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Article

Noether Symmetries of Time Dependent Damped Dynamical Systems: A Geometric Approach

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Abstract

Two damped dynamical systems have been studied for a long time in the literature: The damped Kepler system and the damped Harmonic oscillator. The damped factor is usually in the first order in the velocities and it is either time dependent or coordinate dependent whereas the potential is either autonomous or time and space dependent. One example is the Bateman-Caldirola-Kanai(BCK) model [1] which uses a time dependent Lagrangian/Hamiltonian for studying different aspects of dissipative systems at classical and quantum level. In all damped dynamical systems the role of first integrals is crucial because it simplifies the dynamics and allows for the integration or in general the study of the dynamical equations. To determine the first integrals of a dynamical system there are various methods. To mention a few, the Noether theorem, the specification of the functional form of the first integral I and the subsequent requirement $dI/dt = 0$ which leads to a system of differential equations whose solution provides the first integral, the Prelle-Singer method, and the δ -formalism developed by Katzin and Levin. For most of the damped systems, at least for the cases of the Kepler and the Harmonic Oscillator, there is a Lagrangian which describes the dynamical equations. This means that for these systems one may use the Noether theorem to compute the first integrals. This is the approach of the present work. We consider the time dependent damped dynamical systems described by a Lagrangian $L = A(t)L_0$ where L_0 is an autonomous Lagrangian - usually associated with the undamped system- and prove a Proposition which allows the computation of the Noether integrals in a systematic way using the geometric properties of a Riemannian space. The method can be applied to curved spaces where not much has been done. It is important to note that the metric of this Riemannian space is defined by the kinetic energy of the dynamical system and it is not the metric of the space where the motion of the system occurs. In this way the dynamics of the system “locks” with its own geometry. The purpose of the Proposition is to present an algorithm which applies in definite steps and determines the Noether integrals using standard calculations. The application of the algorithm is demonstrated in the case of Kepler motion. We consider the simple case of Kepler motion with constant damping and a complex one with a time dependent damping of the form $A(t) = \frac{\gamma}{t}$. For the time dependent case we find new results. For example the homothetic vector produces Noether integrals only for the specific values of $\gamma = -1, -\frac{1}{3}$.

Keywords: time dependent noether symmetries; time dependent damped dynamical systems; time dependent lagrangian; damped kepler dynamics; time dependent damped kepler motion; homothetic vectors; time dependent first integrals

1. Introduction

Consider a dynamical system with time dependent damping moving in a Riemannian space under the action of the conservative force $V^i(x)$ with equation:

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k + \phi(t)\dot{x}^i + V^i(x) = 0 \tag{1}$$

Particular forms of such systems have been considered in the literature including the well known damped harmonic oscillator [2]. By a suitable transformation (1) takes the form [3,4]:

$$\ddot{x}^i + \Gamma_{jk}^i \dot{x}^j \dot{x}^k + \omega(s) V^i = 0 \quad (2)$$

which describes a dynamical system with time dependent potential. Equation (2) describes another important dynamical system which is the time dependent harmonic oscillator.

The purpose of the present work is to present a systematic algorithm (Proposition) which computes the first integrals of these dynamical systems and particularly of (1).

It is well known that in order to determine first integrals of motion of dynamical equations there is a number of methods. The common method for Lagrangian systems is the Noether symmetries. Another method which does not require a Lagrangian is to require that the dynamical system admits a quadratic (or higher order) first integral of the form

$$I = K_{ij} \dot{x}^i \dot{x}^j + K_i \dot{x}^i + K \quad (3)$$

and demand $dI/dt = 0$. This requirement results in a system of differential equations among the coefficients K_{ij}, K_i, K and the quantities defining the dynamical equations whose solution provides the first integrals (see [5–7] and references therein). A third method is the Prolle-Singer method [8–10]. Finally it should be mentioned the method of the δ -formalism developed by Katzin and Levine [11], [12,13]. The method we shall follow in the present work is that of the Noether integrals.

It is easy to show that (1) coincides with Lagrange equations of the time dependent Lagrangian

$$L = A(t) L_0 \quad (4)$$

where $L_0 = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - V(x)$ and the conformal factor:

$$A(t) = e^{\int \phi(t) dt} \Leftrightarrow \phi(t) = (\ln A)_{,t}. \quad (5)$$

For example the equation describing the damped harmonic oscillator (with constant damping term) is :

$$\ddot{x}^i + \gamma \dot{x}^i + V^i = 0 \quad (6)$$

where $V(x) = \frac{1}{2} x_i x^i$. The Lagrangian for (6) is [14]:

$$L = e^{\gamma t} L_0$$

where

$$L_0 = \frac{1}{2} \dot{x}^i \dot{x}_i - V(x). \quad (7)$$

It is to be noted that the Lagrangian defined by (1) is different than the Lagrangian $L = \frac{1}{2} \dot{x}^i \dot{x}_i - \omega(t) V(x)$ defined by (2) regardless of the fact that (1) and (2) are related by a change of parameter. Consequently - as will be shown - the Noether integrals of the two Lagrangians are different. This is due to the fact that under transformation of parameter the Lagrangian does not transform in a definite way, that is, as a geometric object.

In order to demonstrate the application of the algorithm in practice we consider the case of the Kepler potential with fixed and with time dependent damping. This dynamical system has been considered in the literature for a long time in various forms although for various damping factors [15–21]. Applying the same algorithm one is possible to determine the Noether symmetries of the damped harmonic oscillator, another case which has been considered extensively in the literature e.g., [22–25]. However for this case the deeper relation of these symmetries and their relation to the geometric structure of the dynamical system has not yet been revealed. Due to its importance the damped harmonic oscillator it is preferable to be discussed as a separate case.

The algorithm which computes the first integrals of dynamical systems (1) is given in the following Proposition (which is proved in the Appendix A).

Proposition 1. Consider a dynamical system with time dependent Lagrangian

$$L = A(t) \left(\frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - V(x^k) \right) \quad (8)$$

moving in a Riemannian space with metric g_{ij} . Lagrange equations for the system are:

$$\ddot{x}^i + \Gamma_{jk}^i \dot{x}^j \dot{x}^k + (\ln A)_{,t} \dot{x}^i + V^i = 0 \quad (9)$$

The Noether point symmetries X of L are generated by the homothetic algebra of g_{ij} as follows;

The general form of X is:

$$X = \xi^0(t) \partial_t + T(t) Y^i(x) \partial_{x^i} \quad (10)$$

where

$$\xi^0(t) = \frac{D - 2\psi}{2(\ln A)_{,t}} T \quad (11)$$

$\psi = 0$ for a KV, $\psi = 1$ for the homothetic vector, $Y^i(x)$ is an element of the homothetic algebra of g_{ij} and the coefficients D, m, C_1 are determined as follows.

Case A: $T_{,t} \neq 0$

Y^i is a gradient vector, $Y^i = M^i$ say, and satisfies the equation

$$Y(V) + DV + mM + C_1 = 0$$

where D, m, C_1 are constants and the function $T(t)$

(a) satisfies the equation

$$T_{,tt} + (\ln A)_{,t} T_{,t} - mT = 0. \quad (12)$$

(b) it is constrained by the condition

$$\left(\frac{D - 2\psi}{2(\ln A)_{,t}} T \right)_{,t} = \frac{D + 2\psi}{2} T. \quad (13)$$

The Noether function $f(t)$ is computed from the relation

$$f = AT_{,t}M + C_1 \int AT dt$$

and the Noether integral is:

$$I = AT \left(-Y^i \dot{x}^i + \frac{D - 2\psi}{2(\ln A)_{,t}} \left(\frac{1}{2} \dot{x}^i \dot{x}_i + V(x) \right) \right) + f(t). \quad (14)$$

Case B: $T_{,t} = 0, T = a_1$

Y^i is gradient or non-gradient and satisfies the relation

$$Y(V) + DV = C_1.$$

The function $A(t)$ satisfies the constraint condition:

$$\left(\frac{D - 2\psi}{2(\ln A)_{,t}} \right)_{,t} = \frac{D + 2\psi}{2}. \quad (15)$$

The Noether function $f(t)$ is computed from the relation

$$f(t) = -C_1 a_1 \int A(t) dt \quad (16)$$

and the Noether integral is:

$$I = A \left(-Y^i \dot{x}^i + \frac{D - 2\psi}{2(\ln A)_{,t}} \left(\frac{1}{2} \dot{x}^i \dot{x}_i + V(x) \right) \right) + f(t). \quad (17)$$

We note that the classifying parameters are the D, m, C_1 . The function T in the case $T_t \neq 0$ is independently determined by the equation $T_{,tt} + (\ln A)_{,t} T_{,t} - mT = 0$ and the constraint (13). The function $A(t)$ is computed from (5) and in the case $T_t = 0$ it is constrained by (15). For the applications it is useful to note that in the case $T_t = (\ln A)_{,t} = 0$ the constraint condition (15) implies that the Killing vectors produce a Noether symmetry only when $D = 0$ and the homothetic vector only for the value $D = -2$.

In practice the application of the Proposition 1 requires the following steps:

- Determine the homothetic algebra of g_{ij}
- Compute the quantity $Y(V)$ for all vectors Y of the homothetic algebra and determine the parameters D, m, C_1
- For $T_t \neq 0$ check the compatibility of the constraint Equation (13) with the T equation for the gradient vectors of the homothetic algebra
- For $T_t = 0$ check the constraint (15) for the conformal function $A(t)$ for all the vectors of the homothetic algebra i.e., the gradient and the non-gradient.
- Proceed to the calculation of the Noether vector and the Noether integral for Case A and Case B using the appropriate formulae.

2. Solution of the Equation $T_{,tt} + (\ln A)_{,t} T_{,t} - mT = 0$

The role of the equation $T_{,tt} + (\ln A)_{,t} T_{,t} - mT = 0$ for $T(t)$ when $T_t \neq 0$ is central for the application of Proposition 1. The solution of this equation is the following.

Case 1. $m = 0$

The equation becomes

$$T_{,tt} + (\ln A)_{,t} T_{,t} = 0$$

and has solution

$$T(t) = C_3 \int \frac{1}{A(t)} dt \quad (18)$$

Case 2. $m \neq 0$

Define $\Delta = \gamma^2 + 4m$.

Case B1: $\Delta > 0$ (Real and distinct roots)

The roots are:

$$r_1 = \frac{-\gamma + \sqrt{\Delta}}{2}, \quad r_2 = \frac{-\gamma - \sqrt{\Delta}}{2}$$

The general solution is:

$$T(t) = C_3 e^{r_1 t} + C_4 e^{r_2 t}$$

Case B2: $\Delta = 0$ (Repeated real root)

In this case:

$$r = -\frac{\gamma}{2}; m = -\frac{\gamma^2}{4}$$

The general solution is:

$$T(t) = (C_3 + C_4 t) e^{-\frac{\gamma}{2} t}$$

Case B3: $\Delta < 0$ (Complex conjugate roots)

The roots are:

$$r = \frac{-\gamma}{2} \pm i\omega$$

where

$$\omega = \frac{\sqrt{-\Delta}}{2}.$$

The general solution is:

$$T(t) = e^{-\frac{\gamma}{2}t} (C_3 \cos(\omega t) + C_4 \sin(\omega t))$$

3. The Case of Euclidian Space E^n

We demonstrate the application of Proposition 1 for the damped Kepler potential which has been considered in the literature for various types of damping (see [21] and references therein). We discuss two cases, one with constant damping and one with a time dependent damping.

We recall that the homothetic algebra of E^n consists of $\frac{n^2+n+2}{2}$ vectors which in Cartesian coordinates are as follows ($I, J = 1, 2, \dots, n$):

$$Y^I = \partial_I : \text{gradient KV with gradient function } x^I$$

$$Y^3 = \varepsilon^{IJ} x_I \partial_{x^J} : \text{non-gradient KV}$$

$$Y^4 = x^i \partial_i : \text{gradient HKV with gradient function } M = \frac{1}{2} x^i x_i$$

3.1. The Kepler Potential with Constant Damping in E^n

The equation for this system is [17,21]:

$$\ddot{x}^i + \gamma \dot{x}^i - \frac{x^i}{(x^i x_i)^{\frac{3}{2}}} = 0$$

where the potential $V(x^i) = -\frac{1}{\sqrt{x_i x^i}}$ and γ is a constant. Using (5) we compute $A(t) = e^{\gamma t}$ therefore the time dependent Lagrangian is:

$$L = e^{\gamma t} \left(\frac{1}{2} \delta_{ij} \dot{x}^i \dot{x}^j - \frac{1}{\sqrt{x_i x^i}} \right)$$

The Hamiltonian is:

$$H = e^{\gamma t} \left(\frac{1}{2} \delta_{ij} \dot{x}^i \dot{x}^j + \frac{1}{\sqrt{x_i x^i}} \right)$$

The homothetic algebra of the metric δ_{ij} consists of the gradient KVs $Y_I = \partial_{x^I}$ $I = 1, 2, \dots, n$ the non-gradient KVs $Y^3 = \varepsilon^{ij} x_i \partial_{x^j}$ and the gradient HV $Y^4 = x^i \partial_{x^i}$ with gradient function $M = \frac{1}{2} x^i x_i$.

For the vectors of the Homothetic algebra we compute the quantity $Y(V)$. The results are shown in Table 1:

$$Y_I(V) = \frac{x^I}{(x^i x_i)^{\frac{3}{2}}} \neq V$$

$$Y^3(V) = 0$$

$$Y^4(V) = \frac{1}{\sqrt{x_i x^i}} = -V.$$

The constraint for $T_{,t} \neq 0$ is

$$Y(V) + DV + mM + C_1 = 0 \quad (19)$$

where $Y^i = M^i$ is a gradient vector and for $T_{,t} = 0$

$$Y(V) + DV = C_1 \quad (20)$$

where Y^i is a gradient or non-gradient vector. From the Table we infer that only the vectors Y^3, Y^4 possibly produce Noether symmetries.

Condition (19) applies only to the gradient HV Y^4 and gives $D = 1, m = C_1 = 0$.

Condition (20) applies to both vectors. For Y^3 gives $D = C_1 = 0$ and for Y^4 , $D = 1, C_1 = 0$.

We continue with each Case separately.

Case A

The gradient HV Y^4 .

For the parameters $D = 1, \psi = 1$ the constraint (13) gives $T_{,t} = -3\gamma T \Rightarrow T_{,tt} = -3\gamma T_{,t}$. For $m = 0$ Equation (12) becomes $T_{,tt} + \gamma T_{,t} = 0$. The two equations are incompatible ($\gamma \neq 0$) therefore the gradient HV does not give a Noether symmetry.

Case B

For this case $T = a_1$ and the constraint is (15).

The non-gradient KV Y^3 .

For $D = \psi = 0$ we compute $\zeta^0 = 0$. The Noether vector is:

$$X = TY^3 = a_1 \epsilon^{ij} x_i \partial_{x_j} \quad (21)$$

Because $C_1 = 0$ the Noether function $f = 0$:

The Noether integral is (conservation of angular momentum along the z - axis)

$$I = a_1 Y^3 \dot{x}_i = a_1 \epsilon^{ij} x_i \dot{x}_j \quad (22)$$

The gradient HV Y^4 .

For the HV Y^4 for Case B we have $D = 1$. Because $(\ln A)_{,t} = 0$ and $D \neq -2$ the vector Y^4 does not produce a Noether symmetry.

We conclude that the damped Kepler potential with constant damping has only one Noether integral which is the angular momentum along the normal axis to the plane of the orbit. This is the expected and well known result. This does not mean that there are no other first integrals for this damping. However they are not point Noether symmetries.

3.2. Kepler Motion with Time Dependent Damping

Consider the Kepler potential with the time dependent damping:

$$\ddot{x}^i + \frac{\gamma}{t} \dot{x}^i - \frac{x^i}{r^3} = 0 \quad (23)$$

where $V(x^i) = -\frac{1}{\sqrt{x_i x^i}}$ and ($\gamma \neq 0$). Using (5) we compute $A(t) = t^\gamma$ ($(\ln A)_{,t} = \gamma/t$) therefore the time dependent Lagrangian is:

$$L = t^\gamma \left(\frac{1}{2} \delta_{ij} \dot{x}^i \dot{x}^j - \frac{1}{\sqrt{x_i x^i}} \right) \quad (24)$$

From Table 1 follows that:

Case A: The gradient HV Y^4 has constants $D = 1, m = C_1 = 0$ and possibly produces a Noether symmetry

Case B: The KV Y^3 produces a Noether symmetry with constants $D = C_1 = 0$ whereas the gradient HV Y^4 has constants $D = 1, C_1 = 0$ and possibly produces a Noether symmetry ($A(t)$ has to satisfy the constraint (15)).

We discuss each case.

Case A

The constraint (13) for the values $D = 1, m = C_1 = 0$ and $\psi = 1$ gives:

$$T_{,t} = -\frac{1+3\gamma}{t}T. \quad (25)$$

Differentiating this we find:

$$tT_{,tt} = -(2+3\gamma)T_{,t}.$$

For $m = 0$ the T Equation (12) gives:

$$T_{,tt} + \frac{\gamma}{t}T_{,t} = 0.$$

Replacing in the equation of the constraint we find:

$$\gamma = -1.$$

We conclude that the homothetic vector gives a Noether symmetry only for the Lagrangian

$$L_1 = \frac{1}{t}L_0 \quad (26)$$

where

$$L_0 = \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j - \frac{1}{\sqrt{x_i x^i}}$$

is the standard autonomous Kepler Lagrangian.

We determine the Noether symmetries of $L_1 = \frac{1}{t}L_0$.

For L_1 we have $A(t) = \frac{1}{t}$, $(\ln A)_{,t} = -\frac{1}{t}$ and Lagrange equation is:

$$\ddot{x}^i - \frac{1}{t}\dot{x}^i - \frac{x^i}{r^3} = 0. \quad (27)$$

We compute T from Equation (25):

$$T_{,t} = \frac{2}{t}T \Rightarrow T = Ct^2$$

where C is a constant. We compute

$$\zeta^0 = \frac{D-2\psi}{2(\ln A)_{,t}}T = \frac{t}{2}T.$$

The Noether vector is:

$$X_1 = \zeta^0\partial_t + Tx^i\partial_i = C\left(\frac{t}{2}\partial_t + x^i\partial_i\right)t^2. \quad (28)$$

For the Noether function from (A20) we have $(M = \frac{1}{2}x^i x_i)$:

$$f = AT_{,t}M + C_1 \int ATdt = Cr^2.$$

Concerning the Noether integral using (14) we find:

$$I = Ct\left[-r\dot{r} + \frac{t}{2}\left(\frac{1}{2}\dot{x}^i\dot{x}_i + \frac{1}{r}\right)\right] + Cr^2. \quad (29)$$

Case B

Vector Y^3 .

For $D = C_1 = 0$ and $\psi = 0$ we compute $\zeta^0 = 0, f = 0$. The Noether vector is

$$X = a_1\varepsilon^{ij}x_i\partial_j$$

and the Noether integral

$$I = -a_1 t^\gamma \varepsilon^{ij} x_i \dot{x}_j$$

Vector Y^4 .

In this case the constraint is (15). For $D = 1$, $\psi = 1$ and $(\ln A)_{,t} = \frac{\gamma}{t}$ we compute $\gamma = -\frac{1}{3}$. Therefore Y^4 produces a Noether symmetry only for the value $\gamma = -\frac{1}{3}$, that is for the Lagrangian

$$L = t^{-1/3} L_0$$

with Lagrange equations

$$\ddot{x}^i - \frac{1}{3t} \dot{x}^i - \frac{x^i}{r^3} = 0. \quad (30)$$

For $D = 1$, $\psi = 1$ we compute $\xi^0 = \frac{3ta_1}{2}$, $f = 0$. The Noether vector is

$$X = \frac{3ta_1}{2} \partial_t + a_1 x^i \partial_i$$

For $C_1 = 0$ we compute $f = 0$.

The Noether integral is:

$$I = a_1 t^\gamma \left[-r\dot{r} + \frac{3t}{2} \left(\frac{1}{2} \frac{dx^i}{dt} \frac{dx_i}{dt} + \frac{1}{r} \right) \right]. \quad (31)$$

We conclude that for general γ the equation $\ddot{x}^i + \frac{\gamma}{t} \dot{x}^i - \frac{x^i}{r^3} = 0$ admits only one integral of motion which is the expected linear integral of angular momentum. However for the specific values $\gamma = -1, -\frac{1}{3}$ the equation (27) admits in addition the quadratic first integrals (29) and (31) respectively.

All Noether integrals have been verified.

4. Relation of a Linearly Damped Dynamical System and a Time Dependent Harmonic Oscillator

Consider Equation (1) and assume a continuous transformation of the parameter $(t, x^i) \rightarrow (s(t), x^i)$. Then (1) becomes:

$$\frac{d^2 x^i}{ds^2} + \Gamma_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds} + \frac{1}{\left(\frac{ds(t)}{dt}\right)^2} \left(\left(\frac{d^2 s(t)}{dt^2} \right) + \phi(t) \frac{ds(t)}{dt} \right) \frac{dx^i}{ds} + \frac{1}{\left(\frac{ds(t)}{dt}\right)^2} V^i = 0. \quad (32)$$

If we define the parameter $s(t)$ by the requirement:

$$\frac{d^2 s(t)}{dt^2} + \phi(t) \frac{ds(t)}{dt} = 0 \quad (33)$$

then the equation of motion becomes:

$$\frac{d^2 x^i}{ds^2} + \Gamma_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds} + \left(\frac{dt}{ds} \right)^2 V^i = 0. \quad (34)$$

or

$$\frac{d^2 x^i}{ds^2} + \Gamma_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds} + \omega(s) V^i = 0. \quad (35)$$

where

$$\omega(s) = \left(\frac{dt}{ds} \right)^2. \quad (36)$$

The general solution of (33) is

$$s(t) = C \int e^{-\int \phi(t) dt} dt. \quad (37)$$

Because the transformation is continuous the function $s(t)$ has an inverse $t(s)$ which can be used to compute explicitly $\omega(s)$ for the damping factor $\phi(t)$. Therefore (1) by a proper transformation of the parameter t takes the form of (35) which has no damping but instead a time dependent potential of the form $\omega(s)V(x)$ where $\omega(s)$ is given by (36).

The following question arises:

“Do the two dynamical Equations (1) and (35) have the same Noether symmetries and consequently the same Noether integrals?”

If they do then one may study one equation and by applying the transformation of the parameter to obtain the results for the other equation.

In order to answer this question we work as follows. Consider the s – space of (35), use the $s(t)$ transformation to derive the corresponding (1) equation in the t – space, or vice versa, determine the Noether symmetries for both equations in the corresponding space and see if they coincide if one is transformed to the other under the t to s transformation.

The obvious (and interesting) case to apply this procedure is that of the damped “oscillator”:

$$\ddot{x}^i + \gamma \dot{x}^i + V^i = 0 \quad (38)$$

where $\phi(t) = \gamma$. For this system the Lagrangian is [14]:

$$L_t = e^{\gamma t} \left(\frac{1}{2} \dot{x}^i \dot{x}_i - V \right).$$

so that $A(t) = e^{\gamma t}$. Using (37) and $\phi(t) = \gamma$ one finds $s = Ce^{-\gamma t}$ (plus an unimportant constant). The inverse transformation is $t = -\frac{1}{\gamma} \ln s$ from which $\frac{dt}{ds} = -\frac{1}{\gamma s}$, therefore $\omega(s) = \frac{1}{\gamma^2 s^2}$. It follows that in the s – space the Equation (38) becomes the undamped “oscillator” with s –dependent potential:

$$\frac{d^2 x^i}{ds^2} + \frac{1}{\gamma^2 s^2} V^i = 0. \quad (39)$$

Using Proposition 1 one shows that the oscillator (38) admits the Noether symmetry $X = T\partial_i$ where $T(t)$ is a solution of the equation (see (12)):

$$T_{,tt} + \gamma T_{,t} + T = 0$$

corresponding to the gradient KVs ∂_i .

In [4] Theorem 2, the authors compute the Noether symmetries of equations of the form (39). Applying this Theorem one finds that the gradient KVs ∂_i define the Noether symmetry $X = T_1(t)\partial_i$ where $T_1(t)$ is a solution of the equation (see (12)):

$$T_{,tt} + \frac{1}{\gamma^2 s^2} + T = 0.$$

Obviously the two vectors are different and - as can be shown - the corresponding Noether integrals are different. Therefore the Noether symmetries of (38) and (39) are different although the two equations are related by a parameter transformation.

This implies that the present study is independent of the corresponding study of [4].

Appendix A

Proof of Proposition 1.

The Noether point symmetry of the Lagrangian L must satisfy the Noether condition:

$$\mathbf{X}^{[1]}L + \frac{d\zeta^0}{dt}L = \frac{df}{dt} \quad (A1)$$

where:

$$\mathbf{X}^{[1]} = \zeta^0(t, x^i) \frac{\partial}{\partial t} + \zeta^i(t, x^i) \frac{\partial}{\partial x^i} + \left(\frac{d\zeta^i}{dt} - \frac{dx^i}{dt} \frac{d\zeta^0}{dt} \right) \frac{\partial}{\partial \dot{x}^i}$$

is the first prolongation of $\mathbf{X}$. Replacing L from (4) one finds:

$$\mathbf{X}^{[1]}(A)L_0 + A\mathbf{X}^{[1]}L_0 + A\frac{d\zeta^0}{dt}L_0 = \frac{df}{dt}. \quad (\text{A2})$$

A standard computation shows that (A2) leads to the following expression:

$$\begin{aligned} & \frac{1}{2}A \left[\left(L_{\zeta}g_{ij} - 2g_{ij}\zeta_{,t}^0 + 2\zeta^0 A_{,t} \right) \dot{x}^i \dot{x}^k - 2g_{ij}\zeta_{,k}^0 \dot{x}^i \dot{x}^j \dot{x}^k + 2 \left(g_{ik}\zeta_{,t}^k \right) \dot{x}^i \right] - \\ & - A\zeta(V) + \frac{A}{2}\zeta_{,t}^0 g_{ij} \dot{x}^i \dot{x}^j + \frac{A}{2}\zeta_{,k}^0 g_{ij} \dot{x}^i \dot{x}^j \dot{x}^k - AV\zeta_{,t}^0 - AV\zeta_{,j}^0 \dot{x}^j - \zeta^0 A_{,t} V(x^i) = f_{,t} + \dot{x}^i f_{,i} \end{aligned}$$

This is an identity, therefore the coefficients of the powers of $\dot{x}^i$ must vanish. This results in the equations:

$$\begin{aligned} (\dot{x}^i)^0 & : A\zeta(V) + (A\zeta^0)_{,t}V = -f_{,t} \\ (\dot{x}^i)^1 & : Ag_{ik}\zeta_{,t}^k - AV\zeta_{,k}^0 = f_{,i} \\ \dot{x}^i \dot{x}^k & : L_{\zeta}g_{ij} = \left(\zeta_{,t}^0 - \zeta^0 \ln(A)_{,t} \right) g_{ij} \\ \dot{x}^i \dot{x}^j \dot{x}^k & : A\zeta_{,k}^0 = 0. \end{aligned}$$

The last equation implies $\zeta^0(t, x^i) = \zeta^0(t)$. Taking this into account, the remaining symmetry conditions become:

$$A\zeta(V) + (A\zeta^0)_{,t}V = -f_{,t} \quad (\text{A3})$$

$$Ag_{ij}\zeta_{,t}^k = f_{,i} \quad (\text{A4})$$

$$L_{\zeta}g_{ij} = \left(\zeta_{,t}^0 - \zeta^0 \ln(A)_{,t} \right) g_{ij}, \quad A(t) \neq 0. \quad (\text{A5})$$

From equation (A5) follows that $\zeta^i(t, x) = T(t)Y^i(x)$ where $Y^i(x)$ is a HV or KV of the metric with homothety factor $\psi = 1$ or 0 respectively, that is, (A5) becomes:

$$L_Y g_{ij} = 2\psi_Y g_{ij} \quad (\text{A6})$$

We have to solve the system of equations:

$$\zeta_{,t}^0 - \zeta^0 \ln(A)_{,t} = 2\psi_Y T \quad (\text{A7})$$

$$AT_Y(V) + (A\zeta^0)_{,t}V = -f_{,t} \quad (\text{A8})$$

$$AT_{,t}Y_i = f_{,i}. \quad (\text{A9})$$

From (A9) follows that for $T_{,t} \neq 0$ the vector Y^i is a gradient vector. We set

$$Y^i = M^i.$$

The Noether integral is

$$\begin{aligned} I &= -\zeta^i \frac{\partial L}{\partial \dot{x}^i} + \zeta^0 H + f \\ I &= -TY^i(x)e^{\gamma t} \dot{x}_i + \zeta^0 H + f \end{aligned} \quad (\text{A10})$$

where $H = \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i - L$ is the Hamiltonian.

From (A3),(A4) we have for the integrability condition of f :

$$\begin{aligned}\frac{\partial}{\partial t}[A(t)T_{,t}Y_i] &= -\frac{\partial}{\partial x^i}[AT\mathbf{Y}(V) + (A\tilde{\zeta}^0)_{,t}V] \\ \frac{\partial \mathbf{Y}(V)}{\partial x^i} + \frac{(A\tilde{\zeta}^0)_{,t}}{AT}V_{,i} + \frac{[A(t)T_{,t}]_t}{AT}Y_i &= 0.\end{aligned}\quad (\text{A11})$$

From (A11) follows ($T \neq 0$):

$$\frac{(A\tilde{\zeta}^0)_{,t}}{AT} = D \Rightarrow (\ln A)_{,t}\tilde{\zeta}^0 + \tilde{\zeta}^0_{,t} = DT \quad (\text{A12})$$

$$\begin{aligned}\frac{[A(t)T_{,t}]_t}{AT} = m &\Rightarrow T_{,tt} + (\ln A)_{,t}T_{,t} - mT = 0 \\ \frac{\partial \mathbf{Y}(V)}{\partial x^i} + DV_{,i} + mY_i &= 0\end{aligned}\quad (\text{A13})$$

For $T_{,t} \neq 0$ Y^i is a gradient, so that $Y^i = M^i$. Then (A13) can be integrated and becomes:

$$\mathbf{Y}(V) + DV + mM + C_1 = 0. \quad (\text{A14})$$

For $T_{,t} = 0$ ($T = a_1 \neq 0$) the $f_{,i} = 0$ and $f(t)$. The Y^i can be non-gradient. In this case we have:

$$Aa_1\mathbf{Y}(V) + (A\tilde{\zeta}^0)_{,t}V = -f_{,t}$$

Using $(A\tilde{\zeta}^0)_{,t} = DAa_1$ we find:

$$Aa_1(\mathbf{Y}(V) + DV) = -f_{,t}.$$

From this we have the three equations:

$$f_{,t} = C_1a_1A \Rightarrow f = -a_1C_1 \int A dt$$

$$\mathbf{Y}(V) + DV = C_1.$$

$$\tilde{\zeta}^0_{,t} = a_1D - (\ln A)_{,t}\tilde{\zeta}^0.$$

The constraint condition.

There is a constraint condition for T and $A(t)$. Using (A7) and (A12) we find:

$$\tilde{\zeta}^0(t) = \frac{D - 2\psi}{2(\ln A)_{,t}}T \quad (\text{A15})$$

$$\tilde{\zeta}^0_{,t}(t) = \frac{D + 2\psi}{2}T \quad (\text{A16})$$

from which follows the important and key constraint condition:

$$\left(\frac{D - 2\psi}{2(\ln A)_{,t}}T\right)_{,t} = \frac{D + 2\psi}{2}T. \quad (\text{A17})$$

In the case $T_{,t} = 0$ this becomes a constraint for $A(t)$:

$$\left(\frac{D - 2\psi}{2(\ln A)_{,t}}\right)_{,t} = \frac{D + 2\psi}{2}. \quad (\text{A18})$$

It is important to note that in the case $T_{,t} = 0$ and $(\ln A)_{,t} = 0$ constraint (A18) gives $D = -2\psi$. This implies that for these cases the Killing vectors produce a Noether symmetry only for $D = 0$ and

for the homothetic vector only for $D = -2$.

The Noether function $f(t, x)$.

Assume $T_{,t} \neq 0$ so that $Y^i = M^i$

From (A9) we have:

$$f = AT_{,t}M + R(t) = AT_{,t}M + R(t)$$

The:

$$(AT_{,t})_{,t} = mAT \Rightarrow AT_{,t} = m \int AT dt - C_2$$

$$m \int AT dt = AT_{,t} + C_2$$

We compute:

$$f_{,t} = (AT_{,t})_{,t}S + R_{,t} = mATS + R_{,t}$$

We use (A8) and find:

$$AT\mathbf{Y}(V) + DATV = -mATS - R_{,t}$$

$$-mATS - R_{,t} = AT(-C_1 - mATS)$$

$$R_{,t} = ATC_1$$

$$R = C_1 \int AT dt$$

Finally:

$$f = AT_{,t}M + C_1 \int AT dt \quad (\text{A19})$$

Assume $T_{,t} = 0 \Rightarrow T = T = a_1$.

It follows that $f_{,i} = 0 \Rightarrow f(t)$. Then:

$$AT\mathbf{Y}(V) + (A\xi^0)_{,t}V = -f_{,t}$$

$$-f_{,t} = AT(\mathbf{Y}(V) + DV)$$

It follows that $(\mathbf{Y}(V) + DV)_{,i} = 0$ therefore $mY_i = 0$ and $m = 0$. Also

$$\mathbf{Y}(V) + DV = C_1$$

Then

$$f(t) = -C_1 a_1 \int A dt \quad (\text{A20})$$

The Noether vector

For $T_{,t} \neq 0$ the Noether vector is:

$$X = \xi^0 \partial_t + TY^i \partial_i = \left(\frac{D-2\psi}{2(\ln A)_{,t}} \partial_t + Y^i \partial_i \right) T \quad (\text{A21})$$

and the Noether integral:

$$I = -\xi^i \frac{\partial L}{\partial \dot{x}^i} + \xi^0 H + f$$

where $H = \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i - L$ is the Hamiltonian. We compute:

$$H = A \left(\frac{1}{2} \dot{x}^i \dot{x}_i + V(x) \right)$$

therefore:

$$I = AT \left(-Y^i \dot{x}^i + \frac{D-2\psi}{2(\ln A)_{,t}} \left(\frac{1}{2} \dot{x}^i \dot{x}_i + V(x) \right) \right) + f. \quad (\text{A22})$$

This completes the proof of the Proposition.

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