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Noncommutative Maccone-Pati Uncertainty Principles

K. Mahesh Krishna *

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Noncommutative Maccone-Pati Uncertainty Principles

K. Mahesh Krishna

School of Mathematics and Natural Sciences, Chanakya University Global Campus, NH-648, Haraluru Village, Devanahalli Taluk, Bengaluru Rural District, Karnataka State, 562 110, India; kmaheshak@gmail.com

Abstract: Let $\mathcal{E}$ be a Hilbert C*-module over a unital C*-algebra $\mathcal{A}$. Let $A : \mathcal{D}(A) \subseteq \mathcal{E} \to \mathcal{E}$ and $B : \mathcal{D}(B) \subseteq \mathcal{E} \to \mathcal{E}$ be possibly unbounded self-adjoint morphisms. Then for all $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\langle x, x\rangle = 1$, we show that (1) $d_x(A)^2 + d_x(B)^2 \geq \mp\langle\{A, B\}x, x\rangle + \langle(A \pm B)x, y\rangle\langle y, (A \pm B)x\rangle \pm \{\langle Ax, x\rangle, \langle Bx, x\rangle\}, \quad \forall y \in \mathcal{E}$ satisfying $\|y\| \leq 1$ and $\langle x, y\rangle = 0$. (2) $d_x(A)^2 + d_x(B)^2 \geq \mp i\langle[A, B]x, x\rangle + \langle(A \pm iB)x, y\rangle\langle y, (A \pm iB)x\rangle \mp i[\langle Ax, x\rangle, \langle Bx, x\rangle], \quad \forall y \in \mathcal{E}$ satisfying $\|y\| \leq 1$ and $\langle x, y\rangle = 0$. where $d_x(A) := \sqrt{\langle Ax, Ax\rangle - \langle Ax, x\rangle^2}$, $[A, B] := AB - BA$, $[\langle Ax, x\rangle, \langle Bx, x\rangle] := \langle Ax, x\rangle\langle Bx, x\rangle - \langle Bx, x\rangle\langle Ax, x\rangle$. We call Inequalities (1) and (2) as noncommutative Maccone-Pati uncertainty principles. They reduce to uncertainty principles derived by Maccone and Pati [*Phys. Rev. Lett., 2014*] whenever $\mathcal{A} = \mathbb{C}$

Keywords: Uncertainty Principle; Hilbert C*-module

MSC: 46L08

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1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and A be a possibly unbounded self-adjoint linear operator defined on domain $\mathcal{D}(A) \subseteq \mathcal{H}$. For $h \in \mathcal{D}(A)$ with $\|h\| = 1$, define the **uncertainty** of A at the point h as

$$\Delta_h(A) := \|Ah - \langle Ah, h\rangle h\| = \sqrt{\|Ah\|^2 - \langle Ah, h\rangle^2}.$$

In 1929, Robertson [1] derived the following mathematical form of the uncertainty principle of Heisenberg derived in 1927 [2]. Recall that for two linear operators $A : \mathcal{D}(A) \subseteq \mathcal{H} \to \mathcal{H}$ and $B : \mathcal{D}(B) \subseteq \mathcal{H} \to \mathcal{H}$, we define the commutator $[A, B] := AB - BA$ and the anti-commutator $\{A, B\} := AB + BA$.

Theorem 1. *[1–4] (**Heisenberg-Robertson Uncertainty Principle**) Let $A : \mathcal{D}(A) \subseteq \mathcal{H} \to \mathcal{H}$ and $B : \mathcal{D}(B) \subseteq \mathcal{H} \to \mathcal{H}$ be self-adjoint linear operators. Then for all $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$, we have*

$$\frac{1}{2}\left(\Delta_h(A)^2 + \Delta_h(B)^2\right) \geq \frac{1}{4}(\Delta_h(A) + \Delta_h(B))^2 \geq \Delta_h(A)\Delta_h(B) \geq \frac{1}{2}|\langle[A, B]h, h\rangle|. \tag{1}$$

In 1930, Schrodinger improved Inequality (1) [5].

Theorem 2. *[5,6] (**Heisenberg-Robertson-Schrodinger Uncertainty Principle**) Let $A : \mathcal{D}(A) \subseteq \mathcal{H} \to \mathcal{H}$ and $B : \mathcal{D}(B) \subseteq \mathcal{H} \to \mathcal{H}$ be self-adjoint linear operators. Then for all $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$, we have*

$$\begin{aligned}\Delta_h(A)\Delta_h(B) \geq |\langle Ah, Bh\rangle - \langle Ah, h\rangle\langle Bh, h\rangle| &= \frac{\sqrt{|\langle[A, B]h, h\rangle|^2 + |\langle\{A, B\}h, h\rangle - 2\langle Ah, h\rangle\langle Bh, h\rangle|^2}}{2} \\ &= \frac{\sqrt{(\langle\{A, B\}h, h\rangle - 2\langle Ah, h\rangle\langle Bh, h\rangle)^2 - \langle[A, B]h, h\rangle^2}}{2}.\end{aligned}$$

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In 2014, Maccone and Pati derived the following uncertainty principles which work for orthogonal vectors [7].

Theorem 3. *[7] (**Maccone-Pati Uncertainty Principles**) Let $A : \mathcal{D}(A) \subseteq \mathcal{H} \to \mathcal{H}$ and $B : \mathcal{D}(B) \subseteq \mathcal{H} \to \mathcal{H}$ be self-adjoint linear operators. Then for all $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$, we have*

(i)

$$\Delta_h(A)^2 + \Delta_h(B)^2 \geq |\langle (A+B)h, k\rangle|^2 - \langle \{A, B\}h, h\rangle + 2\langle Ah, h\rangle\langle Bh, h\rangle,$$
$$\forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h, k\rangle = 0.$$

(ii)

$$\Delta_h(A)^2 + \Delta_h(B)^2 \geq |\langle (A-B)h, k\rangle|^2 + \langle \{A, B\}h, h\rangle - 2\langle Ah, h\rangle\langle Bh, h\rangle,$$
$$\forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h, k\rangle = 0.$$

(iii)

$$\Delta_h(A)^2 + \Delta_h(B)^2 \geq \frac{|\langle (A+B)h, k\rangle|^2 + |\langle (A-B)h, k\rangle|^2}{2}, \quad \forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h, k\rangle = 0.$$

(iv)

$$\Delta_h(A)^2 + \Delta_h(B)^2 \geq -i\langle [A, B]h, h\rangle + |\langle (A+iB)h, k\rangle|^2, \quad \forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h, k\rangle = 0.$$

(v)

$$\Delta_h(A)^2 + \Delta_h(B)^2 \geq i\langle [A, B]h, h\rangle + |\langle (A-iB)h, k\rangle|^2, \quad \forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h, k\rangle = 0.$$

(vi)

$$\Delta_h(A)^2 + \Delta_h(B)^2 \geq \frac{|\langle (A+iB)h, k\rangle|^2 + |\langle (A-iB)h, k\rangle|^2}{2}, \quad \forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h, k\rangle = 0.$$

(vii)

$$\Delta_h(A)\Delta_h(B) \geq \frac{-i\langle [A, B]h, h\rangle}{2 - \left|\left\langle \left(\frac{A}{\Delta_h(A)} + i\frac{B}{\Delta_h(B)}\right)h, k\right\rangle\right|^2}, \quad \forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h, k\rangle = 0.$$

(viii)

$$\Delta_h(A)\Delta_h(B) \geq \frac{i\langle [A, B]h, h\rangle}{2 - \left|\left\langle \left(\frac{A}{\Delta_h(A)} - i\frac{B}{\Delta_h(B)}\right)h, k\right\rangle\right|^2}, \quad \forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h, k\rangle = 0.$$

(ix)

$$\Delta_h(A)\Delta_h(B) \geq \frac{2\langle Ah, h\rangle\langle Bh, h\rangle - \langle \{A, B\}h, h\rangle}{2 - \left|\left\langle \left(\frac{A}{\Delta_h(A)} + \frac{B}{\Delta_h(B)}\right)h, k\right\rangle\right|^2}, \quad \forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h, k\rangle = 0.$$

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(x)

$$\Delta_h(A)\Delta_h(B) \geq \frac{-2\langle Ah,h\rangle\langle Bh,h\rangle + \langle\{A,B\}h,h\rangle}{2-\left|\left\langle\left(\frac{A}{\Delta_h(A)}-\frac{B}{\Delta_h(B)}\right)h,k\right\rangle\right|^2}, \quad \forall k \in \mathcal{H} \text{ satisfying } \|k\| = 1 \text{ and } \langle h,k\rangle = 0.$$

Recently we derived noncommutative version of Theorem 2 in [8]. We therefore ask: What is noncommutative analogues of Theorems 3? For this, we need the notion of noncommutative Hilbert spaces known as Hilbert C*-modules which are first introduced by Kaplansky [9] for modules over commutative C*-algebras and later developed for modules over arbitrary C*-algebras by Paschke [10] and Rieffel [11].

Definition 1. *[9–11] Let $\mathcal{A}$ be a unital C*-algebra. A left module $\mathcal{E}$ over $\mathcal{A}$ is said to be a (left)* ***Hilbert C*-module*** *if there exists a map $\langle\cdot,\cdot\rangle : \mathcal{E}\times\mathcal{E} \to \mathcal{A}$ such that the following conditions hold.*

(i) $\langle x,x\rangle \geq 0$, $\forall x \in \mathcal{E}$. *If* $x \in \mathcal{E}$ *satisfies* $\langle x,x\rangle = 0$, *then* $x = 0$.
(ii) $\langle ax+y,z\rangle = a\langle x,z\rangle + \langle y,z\rangle$, $\forall x,y,z \in \mathcal{E}$, $\forall a \in \mathcal{A}$.
(iii) $\langle x,y\rangle = \langle y,x\rangle^*$, $\forall x,y \in \mathcal{E}$.
(iv) $\mathcal{E}$ *is complete w.r.t. the norm* $\|x\| := \sqrt{\|\langle x,x\rangle\|}$, $\forall x \in \mathcal{E}$.

In the paper, use the following noncommutative Cauchy-Schwarz inequality.

Theorem 4. *[10,11] Let $\mathcal{E}$ be a Hilbert C*-module. Then*

$$\langle x,y\rangle\langle y,x\rangle \leq \|\langle y,y\rangle\|\langle x,x\rangle, \quad \forall x,y \in \mathcal{E}.$$

We derive noncommutative analogues of Theorem 3 in Theorem 5.

2. Noncommutative Maccone-Pati Uncertainty Principles

Let $\mathcal{E}$ be a Hilbert C*-module over a unital C*-algebra $\mathcal{A}$ and A be a possibly unbounded self-adjoint morphism defined on domain $\mathcal{D}(A) \subseteq \mathcal{E}$. For $x \in \mathcal{D}(A)$ with $\langle x,x\rangle = 1$, we define the noncommutative uncertainty of A at the point x as

$$\Delta_x(A) := \|Ax - \langle Ax,x\rangle x\| = \sqrt{\|\langle Ax,Ax\rangle - \langle Ax,x\rangle^2\|}.$$

We define

$$d_x(A) := \sqrt{\langle Ax,Ax\rangle - \langle Ax,x\rangle^2} = \sqrt{\langle Ax-\langle Ax,x\rangle x, Ax-\langle Ax,x\rangle x\rangle}.$$

Then $d_x(A) \in \mathcal{A}$ and

$$\Delta_x(A) = \|d_x(A)\|.$$

Given two elements $a,b \in \mathcal{A}$, we define $[a,b] := ab - ba$ and $\{a,b\} := ab + ba$. Similarly, for two morphisms $A : \mathcal{D}(A) \subseteq \mathcal{E} \to \mathcal{E}$ and $B : \mathcal{D}(B) \subseteq \mathcal{E} \to \mathcal{E}$, we define $[A,B] := AB - BA$ and $\{A,B\} := AB + BA$.

Theorem 5. *(**Noncommutative Maccone-Pati Uncertainty Principles**) Let $\mathcal{E}$ be a Hilbert C*-module over a unital C*-algebra $\mathcal{A}$. Let $A : \mathcal{D}(A) \subseteq \mathcal{E} \to \mathcal{E}$ and $B : \mathcal{D}(B) \subseteq \mathcal{E} \to \mathcal{E}$ be self-adjoint morphisms. Then for all $x \in \mathcal{D}(AB)\cap\mathcal{D}(BA)$ with $\langle x,x\rangle = 1$, we have*

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(i)

$$d_x(A)^2 + d_x(B)^2 \geq -\langle\{A,B\}x,x\rangle + \langle(A+B)x,y\rangle\langle y,(A+B)x\rangle + \{\langle Ax,x\rangle,\langle Bx,x\rangle\},$$
$$\forall y \in \mathcal{E} \textit{ satisfying } \|y\| \leq 1 \textit{ and } \langle x,y\rangle = 0.$$

(ii)

$$\Delta_x(A)^2 + \Delta_x(B)^2 \geq \|-\langle\{A,B\}x,x\rangle + \langle(A+B)x,y\rangle\langle y,(A+B)x\rangle + \{\langle Ax,x\rangle,\langle Bx,x\rangle\}\|,$$
$$\forall y \in \mathcal{E} \textit{ satisfying } \|y\| \leq 1 \textit{ and } \langle x,y\rangle = 0.$$

(iii)

$$d_x(A)^2 + d_x(B)^2 \geq \langle\{A,B\}x,x\rangle + \langle(A-B)x,y\rangle\langle y,(A-B)x\rangle - \{\langle Ax,x\rangle,\langle Bx,x\rangle\},$$
$$\forall y \in \mathcal{E} \textit{ satisfying } \|y\| \leq 1 \textit{ and } \langle x,y\rangle = 0.$$

(iv)

$$\Delta_x(A)^2 + \Delta_x(B)^2 \geq \|\langle\{A,B\}x,x\rangle + \langle(A-B)x,y\rangle\langle y,(A-B)x\rangle - \{\langle Ax,x\rangle,\langle Bx,x\rangle\}\|,$$
$$\forall y \in \mathcal{E} \textit{ satisfying } \|y\| \leq 1 \textit{ and } \langle x,y\rangle = 0.$$

(v)

$$d_x(A)^2 + d_x(B)^2 \geq \frac{\langle(A+B)x,y\rangle\langle y,(A+B)x\rangle + \langle(A-B)x,y\rangle\langle y,(A-B)x\rangle}{2},$$
$$\forall y \in \mathcal{E} \textit{ satisfying } \|y\| \leq 1 \textit{ and } \langle x,y\rangle = 0.$$

(vi)

$$\Delta_x(A)^2 + \Delta_x(B)^2 \geq \frac{\|\langle(A+B)x,y\rangle\langle y,(A+B)x\rangle + \langle(A-B)x,y\rangle\langle y,(A-B)x\rangle\|}{2},$$
$$\forall y \in \mathcal{E} \textit{ satisfying } \|y\| \leq 1 \textit{ and } \langle x,y\rangle = 0.$$

(vii)

$$d_x(A)^2 + d_x(B)^2 \geq -i\langle[A,B]x,x\rangle + \langle(A+iB)x,y\rangle\langle y,(A+iB)x\rangle - i[\langle Ax,x\rangle,\langle Bx,x\rangle],$$
$$\forall y \in \mathcal{E} \textit{ satisfying } \|y\| \leq 1 \textit{ and } \langle x,y\rangle = 0.$$

(viii)

$$\Delta_x(A)^2 + \Delta_x(B)^2 \geq \|-i\langle[A,B]x,x\rangle + \langle(A+iB)x,y\rangle\langle y,(A+iB)x\rangle - i[\langle Ax,x\rangle,\langle Bx,x\rangle]\|,$$
$$\forall y \in \mathcal{E} \textit{ satisfying } \|y\| \leq 1 \textit{ and } \langle x,y\rangle = 0.$$

(ix)

$$d_x(A)^2 + d_x(B)^2 \geq i\langle[A,B]x,x\rangle + \langle(A-iB)x,y\rangle\langle y,(A-iB)x\rangle + i[\langle Ax,x\rangle,\langle Bx,x\rangle],$$
$$\forall y \in \mathcal{E} \textit{ satisfying } \|y\| \leq 1 \textit{ and } \langle x,y\rangle = 0.$$

(x)

$$\Delta_x(A)^2 + \Delta_x(B)^2 \geq \|i\langle[A,B]x,x\rangle + \langle(A-iB)x,y\rangle\langle y,(A-iB)x\rangle + i[\langle Ax,x\rangle,\langle Bx,x\rangle]\|,$$
$$\forall y \in \mathcal{E} \textit{ satisfying } \|y\| \leq 1 \textit{ and } \langle x,y\rangle = 0.$$

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(xi)

$$d_x(A)^2 + d_x(B)^2 \geq \frac{\langle (A+iB)x, y\rangle\langle y, (A+iB)x\rangle + \langle (A-iB)x, y\rangle\langle y, (A-iB)x\rangle}{2},$$
$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y\rangle = 0.$$

(xii)

$$\Delta_x(A)^2 + \Delta_x(B)^2 \geq \frac{\|\langle (A+iB)x, y\rangle\langle y, (A+iB)x\rangle + \langle (A-iB)x, y\rangle\langle y, (A-iB)x\rangle\|}{2},$$
$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y\rangle = 0.$$

Proof. Let $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\langle x, x\rangle = 1$. Let $y \in \mathcal{E}$ be such that $\|y\| \leq 1$ and $\langle x, y\rangle = 0$.

(i)

$$\begin{aligned}
&\langle (A+B)x, y\rangle\langle y, (A+B)x\rangle \\
&= \langle Ax - \langle Ax, x\rangle x + Bx - \langle Bx, x\rangle x, y\rangle\langle y, Ax - \langle Ax, x\rangle x + Bx - \langle Bx, x\rangle x\rangle \\
&\leq \|\langle y, y\rangle\|\langle (Ax - \langle Ax, x\rangle x) + (Bx - \langle Bx, x\rangle x), (Ax - \langle Ax, x\rangle x) + (Bx - \langle Bx, x\rangle x)\rangle \\
&\leq \langle Ax - \langle Ax, x\rangle x, Ax - \langle Ax, x\rangle x\rangle + \langle Bx - \langle Bx, x\rangle x, Bx - \langle Bx, x\rangle x\rangle \\
&\quad + \langle Bx - \langle Bx, x\rangle x, Ax - \langle Ax, x\rangle x\rangle + \langle Ax - \langle Ax, x\rangle x, Bx - \langle Bx, x\rangle x\rangle \\
&= d_x(A)^2 + d_x(B)^2 + \langle Bx, Ax\rangle - \langle Bx, x\rangle\langle Ax, x\rangle + \langle Ax, Bx\rangle - \langle Ax, x\rangle\langle Bx, x\rangle \\
&= d_x(A)^2 + d_x(B)^2 + \langle \{A, B\}x, x\rangle - \{\langle Ax, x\rangle, \langle Bx, x\rangle\}.
\end{aligned}$$

Therefore

$$d_x(A)^2 + d_x(B)^2 \geq -\langle \{A, B\}x, x\rangle + \langle (A+B)x, y\rangle\langle y, (A+B)x\rangle + \{\langle Ax, x\rangle, \langle Bx, x\rangle\}.$$

(ii) Take norm on (i).

(iii) Similar to (i).

(iv) Take norm on (iii).

(v)

$$\begin{aligned}
&\langle (A+B)x, y\rangle\langle y, (A+B)x\rangle + \langle (A-B)x, y\rangle\langle y, (A-B)x\rangle = 2\langle Ax, y\rangle\langle y, Ax\rangle + 2\langle Bx, y\rangle\langle y, Bx\rangle \\
&= 2\langle Ax - \langle Ax, x\rangle x, y\rangle\langle y, Ax - \langle Ax, x\rangle x\rangle + 2\langle Bx - \langle Bx, x\rangle x, y\rangle\langle y, Bx - \langle Bx, x\rangle x\rangle \\
&\leq 2\|\langle y, y\rangle\|\langle Ax - \langle Ax, x\rangle x, Ax - \langle Ax, x\rangle x\rangle + 2\|\langle y, y\rangle\|\langle Bx - \langle Bx, x\rangle x, Bx - \langle Bx, x\rangle x\rangle \\
&\leq 2d_x(A)^2 + 2d_x(B)^2.
\end{aligned}$$

(vi) Take norm on (v).

(vii)

$$\begin{aligned}
&\langle (A+iB)x, y\rangle\langle y, (A+iB)x\rangle = \langle Ax + iBx, y\rangle\langle y, Ax + iBx\rangle \\
&= \langle Ax - \langle Ax, x\rangle x + i(Bx - \langle Bx, x\rangle x), y\rangle\langle y, Ax - \langle Ax, x\rangle x + i(Bx - \langle Bx, x\rangle x)\rangle \\
&\leq \|\langle y, y\rangle\|\langle (Ax - \langle Ax, x\rangle x) + i(Bx - \langle Bx, x\rangle x), (Ax - \langle Ax, x\rangle x) + i(Bx - \langle Bx, x\rangle x)\rangle \\
&\leq \langle Ax - \langle Ax, x\rangle x, Ax - \langle Ax, x\rangle x\rangle + \langle Bx - \langle Bx, x\rangle x, Bx - \langle Bx, x\rangle x\rangle \\
&\quad + i\langle Bx - \langle Bx, x\rangle x, Ax - \langle Ax, x\rangle x\rangle - i\langle Ax - \langle Ax, x\rangle x, Bx - \langle Bx, x\rangle x\rangle \\
&= d_x(A)^2 + d_x(B)^2 + i(\langle Bx, Ax\rangle - \langle Bx, x\rangle\langle Ax, x\rangle) - i(\langle Ax, Bx\rangle - \langle Ax, x\rangle\langle Bx, x\rangle) \\
&= d_x(A)^2 + d_x(B)^2 + i\langle [A, B]x, x\rangle + i[\langle Ax, x\rangle, \langle Bx, x\rangle].
\end{aligned}$$

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Therefore

$$d_x(A)^2 + d_x(B)^2 \geq -i\langle [A,B]x, x\rangle + \langle (A+iB)x, y\rangle\langle y, (A+iB)x\rangle - i[\langle Ax, x\rangle, \langle Bx, x\rangle].$$

(viii) Take norm on (vii).
(ix) Similar to (vii).
(x) Take norm on (ix).
(xi) Add (vii) and (ix).
(xii) Add (viii) and (ix), then take norm.

□

Corollary 1. *Let $\mathcal{E}$ be a Hilbert C*-module over a unital commutative C*-algebra $\mathcal{A}$. Let $A : \mathcal{D}(A) \to \mathcal{E}$ and $B : \mathcal{D}(B) \to \mathcal{E}$ be self-adjoint morphisms. Then for all $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\langle x, x\rangle = 1$, we have*

(i) *If $\mathcal{A}$ is commutative, then*

$$d_x(A)^2 + d_x(B)^2 \geq -i\langle [A,B]x, x\rangle + \langle (A+iB)x, y\rangle\langle y, (A+iB)x\rangle,$$
$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y\rangle = 0.$$

(ii) *If $\mathcal{A}$ is commutative, then*

$$\Delta_x(A)^2 + \Delta_x(B)^2 \geq \| -i\langle [A,B]x, x\rangle + \langle (A+iB)x, y\rangle\langle y, (A+iB)x\rangle\|,$$
$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y\rangle = 0.$$

We also have following uncertainty principles.

Theorem 6. (***Noncommutative Maccone-Pati Uncertainty Principles***) *Let $\mathcal{E}$ be a Hilbert C*-module over a unital C*-algebra $\mathcal{A}$. Let $A : \mathcal{D}(A) \to \mathcal{E}$ and $B : \mathcal{D}(B) \to \mathcal{E}$ be self-adjoint morphisms. Then for all $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\langle x, x\rangle = 1$, we have*

(i)

$$\Delta_x(A)^2 + \Delta_x(B)^2 \geq \langle y, (A+B)x\rangle\langle (A+B)x, y\rangle - \|\langle \{A,B\}x, x\rangle - \{\langle Ax, x\rangle, \langle Bx, x\rangle\}\|,$$
$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y\rangle = 0.$$

(ii)

$$\Delta_x(A)^2 + \Delta_x(B)^2 \geq \langle y, (A-B)x\rangle\langle (A-B)x, y\rangle - \|\langle \{A,B\}x, x\rangle - \{\langle Ax, x\rangle, \langle Bx, x\rangle\}\|$$
$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y\rangle = 0.$$

(iii)

$$\Delta_x(A)^2 + \Delta_x(B)^2 \geq \frac{\|\langle y, (A+B)x\rangle\langle (A+B)x, y\rangle + \langle (A-B)x, y\rangle\langle y, (A-B)x\rangle\|}{2},$$
$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y\rangle = 0.$$

(iv)

$$\Delta_x(A)^2 + \Delta_x(B)^2 \geq \langle y, (A+iB)x\rangle\langle (A+iB)x, y\rangle - \|\langle [A,B]x, x\rangle + [\langle Ax, x\rangle, \langle Bx, x\rangle]\|,$$
$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y\rangle = 0.$$

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(v)

$$\Delta_x(A)^2 + \Delta_x(B)^2 \geq \langle y, (A - iB)x\rangle\langle (A - iB)x, y\rangle - \|\langle [A,B]x, x\rangle + [\langle Ax, x\rangle, \langle Bx, x\rangle]\|,$$
$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y\rangle = 0.$$

Proof. Let $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\langle x, x\rangle = 1$. Let $y \in \mathcal{E}$ be such that $\|y\| \leq 1$ and $\langle x, y\rangle = 0$.

(i)

$$\begin{aligned}
&\langle y, (A+B)x\rangle\langle (A+B)x, y\rangle \\
&= \langle y, Ax - \langle Ax, x\rangle x + (Bx - \langle Bx, x\rangle x)\rangle\langle Ax - \langle Ax, x\rangle x + (Bx - \langle Bx, x\rangle x), y\rangle \\
&\leq \|\langle (Ax - \langle Ax, x\rangle x) + (Bx - \langle Bx, x\rangle x), (Ax - \langle Ax, x\rangle x) + (Bx - \langle Bx, x\rangle x)\rangle\|\langle y, y\rangle \\
&\leq \|\langle (Ax - \langle Ax, x\rangle x) + (Bx - \langle Bx, x\rangle x), (Ax - \langle Ax, x\rangle x) + (Bx - \langle Bx, x\rangle x)\rangle\|\|\langle y, y\rangle\| \\
&\leq \|\langle Ax - \langle Ax, x\rangle x, Ax - \langle Ax, x\rangle x\rangle + \langle Bx - \langle Bx, x\rangle x, Bx - \langle Bx, x\rangle x\rangle \\
&\quad + \langle Bx - \langle Bx, x\rangle x, Ax - \langle Ax, x\rangle x\rangle + \langle Ax - \langle Ax, x\rangle x, Bx - \langle Bx, x\rangle x\rangle\| \\
&= \|d_x(A)^2 + d_x(B)^2 + (\langle Bx, Ax\rangle - \langle Bx, x\rangle\langle Ax, x\rangle) + \langle Ax, Bx\rangle - \langle Ax, x\rangle\langle Bx, x\rangle\| \\
&\leq \|d_x(A)^2 + d_x(B)^2\| + \|\langle \{A,B\}x, x\rangle - \{\langle Ax, x\rangle, \langle Bx, x\rangle\}\|.
\end{aligned}$$

Therefore

$$\Delta_x(A)^2 + \Delta_x(B)^2 \geq \|d_x(A)^2 + d_x(B)^2\| \geq \langle y, (A+B)x\rangle\langle (A+B)x, y\rangle - \|\langle \{A,B\}x, x\rangle - \{\langle Ax, x\rangle, \langle Bx, x\rangle\}\|.$$

(ii) Similar to (i).

(iii)

$$\begin{aligned}
&\langle y, (A+B)x\rangle\langle (A+B)x, y\rangle + \langle y, (A-B)x\rangle\langle (A-B)x, y\rangle = 2\langle y, Ax\rangle\langle Ax, y\rangle + 2\langle y, Bx\rangle\langle Bx, y\rangle \\
&= 2\langle y, Ax - \langle Ax, x\rangle x\rangle\langle Ax - \langle Ax, x\rangle x, y\rangle + 2\langle y, Bx - \langle Bx, x\rangle x\rangle\langle Bx - \langle Bx, x\rangle x, y\rangle \\
&\leq 2\|\langle Ax - \langle Ax, x\rangle x, Ax - \langle Ax, x\rangle x\rangle\|\langle y, y\rangle + 2\|\langle Bx - \langle Bx, x\rangle x, Bx - \langle Bx, x\rangle x\rangle\|\langle y, y\rangle \\
&\leq 2\Delta_x(A)^2 + 2\Delta_x(B)^2.
\end{aligned}$$

(iv)

$$\begin{aligned}
&\langle y, (A+iB)x\rangle\langle (A+iB)x, y\rangle = \langle y, Ax + iBx\rangle\langle Ax + iBx, y\rangle \\
&= \langle y, Ax - \langle Ax, x\rangle x + i(Bx - \langle Bx, x\rangle x)\rangle\langle Ax - \langle Ax, x\rangle x + i(Bx - \langle Bx, x\rangle x), y\rangle \\
&\leq \|\langle (Ax - \langle Ax, x\rangle x) + i(Bx - \langle Bx, x\rangle x), (Ax - \langle Ax, x\rangle x) + i(Bx - \langle Bx, x\rangle x)\rangle\|\langle y, y\rangle \\
&\leq \|\langle Ax - \langle Ax, x\rangle x, Ax - \langle Ax, x\rangle x\rangle + \langle Bx - \langle Bx, x\rangle x, Bx - \langle Bx, x\rangle x\rangle \\
&\quad + i\langle Bx - \langle Bx, x\rangle x, Ax - \langle Ax, x\rangle x\rangle - i\langle Ax - \langle Ax, x\rangle x, Bx - \langle Bx, x\rangle x\rangle\| \\
&= \|d_x(A)^2 + d_x(B)^2 + i(\langle Bx, Ax\rangle - \langle Bx, x\rangle\langle Ax, x\rangle) - i(\langle Ax, Bx\rangle - \langle Ax, x\rangle\langle Bx, x\rangle)\| \\
&= \|d_x(A)^2 + d_x(B)^2\| + \|(\langle Bx, Ax\rangle - \langle Bx, x\rangle\langle Ax, x\rangle) - (\langle Ax, Bx\rangle - \langle Ax, x\rangle\langle Bx, x\rangle)\| \\
&= \|d_x(A)^2 + d_x(B)^2\| + \|\langle [A,B]x, x\rangle + [\langle Ax, x\rangle, \langle Bx, x\rangle]\|.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\Delta_x(A)^2 + \Delta_x(B)^2 \geq \|d_x(A)^2 + d_x(B)^2\| \\
&\geq \langle y, (A+iB)x\rangle\langle (A+iB)x, y\rangle - \|\langle [A,B]x, x\rangle + [\langle Ax, x\rangle, \langle Bx, x\rangle]\|.
\end{aligned}$$

(v) Similar to (iv).

□

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Theorem 7. *Let $\mathcal{E}$ be a Hilbert C^*-module over a unital commutative C^*-algebra $\mathcal{A}$. Let $A : \mathcal{D}(A) \to \mathcal{E}$ and $B : \mathcal{D}(B) \to \mathcal{E}$ be self-adjoint morphisms. Then for all $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\langle x, x\rangle = 1$, we have*

(i) *If $d_x(A), d_x(B)$ are invertible, then*

$$d_x(A)d_x(B) \geq \frac{-i\langle [A,B]x, x\rangle}{2 - \left\langle \left(\frac{A}{d_x(A)} + \frac{iB}{d_x(B)}\right)x, y\right\rangle \left\langle y, \left(\frac{A}{d_x(A)} + \frac{iB}{d_x(B)}\right)x\right\rangle},$$
$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y\rangle = 0.$$

(ii)

$$\Delta_x(A)\Delta_x(B) \geq \frac{-i\langle [A,B]x, x\rangle}{2 - \left\langle \left(\frac{A}{\Delta_x(A)} + \frac{iB}{\Delta_x(B)}\right)x, y\right\rangle \left\langle y, \left(\frac{A}{\Delta_x(A)} + \frac{iB}{\Delta_x(B)}\right)x\right\rangle},$$
$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y\rangle = 0.$$

(iii) *If $d_x(A), d_x(B)$ are invertible, then*

$$d_x(A)d_x(B) \geq \frac{i\langle [A,B]x, x\rangle}{2 - \left\langle \left(\frac{A}{d_x(A)} - \frac{iB}{d_x(B)}\right)x, y\right\rangle \left\langle y, \left(\frac{A}{d_x(A)} - \frac{iB}{d_x(B)}\right)x\right\rangle},$$
$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y\rangle = 0.$$

(iv)

$$\Delta_x(A)\Delta_x(B) \geq \frac{i\langle [A,B]x, x\rangle}{2 - \left\langle \left(\frac{A}{\Delta_x(A)} - \frac{iB}{\Delta_x(B)}\right)x, y\right\rangle \left\langle y, \left(\frac{A}{\Delta_x(A)} - \frac{iB}{\Delta_x(B)}\right)x\right\rangle},$$
$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y\rangle = 0.$$

(v) *If $d_x(A), d_x(B)$ are invertible, then*

$$d_x(A)d_x(B) \geq \frac{2\langle Ax, x\rangle\langle Bx, x\rangle - \langle \{A,B\}x, x\rangle}{2 - \left\langle \left(\frac{A}{d_x(A)} + \frac{B}{d_x(B)}\right)x, y\right\rangle \left\langle y, \left(\frac{A}{d_x(A)} + \frac{B}{d_x(B)}\right)x\right\rangle},$$
$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y\rangle = 0.$$

(vi)

$$\Delta_x(A)\Delta_x(B) \geq \frac{2\langle Ax, x\rangle\langle Bx, x\rangle - \langle \{A,B\}x, x\rangle}{2 - \left\langle \left(\frac{A}{\Delta_x(A)} + \frac{B}{\Delta_x(B)}\right)x, y\right\rangle \left\langle y, \left(\frac{A}{\Delta_x(A)} + \frac{B}{\Delta_x(B)}\right)x\right\rangle},$$
$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y\rangle = 0.$$

(vii) *If $d_x(A), d_x(B)$ are invertible, then*

$$d_x(A)d_x(B) \geq \frac{-2\langle Ax, x\rangle\langle Bx, x\rangle + \langle \{A,B\}x, x\rangle}{2 - \left\langle \left(\frac{A}{d_x(A)} - \frac{B}{d_x(B)}\right)x, y\right\rangle \left\langle y, \left(\frac{A}{d_x(A)} - \frac{B}{d_x(B)}\right)x\right\rangle},$$
$$\forall y \in \mathcal{E} \text{ satisfying } \|y\| \leq 1 \text{ and } \langle x, y\rangle = 0.$$

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(viii)

$$\Delta_x(A)\Delta_x(B) \geq \frac{-2\langle Ax,x\rangle\langle Bx,x\rangle + \langle\{A,B\}x,x\rangle}{2-\left\langle\left(\frac{A}{\Delta_x(A)}-\frac{B}{\Delta_x(B)}\right)x,y\right\rangle\left\langle y,\left(\frac{A}{\Delta_x(A)}-\frac{B}{\Delta_x(B)}\right)x\right\rangle},$$
$$\forall y\in\mathcal{E} \textit{ satisfying } \|y\|\leq 1 \textit{ and } \langle x,y\rangle = 0.$$

Proof. Let $x \in \mathcal{D}(AB)\cap\mathcal{D}(BA)$ with $\langle x,x\rangle = 1$. Let $y\in\mathcal{E}$ be such that $\|y\|\leq 1$ and $\langle x,y\rangle = 0$.

(i) Since $\mathcal{A}$ is commutative and $d_x(A), d_x(B)$ are invertible,

$$\begin{aligned}
&\left\langle\left(\frac{A}{d_x(A)}+\frac{iB}{d_x(B)}\right)x,y\right\rangle\left\langle y,\left(\frac{A}{d_x(A)}+\frac{iB}{d_x(B)}\right)x\right\rangle\\
&=\left\langle\frac{Ax-\langle Ax,x\rangle x}{d_x(A)}+\frac{i(Bx-\langle Bx,x\rangle x)}{d_x(B)},y\right\rangle\left\langle y,\frac{Ax-\langle Ax,x\rangle x}{d_x(A)}+\frac{i(Bx-\langle Bx,x\rangle x)}{d_x(B)}\right\rangle\\
&\leq\|\langle y,y\rangle\|\left\langle\frac{Ax-\langle Ax,x\rangle x}{d_x(A)}+\frac{i(Bx-\langle Bx,x\rangle x)}{d_x(B)},\frac{Ax-\langle Ax,x\rangle x}{d_x(A)}+\frac{i(Bx-\langle Bx,x\rangle x)}{d_x(B)}\right\rangle\\
&\leq\left\langle\frac{Ax-\langle Ax,x\rangle x}{d_x(A)},\frac{Ax-\langle Ax,x\rangle x}{d_x(A)}\right\rangle+\left\langle\frac{Bx-\langle Bx,x\rangle x}{d_x(B)},\frac{Bx-\langle Bx,x\rangle x}{d_x(B)}\right\rangle\\
&\quad+i\left\langle\frac{Bx-\langle Bx,x\rangle x}{d_x(B)},\frac{Ax-\langle Ax,x\rangle x}{d_x(A)}\right\rangle-i\left\langle\frac{Ax-\langle Ax,x\rangle x}{d_x(A)},\frac{Bx-\langle Bx,x\rangle x}{d_x(B)}\right\rangle\\
&=1+1+\frac{i\langle Bx,Ax\rangle-i\langle Bx,x\rangle\langle Ax,x\rangle}{d_x(A)d_x(B)}+\frac{-i\langle Ax,Bx\rangle+i\langle Ax,x\rangle\langle Bx,x\rangle}{d_x(A)d_x(B)}\\
&=2+\frac{i\langle[A,B]x,x\rangle}{d_x(A)d_x(B)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{-i\langle[A,B]x,x\rangle}{d_x(A)d_x(B)}\leq 2-\left\langle\left(\frac{A}{d_x(A)}+\frac{iB}{d_x(B)}\right)x,y\right\rangle\left\langle y,\left(\frac{A}{d_x(A)}+\frac{iB}{d_x(B)}\right)x\right\rangle\\
&\implies\frac{-i\langle[A,B]x,x\rangle}{2-\left\langle\left(\frac{A}{d_x(A)}+\frac{iB}{d_x(B)}\right)x,y\right\rangle\left\langle y,\left(\frac{A}{d_x(A)}+\frac{iB}{d_x(B)}\right)x\right\rangle}\leq d_x(A)d_x(B).
\end{aligned}$$

(ii) Since $\mathcal{A}$ is commutative,

$$\begin{aligned}
&\left\langle\left(\frac{A}{\Delta_x(A)}+\frac{iB}{\Delta_x(B)}\right)x,y\right\rangle\left\langle y,\left(\frac{A}{\Delta_x(A)}+\frac{iB}{\Delta_x(B)}\right)x\right\rangle\\
&=\left\langle\frac{Ax-\langle Ax,x\rangle x}{\Delta_x(A)}+\frac{i(Bx-\langle Bx,x\rangle x)}{\Delta_x(B)},y\right\rangle\left\langle y,\frac{Ax-\langle Ax,x\rangle x}{\Delta_x(A)}+\frac{i(Bx-\langle Bx,x\rangle x)}{\Delta_x(B)}\right\rangle\\
&\leq\|\langle y,y\rangle\|\left\langle\frac{Ax-\langle Ax,x\rangle x}{\Delta_x(A)}+\frac{i(Bx-\langle Bx,x\rangle x)}{\Delta_x(B)},\frac{Ax-\langle Ax,x\rangle x}{\Delta_x(A)}+\frac{i(Bx-\langle Bx,x\rangle x)}{\Delta_x(B)}\right\rangle\\
&\leq\left\langle\frac{Ax-\langle Ax,x\rangle x}{\Delta_x(A)},\frac{Ax-\langle Ax,x\rangle x}{\Delta_x(A)}\right\rangle+\left\langle\frac{Bx-\langle Bx,x\rangle x}{\Delta_x(B)},\frac{Bx-\langle Bx,x\rangle x}{\Delta_x(B)}\right\rangle\\
&\quad+i\left\langle\frac{Bx-\langle Bx,x\rangle x}{\Delta_x(B)},\frac{Ax-\langle Ax,x\rangle x}{\Delta_x(A)}\right\rangle-i\left\langle\frac{Ax-\langle Ax,x\rangle x}{\Delta_x(A)},\frac{Bx-\langle Bx,x\rangle x}{\Delta_x(B)}\right\rangle\\
&=1+1+\frac{i\langle Bx,Ax\rangle-i\langle Bx,x\rangle\langle Ax,x\rangle}{\Delta_x(A)\Delta_x(B)}+\frac{-i\langle Ax,Bx\rangle+i\langle Ax,x\rangle\langle Bx,x\rangle}{\Delta_x(A)\Delta_x(B)}\\
&=2+\frac{i\langle[A,B]x,x\rangle}{\Delta_x(A)\Delta_x(B)}.
\end{aligned}$$

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Therefore

$$\frac{-i\langle [A,B]x,x\rangle}{\Delta_x(A)\Delta_x(B)} \leq 2 - \left\langle \left(\frac{A}{\Delta_x(A)} + \frac{iB}{\Delta_x(B)}\right)x, y\right\rangle \left\langle y, \left(\frac{A}{\Delta_x(A)} + \frac{iB}{\Delta_x(B)}\right)x\right\rangle$$
$$\implies \frac{-i\langle [A,B]x,x\rangle}{2 - \left\langle \left(\frac{A}{\Delta_x(A)} + \frac{iB}{\Delta_x(B)}\right)x, y\right\rangle \left\langle y, \left(\frac{A}{\Delta_x(A)} + \frac{iB}{\Delta_x(B)}\right)x\right\rangle} \leq \Delta_x(A)\Delta_x(B).$$

(iii) Similar to (i).

(iv) Similar to (ii).

(v) Since $\mathcal{A}$ is commutative and $d_x(A)$, $d_x(B)$ are invertible,

$$\begin{aligned}
&\left\langle \left(\frac{A}{d_x(A)} + \frac{B}{d_x(B)}\right)x, y\right\rangle \left\langle y, \left(\frac{A}{d_x(A)} + \frac{B}{d_x(B)}\right)x\right\rangle \\
&= \left\langle \frac{Ax - \langle Ax,x\rangle x}{d_x(A)} + \frac{Bx - \langle Bx,x\rangle x}{d_x(B)}, y\right\rangle \left\langle y, \frac{Ax - \langle Ax,x\rangle x}{d_x(A)} + \frac{Bx - \langle Bx,x\rangle x}{d_x(B)}\right\rangle \\
&\leq \|\langle y,y\rangle\| \left\langle \frac{Ax - \langle Ax,x\rangle x}{d_x(A)} + \frac{Bx - \langle Bx,x\rangle x}{d_x(B)}, \frac{Ax - \langle Ax,x\rangle x}{d_x(A)} + \frac{Bx - \langle Bx,x\rangle x}{d_x(B)}\right\rangle \\
&\leq \left\langle \frac{Ax - \langle Ax,x\rangle x}{d_x(A)}, \frac{Ax - \langle Ax,x\rangle x}{d_x(A)}\right\rangle + \left\langle \frac{Bx - \langle Bx,x\rangle x}{d_x(B)}, \frac{Bx - \langle Bx,x\rangle x}{d_x(B)}\right\rangle \\
&\quad + \left\langle \frac{Bx - \langle Bx,x\rangle x}{d_x(B)}, \frac{Ax - \langle Ax,x\rangle x}{d_x(A)}\right\rangle + \left\langle \frac{Ax - \langle Ax,x\rangle x}{d_x(A)}, \frac{Bx - \langle Bx,x\rangle x}{d_x(B)}\right\rangle \\
&= 1 + 1 + \frac{\langle Bx, Ax\rangle - \langle Bx,x\rangle\langle Ax,x\rangle}{d_x(A)d_x(B)} + \frac{\langle Ax, Bx\rangle - \langle Ax,x\rangle\langle Bx,x\rangle}{d_x(A)d_x(B)} \\
&= 2 + \frac{-2\langle Ax,x\rangle\langle Bx,x\rangle + \langle \{A,B\}x,x\rangle}{d_x(A)d_x(B)}.
\end{aligned}$$

Therefore

$$-\frac{-2\langle Ax,x\rangle\langle Bx,x\rangle + \langle \{A,B\}x,x\rangle}{d_x(A)d_x(B)} \leq 2 - \left\langle \left(\frac{A}{d_x(A)} + \frac{B}{d_x(B)}\right)x, y\right\rangle \left\langle y, \left(\frac{A}{d_x(A)} + \frac{B}{d_x(B)}\right)x\right\rangle$$
$$\implies \frac{2\langle Ax,x\rangle\langle Bx,x\rangle - \langle \{A,B\}x,x\rangle}{2 - \left\langle \left(\frac{A}{d_x(A)} + \frac{B}{d_x(B)}\right)x, y\right\rangle \left\langle y, \left(\frac{A}{d_x(A)} + \frac{B}{d_x(B)}\right)x\right\rangle} \leq d_x(A)d_x(B).$$

(vi) Since $\mathcal{A}$ is commutative,

$$\begin{aligned}
&\left\langle \left(\frac{A}{\Delta_x(A)} + \frac{B}{\Delta_x(B)}\right)x, y\right\rangle \left\langle y, \left(\frac{A}{\Delta_x(A)} + \frac{B}{\Delta_x(B)}\right)x\right\rangle \\
&= \left\langle \frac{Ax - \langle Ax,x\rangle x}{\Delta_x(A)} + \frac{Bx - \langle Bx,x\rangle x}{\Delta_x(B)}, y\right\rangle \left\langle y, \frac{Ax - \langle Ax,x\rangle x}{\Delta_x(A)} + \frac{Bx - \langle Bx,x\rangle x}{\Delta_x(B)}\right\rangle \\
&\leq \|\langle y,y\rangle\| \left\langle \frac{Ax - \langle Ax,x\rangle x}{\Delta_x(A)} + \frac{Bx - \langle Bx,x\rangle x}{\Delta_x(B)}, \frac{Ax - \langle Ax,x\rangle x}{\Delta_x(A)} + \frac{Bx - \langle Bx,x\rangle x}{\Delta_x(B)}\right\rangle \\
&\leq \left\langle \frac{Ax - \langle Ax,x\rangle x}{\Delta_x(A)}, \frac{Ax - \langle Ax,x\rangle x}{\Delta_x(A)}\right\rangle + \left\langle \frac{Bx - \langle Bx,x\rangle x}{\Delta_x(B)}, \frac{Bx - \langle Bx,x\rangle x}{\Delta_x(B)}\right\rangle \\
&\quad + \left\langle \frac{Bx - \langle Bx,x\rangle x}{\Delta_x(B)}, \frac{Ax - \langle Ax,x\rangle x}{\Delta_x(A)}\right\rangle + \left\langle \frac{Ax - \langle Ax,x\rangle x}{\Delta_x(A)}, \frac{Bx - \langle Bx,x\rangle x}{\Delta_x(B)}\right\rangle \\
&= 1 + 1 + \frac{\langle Bx, Ax\rangle - \langle Bx,x\rangle\langle Ax,x\rangle}{\Delta_x(A)\Delta_x(B)} + \frac{\langle Ax, Bx\rangle - \langle Ax,x\rangle\langle Bx,x\rangle}{\Delta_x(A)\Delta_x(B)} \\
&= 2 + \frac{2\langle Ax,x\rangle\langle Bx,x\rangle + \langle \{A,B\}x,x\rangle}{\Delta_x(A)\Delta_x(B)}.
\end{aligned}$$

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Therefore

$$-\frac{-2\langle Ax,x\rangle\langle Bx,x\rangle+\langle\{A,B\}x,x\rangle}{\Delta_x(A)\Delta_x(B)}\leq 2-\left\langle\left(\frac{A}{\Delta_x(A)}+\frac{B}{\Delta_x(B)}\right)x,y\right\rangle\left\langle y,\left(\frac{A}{\Delta_x(A)}+\frac{B}{\Delta_x(B)}\right)x\right\rangle$$

$$\implies \frac{2\langle Ax,x\rangle\langle Bx,x\rangle-\langle\{A,B\}x,x\rangle}{2-\left\langle\left(\frac{A}{\Delta_x(A)}+\frac{B}{\Delta_x(B)}\right)x,y\right\rangle\left\langle y,\left(\frac{A}{\Delta_x(A)}+\frac{B}{\Delta_x(B)}\right)x\right\rangle}\leq \Delta_x(A)\Delta_x(B).$$

(vii) Similar to (v).

(viii) Similar to (vi).

□

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