

Short Note

Not peer-reviewed version

Non-Archimedean Hudzik-Landes- Dragomir-Kato-Saito-Tamura Inequality

[K. Mahesh Krishna](#) *

Posted Date: 16 March 2026

doi: 10.20944/preprints202601.1097.v2

Keywords: normed linear space; triangle inequality; non-Archimedean linear space; ultra-norm



Preprints.org is a free multidisciplinary platform providing preprint service that is dedicated to making early versions of research outputs permanently available and citable. Preprints posted at Preprints.org appear in Web of Science, Crossref, Google Scholar, Scilit, Europe PMC.

Copyright: This open access article is published under a [Creative Commons CC BY 4.0 license](#), which permit the free download, distribution, and reuse, provided that the author and preprint are cited in any reuse.

Disclaimer/Publisher's Note: The statements, opinions, and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions, or products referred to in the content.

Short Note

Non-Archimedean Hudzik-Landes-Dragomir-Kato-Saito-Tamura Inequality

K. Mahesh Krishna

School of Mathematics and Natural Sciences, Chanakya University Global Campus, NH-648, Haraluru Village, Devanahalli Taluk, Bengaluru North District, Karnataka State, 562 110, India; kmaheshak@gmail.com

Abstract

In 1992, Hudzik and Landes [Math. Ann.] derived a breakthrough generalization of the triangle inequality for two nonzero elements in normed linear spaces, which was generalized to finitely many nonzero elements independently in 2006 by Dragomir [Bull. Aust. Math. Soc.] and in 2007 by Kato, Saito and Tamura [Math. Inequal. Appl]. We derive a non-Archimedean version of Hudzik-Landes-Dragomir-Kato-Saito-Tamura inequality.

Keywords: normed linear space; triangle inequality; non-Archimedean linear space; ultra-norm

MSC: 12J25; 46S10

1. Introduction

Let $\mathcal{X}$ be a normed linear space (NLS). From the definition of the norm, we have the triangle inequality

$$\|x + y\| \leq \|x\| + \|y\|, \quad \forall x, y \in \mathcal{X}. \tag{1}$$

In 1992, Hudzik and Landes derived a breakthrough generalization of Inequality (1) which is valid for any two nonzero elements in a **NLS** [2].

Theorem 1. [2] (*Hudzik-Landes Inequality*) Let $\mathcal{X}$ be a NLS. Then for all $x, y \in \mathcal{X} \setminus \{0\}$,

$$\|x + y\| \leq \|x\| + \|y\| - \left(2 - \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right) \min\{\|x\|, \|y\|\}. \tag{2}$$

We note that, in 2006, Maligranda independently derived Inequality (2) [4]. It is natural to ask for a generalization of Inequality (2) to more than two non-zero vectors. This is done independently by Dragomir in 2006 [1] and by Kato, Saito and Tamura in 2007 [3].

Theorem 2. [1,3] (*Hudzik-Landes-Dragomir-Kato-Saito-Tamura Inequality*) Let $\mathcal{X}$ be a NLS and $n \in \mathbb{N}$. Then for all $x_1, \dots, x_n \in \mathcal{X} \setminus \{0\}$, we have

$$\left\|\sum_{j=1}^n x_j\right\| \leq \sum_{j=1}^n \|x_j\| - \left(n - \left\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\right\|\right) \min_{1 \leq k \leq n} \|x_k\|.$$

It is natural and important to ask what are non-Archimedean versions of Theorems 1 and 2? We answer the question by deriving non-Archimedean version of Hudzik-Landes-Dragomir-Kato-Saito-Tamura Inequality (Theorem 4).

2. Non-Archimedean Hudzik-Landes-Dragomir-Kato-Saito-Tamura Inequality

Let $\mathbb{K}$ be a field. Recall that a map $|\cdot| : \mathbb{K} \rightarrow [0, \infty)$ is said to be a non-Archimedean valuation if following conditions holds.

- (i) If $\lambda \in \mathbb{K}$ is such that $|\lambda| = 0$, then $\lambda = 0$.
- (ii) $|\lambda\mu| = |\lambda||\mu|$ for all $\lambda, \mu \in \mathbb{K}$.
- (iii) (Ultra-triangle inequality) $|\lambda + \mu| \leq \max\{|\lambda|, |\mu|\}$ for all $\lambda, \mu \in \mathbb{K}$.

In this case, $\mathbb{K}$ is called as non-Archimedean valued field [6]. Let $\mathcal{X}$ be a vector space over a non-Archimedean valued field $\mathbb{K}$ with valuation $|\cdot|$. Recall that a map $\|\cdot\| : \mathcal{X} \rightarrow [0, \infty)$ is said to be a non-Archimedean norm if following conditions holds.

- (i) If $x \in \mathcal{X}$ is such that $\|x\| = 0$, then $x = 0$.
- (ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{K}$, for all $x \in \mathcal{X}$.
- (iii) (Ultra-norm inequality) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in \mathcal{X}$.

In this case, $\mathcal{X}$ is called as non-Archimedean linear space (NALS) [5]. We first derive non-Archimedean version of Inequality (2).

Theorem 3. (Non-Archimedean Hudzik-Landes Inequality) Let $\mathcal{X}$ be a NALS over $\mathbb{K}$. Then for all $x, y \in \mathcal{X} \setminus \{0\}$ with $\|x\|, \|y\| \in \mathbb{K}$ it holds

$$\|x + y\| \leq \min\left\{\|x\| \max\left\{\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|, \left|\frac{1}{\|x\|} - \frac{1}{\|y\|}\right|\|y\|\right\}, \|y\| \max\left\{\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|, \left|\frac{1}{\|x\|} - \frac{1}{\|y\|}\right|\|x\|\right\}\right\}.$$

Proof. Let $x, y \in \mathcal{X} \setminus \{0\}$ with $\|x\|, \|y\| \in \mathbb{K}$. Then

$$\begin{aligned}\|x + y\| &= \left\| \|x\| \left(\frac{x}{\|x\|} + \frac{y}{\|y\|} \right) + \left(1 - \frac{\|x\|}{\|y\|} \right) y \right\| \\ &\leq \max\left\{ \left\| \|x\| \left(\frac{x}{\|x\|} + \frac{y}{\|y\|} \right) \right\|, \left\| \left(1 - \frac{\|x\|}{\|y\|} \right) y \right\| \right\} \\ &= \max\left\{ \|x\| \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|, \left\| x \right\| \left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| \|y\| \right\}\end{aligned}$$

and

$$\begin{aligned}\|x + y\| &= \left\| \left(1 - \frac{\|y\|}{\|x\|} \right) x + \|y\| \left(\frac{x}{\|x\|} + \frac{y}{\|y\|} \right) \right\| \\ &\leq \max\left\{ \left\| \left(1 - \frac{\|y\|}{\|x\|} \right) x \right\|, \left\| \|y\| \left(\frac{x}{\|x\|} + \frac{y}{\|y\|} \right) \right\| \right\} \\ &\leq \max\left\{ \|y\| \left| \frac{1}{\|y\|} - \frac{1}{\|x\|} \right| \|x\|, \|y\| \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| \right\}.\end{aligned}$$

Therefore

$$\|x + y\| \leq \|x\| \max\left\{ \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|, \left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| \|y\| \right\} \quad (3)$$

and

$$\|x + y\| \leq \|y\| \max\left\{ \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|, \left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| \|x\| \right\}. \quad (4)$$

Inequalities (3) and (4) give

$$\|x + y\| \leq \min\left\{ \|x\| \max\left\{ \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|, \left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| \|y\| \right\}, \|y\| \max\left\{ \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|, \left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| \|x\| \right\} \right\}.$$

□

Note the additional assumption $\|x\|, \|y\| \in \mathbb{K}$ in the previous theorem. The reason is that, since the norm is a real number, we generally do not have a guarantee that it belongs to the given non-Archimedean field. Now we derive non-Archimedean version of Theorem 2.

Theorem 4. (Non-Archimedean Hudzik-Landes-Dragomir-Kato-Saito-Tamura Inequality) Let $\mathcal{X}$ be a NALS over $\mathbb{K}$ and $n \in \mathbb{N}$. Then for all $x_1, \dots, x_n \in \mathcal{X} \setminus \{0\}$ with $\|x_1\|, \dots, \|x_n\| \in \mathbb{K}$ it holds

$$\left\| \sum_{j=1}^n x_j \right\| \leq \min_{1 \leq k \leq n} \left\{ \|x_k\| \max \left\{ \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\|, \max_{1 \leq j \leq n} \left| \frac{1}{\|x_j\|} - \frac{1}{\|x_k\|} \right| \|x_j\| \right\} \right\}.$$

Proof. Let $x_1, \dots, x_n \in \mathcal{X} \setminus \{0\}$ with $\|x_1\|, \dots, \|x_n\| \in \mathbb{K}$. Let $1 \leq k \leq n$ be fixed. Then

$$\begin{aligned} \left\| \sum_{j=1}^n x_j \right\| &= \left\| \sum_{j=1}^n \frac{\|x_k\| x_j}{\|x_j\|} + \sum_{j=1}^n \left(1 - \frac{\|x_k\|}{\|x_j\|} \right) x_j \right\| \\ &\leq \max \left\{ \left\| \sum_{j=1}^n \frac{\|x_k\| x_j}{\|x_j\|} \right\|, \left\| \sum_{j=1}^n \left(1 - \frac{\|x_k\|}{\|x_j\|} \right) x_j \right\| \right\} \\ &= \max \left\{ \|x_k\| \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\|, \left\| \sum_{j=1}^n \left(\frac{1}{\|x_k\|} - \frac{1}{\|x_j\|} \right) x_j \right\| \right\} \\ &= \|x_k\| \max \left\{ \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\|, \left\| \sum_{j=1}^n \left(\frac{1}{\|x_k\|} - \frac{1}{\|x_j\|} \right) x_j \right\| \right\} \\ &\leq \|x_k\| \max \left\{ \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\|, \max_{1 \leq j \leq n} \left| \frac{1}{\|x_k\|} - \frac{1}{\|x_j\|} \right| \|x_j\| \right\} \\ &= \|x_k\| \max \left\{ \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\|, \max_{1 \leq j \leq n} \left| \frac{1}{\|x_k\|} - \frac{1}{\|x_j\|} \right| \|x_j\| \right\}. \end{aligned}$$

By varying k and taking minimum in the right side of previous inequality gives

$$\left\| \sum_{j=1}^n x_j \right\| \leq \min_{1 \leq k \leq n} \left\{ \|x_k\| \max \left\{ \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\|, \max_{1 \leq j \leq n} \left| \frac{1}{\|x_j\|} - \frac{1}{\|x_k\|} \right| \|x_j\| \right\} \right\}.$$

□

Now we derive continuous version of Theorem 4.

Theorem 5. Let $\mathcal{X}$ be a NALS over $\mathbb{K}$ and (Ω, μ) be a non-Archimedean measure space. Let $\phi : \Omega \rightarrow \mathcal{X} \setminus \{0\}$ be a measurable function such that $\|\phi(\alpha)\| \in \mathbb{K}$ for every $\alpha \in \Omega$. Then

$$\left\| \int_{\Omega} \phi(\alpha) d\mu(\alpha) \right\| \leq \inf_{\beta \in \Omega} \left\{ \|\phi(\beta)\| \max \left\{ \left\| \int_{\Omega} \frac{\phi(\alpha)}{\|\phi(\alpha)\|} d\mu(\alpha) \right\|, \sup_{\alpha \in \Omega} \left| \frac{1}{\|\phi(\alpha)\|} - \frac{1}{\|\phi(\beta)\|} \right| \|\phi(\alpha)\| \right\} \right\}.$$

Proof. Let $\phi : \Omega \rightarrow \mathcal{X} \setminus \{0\}$ be a measurable function such that $\|\phi(\alpha)\| \in \mathbb{K}$ for every $\alpha \in \Omega$. Let $\beta \in \Omega$ be fixed. Then

$$\begin{aligned} \left\| \int_{\Omega} \phi(\alpha) d\mu(\alpha) \right\| &= \left\| \int_{\Omega} \frac{\|\phi(\beta)\|\phi(\alpha)}{\|\phi(\alpha)\|} d\mu(\alpha) + \int_{\Omega} \left(1 - \frac{\|\phi(\beta)\|}{\|\phi(\alpha)\|}\right) \phi(\alpha) d\mu(\alpha) \right\| \\ &\leq \max \left\{ \left\| \int_{\Omega} \frac{\|\phi(\beta)\|\phi(\alpha)}{\|\phi(\alpha)\|} d\mu(\alpha) \right\|, \left\| \int_{\Omega} \left(1 - \frac{\|\phi(\beta)\|}{\|\phi(\alpha)\|}\right) \phi(\alpha) d\mu(\alpha) \right\| \right\} \\ &= \max \left\{ \|\phi(\beta)\| \left\| \int_{\Omega} \frac{\phi(\alpha)}{\|\phi(\alpha)\|} d\mu(\alpha) \right\|, \left\| \int_{\Omega} \left(\frac{1}{\|\phi(\beta)\|} - \frac{1}{\|\phi(\alpha)\|} \right) \phi(\alpha) d\mu(\alpha) \right\| \right\} \\ &= \|\phi(\beta)\| \max \left\{ \left\| \int_{\Omega} \frac{\phi(\alpha)}{\|\phi(\alpha)\|} d\mu(\alpha) \right\|, \left\| \int_{\Omega} \left(\frac{1}{\|\phi(\beta)\|} - \frac{1}{\|\phi(\alpha)\|} \right) \phi(\alpha) d\mu(\alpha) \right\| \right\} \\ &\leq \|\phi(\beta)\| \max \left\{ \left\| \int_{\Omega} \frac{\phi(\alpha)}{\|\phi(\alpha)\|} d\mu(\alpha) \right\|, \sup_{\alpha \in \Omega} \left\| \left(\frac{1}{\|\phi(\beta)\|} - \frac{1}{\|\phi(\alpha)\|} \right) \phi(\alpha) \right\| \right\} \\ &= \|\phi(\beta)\| \max \left\{ \left\| \int_{\Omega} \frac{\phi(\alpha)}{\|\phi(\alpha)\|} d\mu(\alpha) \right\|, \sup_{\alpha \in \Omega} \left| \frac{1}{\|\phi(\beta)\|} - \frac{1}{\|\phi(\alpha)\|} \right| \|\phi(\alpha)\| \right\}. \end{aligned}$$

By varying β and taking infimum in the previous inequality gives

$$\left\| \int_{\Omega} \phi(\alpha) d\mu(\alpha) \right\| \leq \inf_{\beta \in \Omega} \left\{ \|\phi(\beta)\| \max \left\{ \left\| \int_{\Omega} \frac{\phi(\alpha)}{\|\phi(\alpha)\|} d\mu(\alpha) \right\|, \sup_{\alpha \in \Omega} \left| \frac{1}{\|\phi(\alpha)\|} - \frac{1}{\|\phi(\beta)\|} \right| \|\phi(\alpha)\| \right\} \right\}.$$

□

3. Conclusions

- (1) In 1992, Hudzik and Landes improved centuries old triangle inequality in normed linear spaces [2].
- (2) In 2006, Dragomir extended Hudzik-Landes inequality for more than two vectors [1].
- (3) In 2007, Kato, Saito and Tamura extended Hudzik-Landes inequality without knowing the work of Dragomir [3].
- (4) In this article, we extended centuries old ultra-norm inequality.

References

1. Sever S. Dragomir. Bounds for the normalised Jensen functional. *Bull. Aust. Math. Soc.*, 74(3):471–478, 2006.
2. Henryk Hudzik and Thomas R. Landes. Characteristic of convexity of Köthe function spaces. *Math. Ann.*, 294(1):117–124, 1992.
3. Mikio Kato, Kichi-Suke Saito, and Takayuki Tamura. Sharp triangle inequality and its reverse in Banach spaces. *Math. Inequal. Appl.*, 10(2):451–460, 2007.
4. Lech Maligranda. Simple norm inequalities. *Am. Math. Mon.*, 113(3):256–260, 2006.
5. C. Perez-Garcia and W. H. Schikhof. *Locally convex spaces over non-Archimedean valued fields*, volume 119 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge University Press, 2010.
6. W. H. Schikhof. *Ultrametric calculus. An introduction to p-adic analysis*, volume 4 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge University Press, 2006.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.