. \tag{34}$$

Since the maximal variation on the interval $(x_0 - \delta, x_0 + \delta)$ occurs at the right endpoint, one obtains

$$\delta_f(\varepsilon, x_0) \quad = \quad \log\big(1 + \varepsilon e^{-x_0}\big). \tag{35}$$

Hence, with $x_n = n$,

$$\delta_f(\varepsilon, x_n) \quad = \quad \log\big(1 + \varepsilon e^{-n}\big) \longrightarrow 0. \tag{36}$$

By part (i), e^x is not uniformly continuous on $\mathbb{R}$.

Example 5 (Application of part (ii): proving uniform continuity from a positive lower bound). *Let $f(x) = \arctan x$. We have*

$$|\arctan x - \arctan y| \quad \leq \quad |x - y| \qquad (x, y \in \mathbb{R}). \tag{37}$$

Therefore every pointwise radius satisfies

$$\delta_f(\varepsilon, x) \quad \geq \quad \varepsilon \qquad \text{for all } x \in \mathbb{R}. \tag{38}$$

Taking $m(\varepsilon) = \varepsilon$, part (ii) gives $\delta_f^U(\varepsilon) \geq \varepsilon > 0$, so $\arctan x$ is uniformly continuous on $\mathbb{R}$.

Example 6 (Application of part (iii): locating a geometric bottleneck). *Consider the continuous piecewise linear function*

$$f(x) \quad = \quad \begin{cases} x - 1, & x \leq -1, \\ 2x, & -1 \leq x \leq 1, \\ x + 1, & x \geq 1. \end{cases} \tag{39}$$

At the base point $x_* = 0$, one has $f(0) = 0$ and, for $|x| < \delta \leq 1$,

$$|f(x) - f(0)| \quad = \quad |2x| < \varepsilon \tag{40}$$

precisely when $|x| < \varepsilon/2$. Hence

$$\delta_f(\varepsilon, 0) \quad = \quad \frac{\varepsilon}{2}. \tag{41}$$

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On the other hand, the function is globally 2-Lipschitz, so $\delta_f^U(\varepsilon) \geq \varepsilon/2$. *Since* $\delta_f^U(\varepsilon) \leq \delta_f(\varepsilon, 0) = \varepsilon/2$, *we conclude that*

$$\delta_f^U(\varepsilon) \quad = \quad \delta_f(\varepsilon, 0) = \frac{\varepsilon}{2}. \tag{42}$$

Thus the origin is a bottleneck point: the steepest local part of the graph controls the global uniform radius.

Example 7 (Application of part (iv): computing the uniform radius from an explicit pointwise formula)**.** *Let* $f(x) = |x|$. *By the reverse triangle inequality,*

$$\big||x| - |x_0|\big| \quad \leq \quad |x - x_0|. \tag{43}$$

Hence every $\delta \leq \varepsilon$ *is admissible at every base point* x_0, *which shows that*

$$\delta_f(\varepsilon, x_0) \quad \geq \quad \varepsilon \qquad \textit{for all } x_0 \in \mathbb{R}. \tag{44}$$

Conversely, if $\delta > \varepsilon$, *choose* $x = x_0 + \varepsilon$ *when* $x_0 \geq 0$, *or choose* $x = x_0 - \varepsilon$ *when* $x_0 < 0$; *then* $|x - x_0| < \delta$ *while*

$$\big||x| - |x_0|\big| \quad = \quad \varepsilon. \tag{45}$$

Therefore

$$\delta_f(\varepsilon, x_0) \quad = \quad \varepsilon \qquad \textit{for all } x_0 \in \mathbb{R}, \tag{46}$$

and part (iv) yields

$$\delta_f^U(\varepsilon) \quad = \quad \inf_{x_0 \in \mathbb{R}} \delta_f(\varepsilon, x_0) = \varepsilon. \tag{47}$$

Thus the uniform radius is recovered by minimizing the explicit pointwise formula.

3. Examples and Explicit Computations

In this section we illustrate the general theory by working out two concrete families of functions in detail. First, in Section 3.1 we study the quadratic polynomial $f(x) = ax^2 + bx + c$ and compute an explicit closed-form expression for the pointwise radius of continuity $\delta_f(\varepsilon, x_0)$ in terms of ε, x_0, and the coefficients a, b, c. This formula allows us to analyze how the radius behaves as the base point x_0 moves along the real line, showing in particular that it decays at infinity and attains its maximal value at the vertex $x_0 = -b/(2a)$. We also identify a simple symmetry condition on the coefficients under which the radius becomes an even function of x_0. Second, in Section 3.2 we turn to a probabilistic example and consider the normal density with parameters μ and σ^2. Using an explicit Lipschitz constant for this density, we derive a closed-form expression for the uniform radius of continuity $\delta_f^U(\varepsilon)$, which exhibits a piecewise behaviour in ε: linear growth up to a threshold followed by saturation at infinity. The resulting formula makes transparent how the uniform radius depends on the scale parameter σ, and thereby demonstrates how our radius-of-continuity framework captures both geometric and probabilistic features of a function in a quantitative way.

3.1. Radius of Pointwise Continuity of the Quadratic Polynomial

Example 7. *Let* $f(x) = ax^2 + bx + c (a \neq 0)$ *be the quadratic polynomial. We calculate* $\delta_f(\varepsilon, x_0)$ *in the following three steps:*

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Step 1 — Increment at x_0**.**

$$\begin{aligned} f(x_0+h)-f(x_0) &= a\left[(x_0+h)^2-x_0^2\right]+b\,h \\ &= a(2x_0h+h^2)+b\,h \\ &= ah^2+(2ax_0+b)h. \end{aligned} \tag{48}$$

$$B \;:=\; 2ax_0+b. \tag{49}$$

Step 2 — Exact oscillation on $|h|<\delta$**.**

$$\begin{aligned} \sup_{|h|\le\delta} |ah^2+Bh| &= \max\Big\{|a\delta^2+B\delta|,\ |a\delta^2-B\delta|\Big\} \\ &= |a|\,\delta^2+|B|\,\delta. \end{aligned} \tag{50}$$

$$\omega_f(x_0;\delta) \;=\; |a|\,\delta^2+\big|\,2ax_0+b\,\big|\,\delta. \tag{51}$$

Step 3 — Solve $\omega_f(x_0;\delta)<\varepsilon$**.**

$$\begin{aligned} \delta_f(\varepsilon,x_0) &= \sup\Big\{\delta>0:\ |a|\,\delta^2+\big|\,2ax_0+b\,\big|\,\delta<\varepsilon\Big\} \\ &= \sup\Big\{\delta>0: |a|\,\delta^2+\big|\,2ax_0+b\,\big|\,\delta-\varepsilon\le 0\Big\} \\ &= \frac{-\big|\,2ax_0+b\,\big|+\sqrt{\big|\,2ax_0+b\,\big|^2+4|a|\varepsilon}}{2|a|} \\ &= \frac{2\varepsilon}{\big|\,2ax_0+b\,\big|+\sqrt{\big|\,2ax_0+b\,\big|^2+4|a|\varepsilon}}. \end{aligned} \tag{52}$$

Behavior and Evaluations

We have:

1. *Asymptotic Equivalency:*

$$\delta_f(\varepsilon,x_0) \;\sim\; \frac{\varepsilon}{2\,|a|\,|x_0|}: x_0\to\pm\infty. \tag{53}$$

2. *Left/Right Branch:*

$$\lim_{x_0\to\pm\infty}\delta_f(\varepsilon,x_0) \;=\; 0. \tag{54}$$

3. *Vertex:*

$$\delta_f\!\left(\varepsilon,-\frac{b}{2a}\right) \;=\; \sqrt{\frac{\varepsilon}{|a|}}. \tag{55}$$

4. *Symmetry Condition:*

$$\forall\varepsilon\in\mathbb{R}^+\forall x_0\in\mathbb{R}:\delta_f(\varepsilon,-x_0)=\delta_f(\varepsilon,x_0) \quad\Longleftrightarrow\quad b=0. \tag{56}$$

5. *Figure 3 presents the radius of pointwise continuity for the quadratic polynomial function.*

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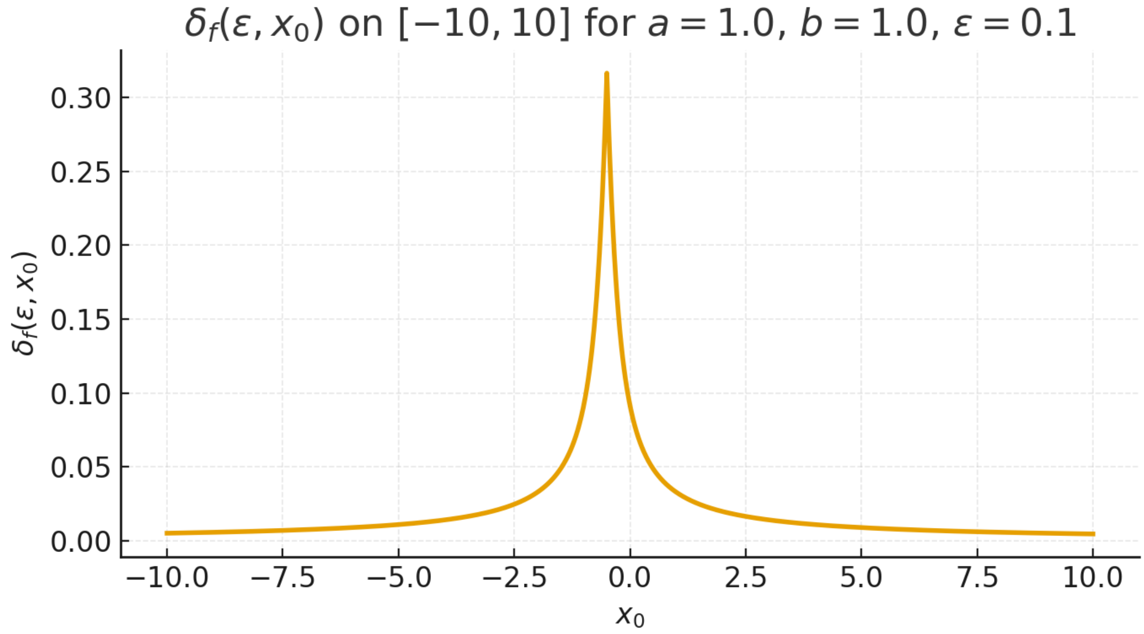

Figure 3. Pointwise radius of continuity $\delta_f(\varepsilon, x_0)$ for the quadratic $f(x) = x^2 + x + 1$ on $[-10, 10]$ with $\varepsilon = 0.1$. The anti-symmetric curve attains its maximum at the vertex $x_0 = -\frac{1}{2}$ and decays rapidly as $|x_0| \to +\infty$, illustrating how the admissible neighbourhood size shrinks away from the point of minimal slope.

Remark 8. *The quadratic polynomial f is not uniformly continuous. To see this using equations (17) and (52) we have:*

$$\delta_f^U(\varepsilon) = \inf_{x_0 \in \mathbb{R}} \delta_f(\varepsilon, x_0) = \inf_{x_0 \in \mathbb{R}} \frac{2\varepsilon}{|\,2ax_0 + b\,| + \sqrt{|\,2ax_0 + b\,|^2 + 4|a|\varepsilon}} = 0. \tag{57}$$

3.2. Radius of Uniform Continuity of the General Normal Probability Density Function

Example 8. *Let $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big)$ be the probability density of the normal distribution $\mathcal{N}(\mu, \sigma^2)$. We calculate $\delta_f^U(\varepsilon)$ as follows:*

Step 1 (Function and derivative).

$$f'(x) \;=\; -\frac{x-\mu}{\sigma^2}\, f(x) \;=\; -\frac{x-\mu}{\sqrt{2\pi}\,\sigma^3} \exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big). \tag{58}$$

Step 2 (Global Lipschitz constant). *Let $L := \sup_{x \in \mathbb{R}} |f'(x)|$. Set $t = (x-\mu)/\sigma$. For $t \geq 0$, $g(t) := te^{-t^2/2}$ has*

$$g'(t) = e^{-t^2/2}(1 - t^2) = 0 \iff t = 1,$$

and g is maximized at $t = 1$ with $g(1) = e^{-1/2}$. Hence

$$L \;=\; \frac{1}{\sqrt{2\pi}\,\sigma^2}\, e^{-1/2} \;=\; \frac{1}{\sqrt{2\pi e}\,\sigma^2}. \tag{59}$$

Step 3 (Uniform oscillation upper bound). *By the mean value theorem, for all $x \in \mathbb{R}$ and $|h| < \delta$,*

$$|f(x+h) - f(x)| \;\leq\; L|h| \;\leq\; L\delta.$$

Taking the sup over x and $|h| < \delta$ gives

$$\omega_f^U(\delta) \;\leq\; L\delta. \tag{60}$$

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Step 4 (Sharpness—matching lower bound). *Choosing x near $\mu \pm \sigma$ (where $|f'|$ is maximal) and $|h| \downarrow 0$ with the sign of f',*

$$\frac{|f(x+h)-f(x)|}{|h|} \longrightarrow |f'(x)| \approx L,$$

so for any $\eta > 0$ and small enough δ,

$$\omega_f^U(\delta) \geq (L-\eta)\delta.$$

Together with (60),

$$\omega_f^U(\delta) = L\delta \qquad (\delta > 0). \tag{61}$$

Step 5 (Saturation ceiling). $\max(f) - \min(f) = f(\mu) - 0 = 1/(\sqrt{2\pi}\,\sigma)$. *Thus*

$$\omega_f^U(\delta) = \min\Big\{L\delta, \frac{1}{\sqrt{2\pi}\,\sigma}\Big\}. \tag{62}$$

Step 6 (Solve for δ). *By definition and (62),*

$$\begin{aligned}\delta_f^U(\varepsilon) &= \sup\{\delta > 0 : \omega_f^U(\delta) \leq \varepsilon\} \\ &= \sup\Big\{\delta > 0 : \min\{L\delta, \tfrac{1}{\sqrt{2\pi}\,\sigma}\} \leq \varepsilon\Big\} \\ &= \begin{cases}(\sqrt{2\pi e}\,\sigma^2)\,\varepsilon, & 0 < \varepsilon < \dfrac{1}{\sqrt{2\pi}\,\sigma}, \\ +\infty, & \varepsilon \geq \dfrac{1}{\sqrt{2\pi}\,\sigma}.\end{cases}\end{aligned} \tag{63}$$

Behavior and Evaluations

We have:

1. *Asymptotic (small ε):*

$$\delta_f^U(\varepsilon) \sim \sqrt{2\pi e}\,\sigma^2\,\varepsilon : \varepsilon \downarrow 0. \tag{64}$$

2. *Parameter dependence:*
   The radius is independent of μ. For fixed $\varepsilon < \varepsilon_* = 1/(\sqrt{2\pi}\,\sigma)$, $\delta_f^U(\varepsilon)$ grows quadratically *in σ ($\propto \sigma^2$). The threshold $\varepsilon_* = 1/(\sqrt{2\pi}\,\sigma)$ decreases like $1/\sigma$, and the break radius $\delta_* = \sigma\sqrt{e}$ increases* linearly *in σ.*
3. *Monotonicity and jump:*
   δ_f^U is strictly increasing and linear on $(0, \varepsilon_*)$ with slope $\sqrt{2\pi e}\,\sigma^2$. It has a jump at $\varepsilon = \varepsilon_*$:

$$\lim_{\varepsilon \uparrow \varepsilon_*} \delta_f^U(\varepsilon) = \delta_* = \sigma\sqrt{e}, \qquad \delta_f^U(\varepsilon_*) = +\infty. \tag{65}$$

4. *Scaling identity:*
   If $f_{(\mu,\sigma)}$ is the pdf for $\mathcal{N}(\mu, \sigma^2)$ distribution and $f_{(0,1)}$ is pdf for the standard normal distribution, then for $0 < \varepsilon < \varepsilon_*$:

$$\delta_{f_{(\mu,\sigma)}}^U(\varepsilon) = \sigma^2\,\delta_{f_{(0,1)}}^U(\sigma\,\varepsilon). \tag{66}$$

5. *Figure 4 presents the radius of uniform continuity for the normal probability density function.*

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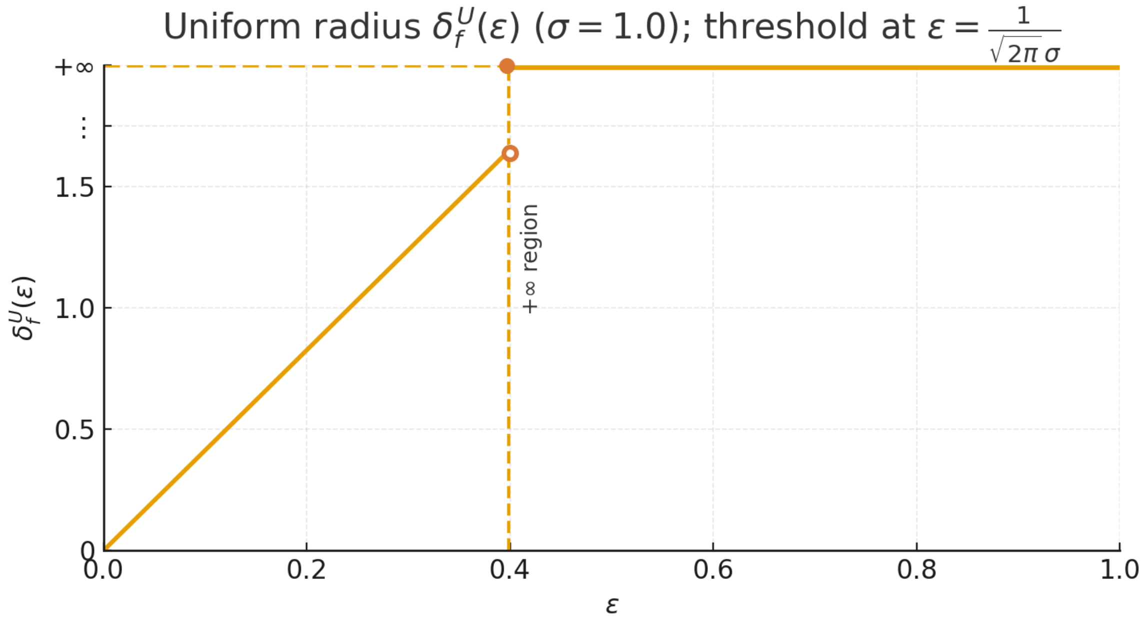

Figure 4. Uniform radius of continuity $\delta_f^U(\varepsilon)$ for the Gaussian density $f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-x^2/(2\sigma^2)}$ with $\sigma = 1$. For small ε the radius grows linearly until the critical value $\varepsilon = 1/(\sqrt{2\pi}\sigma)$; beyond this threshold the ε-band already covers the full range of f and $\delta_f^U(\varepsilon)$ becomes infinite.

4. Discussion

4.1. Summary of the Radius-of-Continuity Viewpoint

In this paper we proposed the *radius of continuity* as a quantitative refinement of the classical ε–δ definition for real-valued functions on $\mathbb{R}$. For each function f, point x_0, and accuracy level $\varepsilon > 0$, the pointwise radius $\delta_f(\varepsilon, x_0)$ captures the largest symmetric neighbourhood on which the deviation $|f(x) - f(x_0)|$ remains below ε. On the global side, the uniform radius $\delta_f^U(\varepsilon)$ measures how far we may move anywhere in the domain while enforcing the same tolerance uniformly in x_0. We showed that both radii admit natural descriptions in terms of pointwise and uniform oscillation, which clarify their structural properties and their behaviour under algebraic operations such as sums, products, and compositions. The resulting invariants distinguish constant functions from non-constant ones and allow us to compare the “stability” of different functions on a common quantitative scale. The explicit computations for a quadratic polynomial and for the normal density illustrate how these abstract notions lead to concrete formulas that encode geometric and probabilistic information about the underlying functions.

4.2. Relation to Classical Moduli of Continuity and $\varepsilon - -\delta$ Proofs

From a conceptual perspective, the radii $\delta_f(\varepsilon, x_0)$ and $\delta_f^U(\varepsilon)$ can be viewed as inverse objects to the pointwise and uniform moduli of continuity. While a modulus assigns, for each spatial scale δ, an upper bound on $|f(x) - f(y)|$, our radii fix a vertical tolerance ε and ask for the largest admissible horizontal scale. Thus the radius-of-continuity framework repackages the traditional ε–δ quantifiers into numerical invariants that are often easier to visualize and to compute explicitly. At the same time, the proofs in Section 2 make clear that the standard techniques used in real analysis ε–δ arguments—such as bounding slopes, factoring expressions, or exploiting Lipschitz estimates—extend seamlessly to the analysis of radii. In this sense the present approach does not replace the classical theory, but rather sits on top of it: the same inequalities that certify continuity now yield quantitative control of δ_f and δ_f^U, providing a bridge between elementary real analysis and more metric-oriented notions of regularity [11,12].

4.3. Future Work

Several directions for further investigation emerge naturally from this study. First, it would be interesting to develop a more systematic comparison between the pointwise and uniform radii, for example by identifying classes of functions for which $\delta_f^U(\varepsilon)$ can be reconstructed from the family $\{\delta_f(\varepsilon, x_0)\}_{x_0}$. Second, in the special case of absolutely continuous functions, one may hope to express radii in terms of the derivative, leading to integral or differential characterizations that parallel the usual connections between absolute continuity and bounded variation [13]. Third, the definitions extend verbatim to general metric spaces, and it remains to explore how the underlying geometry (for instance, doubling properties or curvature bounds) constrains the possible profiles of δ_f and δ_f^U. Finally, higher dimensional examples—such as multivariate polynomials, radial functions, or densities on $\mathbb{R}^d$—could provide additional insight into how radii interact with phenomena like anisotropy and directional derivatives, and how they might be used in applications that require explicit control over perturbations in several variables.

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