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Article

Fuzzy, Neutrosophic, Plithogenic, Rough, Granular, and Functorial Ordered and Ranking Numbers

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Abstract

This paper develops a unified framework for uncertainty numbers by introducing *ordered* and *ranking* structures across six paradigms: fuzzy, neutrosophic, plithogenic, rough, granular, and functorial numbers. We define ordered variants via monotone boundary curves, chains of approximations, and diagrammatic (functorial) morphisms, and we prove that ordered neutrosophic and plithogenic numbers strictly generalize ordered fuzzy numbers. On the comparative side, we specify ranking functionals that are monotone with respect to natural dominance relations and show that ranking neutrosophic numbers recover classical rankings of fuzzy numbers, while ranking plithogenic numbers subsume both. We also formalize ordered and ranking versions of rough and granular numbers and establish retention of their native structures under projection. Finally, we introduce *Ordered/Ranking Functorial Numbers*, which organize all models as semiring-valued diagrams, yielding embedding and stability theorems and illustrative examples.

Keywords: fuzzy number; neutrosophic number; plithogenic number; rough number; granular number; functorial number

1. Preliminaries

This section gathers the numerical uncertainty models used in the paper. Unless stated otherwise, functions are defined on the real line $\mathbb{R}$.

1.1. Fuzzy Number and Neutrosophic Number

A fuzzy number represents an imprecise real quantity using a membership function $\mu : \mathbb{R} \rightarrow [0, 1]$ that is normal, convex, upper semicontinuous, and has compact support [1]. Related concepts also include the *intuitionistic fuzzy number* [2,3] and the *picture fuzzy number* [4,5], both of which have been extensively investigated as important extensions of fuzzy numerical representations. A neutrosophic number extends this concept by assigning to each $x \in \mathbb{R}$ three degrees: truth $T(x)$, indeterminacy $I(x)$, and falsity $F(x)$ [6,7]. Related concepts include the *trapezoidal neutrosophic number* [8–11] and the *triangular neutrosophic number* [12–15], which are widely studied in the literature.

Definition 1 (Fuzzy Number). [1] A fuzzy number is a fuzzy set A on $\mathbb{R}$ with membership $\mu_A : \mathbb{R} \rightarrow [0, 1]$ satisfying:

- (i) **Normality:** $\max_{x \in \mathbb{R}} \mu_A(x) = 1$.
- (ii) **Convexity:** for all $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$,

$$\mu_A(\lambda x + (1 - \lambda)y) \geq \min\{\mu_A(x), \mu_A(y)\}.$$

- (iii) **Upper semi-continuity:** μ_A is u.s.c. on $\mathbb{R}$.
- (iv) **Compact support:** $\{x \in \mathbb{R} \mid \mu_A(x) > 0\}$ is compact.

Definition 2 (Single-Valued Neutrosophic Number on $\mathbb{R}$). [6,7] A single-valued neutrosophic number (SVNN) on $\mathbb{R}$ is a triple $N = (T, I, F)$ with $T, I, F : \mathbb{R} \rightarrow [0, 1]$. For each $x \in \mathbb{R}$, $T(x)$, $I(x)$, and $F(x)$ denote, respectively, the degrees of truth, indeterminacy, and falsity, with

$$0 \leq T(x) + I(x) + F(x) \leq 3.$$

A neutrosophic number is normal if $\sup_x T(x) = 1$ and $\inf_x F(x) = 0$, convex if each α -cut of T is convex while the β -superlevel of F is convex (dually), and well-posed if T, I, F are u.s.c. and have compact supports (whenever appropriate).

Example 1 (Canonical triangular instances). A triangular fuzzy number with mode m and spreads $a, b > 0$ is

$$\mu_A(x) = \begin{cases} \frac{x - (m - a)}{a}, & x \in [m - a, m], \\ \frac{(m + b) - x}{b}, & x \in (m, m + b], \\ 0, & \text{otherwise.} \end{cases}$$

A triangular SVNN can be specified by (T, I, F) where T is triangular as above, I is a (possibly wider) triangular cap centered at m , and F is the pointwise reflection $F(x) = 1 - T(x)$ (or a wider trapezoid), yielding a compact, u.s.c. triple.

1.2. Plithogenic Number

Plithogenic theory augments membership with explicit attributes and a contradiction degree between attribute values, guiding aggregation and reasoning [16,17].

Definition 3 (Plithogenic Number). [16] Fix an attribute v with value domain V and a symmetric contradiction map $c : V \times V \rightarrow [0, 1]$ with $c(a, a) = 0$. A plithogenic number on $\mathbb{R}$ is a family $\{\mu_a\}_{a \in V}$ of membership functions $\mu_a : \mathbb{R} \rightarrow [0, 1]$ together with an aggregation operator Agg (t -norm/ t -conorm-based) that depends on c :

$$\mu_{\text{pl}}(x) = \text{Agg}(\{\mu_a(x)\}_{a \in V}; c) \in [0, 1], \quad x \in \mathbb{R}.$$

It is normal/convex/u.s.c./compactly supported when μ_{pl} inherits these properties from $\{\mu_a\}_{a \in V}$ under Agg .

Example 2 (Two-attribute aggregation). Let $V = \{\text{cost}, \text{performance}\}$, with triangular $\mu_{\text{cost}}, \mu_{\text{perf}}$ on $\mathbb{R}$. Suppose $c(\text{cost}, \text{perf}) = \gamma \in (0, 1)$ and define

$$\mu_{\text{pl}}(x) = (1 - \gamma) \min\{\mu_{\text{cost}}(x), \mu_{\text{perf}}(x)\} \oplus \gamma \max\{\mu_{\text{cost}}(x), \mu_{\text{perf}}(x)\},$$

where $\oplus$ is a t -conorm (e.g., $\max$). Then μ_{pl} is a normal, convex, u.s.c. membership on $\mathbb{R}$ whenever the constituents are, yielding a well-posed plithogenic number.

1.3. Rough Number

A rough number is an interval of lower and upper mean attribute values over rough approximations under an equivalence relation [18–20]. Extensions of the rough number, such as the Fuzzy Rough Number and Neutrosophic Rough Number, have also been studied[21–26].

Definition 4 (Universe and Equivalence Classes). (cf.[18]) Let U be a nonempty finite set (the universe). Let

$$R \subseteq U \times U$$

be an equivalence relation on U . For each $x \in U$, denote its equivalence class by

$$[x]_R = \{y \in U \mid (x, y) \in R\}.$$

Definition 5 (Lower and Upper Approximations). (cf.[18]) For any $G \subseteq U$, define its lower approximation and upper approximation with respect to R by

$$\underline{G} = \{ x \in U \mid [x]_R \subseteq G \}, \quad \overline{G} = \{ x \in U \mid [x]_R \cap G \neq \emptyset \}.$$

Here $\underline{G}$ collects all elements that definitely belong to G , while $\overline{G}$ collects those that possibly belong.

Definition 6 (Rough Number). (cf.[18,27,28]) Let $R = \{G_1, G_2, \dots, G_t\}$ be an ordered partition of U with $G_1 < \dots < G_t$. For each class G_q , define:

$$\text{Lim}^L(G_q) = \frac{1}{|\underline{G_q}|} \sum_{y \in \underline{G_q}} R(y), \quad \text{Lim}^U(G_q) = \frac{1}{|\overline{G_q}|} \sum_{y \in \overline{G_q}} R(y).$$

Then the rough number of G_q is the interval

$$\text{RN}(G_q) = [\text{Lim}^L(G_q), \text{Lim}^U(G_q)].$$

Example 3 (Rough Number — Affordable houses by neighborhood (fully worked)). *Universe and equivalence.* Let $U = \{h_1, \dots, h_8\}$ be eight houses. Neighborhood (indiscernibility) classes are

$$C_1 = \{h_1, h_2, h_3\}, \quad C_2 = \{h_4, h_5\}, \quad C_3 = \{h_6, h_7, h_8\}.$$

Let the numeric attribute be the listing price (in \$1000):

house	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8
price $a(\cdot)$	360	390	420	380	395	440	430	470

Target set (“affordable”). Define

$$G = \{x \in U : a(x) \leq 400\} = \{h_1, h_2, h_4, h_5\}.$$

Lower/upper approximations w.r.t. neighborhood classes:

$$\underline{G} = \{x : [x] \subseteq G\} = C_2 = \{h_4, h_5\}, \quad \overline{G} = \{x : [x] \cap G \neq \emptyset\} = C_1 \cup C_2 = \{h_1, h_2, h_3, h_4, h_5\}.$$

Rough number (mean prices). Using $\text{Lim}^L(G) = \frac{1}{|\underline{G}|} \sum_{y \in \underline{G}} a(y)$, $\text{Lim}^U(G) = \frac{1}{|\overline{G}|} \sum_{y \in \overline{G}} a(y)$, we compute

$$\begin{aligned} \text{Lim}^L(G) &= \frac{380 + 395}{2} = \frac{775}{2} = 387.5, \\ \text{Lim}^U(G) &= \frac{360 + 390 + 420 + 380 + 395}{5} = \frac{1945}{5} = 389.0. \end{aligned}$$

Hence the rough number of “affordable” is

$$\text{RN}(G) = [387.5, 389.0] \text{ } (\$1000).$$

Based on neighborhood indistinguishability, the definitely-affordable mean price is \$387.5k (from C_2 only), while the possibly-affordable mean rises to \$389.0k when including mixed neighborhoods (C_1) that contain both affordable and non-affordable houses.

1.4. Ordered Fuzzy Number

An ordered fuzzy number represents uncertainty by two continuous boundary functions over $[0,1]$, producing increasing and decreasing membership curves [29,30].

Definition 7 (Ordered Fuzzy Number). [31–34] An ordered fuzzy number (also called an ordered fuzzy real) is an ordered pair

$$A = (\mu_A^\uparrow, \mu_A^\downarrow),$$

where the functions $\mu_A^\uparrow, \mu_A^\downarrow: [0, 1] \rightarrow \mathbb{R}$ are continuous. The corresponding membership curves are

$$\text{up}_A = \{(\mu_A^\uparrow(y), y) : y \in [0, 1]\}, \quad \text{down}_A = \{(\mu_A^\downarrow(y), y) : y \in [0, 1]\}.$$

Example 4 (Ordered Fuzzy Number: commute time). A commuter says the “most typical” door-to-door time is $m = 45$ minutes, rarely below 30 or above 60. As an ordered fuzzy number with triangular α -cuts, take

$$\mu^\uparrow(y) = L(y) = 30 + 15y, \quad \mu^\downarrow(y) = R(y) = 60 - 15y, \quad y \in [0, 1],$$

so the α -cut is $[L(\alpha), R(\alpha)] = [30 + 15\alpha, 60 - 15\alpha]$.

1.5. Ranking Fuzzy Number

A ranking fuzzy number is a fuzzy numerical entity whose order is determined by a ranking function or relation that induces a total preorder among fuzzy numbers, typically through scoring integrals or possibility measures[35–39]. This ordering is consistent with the natural order on real numbers, ensuring that when the fuzzy numbers degenerate to crisp values, their ranking coincides with the classical numerical comparison.

Definition 8 (Ranking of fuzzy numbers). Let $\mathcal{F}$ be a class of fuzzy numbers on $\mathbb{R}$ (normalized, convex, u.s.c.). A ranking is either

- (i) a map $R: \mathcal{F} \rightarrow \mathbb{R}$ inducing a total preorder $A \preceq_R B \iff R(A) \leq R(B)$, or
- (ii) a complete, transitive binary relation $\preceq$ on $\mathcal{F}$ itself.

It is order-consistent if for crisp $a \leq b$ (seen as degenerate fuzzy numbers δ_a, δ_b) we have $\delta_a \preceq \delta_b$. Typical realizations compute R by integrating the possibility distributions against a user viewpoint (satisfaction-function approach), or by possibility/necessity measures in possibility theory.¹

Example 5 (Ranking Fuzzy Number: choosing a laptop). A buyer scores two laptops by overall satisfaction (in points) as triangular fuzzy numbers:

$$A = \text{Tri}(70, 80, 90), \quad B = \text{Tri}(65, 78, 92).$$

Using the centroid ranking $R(\text{Tri}(a, m, b)) = \frac{a+m+b}{3}$,

$$R(A) = \frac{70+80+90}{3} = 80 \quad \text{and} \quad R(B) = \frac{65+78+92}{3} = 78.\bar{3},$$

so $A \succ B$.

1.6. Grey Number

A grey number denotes an unknown real value constrained between known bounds, represented as interval $[g^-, g^+]$, modeling incomplete information (cf.[40–45]).

Definition 9 (Grey Number). [46,47] A grey number is a real quantity constrained by known lower and upper bounds $g^- \leq x \leq g^+$, while the exact position x within $[g^-, g^+]$ is unknown; we denote it by $g^\pm := [g^-, g^+]$.

¹ See, e.g., satisfaction-function ranking and viewpoint dependence; and possibility-theoretic comparisons.

Example 6 (Grey Number: monthly electric bill). *Before the meter is read, the household estimates the bill will be between \$80 and \$120. The unknown true amount x is modeled as the grey number*

$$g^{\pm} = [g^-, g^+] = [80, 120].$$

1.7. Granular Numbers

A granular number encodes a connected set of possible values, typically as center and radius, optionally weighted, capturing structured imprecision [48–51].

Definition 10 (Granular Number (G-number)). *A granular number is an extension of an ordinary number: instead of a single value, it denotes a connected set of possible values. A unified representation uses a center and radius, $X = G(c_X, r_X)$, optionally augmented by a granule weight $A(X)$ as $X = G_{A(X)}(c_X, r_X)$.*

Example 7 (Granular Number: thermostat comfort band). *An office targets 22°C with a tolerated band of $\pm 1^\circ\text{C}$ and sensor reliability weight $A = 0.8$. As a granular number,*

$$X = G_{0.8}(c_X, r_X) = G_{0.8}(22, 1),$$

representing the connected set $[21, 23]$ with weight 0.8.

1.8. Interval Number

An interval number is a closed real interval $[a, b]$, representing all possible values between both lower and upper bounds, inclusively [52,53].

Definition 11 (Interval Number). *An interval number is a nonempty closed interval of the real line,*

$$A = [a_L, a_R] = \{a \in \mathbb{R} \mid a_L \leq a \leq a_R\},$$

with $a_L \leq a_R$. The degenerate case $a_L = a_R$ identifies $A = [a, a]$ with the real number a . Equivalently, one writes $A = \langle m(A), w(A) \rangle$ with midpoint and half-width

$$m(A) = \frac{a_L + a_R}{2}, \quad w(A) = \frac{a_R - a_L}{2}.$$

Example 8 (Real-Life Example). • *Thermometer with $\pm 0.5^\circ\text{C}$ accuracy reading 22°C : interval $[21.5, 22.5]^\circ\text{C}$.*

- *Commute time estimate between 35 and 50 minutes: interval $[35, 50]$ min.*
- *Price quote with possible fluctuation between \$480 and \$520: interval $[480, 520]$ USD.*
- *Board length 200 cm with ± 0.2 cm tolerance: interval $[199.8, 200.2]$ cm.*

1.9. Functorial Numbers

A functorial number assigns each object a commutative semiring naturally, with operations preserved along morphisms[54].

Definition 12 (Functorial Number). *Let $\mathcal{C}$ be a category with finite products. A Functorial Number is a tuple*

$$(N, \oplus, \otimes, 0, 1)$$

where $N : \mathcal{C} \rightarrow \mathbf{Set}$ is a functor, $\oplus, \otimes : N \times N \Rightarrow N$ and $0, 1 : \Delta 1 \Rightarrow N$ are natural transformations, such that for each object X ,

$$(N(X), \oplus_X, \otimes_X, 0_X, 1_X)$$

is a commutative semiring, and for every morphism $f : X \rightarrow Y$ and $a, b \in N(X)$,

$$N(f)(a \oplus_X b) = N(f)a \oplus_Y N(f)b, \quad N(f)(a \otimes_X b) = N(f)a \otimes_Y N(f)b, \quad N(f)(0_X) = 0_Y, \quad N(f)(1_X) = 1_Y.$$

Equivalently, a Functorial Number is a functor $\tilde{N} : \mathcal{C} \rightarrow \mathbf{CRig}$ (commutative semirings and homomorphisms), composed with the forgetful functor to **Set**.

Example 9 (Free polynomial functor (nontrivial)). Let $\mathcal{C} = \mathbf{Set}$ (which has finite products). Define

$$N(X) := \mathbb{N}[X]$$

the commutative polynomial semiring with variables indexed by the set X ; addition and multiplication are the usual polynomial $+$ and $\cdot$, and

$$\oplus_X := +, \quad \otimes_X := \cdot, \quad 0_X := 0, \quad 1_X := 1.$$

For a function $f : X \rightarrow Y$, let $N(f) : \mathbb{N}[X] \rightarrow \mathbb{N}[Y]$ be the unique semiring homomorphism sending each variable $x \in X$ to the variable $f(x) \in Y$ and acting linearly/multiplicatively on monomials. Then for all $a, b \in \mathbb{N}[X]$,

$$N(f)(a \oplus_X b) = N(f)(a + b) = N(f)a + N(f)b = (N(f)a) \oplus_Y (N(f)b),$$

$$N(f)(a \otimes_X b) = N(f)(a \cdot b) = (N(f)a) \cdot (N(f)b) = (N(f)a) \otimes_Y (N(f)b), \quad N(f)(0_X) = 0_Y, \quad N(f)(1_X) = 1_Y.$$

Hence $X \mapsto (\mathbb{N}[X], +, \cdot, 0, 1)$ and $f \mapsto N(f)$ define a functor $\tilde{N} : \mathbf{Set} \rightarrow \mathbf{CRig}$ (and thus a Functorial Number).

Concrete calculation. Let $X = \{u, v\}$, $Y = \{w\}$, and $f : X \rightarrow Y$ with $f(u) = f(v) = w$. For $p = 3u^2 + 2uv + 5 \in \mathbb{N}[X]$,

$$N(f)(p) = 3w^2 + 2w \cdot w + 5 = 5w^2 + 5 \in \mathbb{N}[Y].$$

Example 10 (Constant natural-number semiring (simple but fully functorial)). Let $\mathcal{C}$ be any category with finite products. Define the constant functor $N : \mathcal{C} \rightarrow \mathbf{Set}$ by $N(X) = \mathbb{N}$ for every object X , and $N(f) = \text{id}_{\mathbb{N}}$ for every morphism f . Give each $N(X)$ the standard commutative semiring structure

$$\oplus_X := +, \quad \otimes_X := \times, \quad 0_X := 0, \quad 1_X := 1.$$

Then for all $f : X \rightarrow Y$ in $\mathcal{C}$ and $a, b \in \mathbb{N}$,

$$N(f)(a \oplus_X b) = a + b = (N(f)a) \oplus_Y (N(f)b), \quad N(f)(a \otimes_X b) = a \times b = (N(f)a) \otimes_Y (N(f)b),$$

$$N(f)(0_X) = 0_Y = 0, \quad N(f)(1_X) = 1_Y = 1.$$

Thus N factors as $\mathcal{C} \xrightarrow{!} \mathbf{1} \xrightarrow{\mathbb{N}} \mathbf{CRig} \rightarrow \mathbf{Set}$, so it is a Functorial Number (a functor into **CRig** followed by the forgetful functor).

1.10. Hyperstructure and SuperHyperStructure

Many mathematical and real-world concepts exhibit inherently hierarchical structures. To capture and represent such multi-level relationships in a clear and systematic way, the notions of *hyperstructure* and *superhyperstructure* have been introduced. A *hyperstructure* generalizes a classical algebraic structure by replacing the underlying set S with its powerset $\mathcal{P}(S)$; hyperoperations then combine subsets into (possibly new) subsets, enabling the expression of higher-order relations [55–59]. Related concepts include HyperFuzzy Sets [60–62], HyperAlgebras [63,64], Chemical HyperStructure [65–69], and HyperGraphs [70–73].

Definition 13 (Base set). [74] Let S be a nonempty set, called the base set. Equivalently,

$$S = \{x \mid x \text{ is an element of the underlying universe}\}.$$

All further constructions—such as $\mathcal{P}(S)$ or its iterates $\mathcal{P}_n(S)$ —are formed from elements of S .

Definition 14 (Powerset). [75] The powerset of a set S , denoted $\mathcal{P}(S)$, is the collection of all subsets of S :

$$\mathcal{P}(S) = \{ A \mid A \subseteq S \},$$

including both the empty set $\emptyset$ and S itself.

Definition 15 (Hyperoperation). (cf. [57,76,77]) A hyperoperation is a generalization of a binary operation in which the result of combining two inputs is a set (not necessarily a singleton). Formally, for a set S , a hyperoperation $\circ$ is a map

$$\circ : S \times S \longrightarrow \mathcal{P}(S).$$

Definition 16 (Hyperstructure). (cf. [78–82]) A hyperstructure replaces the base set S by its powerset and is given by

$$\mathcal{H} = (\mathcal{P}(S), \circ),$$

where $\circ$ is a hyperoperation acting on subsets of S (i.e., $\circ : \mathcal{P}(S) \times \mathcal{P}(S) \rightarrow \mathcal{P}(S)$). This framework allows one to combine collections of elements into other collections.

Example 11 (Hyperstructure — Team Skills with Emergent Capabilities (real-life)). **Base set.** Let

$$S = \{\text{FE}, \text{BE}, \text{DS}, \text{DO}, \text{API}, \text{MLOps}, \text{FullStack}\},$$

where FE=frontend, BE=backend, DS=data science, DO=devops, API=API integration, MLOps=ML operations, FullStack=full-stack capability.

Synergy map. Define $\sigma : S \times S \rightarrow \mathcal{P}(S)$ by

$$\sigma(\text{FE}, \text{BE}) = \{\text{FullStack}\}, \quad \sigma(\text{BE}, \text{API}) = \{\text{API}\}, \quad \sigma(\text{DS}, \text{DO}) = \{\text{MLOps}\},$$

and $\sigma(a, b) = \emptyset$ otherwise. Extend to subsets $A, B \subseteq S$ by

$$A \circ B := A \cup B \cup \bigcup_{(a,b) \in A \times B} \sigma(a, b) \in \mathcal{P}(S).$$

Then $\mathcal{H} = (\mathcal{P}(S), \circ)$ is a hyperstructure.

Concrete composition. For

$$A = \{\text{FE}, \text{BE}\}, \quad B = \{\text{DS}, \text{DO}\},$$

we get

$$\begin{aligned} A \circ B &= \{\text{FE}, \text{BE}, \text{DS}, \text{DO}\} \cup \sigma(\text{FE}, \text{DS}) \cup \sigma(\text{FE}, \text{DO}) \\ &\quad \cup \underbrace{\sigma(\text{BE}, \text{DS})}_{\emptyset} \cup \underbrace{\sigma(\text{BE}, \text{DO})}_{\emptyset} \\ &= \{\text{FE}, \text{BE}, \text{DS}, \text{DO}, \text{FullStack}, \text{MLOps}\}. \end{aligned}$$

Thus combining two teams (skill sets) yields a set of attainable capabilities, including emergent ones FullStack, MLOps.

Example 12 (Hyperstructure — Document Tagging with Cross-Topic Emergence). **Base set.** Let

$$S = \{\text{AI}, \text{Health}, \text{Finance}, \text{Cloud}, \text{MedTech}, \text{FinTech}\}.$$

Cross-tag rule. Define $\sigma : S \times S \rightarrow \mathcal{P}(S)$ by

$$\sigma(\text{AI}, \text{Health}) = \{\text{MedTech}\},$$

$$\sigma(\text{AI}, \text{Finance}) = \{\text{FinTech}\},$$

$$\sigma(a, b) = \emptyset \text{ otherwise,}$$

and (symmetrically) $\sigma(b, a) = \sigma(a, b)$. For $A, B \subseteq S$ let the hyperoperation

$$A \circ B := A \cup B \cup \bigcup_{(a,b) \in A \times B} \sigma(a, b) \in \mathcal{P}(S).$$

Concrete composition. If a report has tag set

$$A = \{\text{AI}, \text{Cloud}\}, \quad B = \{\text{Health}\},$$

then

$$A \circ B = \{\text{AI}, \text{Cloud}, \text{Health}, \text{MedTech}\},$$

so $\mathcal{H} = (\mathcal{P}(S), \circ)$ promotes emergent cross-tags (e.g. MedTech) when topics are combined.

A *SuperHyperStructure* extends the concept of a hyperstructure by recursively applying the powerset construction n times. In this framework, operations act on nested collections, thereby enabling the modeling of hierarchical, multi-level interactions [57,59,83–86]. Related notions in the literature include the *SuperHyperGraph* [87–89] and the *SuperHyperUncertain Set* [90–93].

Definition 17 (Iterated powersets). (cf. [64,78,83,94]) For $n \geq 1$, define recursively

$$\mathcal{P}_1(S) = \mathcal{P}(S), \quad \mathcal{P}_{n+1}(S) = \mathcal{P}(\mathcal{P}_n(S)).$$

Similarly, the nonempty iterated powersets are

$$\mathcal{P}_1^*(S) = \mathcal{P}(S) \setminus \{\emptyset\}, \quad \mathcal{P}_{n+1}^*(S) = \mathcal{P}^*(\mathcal{P}_n^*(S)),$$

where $\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\emptyset\}$.

Definition 18 (SuperHyperOperation). [78] Let H be a nonempty set. Define recursively, for each integer $k \geq 0$,

$$\mathcal{P}^0(H) = H, \quad \mathcal{P}^{k+1}(H) = \mathcal{P}(\mathcal{P}^k(H)).$$

Fix $m, n \geq 0$. An (m, n) -SuperHyperOperation is a map

$$\circ^{(m,n)} : H^m \longrightarrow \mathcal{P}_*^n(H),$$

where $\mathcal{P}_*^n(H)$ denotes the n -th iterated powerset of H , either excluding the empty set (classical type) or including it (Neutrosophic type). In the former case we call $\circ^{(m,n)}$ a classical-type (m, n) -SuperHyperOperation; in the latter, a Neutrosophic (m, n) -SuperHyperOperation.

Definition 19 (SuperHyperStructure of order (m, n)). (cf. [57,95–97]) Let S be a nonempty set and $m, n \geq 0$. A (m, n) -SuperHyperStructure on S of arity k is a map

$$\star : \underbrace{\mathcal{P}^m(S) \times \cdots \times \mathcal{P}^m(S)}_{k \text{ factors}} \longrightarrow \mathcal{P}^n(S).$$

When $m = 0$ and $n = 0$, this recovers an ordinary k -ary operation on S ; when $m = 0$, $n = 1$, it is a k -ary hyperoperation; and when $k = 1$, it is an (m, n) -superhyperfunction.

Example 13 (SuperHyperStructure — Weekly Meal Planner ($m=1, n=2, k=2$)). **Base set.** Let the set of dishes be

$$S = \{\text{Salad, Pasta, Soup, StirFry, Curry}\}.$$

We build $\mathcal{P}(S)$ (sets of dishes) and $\mathcal{P}^2(S) = \mathcal{P}(\mathcal{P}(S))$ (sets of menus, each menu being a set of daily dish-sets).

Operation. Define

$$\star : \mathcal{P}(S) \times \mathcal{P}(S) \longrightarrow \mathcal{P}^2(S)$$

that maps two inputs A (available/preferred dishes) and B (supplemental/rotation dishes) to a set of weekly menus. For example, with

$$A = \{\text{Salad, Soup, StirFry}\}, \quad B = \{\text{Pasta, Curry}\},$$

set

$$M_1 = \{\{\text{Salad}\}, \{\text{Pasta}\}, \{\text{Soup}\}, \{\text{StirFry}\}, \{\text{Curry}\}\},$$

$$M_2 = \{\{\text{Soup}\}, \{\text{Curry}\}, \{\text{Salad}\}, \{\text{Pasta}\}, \{\text{StirFry}\}\},$$

and define

$$\star(A, B) = \{M_1, M_2\} \in \mathcal{P}^2(S).$$

Type check. Domain $(\mathcal{P}^1(S))^2$ and codomain $\mathcal{P}^2(S)$ match the $(m, n) = (1, 2)$ SuperHyperStructure specification; here $k = 2$.

Example 14 (SuperHyperStructure — Software Release Blueprints ($m=1, n=2, k=2$)). **Base set.** Let features/modules be

$$S = \{\text{Auth, Payments, Search, Analytics, Export}\}.$$

Then $\mathcal{P}(S)$ are feature sets (candidate bundles), and $\mathcal{P}^2(S)$ are sets of blueprints (each blueprint is a set of bundles, e.g. per milestone).

Operation. Define

$$\star : \mathcal{P}(S) \times \mathcal{P}(S) \longrightarrow \mathcal{P}^2(S),$$

where input $C \subseteq S$ are core features and $O \subseteq S$ are optional features. For instance

$$C = \{\text{Auth, Payments}\}, \quad O = \{\text{Search, Analytics, Export}\}.$$

Construct two blueprints (milestone plans):

$$B_1 = \{\{\text{Auth, Payments}\}, \{\text{Search}\}, \{\text{Analytics}\}, \{\text{Export}\}\},$$

$$B_2 = \{\{\text{Auth, Payments, Search}\}, \{\text{Analytics, Export}\}\}.$$

Set

$$\star(C, O) = \{B_1, B_2\} \in \mathcal{P}^2(S).$$

Each blueprint is a set of sets of features (e.g. phases), so $\star$ indeed maps $(\mathcal{P}^1(S))^2 \rightarrow \mathcal{P}^2(S)$, a concrete $(m, n) = (1, 2)$ SuperHyperStructure capturing hierarchical release planning.

2. Review and Results: Some Uncertain Number

This section investigates the concept of the *Uncertain Number* and explores whether it can be extended to define both the *Ordered Uncertain Number* and the *Ranking Uncertain Number*.

2.1. Ordered Neutrosophic and Plithogenic Number

An ordered neutrosophic number comprises three ordered membership curves for truth, indeterminacy, and falsity, producing convex, levelwise reconstructed degree functions. An ordered plithogenic number collects ordered membership curves per attribute, aggregating them via contradiction-aware operators to yield an ordered curve.

Definition 20 (Ordered Membership Curve). *A pair of continuous functions $(L, R) : [0, 1] \rightarrow \mathbb{R}^2$ is an ordered membership curve if*

$$L \text{ is nondecreasing, } R \text{ is nonincreasing, } \forall y \in [0, 1] \ L(y) \leq R(y).$$

It induces a (normal, convex, u.s.c.) fuzzy membership by the standard inverse- α -cut reconstruction

$$\mu_{(L,R)}(x) := \sup\{y \in [0, 1] \mid L(y) \leq x \leq R(y)\} \in [0, 1].$$

Definition 21 (Ordered Neutrosophic Number (ONN)). *An ordered neutrosophic number is a triple*

$$\mathbf{N} = (A_T, A_I, A_F),$$

where each $A_\bullet = (L_\bullet, R_\bullet)$ is an ordered membership curve (Def. 20). Writing μ_T, μ_I, μ_F for the reconstructed memberships, the associated (single-valued) neutrosophic number on $\mathbb{R}$ is the triplet of degrees $(T, I, F) = (\mu_T, \mu_I, \mu_F)$ with the usual neutrosophic bound $0 \leq T(x) + I(x) + F(x) \leq 3$ for all $x \in \mathbb{R}$.²

Example 15 (Ordered Neutrosophic Number — Safe cruising speed on a wet highway). *Let the universe be the speed $x \in [0, 140]$ (km/h). We model the neutrosophic triplet $\mathbf{N} = (A_T, A_I, A_F)$ of (T) safe, (I) context-dependent, and (F) unsafe (too fast) degrees by ordered membership curves $A_\bullet = (L_\bullet, R_\bullet)$ (Def. 20), each yielding a normal, convex fuzzy set via inverse α -cuts:*

$$\mu_\bullet(x) = \sup\{y \in [0, 1] \mid L_\bullet(y) \leq x \leq R_\bullet(y)\}.$$

(T) Safe speed. *Triangular OFN with support $[40, 100]$, mode 75:*

$$L_T(y) = 40 + 35y, \quad R_T(y) = 100 - 25y.$$

(I) Context dependence (traffic/spray variability). *Trapezoid with “core” $[65, 85]$ and support $[58, 92]$:*

$$L_I(y) = 58 + 7y, \quad R_I(y) = 92 - 7y.$$

(F) Unsafe (too fast). *Triangular OFN with support $[95, 140]$, mode 120:*

$$L_F(y) = 95 + 25y, \quad R_F(y) = 140 - 20y.$$

Each $L_\bullet$ is nondecreasing, $R_\bullet$ nonincreasing, and $L_\bullet(y) \leq R_\bullet(y)$, so (A_T, A_I, A_F) is an ONN (Def. 21). A numerical readout at $x = 90$ km/h:

$$\mu_T(90) = \frac{100 - 90}{100 - 75} = \frac{10}{25} = 0.40, \quad \mu_I(90) = \frac{92 - 90}{92 - 85} = \frac{2}{7} \approx 0.286, \quad \mu_F(90) = 0,$$

whence $(T, I, F) \approx (0.40, 0.286, 0)$ and $T + I + F \leq 3$ as required.

² The bound is automatically satisfied if each $\mu_\bullet$ is a fuzzy membership; no further coupling among (T, I, F) is required here.

Theorem 1 (ONN generalizes OFN). *Let $\mathcal{O}$ be the class of OFNs and $\mathcal{N}$ the class of ONNs. The assignment*

$$\iota: \mathcal{O} \longrightarrow \mathcal{N}, \quad \iota((L, R)) := ((L, R), (L_0, R_0), (L^c, R^c)),$$

is an injective embedding for any fixed admissible choices of: (i) a constant “null” curve (L_0, R_0) with $L_0(y) = R_0(y) = x_0$ (crisp point x_0), and (ii) a complement curve (L^c, R^c) representing a chosen fuzzy complement of $\mu_{(L, R)}$.³ Moreover, the projection

$$\pi_T: \mathcal{N} \rightarrow \mathcal{O}, \quad \pi_T((A_T, A_I, A_F)) := A_T,$$

satisfies $\pi_T \circ \iota = \text{id}_{\mathcal{O}}$.

Proof. For any OFN $A = (L, R)$, $\iota(A)$ is an ONN by Definition 21, since each component is an ordered curve. If $\iota(L_1, R_1) = \iota(L_2, R_2)$, then their T -components coincide, hence $(L_1, R_1) = (L_2, R_2)$; thus ι is injective. By construction, $\pi_T(\iota(L, R)) = (L, R)$, so $\pi_T \circ \iota = \text{id}$. Therefore every OFN appears as a special case of an ONN (with fixed neutral/complement components), i.e. ONN generalizes OFN. $\square$

Definition 22 (Ordered Plithogenic Number (OPN)). *Fix a (finite) set V of attribute values and a symmetric contradiction degree $c: V \times V \rightarrow [0, 1]$ with $c(a, a) = 0$. An ordered plithogenic number is a tuple*

$$P = (V, \{A_a\}_{a \in V}, c, \text{Agg}),$$

where each $A_a = (L_a, R_a)$ is an ordered membership curve (Def. 20), and Agg is a levelwise aggregation scheme producing either

- (i) *an aggregated OFN $A_\star = (L_\star, R_\star)$ by continuous, monotone maps*

$$L_\star(y) = \text{Agg}^\downarrow(\{L_a(y)\}_{a \in V}; c), \quad R_\star(y) = \text{Agg}^\uparrow(\{R_a(y)\}_{a \in V}; c),$$

with $L_\star(y) \leq R_\star(y)$ for all y , or

- (ii) *the family view $\{A_a\}_{a \in V}$ together with Agg as part of the structure.*

Typical choices are t -norm/ t -conorm-based mixtures whose weights depend on c (e.g., more conorm when contradiction is higher). Monotonicity of $\text{Agg}^\downarrow, \text{Agg}^\uparrow$ in each argument guarantees that $A_\star$ is an OFN whenever the A_a are.

Example 16 (Ordered Plithogenic Number — Daily salt intake aggregated across stakeholders). *Let $x \in [0, 5000]$ be daily sodium intake (mg). We consider $V = \{\text{Cardio}, \text{Coach}, \text{Chef}\}$: a cardiologist, an athletics coach, and a chef. Each $a \in V$ provides an ordered membership curve $A_a = (L_a, R_a)$ encoding the acceptability of x for that perspective; the family together with a contradiction map c and a levelwise aggregator Agg forms an OPN (Def. 22).*

Ordered curves (OFNs).

$$\begin{array}{ll} \text{Cardio:} & L_{\text{Cardio}}(y) = 500 + 1000y & R_{\text{Cardio}}(y) = 2500 - 1000y \\ \text{Coach:} & L_{\text{Coach}}(y) = 1000 + 800y & R_{\text{Coach}}(y) = 3500 - 500y \\ \text{Chef:} & L_{\text{Chef}}(y) = 1500 + 1000y & R_{\text{Chef}}(y) = 5000 - 1000y \end{array}$$

Shapes: *Cardio—triangular (mode 1500); Coach—trapezoid (core [1800, 3000]); Chef—trapezoid (core [2500, 4000]).*

³ For instance, one may take (L^c, R^c) as the ordered curve of Zadeh’s complement $1 - \mu_{(L, R)}$, whenever that complement is again representable by an ordered curve (e.g. for symmetric triangular/trapezoidal cases). Alternatively, any fixed admissible (L^c, R^c) can be used—the embedding does not depend on the particular choice.

Contradiction degrees. Symmetric $c : V \times V \rightarrow [0, 1]$ with $c(a, a) = 0$; for instance

$$c(\text{Cardio}, \text{Chef}) = 0.90, \quad c(\text{Cardio}, \text{Coach}) = 0.60, \quad c(\text{Coach}, \text{Chef}) = 0.40.$$

Set the global mixing weight $\gamma := \frac{1}{3}(0.90 + 0.60 + 0.40) = \frac{19}{30} \approx 0.633$.

Levelwise aggregation. Define monotone maps (for each $y \in [0, 1]$)

$$L_*(y) := (1 - \gamma) \min_{a \in V} L_a(y) + \gamma \frac{L_{\text{Cardio}}(y) + L_{\text{Coach}}(y) + L_{\text{Chef}}(y)}{3},$$

$$R_*(y) := (1 - \gamma) \max_{a \in V} R_a(y) + \gamma \frac{R_{\text{Cardio}}(y) + R_{\text{Coach}}(y) + R_{\text{Chef}}(y)}{3}.$$

Then $A_* = (L_*, R_*)$ is an aggregated OFN, and

$$P = (V, \{A_a\}_{a \in V}, c, \text{Agg} = (L_*, R_*))$$

is an ordered plithogenic number.

Numerical slice at $y = 0.8$.

$$\begin{aligned} L_{\text{Cardio}}(0.8) &= 500 + 1000(0.8) = 1300, & R_{\text{Cardio}}(0.8) &= 2500 - 1000(0.8) = 1700, \\ L_{\text{Coach}}(0.8) &= 1000 + 800(0.8) = 1640, & R_{\text{Coach}}(0.8) &= 3500 - 500(0.8) = 3100, \\ L_{\text{Chef}}(0.8) &= 1500 + 1000(0.8) = 2300, & R_{\text{Chef}}(0.8) &= 5000 - 1000(0.8) = 4200. \end{aligned}$$

Thus $\min L = 1300$, $\max R = 4200$, and

$$\bar{L} = \frac{1300+1640+2300}{3} = 1746.7, \quad \bar{R} = \frac{1700+3100+4200}{3} = 3000.$$

With $\gamma \approx 0.633$,

$$L_*(0.8) \approx (1 - 0.633) \cdot 1300 + 0.633 \cdot 1746.7 \approx 1596, \quad R_*(0.8) \approx (1 - 0.633) \cdot 4200 + 0.633 \cdot 3000 \approx 3456.$$

Hence the aggregated α -cut at level 0.8 is $[L_*(0.8), R_*(0.8)] \approx [1596, 3456]$ mg. For an intake $x = 2400$ mg, $2400 \in [1596, 3456]$, so the aggregated membership satisfies $\mu_{(L_*, R_*)}(2400) \geq 0.8$, illustrating how contradiction-aware aggregation yields a single ordered fuzzy number summarizing the stakeholders' views.

Theorem 2 (OPN generalizes OFN). Let $\mathcal{P}$ be the class of OPNs and $\mathcal{O}$ the class of OFNs. The map

$$j : \mathcal{O} \rightarrow \mathcal{P}, \quad j((L, R)) := (\{v_0\}, \{A_{v_0} = (L, R)\}, c \equiv 0, \text{Agg} = \text{Id}),$$

is an injective embedding whose aggregated view returns (L, R) .

Proof. With a singleton attribute set $V = \{v_0\}$, zero contradiction, and identity aggregation, Definition 22 reduces to the given OFN. Injectivity is immediate. The aggregated curve is (L, R) by construction. $\square$

Theorem 3 (OPN generalizes ONN). Let $\mathcal{N}$ be the class of ONNs and $\mathcal{P}$ the class of OPNs. There is an injective embedding

$$\kappa : \mathcal{N} \rightarrow \mathcal{P}, \quad \kappa((A_T, A_I, A_F)) := (\{T, I, F\}, \{A_T, A_I, A_F\}, c, \text{Agg} = \text{Id}),$$

for any fixed contradiction map c on $\{T, I, F\}$ (e.g. user-specified). In the family view, κ is the identity on components.

Proof. Take $V = \{T, I, F\}$ and place the three ordered curves of the ONN as the attribute-indexed family. With identity aggregation, Definition 22 reproduces exactly the ONN data. Distinct ONNs yield distinct families, hence injectivity. $\square$

2.2. Ordered Rough Numbers

An ordered rough number maps indices to rough intervals from lower and upper approximations, using averages along an increasing chain.

Definition 23 (Rough model and notation). Let U be a nonempty finite universe, $E \subseteq U \times U$ an equivalence (indiscernibility) relation with quotient (granules) $U/E = \{C_1, \dots, C_s\}$, and let $f : U \rightarrow \mathbb{R}$ be a numeric attribute (valuation). For $A \subseteq U$, the lower/upper E -approximations are

$$\underline{A} := \{x \in U : [x]_E \subseteq A\}, \quad \overline{A} := \{x \in U : [x]_E \cap A \neq \emptyset\}.$$

For a nonempty finite $S \subseteq U$, write $\text{avg}_f(S) := \frac{1}{|S|} \sum_{x \in S} f(x)$. The (interval-valued) rough number induced by A is

$$\text{RN}_f(A) := [L(A), U(A)] := [\text{avg}_f(\underline{A}), \text{avg}_f(\overline{A})],$$

whenever $\underline{A}$ and $\overline{A}$ are nonempty.⁴

Definition 24 (Ordered Rough Number (ORN)). Fix a totally ordered index set I (e.g. $I = \{1, \dots, t\}$). An ordered rough number on (U, E, f) is specified by an increasing chain of target sets $(A_i)_{i \in I}$, $A_i \subseteq U$, with $i \leq j \Rightarrow A_i \subseteq A_j$, and such that $\underline{A_i}, \overline{A_i} \neq \emptyset$ for all i . Its ordered interval profile is the pair of maps

$$L(i) := \text{avg}_f(\underline{A_i}), \quad U(i) := \text{avg}_f(\overline{A_i}), \quad (i \in I),$$

and we write

$$\text{ORN}_f((A_i)_{i \in I}) := ([L(i), U(i)])_{i \in I}.$$

Each section $[L(i), U(i)]$ is the rough number associated to A_i .

Example 17 (Ordered Rough Number — Delivery-time thresholds with explicit calculations). Consider parcel deliveries from three suppliers over one week. Universe $U = \{x_1, \dots, x_8\}$ lists all recorded deliveries; the equivalence relation E groups deliveries by supplier into granules:

$$C_1 = \{x_1, x_2, x_3\}, \quad C_2 = \{x_4, x_5, x_6\}, \quad C_3 = \{x_7, x_8\}.$$

Let $f : U \rightarrow \mathbb{R}$ be the door-to-door delivery time (minutes) with values

$$\begin{aligned} f(x_1) &= 30, f(x_2) = 32, f(x_3) = 34 && (\text{Supplier 1}), \\ f(x_4) &= 40, f(x_5) = 45, f(x_6) = 55 && (\text{Supplier 2}), \\ f(x_7) &= 60, f(x_8) = 65 && (\text{Supplier 3}). \end{aligned}$$

Define an increasing chain of target sets by delivery-time thresholds

$$A_1 = \{x \in U : f(x) \leq 40\}, \quad A_2 = \{x \in U : f(x) \leq 55\}, \quad A_3 = \{x \in U : f(x) \leq 65\} = U.$$

For each i , compute lower/upper E -approximations $\underline{A_i} = \{x : [x]_E \subseteq A_i\}$, $\overline{A_i} = \{x : [x]_E \cap A_i \neq \emptyset\}$, and the ordered rough interval $[L(i), U(i)] = [\text{avg}_f(\underline{A_i}), \text{avg}_f(\overline{A_i})]$.

⁴ If an approximation is empty, the mean is undefined; throughout we restrict to indices for which both are nonempty.

1) Level $i = 1$ ($\tau_1 = 40$): $A_1 = \{x_1, x_2, x_3, x_4\}$.

$$\underline{A}_1 = C_1 = \{x_1, x_2, x_3\}, \quad \overline{A}_1 = C_1 \cup C_2 = \{x_1, \dots, x_6\}.$$

Sums and means:

$$\sum_{C_1} f = 30 + 32 + 34 = 96, \quad \text{avg}_f(C_1) = 96/3 = 32.000,$$

$$\sum_{C_1 \cup C_2} f = (96) + (40 + 45 + 55) = 96 + 140 = 236, \quad \text{avg}_f(C_1 \cup C_2) = 236/6 = \frac{118}{3} \approx 39.333\dot{3}.$$

Hence

$$[L(1), U(1)] = [32.000, 118/3] \approx [32.000, 39.333\dot{3}].$$

2) Level $i = 2$ ($\tau_2 = 55$): $A_2 = \{x_1, \dots, x_6\} = C_1 \cup C_2$.

$$\underline{A}_2 = C_1 \cup C_2, \quad \overline{A}_2 = C_1 \cup C_2,$$

so

$$L(2) = U(2) = \text{avg}_f(C_1 \cup C_2) = 236/6 = \frac{118}{3} \approx 39.333\dot{3}.$$

Thus $[L(2), U(2)] = [118/3, 118/3]$.

3) Level $i = 3$ ($\tau_3 = 65$): $A_3 = U = C_1 \cup C_2 \cup C_3$.

$$\sum_U f = (96) + (140) + (60 + 65) = 96 + 140 + 125 = 361, \quad |U| = 8,$$

$$\text{avg}_f(U) = 361/8 = 45.125.$$

Hence

$$[L(3), U(3)] = [45.125, 45.125].$$

Collecting the ordered rough profile

$$\text{ORN}_f((A_i)) = ([L(i), U(i)])_{i=1}^3 = ([32.000, 118/3], [118/3, 118/3], [45.125, 45.125]).$$

Numerically,

$$[32.000, 39.333\dot{3}], \quad [39.333\dot{3}, 39.333\dot{3}], \quad [45.125, 45.125].$$

As the threshold increases, both endpoints are nondecreasing, illustrating the ordered rough number defined by this real logistics scenario.

Theorem 4 (Rough-number structure at each index). Let $\text{ORN}_f((A_i))$ be an ordered rough number on (U, E, f) . For every $i \in I$, the interval $[L(i), U(i)]$ equals $\text{RN}_f(A_i)$ in the sense of Definition 23. Hence an ORN is a family of (ordinary) rough numbers indexed by I .

Proof. By Definition 24, $L(i) = \text{avg}_f(\underline{A}_i)$ and $U(i) = \text{avg}_f(\overline{A}_i)$ whenever both approximations are nonempty. This is exactly the construction of $\text{RN}_f(A_i)$ in Definition 23. $\square$

Theorem 5 (Monotonicity under threshold chains). Assume the following compatibility conditions:

- (a) **Classwise constancy:** f is constant on each E -class $C \in U/E$ (i.e., $x, y \in C \Rightarrow f(x) = f(y)$).
- (b) **Threshold chain:** There exist real numbers $\tau_1 < \tau_2 < \dots < \tau_t$ such that

$$A_i = \{x \in U : f(x) \leq \tau_i\}, \quad i = 1, \dots, t.$$

Then for the ordered rough number $\text{ORN}_f((A_i)_{i=1}^t)$ we have, for all $i < t$,

$$L(i) \leq L(i+1) \quad \text{and} \quad U(i) \leq U(i+1).$$

In particular, $i \mapsto [L(i), U(i)]$ is coordinatewise nondecreasing.

Proof. By classwise constancy, if a class C intersects A_i then $C \subseteq A_i$, because all elements of C share the same f -value. Hence $\underline{A}_i = \overline{A}_i = \bigcup \{C \in U/E : f(C) \leq \tau_i\}$, where $f(C)$ denotes the (common) value on C . Consequently,

$$L(i) = U(i) = \text{avg}_f\left(\bigcup_{f(C) \leq \tau_i} C\right).$$

As i increases, the indexing threshold τ_i increases, so the union above is nested: $\underline{A}_i = \overline{A}_i \subseteq \underline{A}_{i+1} = \overline{A}_{i+1}$. Moreover, every newly added class C satisfies $f(C) \in (\tau_i, \tau_{i+1}]$, hence $f(C) \geq \tau_i \geq \text{avg}_f(\underline{A}_i)$ (the last inequality holds because all values in $\underline{A}_i$ are $\leq \tau_i$). Averages over finite sets are monotone under adding elements whose value is at least the current average, therefore $L(i) \leq L(i+1)$ and, since $L(i) = U(i)$ for all i , also $U(i) \leq U(i+1)$. $\square$

2.3. Ordered Granular Numbers

An ordered granular number is a sequence of granular intervals, indexed monotonically, whose endpoints vary consistently, forming inclusion-nested uncertainty bands.

Definition 25 (Ordered Granular Number (OGN)). Let $(I, \leq)$ be a nonempty totally ordered index set (e.g. $I = \{1, \dots, t\}$). An ordered granular number is a family

$$\text{OGN} := (G_{a(i)}(c(i), r(i)))_{i \in I}, \quad c(i) \in \mathbb{R}, r(i) \in [0, \infty), a(i) \in [0, 1],$$

such that each section $G(c(i), r(i))$ is a granular number. We call **OGN** inclusion-monotone if for all $i \leq j$,

$$G(c(i), r(i)) \subseteq G(c(j), r(j)) \iff |c(j) - c(i)| \leq r(j) - r(i). \quad (1)$$

(Equivalently, $[c(i) - r(i), c(i) + r(i)] \subseteq [c(j) - r(j), c(j) + r(j)]$.)

Example 18 (Ordered Granular Number — Project budget with widening contingency). A software project maintains an inclusion-nested sequence of budget bands (center $\pm$ radius), each with a confidence weight $A \in [0, 1]$. Let the baseline estimate be $c = 100,000$ USD. Define the ordered granular family $\text{OGN} = (G_{A(i)}(c(i), r(i)))_{i=1}^3$ by

Index i	Name	$c(i)$ [USD]	$r(i)$ [USD]	$A(i)$
1	Baseline	100,000	5,000	0.90
2	+ 10% contingency	100,000	12,000	0.75
3	+ 20% contingency	100,000	20,000	0.60

Each granular section is the closed interval $G(c(i), r(i)) = [c(i) - r(i), c(i) + r(i)]$. Hence

$$G_1 = [95,000, 105,000], \quad G_2 = [88,000, 112,000], \quad G_3 = [80,000, 120,000].$$

We have strict nesting $G_1 \subset G_2 \subset G_3$. The inclusion-monotonicity condition $|c(j) - c(i)| \leq r(j) - r(i)$ holds for all $i \leq j$ because the center is constant (left side = 0) and radii increase (right side > 0). Endpoint monotonicities follow:

$$L(1) = 95,000 \geq L(2) = 88,000 \geq L(3) = 80,000, \quad U(1) = 105,000 \leq U(2) = 112,000 \leq U(3) = 120,000.$$

Thus

$$\mathbf{OGN} = (G_{0.90}(100,000, 5,000), G_{0.75}(100,000, 12,000), G_{0.60}(100,000, 20,000))$$

is an ordered granular number representing increasingly conservative budget bands with decreasing confidence weights, a common real-life planning practice.

Theorem 6 (Granular structure of an OGN). *Let $\mathbf{OGN} = (G_{a(i)}(c(i), r(i)))_{i \in I}$ be an ordered granular number. Then, for every $i \in I$, its section $G(c(i), r(i)) = [c(i) - r(i), c(i) + r(i)]$ is a granular number. Hence an OGN is a family of granular numbers indexed by I .*

Proof. By Definition 25 each component $G(c(i), r(i))$ is, by construction, a closed ball in $(\mathbb{R}, |\cdot|)$, i.e. a granular number. The conclusion follows immediately. $\square$

Theorem 7 (Nested chain and endpoint monotonicity). *Assume $\mathbf{OGN}$ is inclusion-monotone in the sense of (1). Then for all $i \leq j$,*

$$[c(i) - r(i), c(i) + r(i)] \subseteq [c(j) - r(j), c(j) + r(j)].$$

Consequently, the left endpoints $L(i) := c(i) - r(i)$ form a nonincreasing map ($i \leq j \Rightarrow L(i) \geq L(j)$) and the right endpoints $U(i) := c(i) + r(i)$ form a nondecreasing map ($i \leq j \Rightarrow U(i) \leq U(j)$).

Proof. The inclusion is exactly (1) rewritten for intervals. If $[L(i), U(i)] \subseteq [L(j), U(j)]$ then $L(i) \geq L(j)$ and $U(i) \leq U(j)$, yielding the endpoint monotonicities. $\square$

2.4. Ordered Functorial Numbers

An ordered functorial number is a diagram of semiring-valued functors with natural, order-indexed embeddings, preserving operations across objects and morphisms.

Definition 26 (Commutative semirings and functor category). *Let $\mathbf{CRig}$ denote the category of commutative semirings (objects) and semiring homomorphisms (arrows). Let I be a (small) totally ordered set, viewed as a thin category. The functor category $\mathbf{CRig}^I$ has as objects the I -diagrams $A : I \rightarrow \mathbf{CRig}$ and as arrows the natural transformations.*

Definition 27 (Ordered Functorial Number (OFN)). *Fix a base category $\mathcal{C}$ with finite products and a total order I . An Ordered Functorial Number is a functor*

$$\mathbb{N} : \mathcal{C} \longrightarrow \mathbf{CRig}^I.$$

Equivalently, it is the same data as a family $\{N_i\}_{i \in I}$ of Functorial Numbers $N_i : \mathcal{C} \rightarrow \mathbf{CRig}$ together with, for all $i \leq j$ in I , a natural transformation of functors (in $\mathcal{C}$)

$$m_{ij} : N_i \Longrightarrow N_j,$$

whose components are semiring homomorphisms, satisfying $m_{ii} = \text{id}$ and $m_{jk} \circ m_{ij} = m_{ik}$ for $i \leq j \leq k$. If each component $m_{ij}(X)$ is a monomorphism in $\mathbf{CRig}$, we say the OFN is inclusion-monotone.

Example 19 (Ordered Functorial Number — Household shopping baskets over months). *Let the base category be $\mathcal{C} = \mathbf{Set}$ and the index (time) order be $I = \{1 < 2 < 3\}$ (months). Define an Ordered Functorial Number (Def. 27)*

$$\mathbb{N} : \mathbf{Set} \longrightarrow \mathbf{CRig}^I, \quad X \longmapsto (N_1(X) \xrightarrow{m_{12}} N_2(X) \xrightarrow{m_{23}} N_3(X)),$$

by taking, for every set X ,

$$N_i(X) := \mathbb{N}[X] \quad (\text{the commutative polynomial semiring in variables } X),$$

with $+$ and $\cdot$ the usual polynomial addition/multiplication, and with

$$m_{12}(X) = \text{id}_{\mathbb{N}[X]}, \quad m_{23}(X) = \text{id}_{\mathbb{N}[X]}$$

(so the diagram is inclusion-monotone via identities). For a function $f : X \rightarrow Y$, let $N_i(f) : \mathbb{N}[X] \rightarrow \mathbb{N}[Y]$ be the unique semiring homomorphism that renames variables $x \mapsto f(x)$; this makes $\mathbb{N}$ a functor into **CRig**¹.

Concrete data. Take a two-SKU set $X = \{\text{milk}, \text{bread}\}$ and record two monthly “basket polynomials”

$$p_1 = 3 \text{ milk} + 2 \text{ bread} \in N_1(X), \quad p_2 = 1 \text{ milk} + 1 \text{ bread} \in N_2(X).$$

Pooling baskets corresponds to semiring addition:

$$p_1 \oplus p_2 = (3+1) \text{ milk} + (2+1) \text{ bread} = 4 \text{ milk} + 3 \text{ bread} \in N_2(X).$$

Bundling (e.g., counting ordered pairs of items) corresponds to semiring multiplication:

$$\begin{aligned} p_1 \otimes p_2 &= (3 \text{ milk} + 2 \text{ bread}) \cdot (1 \text{ milk} + 1 \text{ bread}) \\ &= 3 \text{ milk}^2 + 3 \text{ milkbread} + 2 \text{ breadmilk} + 2 \text{ bread}^2 \\ &= 3 \text{ milk}^2 + 5 \text{ milkbread} + 2 \text{ bread}^2 \in N_3(X), \end{aligned}$$

where commutativity merges the mixed terms.

Naturality check (category aggregation). Let $Y = \{\text{grocery}\}$ and $f : X \rightarrow Y$ map both SKUs to grocery. Then

$$N_i(f)(p_1) = 3 \text{ g} + 2 \text{ g} = 5 \text{ g}, \quad N_i(f)(p_2) = 1 \text{ g} + 1 \text{ g} = 2 \text{ g},$$

so

$$N_i(f)(p_1 \oplus p_2) = 7 \text{ g} = N_i(f)(p_1) \oplus N_i(f)(p_2),$$

and

$$N_i(f)(p_1 \otimes p_2) = 3 \text{ g}^2 + 5 \text{ g}^2 + 2 \text{ g}^2 = 10 \text{ g}^2 = (5 \text{ g}) \cdot (2 \text{ g}) = N_i(f)(p_1) \otimes N_i(f)(p_2).$$

Thus semiring operations and the time-indexed embeddings are preserved along f , as required by an Ordered Functorial Number.

Theorem 8 (OFN carries Functorial Number structure at every level). Let $\mathbb{N} : \mathcal{C} \rightarrow \mathbf{CRig}^I$ be an OFN and let $\text{ev}_i : \mathbf{CRig}^I \rightarrow \mathbf{CRig}$ be the evaluation functor at $i \in I$. Then the composite

$$N_i := \text{ev}_i \circ \mathbb{N} : \mathcal{C} \longrightarrow \mathbf{CRig}$$

is a Functorial Number for every $i \in I$. Moreover, for $i \leq j$ the morphism $m_{ij} := \text{ev}_{i \leq j} \circ \mathbb{N}$ is a natural transformation $N_i \Rightarrow N_j$ whose components are semiring homomorphisms.

Proof. By functoriality of $\mathbb{N}$ and of evaluation, N_i is a functor into **CRig**, hence a Functorial Number. Naturality and the semiring-hom property of m_{ij} are inherited from the arrow part of the diagram $\mathbb{N}(X) : I \rightarrow \mathbf{CRig}$ for each X . $\square$

Definition 28 (Ambient semirings for classical ordered models). We shall use the following commutative semirings:

- (a) The pair semiring $\mathbb{R}^2$ with componentwise $+$ and $\cdot$ and units $(0, 0)$, $(1, 1)$.
- (b) The triple semiring $(\mathbb{R}_{\geq 0})^3$ with componentwise $+$ and $\cdot$.

(c) For a poset (thin category) J , we write $\mathbf{CRig}^J$ for J -diagrams in $\mathbf{CRig}$.

Theorem 9 (OFN generalizes ordered fuzzy numbers). *The assignment $A \mapsto (\mathbb{N}_A, s_A)$ defines a faithful embedding from ordered fuzzy numbers into OFNs with designated sections over $(\mathcal{C}, I) = (1, [0, 1])$.*

Proof. Given $(\mathbb{N}_A, s_A)$ one recovers A by $\mu_A^\uparrow(y) = \pi_1(s_A(y))$, $\mu_A^\downarrow(y) = \pi_2(s_A(y))$. Morphisms (e.g. equality) are preserved and reflected because the diagram is constant and sections determine A pointwise in y . $\square$

Theorem 10 (OFN generalizes ordered neutrosophic numbers). *The mapping $N \mapsto (\mathbb{N}_N, s_N)$ embeds ordered neutrosophic numbers into OFNs with designated sections over $(1, [0, 1])$.*

Proof. As above, N is recovered from s_N componentwise. Faithfulness follows because equality of sections implies equality of the three curves (T, I, F) . $\square$

Theorem 11 (OFN generalizes ordered rough numbers). *The assignment $([L(i), U(i)])_{i \in J} \mapsto (\mathbb{N}_R, s_R)$ embeds ordered rough numbers into OFNs with designated sections over $(1, J)$.*

Proof. Pointwise recovery $L(i) = \pi_1(s_R(i))$, $U(i) = \pi_2(s_R(i))$ yields faithfulness. The order on J is carried by the index category $I = J$. $\square$

2.5. Ranking Neutrosophic and Plithogenic Number

Let $X \subseteq \mathbb{R}$ be the universe of real evaluations, endowed with a probability weight $w : X \rightarrow [0, 1]$ such that $\int_X w(x) dx = 1$. All functions below are assumed measurable and bounded.

Definition 29 (Ranking Neutrosophic Number (RNN)). (cf.[98–100]) *Fix a continuous neutrosophic kernel $v : [0, 1]^3 \rightarrow \mathbb{R}$ that is nondecreasing in T , nonincreasing in I and in F . Define the neutrosophic score*

$$R_N(N) := \int_X v(T(x), I(x), F(x)) w(x) dx \in \mathbb{R},$$

and the induced preorder $N_1 \preceq_N N_2 \iff R_N(N_1) \leq R_N(N_2)$.

Example 20 (Ranking Neutrosophic Number — Hiring two candidates with explicit scoring). *Let $X = \{\text{CV}, \text{Interview}, \text{Coding}\}$ be three evaluation contexts, with uniform weight $w(x) = \frac{1}{3}$. For a single-valued neutrosophic number $N = (T, I, F)$ on X , use the ranking kernel*

$$v(t, i, f) = t - 0.5i - f,$$

which is nondecreasing in t and nonincreasing in i, f (so it fits Def. of RNN). The score is

$$R_N(N) = \sum_{x \in X} v(T(x), I(x), F(x)) w(x).$$

Candidate A.

x	$T(x)$	$I(x)$	$F(x)$
CV	0.80	0.10	0.10
Interview	0.70	0.20	0.20
Coding	0.60	0.20	0.30

Pointwise values:

$$v_{\text{CV}} = 0.80 - 0.5(0.10) - 0.10 = 0.80 - 0.05 - 0.10 = 0.65,$$

$$v_{\text{Int}} = 0.70 - 0.5(0.20) - 0.20 = 0.70 - 0.10 - 0.20 = 0.40,$$

$$v_{\text{Code}} = 0.60 - 0.5(0.20) - 0.30 = 0.60 - 0.10 - 0.30 = 0.20.$$

Hence

$$R_N(A) = \frac{1}{3} (0.65 + 0.40 + 0.20) = \frac{1}{3} \cdot 1.25 \approx 0.416\bar{6}.$$

Candidate B.

x	$T(x)$	$I(x)$	$F(x)$
CV	0.70	0.20	0.20
Interview	0.80	0.10	0.10
Coding	0.55	0.25	0.30

Pointwise values:

$$\begin{aligned} v_{CV} &= 0.70 - 0.5(0.20) - 0.20 = 0.40, \\ v_{Int} &= 0.80 - 0.5(0.10) - 0.10 = 0.65, \\ v_{Code} &= 0.55 - 0.5(0.25) - 0.30 = 0.55 - 0.125 - 0.30 = 0.125. \end{aligned}$$

Thus

$$R_N(B) = \frac{1}{3} (0.40 + 0.65 + 0.125) = \frac{1}{3} \cdot 1.175 \approx 0.391\bar{6}.$$

Ranking: $R_N(A) \approx 0.4167 > R_N(B) \approx 0.3917$, so $A \succ B$ under this neutrosophic ranking.

Theorem 12 (RNN generalizes Ranking Fuzzy Number). *Let R_F be given by a satisfaction kernel s as above. Choose any $\alpha, \beta \geq 0$ and define*

$$v(t, i, f) := s(t) - \alpha i - \beta |f - (1 - t)|.$$

Then for every fuzzy number A ,

$$R_N(\iota(A)) = \int_X s(\mu_A(x)) w(x) dx = R_F(A),$$

hence the restriction of $\preceq_N$ to $\iota(\mathcal{F})$ coincides with $\preceq_F$.

Proof. For $N = \iota(A)$ we have $(T, I, F) = (\mu_A, 0, 1 - \mu_A)$. Pointwise, $v(T(x), I(x), F(x)) = s(\mu_A(x)) - \alpha \cdot 0 - \beta \cdot |(1 - \mu_A(x)) - (1 - \mu_A(x))| = s(\mu_A(x))$. Integrating against w yields $R_N(\iota(A)) = R_F(A)$. Therefore $A \preceq_F B$ iff $R_F(A) \leq R_F(B)$ iff $R_N(\iota(A)) \leq R_N(\iota(B))$ iff $\iota(A) \preceq_N \iota(B)$. $\square$

Definition 30 (Ranking Plithogenic Number (RPN)). *Fix a continuous plithogenic kernel*

$$\Psi : [0, 1]^V \times [0, 1]^{V \times V} \longrightarrow \mathbb{R},$$

which is nondecreasing in the coordinates of beneficial attributes and nonincreasing in the coordinates of adverse attributes (as declared by the modeler), and depends continuously on c . Define the plithogenic score

$$R_P(P) := \int_X \Psi((\mu_a(x))_{a \in V}, c) w(x) dx \in \mathbb{R},$$

and the induced preorder $P_1 \preceq_P P_2 \iff R_P(P_1) \leq R_P(P_2)$.

Example 21 (Ranking Plithogenic Number — Smartphone choice with contradiction-aware score). *Consider attribute values $V = \{\text{Battery}, \text{Camera}, \text{Price}\}$, where Battery and Camera are beneficial and Price is encoded as cheapness (higher is better). Let $X = \{*\}$ be a single usage context with $w(*) = 1$, so the integral reduces to a point evaluation. Use the plithogenic kernel*

$$\Psi((\mu_a)_{a \in V}, c) = \sum_{a \in V} w_a \mu_a - \lambda \sum_{a < b} c(a, b) |\mu_a - \mu_b|,$$

with weights $(w_{\text{Battery}}, w_{\text{Camera}}, w_{\text{Price}}) = (0.40, 0.35, 0.25)$ and $\lambda = 0.30$. Contradiction degrees (symmetric, $c(a, a) = 0$):

$$c(\text{Battery}, \text{Camera}) = 0.20, \quad c(\text{Battery}, \text{Price}) = 0.70, \quad c(\text{Camera}, \text{Price}) = 0.60.$$

Phone P. Scores (in $[0, 1]$):

$$\mu_{\text{Battery}} = 0.85, \quad \mu_{\text{Camera}} = 0.75, \quad \mu_{\text{Price}} = 0.60.$$

Base utility:

$$U_P = 0.40(0.85) + 0.35(0.75) + 0.25(0.60) = 0.34 + 0.2625 + 0.15 = 0.7525.$$

Disagreement penalty:

$$\begin{aligned} c(B, C)|\Delta| &= 0.20 |0.85 - 0.75| = 0.020, \\ c(B, P)|\Delta| &= 0.70 |0.85 - 0.60| = 0.175, \\ c(C, P)|\Delta| &= 0.60 |0.75 - 0.60| = 0.090, \\ \text{sum} &= 0.285, \quad \lambda \cdot \text{sum} = 0.30 \times 0.285 = 0.0855. \end{aligned}$$

Therefore

$$R_P(P) = \Psi(\mu, c) = 0.7525 - 0.0855 = 0.6670.$$

Phone Q. Scores:

$$\mu_{\text{Battery}} = 0.70, \quad \mu_{\text{Camera}} = 0.85, \quad \mu_{\text{Price}} = 0.80.$$

Base utility:

$$U_Q = 0.40(0.70) + 0.35(0.85) + 0.25(0.80) = 0.28 + 0.2975 + 0.20 = 0.7775.$$

Disagreement penalty:

$$\begin{aligned} 0.20 |0.70 - 0.85| &= 0.030, \\ 0.70 |0.70 - 0.80| &= 0.070, \\ 0.60 |0.85 - 0.80| &= 0.030, \\ \text{sum} &= 0.130, \quad \lambda \cdot \text{sum} = 0.30 \times 0.130 = 0.039. \end{aligned}$$

Thus

$$R_P(Q) = 0.7775 - 0.039 = 0.7385.$$

Ranking: $R_P(Q) = 0.7385 > R_P(P) = 0.6670$, hence $Q \succ P$ under the plithogenic ranking with contradiction awareness.

Theorem 13 (RPN generalizes RNN). Let $V = \{T, I, F\}$ and, given a neutrosophic number $N = (T, I, F)$, form the plithogenic instance $P_N = \{\mu_T, \mu_I, \mu_F\}$ with $\mu_T := T$, $\mu_I := I$, $\mu_F := F$ and any contradiction c satisfying $c(T, F) = 1$ and $c(T, I) = c(I, F) \in [0, 1]$. Choose Ψ such that for all (t, i, f) one has $\Psi((t, i, f), c) = v(t, i, f)$. Then $R_P(P_N) = R_N(N)$ and hence $\preceq_P$ restricted to the image of this embedding coincides with $\preceq_N$.

Proof. By the stated choice of Ψ , pointwise $\Psi((\mu_T(x), \mu_I(x), \mu_F(x)), c) = v(T(x), I(x), F(x))$. Integrating against w yields $R_P(P_N) = R_N(N)$; the induced preorders match. $\square$

Theorem 14 (RPN generalizes Ranking Fuzzy Number). Embed a fuzzy number A as a plithogenic instance with either

(a) $V = \{T\}$ and $\mu_T := \mu_A$, or

(b) $V = \{T, F\}$, $\mu_T := \mu_A$, $\mu_F := 1 - \mu_A$, and $c(T, F) = 1$.

Choose Ψ so that $\Psi((t), c) = s(t)$ in case (a), or $\Psi((t, f), c) = s(t)$ in case (b). Then $R_P(P_A) = R_F(A)$ and $\preceq_P$ restricted to these embeddings coincides with $\preceq_F$.

Proof. In case (a), pointwise equality $\Psi((\mu_A(x)), c) = s(\mu_A(x))$ gives $R_P(P_A) = \int_X s(\mu_A(x))w(x)dx = R_F(A)$. In case (b) we again have $\Psi((\mu_A(x), 1 - \mu_A(x)), c) = s(\mu_A(x))$, hence the same identity; pre-orders coincide because both compare the same real scores. $\square$

2.6. Ranking Rough Number

Ranking Rough Number assigns a monotone score to a rough number's endpoints, inducing a total preorder while preserving the underlying rough-number interval structure.

Definition 31 (Ranking Rough Number (RRN)). Fix a continuous scoring kernel $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ that is nondecreasing in each argument. For a rough number $RN(G_q) = [L_q, U_q]$ (with $L_q \leq U_q$), define its ranking score

$$R_{RN}(G_q) := \varphi(L_q, U_q).$$

The induced preorder on rough numbers is

$$RN(G_i) \preceq_{\varphi} RN(G_j) \iff R_{RN}(G_i) \leq R_{RN}(G_j).$$

Typical choices include the convex/penalized form $\varphi(L, U) = (1 - \lambda)L + \lambda U - \gamma(U - L)$ with $\lambda \in [0, 1]$, $\gamma \geq 0$.

Example 22 (Ranking Rough Number — Choosing a shipping policy from rough delivery times). We compare two shipping policies (Priority vs. Economy) using rough numbers built from a small, real dataset.

Data and indiscernibility. Universe $U = \{x_1, \dots, x_{10}\}$ are 10 recent deliveries with door-to-door delays $f : U \rightarrow \mathbb{R}_{\geq 0}$ (minutes). Deliveries are indiscernible by courier (equivalence classes):

$$\begin{aligned} C_1 &= \{x_1, x_2, x_3\} & f &= \{12, 18, 25\}, & \sum f &= 55, & \bar{f}(C_1) &= 55/3 = 18.\bar{3}, \\ C_2 &= \{x_4, x_5, x_6\} & f &= \{10, 15, 20\}, & \sum f &= 45, & \bar{f}(C_2) &= 45/3 = 15, \\ C_3 &= \{x_7, x_8, x_9, x_{10}\} & f &= \{30, 35, 28, 40\}, & \sum f &= 133, & \bar{f}(C_3) &= 133/4 = 33.25. \end{aligned}$$

Totals: $\sum_U f = 55 + 45 + 133 = 233$, $|U| = 10$, hence $\bar{f}(U) = 233/10 = 23.3$.

Policies as target sets.

$$\begin{aligned} \text{Priority } (G_P) &: \{x_1, x_2\} \cup C_2 \cup \{x_7, x_8\}, \\ \text{Economy } (G_E) &: C_1 \cup \{x_7, x_8\}. \end{aligned}$$

Lower/upper E-approximations:

	$\underline{A}$	$\bar{A}$
$A = G_P$	C_2	$C_1 \cup C_2 \cup C_3 = U$
$A = G_E$	C_1	$C_1 \cup C_3$

Rough numbers $RN_f(A) = [L(A), U(A)]$ with $L = \bar{f}(\underline{A})$, $U = \bar{f}(\bar{A})$:

$$RN_f(G_P) = [\bar{f}(C_2), \bar{f}(U)] = [15, 23.3], \quad RN_f(G_E) = [\bar{f}(C_1), \bar{f}(C_1 \cup C_3)].$$

Compute $\bar{f}(C_1 \cup C_3) = \frac{55 + 133}{3 + 4} = \frac{188}{7} \approx 26.857142857$, hence

$$RN_f(G_E) = [18.\bar{3}, 26.857142857].$$

Ranking kernel and scores. Use the monotone, penalized midpoint

$$\varphi(L, U) = (1 - \lambda)L + \lambda U - \gamma(U - L), \quad \lambda = 0.5, \gamma = 0.2.$$

Priority $[L, U] = [15, 23.3]$:

$$\varphi(15, 23.3) = 0.5(15) + 0.5(23.3) - 0.2(23.3 - 15) = 7.5 + 11.65 - 0.2(8.3) = 19.15 - 1.66 = 17.49.$$

Economy $[L, U] = [18.\bar{3}, 26.857142857]$:

$$\begin{aligned} \varphi &= 0.5(18.\bar{3}) + 0.5(26.857142857) - 0.2(26.857142857 - 18.\bar{3}) \\ &= 9.166666667 + 13.428571429 - 0.2(8.523809524) \\ &= 22.595238096 - 1.704761905 = 20.890476191. \end{aligned}$$

Since $\varphi(G_P) = 17.49 < 20.8905 = \varphi(G_E)$, we have

$$\text{RN}_f(G_P) \preceq_{\varphi} \text{RN}_f(G_E),$$

so Priority is preferred (smaller/faster and tighter).

Theorem 15 (RRN retains the rough-number structure). For every $G_q \subseteq U$, the object

$$\text{RRN}(G_q) := \left(\underbrace{[L_q, U_q]}_{\text{RN}(G_q)}, \underbrace{\text{RN}_{\text{RN}}(G_q)}_{\text{score}} \right)$$

has as its first component a rough number $\text{RN}(G_q)$ computed from (U, R) and G_q . Consequently, the collection of all RRN's projects onto the class of rough numbers; i.e., forgetting the score recovers exactly the standard rough-number structure.

Proof. By construction, $L_q = \text{Lim}^L(G_q)$ and $U_q = \text{Lim}^U(G_q)$ are computed from the lower/upper approximations $\underline{G}_q, \overline{G}_q$ under the same (U, R) and attribute $R(\cdot)$. Hence $[L_q, U_q]$ satisfies the definition of a rough number. The added scalar $\text{RN}_{\text{RN}}(G_q) = \varphi(L_q, U_q)$ does not modify G_q nor $\overline{G}_q$, therefore the first component of $\text{RRN}(G_q)$ is exactly $\text{RN}(G_q)$. The projection $([\cdot, \cdot], \cdot) \mapsto [\cdot, \cdot]$ maps the class of RRN's onto the class of rough numbers, establishing the claim. $\square$

Lemma 1 (Order consistency under endpoint dominance). Let $[L_1, U_1]$ and $[L_2, U_2]$ be rough numbers with $L_1 \leq L_2$ and $U_1 \leq U_2$. If φ is nondecreasing in each argument, then $[L_1, U_1] \preceq_{\varphi} [L_2, U_2]$.

Proof. Monotonicity of φ yields $\varphi(L_1, U_1) \leq \varphi(L_2, U_2)$, hence the asserted preorder relation. $\square$

2.7. Ranking Granular Numbers

Ranking Granular Numbers maps each granular number's center, radius, and weight to a score, yielding a preference order without altering granular content structure.

Definition 32 (Ranking Granular Number (RGN)). Fix a continuous scoring kernel

$$\varphi : \mathbb{R} \times [0, \infty) \times [0, 1] \longrightarrow \mathbb{R}, \quad (c, r, A) \mapsto \varphi(c, r, A),$$

that is nondecreasing in c , nonincreasing in r , and nondecreasing in A . For a granular number $X = (G(c_X, r_X), A(X))$, define its ranking score

$$\text{RGN}(X) := \varphi(c_X, r_X, A(X)).$$

The induced preorder on granular numbers is

$$X \preceq_{\varphi} Y \iff R_{GN}(X) \leq R_{GN}(Y).$$

Typical choices include risk-penalized, weight-rewarded scores, e.g. $\varphi(c, r, A) = c - \gamma r + \eta A$ with $\gamma, \eta \geq 0$, or $\varphi(c, r, A) = (1 - \lambda)c + \lambda \frac{(c-r)+(c+r)}{2} - \gamma r + \eta A = c - \gamma r + \eta A$ for symmetric intervals.

Example 23 (Ranking Granular Number — Picking a phone by battery-life band). Two phones report lab-tested battery life as granular numbers (center $\pm$ radius, with reliability weight A). Let the monotone score be

$$\varphi(c, r, A) = c - 0.5r + 0.8A,$$

which is nondecreasing in c and A , and nonincreasing in r .

Phone M. Battery band $G(c_M, r_M) = [10 - 1.5, 10 + 1.5] = [8.5, 11.5]$ hours, weight $A_M = 0.8$.

$$R_{GN}(M) = 10 - 0.5(1.5) + 0.8(0.8) = 10 - 0.75 + 0.64 = 9.89.$$

Phone N. Battery band $G(c_N, r_N) = [9.5 - 0.8, 9.5 + 0.8] = [8.7, 10.3]$ hours, weight $A_N = 0.6$.

$$R_{GN}(N) = 9.5 - 0.5(0.8) + 0.8(0.6) = 9.5 - 0.4 + 0.48 = 9.58.$$

Since $R_{GN}(N) \leq R_{GN}(M)$ is false and $R_{GN}(M) > R_{GN}(N)$, we obtain

$$N \preceq_{\varphi} M \text{ and hence } M \succ N.$$

Phone M ranks higher: larger center (expected life), acceptable uncertainty, and stronger reliability weight.

Theorem 16 (RGN retains the granular-number structure). For any granular number $X = (G(c_X, r_X), A(X))$, the object

$$RGN(X) := \left(\underbrace{G(c_X, r_X)}_{\text{granular number}}, \underbrace{A(X)}_{\text{weight}}, \underbrace{R_{GN}(X)}_{\text{score}} \right)$$

has as its first two components exactly the underlying granular number (and its weight). Consequently, the projection $\pi : RGN(X) \mapsto (G(c_X, r_X), A(X))$ maps the class of all RGNs onto the class of granular numbers; forgetting the score recovers the original granular-number structure.

Proof. By definition, $G(c_X, r_X)$ is the connected value set of X and $A(X)$ its weight. The scalar $R_{GN}(X) = \varphi(c_X, r_X, A(X))$ is computed from $(c_X, r_X, A(X))$ and does not alter the set $G(c_X, r_X)$ nor the weight. Hence the first (two) component(s) of $RGN(X)$ coincide(s) with the given granular number (and its weight). The projection π is therefore surjective onto $\mathcal{G}_A$, establishing that RGNs retain the granular-number structure. $\square$

Definition 33 (Benefit dominance on granular numbers). We say $X = (G(c_1, r_1), A_1)$ benefit-dominates $Y = (G(c_2, r_2), A_2)$, written $X \succeq^* Y$, if

$$c_1 \geq c_2, \quad r_1 \leq r_2, \quad A_1 \geq A_2.$$

Lemma 2 (Order consistency under dominance). If $X \succeq^* Y$ and φ is nondecreasing in c , nonincreasing in r , and nondecreasing in A , then $Y \preceq_{\varphi} X$.

Proof. From $c_1 \geq c_2, r_1 \leq r_2, A_1 \geq A_2$ and the monotonicity of φ ,

$$\varphi(c_2, r_2, A_2) \leq \varphi(c_1, r_1, A_1),$$

i.e. $R_{GN}(Y) \leq R_{GN}(X)$, hence $Y \preceq_{\varphi} X$. $\square$

2.8. Ranking Functorial Numbers

Ranking Functorial Numbers adds a natural scoring transformation to functorial-number semirings, comparing elements consistently across objects and morphisms while retaining algebraic operations unchanged.

Notation 17 ((Recall) Functorial Number). Let $\mathcal{C}$ be a category with finite products. A Functorial Number is a functor $\tilde{N} : \mathcal{C} \rightarrow \mathbf{CRig}$ (the category of commutative semirings and homomorphisms). Writing $U : \mathbf{CRig} \rightarrow \mathbf{Set}$ for the forgetful functor, we set $N := U \circ \tilde{N}$, so that each object $X \in \text{Ob}(\mathcal{C})$ carries a commutative semiring

$$(N(X), \oplus_X, \otimes_X, 0_X, 1_X),$$

and for every $f : X \rightarrow Y$ in $\mathcal{C}$, $N(f) : N(X) \rightarrow N(Y)$ is a semiring homomorphism.

Definition 34 (Ranking Functorial Number (RFN)). Fix a totally preordered set $(S, \preceq)$ (e.g. $(\mathbb{R}, \leq)$) and let $\underline{S} : \mathcal{C} \rightarrow \mathbf{Set}$ be the constant functor at the set S . A Ranking Functorial Number on $\mathcal{C}$ is a pair

$$(\tilde{N} : \mathcal{C} \rightarrow \mathbf{CRig}, \rho : N \Rightarrow \underline{S}),$$

where $N := U \circ \tilde{N}$ and $\rho = \{\rho_X : N(X) \rightarrow S\}_{X \in \mathcal{C}}$ is a natural transformation (i.e. for all $f : X \rightarrow Y$, $\rho_Y \circ N(f) = \rho_X$). Each component ρ_X is called a ranking functional on $N(X)$ and induces a total preorder

$$a \preceq_{\rho_X} b \iff \rho_X(a) \preceq \rho_X(b).$$

We say that ρ is monotone w.r.t. the semiring order if, whenever $N(X)$ carries a designated preorder $\sqsubseteq_X$ compatible with $\oplus_X, \otimes_X$ (e.g. a pointwise, cutwise, or dominance order), then $a \sqsubseteq_X b$ implies $\rho_X(a) \preceq \rho_X(b)$.

Example 24 (Ranking Functorial Number — Revenue ranking for coffee shop menus with category aggregation). Let $\mathcal{C}$ be the small category with two objects X (SKUs) and Y (categories), and a single non-identity arrow $f : X \rightarrow Y$ that aggregates SKUs into categories. Concretely,

$$X = \{\text{Espresso, Latte, Muffin}\}, \quad Y = \{\text{Drinks, Food}\},$$

$$f(\text{Espresso}) = \text{Drinks}, \quad f(\text{Latte}) = \text{Drinks}, \quad f(\text{Muffin}) = \text{Food}.$$

Functorial number. Define $\tilde{N} : \mathcal{C} \rightarrow \mathbf{CRig}$ by

$$\tilde{N}(X) = \mathbb{N}[X], \quad \tilde{N}(Y) = \mathbb{N}[Y],$$

the commutative polynomial semirings, and for $f : X \rightarrow Y$ let

$$\tilde{N}(f) : \mathbb{N}[X] \longrightarrow \mathbb{N}[Y]$$

be the unique semiring homomorphism that renames variables along f (i.e., sends a monomial in X to the corresponding monomial in Y by replacing each SKU by its category). Write $N := U \circ \tilde{N}$ for the underlying-set functor.

Ranking functional (natural in X). Fix prices per category

$$p_Y(\text{Drinks}) = 5, \quad p_Y(\text{Food}) = 3 \quad (\text{USD per unit}),$$

and induce SKU prices by pullback $p_X := p_Y \circ f$, i.e.,

$$p_X(\text{Espresso}) = 5, \quad p_X(\text{Latte}) = 5, \quad p_X(\text{Muffin}) = 3.$$

Define for each object $Z \in \{X, Y\}$ a ranking functional

$$\rho_Z : N(Z) = \mathbb{N}[Z] \longrightarrow \mathbb{R}, \quad \rho_Z(P) := \sum_{z \in Z} [\text{coeff of } z \text{ in } P] \cdot p_Z(z),$$

i.e., take only the degree-1 part (linear terms) of P and multiply by the corresponding prices. Then the pair $(\tilde{N}, \rho)$ is an RFN with $S = \mathbb{R}$ and the usual order: for the unique non-identity f we have the naturality identity

$$\rho_Y(N(f)(P)) = \rho_X(P) \quad \text{for all } P \in \mathbb{N}[X],$$

because renaming variables along f preserves linear coefficients, and $p_X = p_Y \circ f$.

Two menus and their scores. Consider two candidate daily menus as (linear) polynomials in $\mathbb{N}[X]$:

$$\begin{aligned} P_A &= 40 \text{ Espresso} + 25 \text{ Latte} + 18 \text{ Muffin}, \\ P_B &= 30 \text{ Espresso} + 35 \text{ Latte} + 10 \text{ Muffin}. \end{aligned}$$

SKU-level ranking (in X):

$$\begin{aligned} \rho_X(P_A) &= 5(40 + 25) + 3(18) = 5 \cdot 65 + 54 = 325 + 54 = 379, \\ \rho_X(P_B) &= 5(30 + 35) + 3(10) = 5 \cdot 65 + 30 = 325 + 30 = 355. \end{aligned}$$

Hence $P_A \succ P_B$ at SKU level.

Naturality check via category aggregation. Applying $N(f)$ collapses SKUs to categories:

$$N(f)(P_A) = 65 \text{ Drinks} + 18 \text{ Food}, \quad N(f)(P_B) = 65 \text{ Drinks} + 10 \text{ Food}.$$

Now evaluate in Y :

$$\begin{aligned} \rho_Y(N(f)(P_A)) &= 5 \cdot 65 + 3 \cdot 18 = 325 + 54 = 379, \\ \rho_Y(N(f)(P_B)) &= 5 \cdot 65 + 3 \cdot 10 = 325 + 30 = 355, \end{aligned}$$

which matches the SKU-level scores, confirming $\rho_Y \circ N(f) = \rho_X$.

Since $355 < 379$, the RFN ranks P_A higher than P_B :

$$P_B \preceq_{\rho_X} P_A \quad \text{and} \quad N(f)(P_B) \preceq_{\rho_Y} N(f)(P_A).$$

Thus, regardless of working at the SKU level or the aggregated category level, the same revenue-based ranking is obtained—exactly the naturality property required of a Ranking Functorial Number.

Theorem 18 (RFN carries the Functorial-Number structure). *Let $(\tilde{N}, \rho)$ be a Ranking Functorial Number. Then the projection onto $\tilde{N}$ forgets only the scoring and leaves intact the commutative semiring data. In particular, $\tilde{N} : \mathcal{C} \rightarrow \mathbf{CRig}$ is a Functorial Number, and $N = U \circ \tilde{N}$ with its $\oplus, \otimes, 0, 1$ is unchanged by adding ρ .*

Proof. By definition, $(\tilde{N}, \rho)$ consists of the functor $\tilde{N} : \mathcal{C} \rightarrow \mathbf{CRig}$ together with a natural transformation $\rho : U \circ \tilde{N} \Rightarrow \underline{S}$. The semiring structure on each fiber $N(X)$ and the homomorphism property of $N(f)$ are supplied solely by $\tilde{N}$, independent of ρ . Hence forgetting ρ yields exactly the underlying Functorial Number. $\square$

Assume $\mathcal{C}$ has a terminal object **1**. Writing $N_1 := N(\mathbf{1})$:

- Definition 35** (Fuzzy, neutrosophic, rough fibers at $\mathbf{1}$). (a) **Fuzzy fiber.** Let N_1 be a chosen class $\mathcal{F}$ of fuzzy numbers on $\mathbb{R}$ endowed with a commutative semiring structure via any standard construction (e.g. α -cut Minkowski sum/product with Dirac $0, 1$ as units). A ranking fuzzy number is a map $R_F : \mathcal{F} \rightarrow S$ inducing a total preorder.
- (b) **Neutrosophic fiber.** Let $N_1 = \mathcal{N}$ be a class of (single-valued) neutrosophic numbers with well-defined $\oplus, \otimes$ (e.g. component-wise t-conorm/t-norm with neutral elements). A ranking neutrosophic number is a map $R_N : \mathcal{N} \rightarrow S$ inducing a total preorder.
- (c) **Rough fiber.** Let $N_1 = \mathcal{RN}$ be (a subclass of) rough numbers, e.g. intervals $[L, U]$ with $0 \leq L \leq U$, operations given by Minkowski arithmetic, and units $[0, 0], [1, 1]$. A ranking rough number is a map $R_R : \mathcal{RN} \rightarrow S$.

Theorem 19 (RFN generalizes fuzzy/neutrosophic/rough rankings). Let $(\tilde{N}, \rho)$ be an RFN on $\mathcal{C}$ and let $\mathbf{1}$ be terminal.

- (i) (Fuzzy) If $N(\mathbf{1}) = \mathcal{F}$ and $\rho_1 = R_F$, then ρ restricted to $\mathbf{1}$ is exactly a ranking of fuzzy numbers.
- (ii) (Neutrosophic) If $N(\mathbf{1}) = \mathcal{N}$ and $\rho_1 = R_N$, then ρ restricted to $\mathbf{1}$ is exactly a ranking of neutrosophic numbers.
- (iii) (Rough) If $N(\mathbf{1}) = \mathcal{RN}$ and $\rho_1 = R_R$, then ρ restricted to $\mathbf{1}$ is exactly a ranking of rough numbers.

Conversely, any classical ranking in (i)–(iii) extends to an RFN by taking a constant functor $\tilde{N}$ (on objects) with the chosen semiring fiber at $\mathbf{1}$ and defining ρ to be constant-on-morphisms with component ρ_1 equal to the given ranking.

Proof. For (i)–(iii), evaluation at the terminal object gives the component $\rho_1 : N(\mathbf{1}) \rightarrow S$, which is, by hypothesis, precisely the corresponding classical ranking map. Naturality imposes no further constraint at $\mathbf{1}$ (there is a unique arrow $X \rightarrow \mathbf{1}$), so the induced preorder on $N(\mathbf{1})$ is exactly the classical one.

For the converse, given any ranking map $R : M \rightarrow S$ on a chosen semiring fiber M (e.g. $\mathcal{F}, \mathcal{N}$, or $\mathcal{RN}$), define $\tilde{N}$ to be the functor that sends every X to the same semiring M and every f to the identity homomorphism on M . Define $\rho_X := R$ for all X . Then $\rho_Y \circ N(f) = R = \rho_X$ for every f , so ρ is natural. Hence $(\tilde{N}, \rho)$ is an RFN whose terminal component is the given classical ranking. $\square$

Lemma 3 (Monotonicity transport). Suppose each $N(X)$ is equipped with a preorder $\sqsubseteq_X$ compatible with $\oplus_X, \otimes_X$ (e.g. dominance for intervals, cutwise order for fuzzy numbers, componentwise order for neutrosophic numbers), and $N(f)$ is monotone w.r.t. these preorders. If ρ is monotone w.r.t. $\sqsubseteq_X$ for each X , then for all $f : X \rightarrow Y$ and $a \sqsubseteq_X b$ one has $\rho_Y(N(f)a) \preceq \rho_Y(N(f)b)$, so functorial transport preserves ranking inequalities.

Proof. By monotonicity of $N(f)$ and $\rho_Y, a \sqsubseteq_X b \Rightarrow N(f)a \sqsubseteq_Y N(f)b \Rightarrow \rho_Y(N(f)a) \preceq \rho_Y(N(f)b)$. $\square$

3. Additional Review and Results: Some HyperUncertain Number

3.1. HyperInterval and SuperHyperInterval Number

A HyperInterval Number is a set-valued generalization of interval numbers, closed under hyperoperations, robustly modeling uncertainty via hyperstructural subset combinations. A SuperHyperInterval Number organizes hyperintervals within iterated powersets, combining them through superhyperoperations, thereby generalizing hyperinterval numbers for hierarchical, multi-level uncertainty.

Notation 20. Let

$$\mathbb{I} := \{ [a, b] \subseteq \mathbb{R} \mid a \leq b, a, b \in \mathbb{R} \}$$

be the set of all (nonempty, closed) real intervals. For $A = [a_L, a_R]$, $B = [b_L, b_R] \in \mathbb{I}$ set

$$\text{Hull}_+(A, B) := [a_L + b_L, a_R + b_R],$$

$$\text{Hull}_\times(A, B) := [\min\{a_L b_L, a_L b_R, a_R b_L, a_R b_R\}, \max\{a_L b_L, a_L b_R, a_R b_L, a_R b_R\}].$$

These are the classical interval–arithmetic sum/product (the outer or convex hulls of the pointwise sum/product sets $\{a+b : a \in A, b \in B\}$ and $\{ab : a \in A, b \in B\}$).

Definition 36 (HyperInterval Number and basic hyperoperations). A HyperInterval Number is an element of $\mathbb{I}$ considered inside the following hyperstructure:

$$(\mathbb{I}, \boxplus, \boxtimes)$$

where, for $A, B \in \mathbb{I}$,

$$A \boxplus B := \{J \in \mathbb{I} \mid J \subseteq \text{Hull}_+(A, B)\}, \quad A \boxtimes B := \{J \in \mathbb{I} \mid J \subseteq \text{Hull}_\times(A, B)\}.$$

Equivalently, $A \boxplus B$ (resp. $A \boxtimes B$) is the set of all closed subintervals of the classical interval sum (resp. product) hull.

Example 25 (HyperInterval Number — Grocery Trip Duration with Choice Sets). A shopper plans a quick grocery run. Two independent choices create uncertainty bands:

- **Travel time (route choice):**

$$H_{\text{travel}} = \{[25, 35] \text{ min}, [30, 45] \text{ min}\}.$$

- **Checkout time (lane choice):**

$$H_{\text{checkout}} = \{[3, 8] \text{ min}, [5, 12] \text{ min}\}.$$

Each set is a HyperInterval Number ($H \in \mathcal{P}^*(\mathcal{I})$). The total time is the hyperaddition

$$H_{\text{total}} = H_{\text{travel}} \boxplus H_{\text{checkout}} = \{[25, 35] + [3, 8], [25, 35] + [5, 12], [30, 45] + [3, 8], [30, 45] + [5, 12]\}.$$

Evaluating the Minkowski sums gives four concrete intervals:

$$H_{\text{total}} = \{[28, 43], [30, 47], [33, 53], [35, 57]\} \text{ minutes}.$$

Thus, the day's “total trip time” is not a single interval but a set of intervals reflecting route/checkout combinations. Operations such as adding a fixed “parking search” buffer $[2, 4]$ minutes simply hyperadd that interval to each element of H_{total} .

Theorem 21 (HyperInterval Numbers form a hyperstructure). With $\boxplus, \boxtimes$ from Definition 36, each map $\mathbb{I} \times \mathbb{I} \rightarrow \mathcal{P}(\mathbb{I})$ is well-defined and never empty; hence $(\mathbb{I}, \boxplus, \boxtimes)$ is a (binary) hyperstructure.

Proof. Fix $A, B \in \mathbb{I}$. By construction, $\text{Hull}_+(A, B) \in \mathbb{I}$ and every closed subinterval $J \subseteq \text{Hull}_+(A, B)$ again lies in $\mathbb{I}$; thus $A \boxplus B \subseteq \mathbb{I}$ and is nonempty (since it contains $\text{Hull}_+(A, B)$). The same argument applies to $\boxtimes$ using $\text{Hull}_\times$. $\square$

Theorem 22 (HyperInterval Numbers generalize Interval Numbers). Define selectors

$$\text{Sel}_+(A, B) := \text{Hull}_+(A, B), \quad \text{Sel}_\times(A, B) := \text{Hull}_\times(A, B).$$

Then $\text{Sel}_+(A, B) \in A \boxplus B$ and $\text{Sel}_\times(A, B) \in A \boxtimes B$ for all $A, B \in \mathbb{I}$. Moreover, for degenerate intervals $[a, a], [b, b]$ (identified with $a, b \in \mathbb{R}$),

$$\text{Sel}_+([a, a], [b, b]) = [a+b, a+b], \quad \text{Sel}_\times([a, a], [b, b]) = [ab, ab].$$

Hence the classical interval arithmetic (and, as a special case, real arithmetic) is obtained by selecting a distinguished element from each hyperresult; in this precise sense, *HyperInterval Numbers* generalize *Interval Numbers*.

Proof. By Definition 36, $A \boxplus B = \{J \in \mathbb{I} : J \subseteq \text{Hull}_+(A, B)\}$, so $\text{Hull}_+(A, B) \in A \boxplus B$; similarly for $\boxtimes$. When $A = [a, a]$ and $B = [b, b]$, $\text{Hull}_+(A, B) = [a+b, a+b]$ and $\text{Hull}_\times(A, B) = [ab, ab]$, yielding the claim. $\square$

Definition 37 (SuperHyperInterval Number and superhyperoperations). Write $\mathcal{P}^*(\mathbb{I}) = \mathcal{P}(\mathbb{I}) \setminus \{\emptyset\}$. A SuperHyperInterval Number is an element $\mathcal{A} \in \mathcal{P}^*(\mathbb{I})$ (a nonempty family of intervals). Define the superhyperoperations

$$\begin{aligned} \boxplus^* : \mathcal{P}^*(\mathbb{I}) \times \mathcal{P}^*(\mathbb{I}) &\longrightarrow \mathcal{P}(\mathcal{P}^*(\mathbb{I})), \\ \boxtimes^* : \mathcal{P}^*(\mathbb{I}) \times \mathcal{P}^*(\mathbb{I}) &\longrightarrow \mathcal{P}(\mathcal{P}^*(\mathbb{I})), \end{aligned}$$

by

$$\mathcal{A} \boxplus^* \mathcal{B} := \{A \boxplus B \mid A \in \mathcal{A}, B \in \mathcal{B}\}, \quad \mathcal{A} \boxtimes^* \mathcal{B} := \{A \boxtimes B \mid A \in \mathcal{A}, B \in \mathcal{B}\}.$$

Thus each output is a set of nonempty sets of intervals, i.e. an element of $\mathcal{P}^2(\mathbb{I})$.

Example 26 (SuperHyperInterval Number — Commute Time Under Weather Scenarios). A commuter's one-way time depends on both route choice and weather scenario. For each scenario, we first obtain a HyperInterval of route-dependent intervals:

Clear (Scenario S_{clear}):

Rain (Scenario S_{rain}):

$$H_{\text{clear}} = \{[25, 35], [30, 45]\} \text{ min.}$$

$$H_{\text{rain}} = \{[35, 50], [40, 60]\} \text{ min.}$$

The SuperHyperInterval collecting these scenario-wise HyperIntervals is

$$\mathbb{H}_{\text{commute}} = \{H_{\text{clear}}, H_{\text{rain}}\} \in \mathcal{P}^*(\mathcal{P}^*(\mathbb{I})).$$

If we also have a stopover option (e.g., coffee pickup) modeled by another HyperInterval $H_{\text{coffee}} = \{[0, 0], [4, 7]\}$ minutes, then the set of scenario-aware total times is obtained via superhyperaddition:

$$\mathbb{H}_{\text{total}} = \mathbb{H}_{\text{commute}} \boxplus \{H_{\text{coffee}}\} = \{H_{\text{clear}} \boxplus H_{\text{coffee}}, H_{\text{rain}} \boxplus H_{\text{coffee}}\}.$$

Thus each scenario (clear vs. rain) yields its own set of feasible intervals for total travel time, preserving the hierarchy (scenario $\rightarrow$ route choices) inherent in real-life decision making.

Theorem 23 (SuperHyperInterval Numbers form a $(1, 2)$ -SuperHyperStructure). Let $S = \mathbb{I}$. The pair of maps

$$\begin{aligned} \boxplus^*, \boxtimes^* : \underbrace{\mathcal{P}^1(S) \times \mathcal{P}^1(S)}_{= \mathcal{P}(S) \times \mathcal{P}(S)} &\longrightarrow \underbrace{\mathcal{P}^2(S)}_{= \mathcal{P}(\mathcal{P}(S))} \end{aligned}$$

from Definition 37 endows S with a $(m, n) = (1, 2)$ two-ary SuperHyperStructure (cf. the general Definition of SuperHyperStructure in the preliminaries).

Proof. Given nonempty $\mathcal{A}, \mathcal{B} \subseteq \mathbb{I}$ and any $A \in \mathcal{A}, B \in \mathcal{B}$, Theorem 21 yields $A \boxplus B \in \mathcal{P}^*(\mathbb{I})$ and $A \boxtimes B \in \mathcal{P}^*(\mathbb{I})$. Therefore the images $\{A \boxplus B\}_{A, B}$ and $\{A \boxtimes B\}_{A, B}$ are nonempty subsets of $\mathcal{P}^*(\mathbb{I})$, i.e. elements of $\mathcal{P}(\mathcal{P}^*(\mathbb{I})) = \mathcal{P}^2(S)$, as required. $\square$

Theorem 24 (SuperHyperInterval Numbers generalize HyperInterval Numbers). *Let $\eta : \mathbb{I} \rightarrow \mathcal{P}^*(\mathbb{I})$ be the singleton embedding $\eta(A) = \{A\}$ and let $\text{Unwrap} : \mathcal{P}(\mathcal{P}^*(\mathbb{I})) \rightarrow \mathcal{P}^*(\mathbb{I})$ be the (set-theoretic) union $\text{Unwrap}(S) = \bigcup S$. Then for all $A, B \in \mathbb{I}$,*

$$\text{Unwrap}(\eta(A) \boxplus^* \eta(B)) = A \boxplus B, \quad \text{Unwrap}(\eta(A) \boxtimes^* \eta(B)) = A \boxtimes B.$$

Consequently, the HyperInterval hyperstructure is recovered from the SuperHyperInterval structure by singleton embedding followed by Unwrap, so SuperHyperInterval Numbers generalize HyperInterval Numbers.

Proof. By Definition 37, $\eta(A) \boxplus^* \eta(B) = \{A \boxplus B\}$ and $\eta(A) \boxtimes^* \eta(B) = \{A \boxtimes B\}$. Taking unions gives $\text{Unwrap}(\{A \boxplus B\}) = A \boxplus B$ and $\text{Unwrap}(\{A \boxtimes B\}) = A \boxtimes B$. $\square$

3.2. HyperGranular and SuperHyperGranular Number

This section introduces two new uncertainty numbers. First, the *HyperGranular Number* (HGN) equips the class of granular numbers with a *hyperoperation*, thereby forming a *hyperstructure* in the sense of hyperalgebra. We prove that HGNs strictly generalize classical granular numbers. Second, the *SuperHyperGranular Number* (SHGN) lifts HGNs to iterated powersets and endows them with an (m, n) -superhyperoperation, forming an (m, n) -SuperHyperStructure. We prove that SHGNs strictly generalize HGNs.

Remark 1. Recall the (weighted) granular number

$$X = G_{a_X}(c_X, r_X), \quad c_X \in \mathbb{R}, \quad r_X \in [0, \infty), \quad a_X \in [0, 1],$$

which represents the connected set (interval)

$$\text{set}(X) = [c_X - r_X, c_X + r_X],$$

optionally annotated with a weight a_X . Let

$$\mathfrak{G} := \left\{ G_a(c, r) \mid c \in \mathbb{R}, r \in [0, \infty), a \in [0, 1] \right\}$$

be the carrier of all (weighted) granular numbers. Classical (deterministic) granular arithmetic uses the Minkowski sum

$$G_a(c, r) \boxplus G_b(d, s) := G_{a \odot b}(c+d, r+s),$$

where, for definiteness, we take the weight-combiner $a \odot b := ab$ (any fixed t -norm would also work).

Definition 38 (HyperGranular Number and its hyper-sum). A HyperGranular Number is any element of the carrier $\mathfrak{G}$. Define a hyper-sum $\oplus_H : \mathfrak{G} \times \mathfrak{G} \rightarrow \mathcal{P}(\mathfrak{G})$ by

$$X \oplus_H Y := \left\{ G_{a_X \odot a_Y}(c_X + c_Y, \rho) \mid \rho \in [\max\{r_X, r_Y\}, r_X + r_Y] \right\}.$$

Example 27 (HyperGranular Number — “Drive + Parking/Walk” Total Time). **Setup.** A commuter models two granular components (minutes):

$$X = G_{a_X}(c_X, r_X) = G_{0.85}(22, 1.0) \quad (\text{drive time}), \quad Y = G_{a_Y}(c_Y, r_Y) = G_{0.90}(8, 0.5) \quad (\text{parking + walk}).$$

Here c is the typical time, r is the tolerance (half-width), and $a \in [0, 1]$ is a confidence/weight. For concreteness, take the weight-combiner $\odot$ to be the product: $a_X \odot a_Y = a_X a_Y$.

Hyper-sum. By Definition 38, the total time is the set

$$X \oplus_H Y = \left\{ G_{a_X a_Y}(c_X + c_Y, \rho) \mid \rho \in [\max\{r_X, r_Y\}, r_X + r_Y] \right\}.$$

Numerically,

$$a_X a_Y = 0.85 \times 0.90 = 0.765, \quad c_X + c_Y = 22 + 8 = 30, \quad \rho \in [\max\{1.0, 0.5\}, 1.0 + 0.5] = [1.0, 1.5].$$

Hence

$$X \oplus_H Y = \left\{ G_{0.765}(30, \rho) \mid \rho \in [1.0, 1.5] \right\}.$$

Each element $G_{0.765}(30, \rho)$ represents a feasible total-time band $[30 - \rho, 30 + \rho]$ minutes. The minimal radius ($\rho = 1.0$) corresponds to strongly correlated or partially overlapping uncertainties (conservative propagation), while the maximal radius ($\rho = 1.5$) corresponds to full, independent accumulation.

Remark 2. While the classical Minkowski sum fixes the radius to $r_X + r_Y$, the hyper-sum returns all admissible radii between a conservative bound $\max\{r_X, r_Y\}$ and the maximal propagation $r_X + r_Y$. Hence, $X \oplus_H Y$ is a set of granular results, not a single one.

Proposition 1 (Closure and interval image). For all $X, Y \in \mathfrak{G}$, $X \oplus_H Y \subseteq \mathfrak{G}$; in particular, every element of $X \oplus_H Y$ is of the form $G_{a_X a_Y}(c_X + c_Y, \rho)$ with $\rho \in [\max\{r_X, r_Y\}, r_X + r_Y]$. Moreover,

$$\text{set}(G_{a_X a_Y}(c_X + c_Y, \rho)) = [c_X + c_Y - \rho, c_X + c_Y + \rho],$$

so the left/right endpoints vary linearly with ρ .

Proof. By Definition 38, $a_X a_Y \in [0, 1]$ and $\rho \geq 0$, hence each produced triple is in $\mathfrak{G}$. The interval formula follows from the definition of $G(\cdot, \cdot)$. $\square$

Definition 39 (Lift to sets). For $A, B \subseteq \mathfrak{G}$, define the set-lift of $\oplus_H$ by

$$A \circ_H B := \bigcup_{X \in A, Y \in B} (X \oplus_H Y) \in \mathcal{P}(\mathfrak{G}).$$

Theorem 25 (HGNs carry a hyperstructure). The pair $\mathcal{H}_{\text{HGN}} := (\mathcal{P}(\mathfrak{G}), \circ_H)$ is a hyperstructure:

$$\circ_H : \mathcal{P}(\mathfrak{G}) \times \mathcal{P}(\mathfrak{G}) \longrightarrow \mathcal{P}(\mathfrak{G}).$$

Proof. By Definition 39, $\circ_H$ maps two subsets of $\mathfrak{G}$ to a union of subsets of $\mathfrak{G}$, hence again a subset of $\mathfrak{G}$. Thus $\circ_H$ has type $\mathcal{P}(\mathfrak{G}) \times \mathcal{P}(\mathfrak{G}) \rightarrow \mathcal{P}(\mathfrak{G})$, which is exactly the requirement (cf. Hyperstructure definition in the preliminaries). $\square$

Theorem 26 (HGN generalizes the classical Granular Number). Let $\iota : \mathfrak{G} \rightarrow \mathcal{P}(\mathfrak{G})$ be the singleton embedding $\iota(X) := \{X\}$. Define a selector $\text{Sel}_{\max}$ that, for any nonempty $S \subseteq \mathfrak{G}$ of the form in Proposition 1, picks the element with the maximal radius $\rho = r_X + r_Y$. Then, for all $X, Y \in \mathfrak{G}$,

$$\text{Sel}_{\max}(\iota(X) \circ_H \iota(Y)) = G_{a_X a_Y}(c_X + c_Y, r_X + r_Y) = X \boxplus Y.$$

Hence, classical granular arithmetic is recovered as a determinization (selection) of the HGN hyper-sum, and HGNs strictly generalize granular numbers.

Proof. By Definitions 38 and 39, $\iota(X) \circ_H \iota(Y) = X \oplus_H Y$ is exactly the set of $G_{a_X a_Y}(c_X + c_Y, \rho)$ with $\rho \in [\max\{r_X, r_Y\}, r_X + r_Y]$. Selecting the unique element with $\rho = r_X + r_Y$ yields the classical Minkowski-type result $G_{a_X a_Y}(c_X + c_Y, r_X + r_Y) = X \boxplus Y$. The map $X \mapsto \iota(X)$ is injective, so the generalization is strict. $\square$

We now lift HGNs to the iterated powerset universe and define an (m, n) -superhyperoperation.

Definition 40 (Iterated wrap/flatten). For $Z \in \mathfrak{G}$ define the wrap (singleton nesting)

$$\text{wrap}^{(1)}(Z) := \{Z\}, \quad \text{wrap}^{(k+1)}(Z) := \{\text{wrap}^{(k)}(Z)\}.$$

For a nested set $A \in \mathcal{P}^m(\mathfrak{G})$ (the m -fold powerset), define the flatten map to atoms

$$\text{flat}^{(1)}(A) := A \subseteq \mathfrak{G}, \quad \text{flat}^{(k+1)}(A) := \bigcup_{B \in A} \text{flat}^{(k)}(B) \subseteq \mathfrak{G}.$$

Definition 41 (SuperHyperGranular Number and (m, n) -superhyper-sum). Fix integers $m \geq 1$ and $n \geq 1$. An (m) -level SuperHyperGranular Number is an element of $\mathcal{P}^m(\mathfrak{G})$. Define the binary (m, n) -superhyper-sum

$$\star^{(m,n)} : \mathcal{P}^m(\mathfrak{G}) \times \mathcal{P}^m(\mathfrak{G}) \longrightarrow \mathcal{P}^n(\mathfrak{G})$$

by

$$A \star^{(m,n)} B := \left\{ \text{wrap}^{(n)}(Z) \mid X \in \text{flat}^{(m)}(A), Y \in \text{flat}^{(m)}(B), Z \in X \oplus_H Y \right\}.$$

Example 28 (SuperHyperGranular Number — Morning Block Under Weather Scenarios). We consider two scenario families, each a set of granular options (thus elements of $\mathcal{P}(\mathfrak{G})$):

$$\begin{aligned} H_{bf}^{fair} &= \{ G_{0.90}(12, 2), G_{0.80}(9, 1) \} \quad (\text{breakfast, fair weather}), \\ H_{cm}^{fair} &= \{ G_{0.85}(35, 5), G_{0.70}(42, 4) \} \quad (\text{commute, fair weather}), \\ H_{bf}^{rain} &= \{ G_{0.85}(13, 3) \} \quad (\text{breakfast, rain}), \\ H_{cm}^{rain} &= \{ G_{0.70}(45, 6), G_{0.60}(55, 7) \} \quad (\text{commute, rain}). \end{aligned}$$

Collect them into level-2 SuperHyperGranular Numbers (sets of sets):

$$\mathbb{A} = \{ H_{bf}^{fair}, H_{bf}^{rain} \} \in \mathcal{P}^2(\mathfrak{G}), \quad \mathbb{B} = \{ H_{cm}^{fair}, H_{cm}^{rain} \} \in \mathcal{P}^2(\mathfrak{G}).$$

We use the product for weight-combination and apply the $(m, n) = (2, 2)$ superhyper-sum from Definition 41:

$$\mathbb{A} \star^{(2,2)} \mathbb{B} = \left\{ \text{wrap}^{(2)}(Z) \mid X \in \text{flat}^{(2)}(\mathbb{A}), Y \in \text{flat}^{(2)}(\mathbb{B}), Z \in X \oplus_H Y \right\}.$$

Representative outputs. Flattening exposes the granular options; pairing any breakfast option X with any commute option Y and hyper-adding yields a wrapped set-of-sets result. Two illustrative pairs:

(i) Fair + Fair. $X = G_{0.90}(12, 2)$, $Y = G_{0.85}(35, 5)$. Then $a_X a_Y = 0.765$, $c_X + c_Y = 47$, and $\rho \in [\max\{2, 5\}, 2+5] = [5, 7]$. Thus

$$\text{wrap}^{(2)}(X \oplus_H Y) = \left\{ \{ G_{0.765}(47, \rho) \mid \rho \in [5, 7] \} \right\},$$

i.e. a level-2 object whose inner set encodes all feasible morning-block bands $[47 - \rho, 47 + \rho]$ minutes.

(ii) Rain + Rain. $X = G_{0.85}(13, 3)$, $Y = G_{0.70}(45, 6)$. Then $a_X a_Y = 0.595$, $c_X + c_Y = 58$, and $\rho \in [\max\{3, 6\}, 3+6] = [6, 9]$. Hence

$$\text{wrap}^{(2)}(X \oplus_H Y) = \left\{ \{ G_{0.595}(58, \rho) \mid \rho \in [6, 9] \} \right\},$$

capturing heavier-uncertainty morning blocks under rain.

The full $\mathbb{A} \star^{(2,2)} \mathbb{B}$ collects all such wrapped hyper-sums over the flattened option sets, preserving the scenario hierarchy (sets of option-sets) while propagating uncertainty via the HGN hyper-sum. Each wrapped component is a family of granular bands whose radii range from conservative coupling ($\rho = \max$) to full accumulation ($\rho = r_X + r_Y$).

Theorem 27 (SHGNs carry an (m, n) -SuperHyperStructure). *For every $m, n \geq 1$, the pair*

$$\mathcal{S}_{\text{SHGN}}^{(m,n)} := \left(\mathcal{P}^m(\mathfrak{G}), \star^{(m,n)} \right)$$

is an (m, n) -SuperHyperStructure (cf. SuperHyperStructure definition in the preliminaries).

Proof. By Definition 41, the output consists of $\text{wrap}^{(n)}(Z)$ with $Z \in X \oplus_H Y \subseteq \mathfrak{G}$, hence each output element lies in $\mathcal{P}^n(\mathfrak{G})$. Therefore $\star^{(m,n)}$ has the required type $\mathcal{P}^m(\mathfrak{G}) \times \mathcal{P}^m(\mathfrak{G}) \rightarrow \mathcal{P}^n(\mathfrak{G})$. No further algebraic axiom is needed for the existence of a SuperHyperStructure. $\square$

Theorem 28 (SHGN generalizes HGN). *Fix $m, n \geq 1$. Define the embedding $\eta_m : \mathfrak{G} \rightarrow \mathcal{P}^m(\mathfrak{G})$ by $\eta_m(Z) := \text{wrap}^{(m)}(Z)$. For all $X, Y \in \mathfrak{G}$,*

$$\eta_m(X) \star^{(m,n)} \eta_m(Y) = \left\{ \text{wrap}^{(n)}(Z) \mid Z \in X \oplus_H Y \right\}.$$

In particular, if $\text{Unwrap}^{(n)}$ removes n layers of wrapping and $\text{Sel}_{\max}$ is the selector from Theorem 26, then

$$\text{Sel}_{\max} \circ \text{Unwrap}^{(n)} \left(\eta_m(X) \star^{(m,n)} \eta_m(Y) \right) = X \boxplus Y.$$

Hence SHGNs strictly generalize HGNs.

Proof. By Definition 41, when $A = \eta_m(X)$ and $B = \eta_m(Y)$ we have $\text{flat}^{(m)}(A) = \{X\}$ and $\text{flat}^{(m)}(B) = \{Y\}$. Thus

$$A \star^{(m,n)} B = \left\{ \text{wrap}^{(n)}(Z) \mid Z \in X \oplus_H Y \right\}.$$

Applying $\text{Unwrap}^{(n)}$ sends each element to $Z \in X \oplus_H Y$, and the selector $\text{Sel}_{\max}$ then returns $X \boxplus Y$ by Theorem 26. Injectivity of η_m is clear, so the generalization is strict. $\square$

3.3. Granular Set, HyperGranular Set and SuperHyperGranular Set

We formalize *Granular Set*, *HyperGranular Set*, and *SuperHyperGranular Set* and show that each carries, respectively, the structure of a Granular Number (GN), HyperGranular Number (HGN), and SuperHyperGranular Number (SHGN).

Notation 29 (Carrier of granular numbers). *Let*

$$\mathfrak{G} := \left\{ G_a(c, r) \mid c \in \mathbb{R}, r \in [0, \infty), a \in [0, 1] \right\}.$$

Each $X = G_a(c, r) \in \mathfrak{G}$ represents the connected set (interval) $\text{set}(X) = [c - r, c + r]$, optionally annotated with weight a . The classical (deterministic) granular sum is

$$X \boxplus Y := G_{a_X a_Y}(c_X + c_Y, r_X + r_Y).$$

The hypergranular sum $X \oplus_H Y$ is as in Def. 38: $X \oplus_H Y = \{ G_{a_X a_Y}(c_X + c_Y, \rho) \mid \rho \in [\max\{r_X, r_Y\}, r_X + r_Y] \}$. Write $\mathcal{P}^(\cdot)$ for the family of nonempty subsets.*

Definition 42 (Granular Set). *A Granular Set is a nonempty set of granular numbers, i.e. an element $A \in \mathcal{P}^*(\mathfrak{G})$. Its extent is*

$$\text{set}(A) := \bigcup_{X \in A} \text{set}(X) \subseteq \mathbb{R}.$$

Define the granular-set sum

$$\boxplus_{\text{set}} : \mathcal{P}^*(\mathfrak{G}) \times \mathcal{P}^*(\mathfrak{G}) \longrightarrow \mathcal{P}^*(\mathfrak{G}), \quad A \boxplus_{\text{set}} B := \{ X \boxplus Y \mid X \in A, Y \in B \}.$$

Example 29 (Granular Set — Menu Design with Calorie Bands). *Setup.* Each dish is a granular number $G_a(c, r)$ whose extent is the calorie interval $\text{set}(G_a(c, r)) = [c - r, c + r]$. Consider two dish families (nonempty subsets of $\mathfrak{G}$):

$$A = \left\{ X_1 = G_{0.95}(500, 50), X_2 = G_{0.80}(650, 75) \right\} \quad (\text{entrees}), \quad B = \left\{ Y = G_{0.90}(150, 25) \right\} \quad (\text{side}).$$

Then $\text{set}(A) = [450, 550] \cup [575, 725]$ and $\text{set}(B) = [125, 175]$.

Granular-set sum (Def. 42). Using the deterministic granular sum $X \boxplus Y = G_{a_X a_Y}(c_X + c_Y, r_X + r_Y)$, we get

$$A \boxplus_{\text{set}} B = \{ X_1 \boxplus Y, X_2 \boxplus Y \} = \{ G_{0.855}(650, 75), G_{0.72}(800, 100) \}.$$

Extents. $\text{set}(X_1 \boxplus Y) = [575, 725]$ and $\text{set}(X_2 \boxplus Y) = [700, 900]$. Hence

$$\text{set}(A \boxplus_{\text{set}} B) = ([450, 550] + [125, 175]) \cup ([575, 725] + [125, 175]) = [575, 725] \cup [700, 900],$$

matching the Minkowski sum of extents (Theorem 30). Operationally: the two entree choices paired with the side yield two feasible menu-calorie bands, each with a clear center, radius, and confidence.

Theorem 30 (Granular Sets carry GN structure). $(\mathcal{P}^*(\mathfrak{G}), \boxplus_{\text{set}})$ is closed and, via the singleton embedding $\iota: \mathfrak{G} \rightarrow \mathcal{P}^*(\mathfrak{G}), \iota(X) = \{X\}$, one has

$$\iota(X) \boxplus_{\text{set}} \iota(Y) = \{X \boxplus Y\}.$$

Hence Granular Sets carry (and extend by union) the deterministic GN operation. Moreover,

$$\text{set}(A \boxplus_{\text{set}} B) = \bigcup_{X \in A, Y \in B} (\text{set}(X) + \text{set}(Y)),$$

the (outer) Minkowski sum of extents.

Proof. Closure is immediate from the definition: for $X \in A, Y \in B, X \boxplus Y \in \mathfrak{G}$, so the resulting collection is nonempty. The singleton statement is tautological. The extent identity follows from $\text{set}(X \boxplus Y) = \text{set}(X) + \text{set}(Y)$ and distributing union over pairwise sums. $\square$

Remark 3 (Associativity and units). Because $\boxplus$ on $\mathfrak{G}$ is associative with unit $G_1(0, 0)$, the induced $\boxplus_{\text{set}}$ is associative with unit $\{G_1(0, 0)\}$. (Proof: pointwise on generators.)

Definition 43 (HyperGranular Set). A HyperGranular Set is again an element of $S := \mathcal{P}^*(\mathfrak{G})$, but endowed with the hyper-set sum

$$\begin{aligned} \oplus_H^{\text{set}}: S \times S &\longrightarrow \mathcal{P}(S), \\ A \oplus_H^{\text{set}} B &:= \left\{ M \in S \mid M \subseteq \bigcup_{X \in A, Y \in B} (X \oplus_H Y) \right\}. \end{aligned}$$

Thus, given two granular sets, we return all nonempty subfamilies of the pairwise hyper-sum union.

Example 30 (HyperGranular Set — Commute Planning with Flexible Coupling). *Setup.* Let $G_a(c, r)$ denote minutes with confidence a , center c , radius r . Define granular choice sets

$$\begin{aligned} A &= \left\{ X_1 = G_{0.85}(22, 1.0), X_2 = G_{0.75}(28, 1.5) \right\} \quad (\text{drive}), \\ B &= \left\{ Y_1 = G_{0.90}(8, 0.5), Y_2 = G_{0.80}(5, 0.5) \right\} \quad (\text{park+walk}). \end{aligned}$$

Hyper-set sum (Def. 43). For each pair (X_i, Y_j) , the hyper-sum $X_i \oplus_H Y_j$ (Def. 38) returns all granular results

$$G_{a_{ij}}(c_{ij}, \rho), \quad a_{ij} = a_{X_i} a_{Y_j}, \quad c_{ij} = c_{X_i} + c_{Y_j}, \quad \rho \in [\max\{r_{X_i}, r_{Y_j}\}, r_{X_i} + r_{Y_j}].$$

Concretely:

(X_i, Y_j)	a_{ij}	c_{ij}	ρ range
(X_1, Y_1)	$0.85 \times 0.90 = 0.765$	$22 + 8 = 30$	$[1.0, 1.5]$
(X_1, Y_2)	$0.85 \times 0.80 = 0.680$	$22 + 5 = 27$	$[1.0, 1.5]$
(X_2, Y_1)	$0.75 \times 0.90 = 0.675$	$28 + 8 = 36$	$[1.5, 2.0]$
(X_2, Y_2)	$0.75 \times 0.80 = 0.600$	$28 + 5 = 33$	$[1.5, 2.0]$

Thus $A \oplus_H^{\text{set}} B$ (Def. 43) consists of all nonempty subfamilies M drawn from the union of these four hyper-sum bands.

A concrete member $M^{\max} \in A \oplus_H^{\text{set}} B$. Selecting the maximal coupling radius in each pair (i.e., $\rho = r_{X_i} + r_{Y_j}$) yields

$$M^{\max} = \{G_{0.765}(30, 1.5), G_{0.680}(27, 1.5), G_{0.675}(36, 2.0), G_{0.600}(33, 2.0)\}.$$

Operationally: $M^{\max}$ encodes four feasible total-commute bands under worst-case uncertainty accumulation, one per (drive, park+walk) choice.

Theorem 31 (HyperGranular Sets carry HGN structure). $(S, \oplus_H^{\text{set}})$ is a hyperstructure: for each $A, B \in S$, $A \oplus_H^{\text{set}} B \subseteq \mathcal{P}(S)$ and is nonempty. Under the singleton embedding $\iota : \mathfrak{G} \rightarrow S$,

$$\iota(X) \oplus_H^{\text{set}} \iota(Y) = \{M \in S \mid M \subseteq X \oplus_H Y\},$$

so selecting (e.g. by the maximal-radius selector $\text{Sel}_{\max}$) an element of this image recovers the HGN result:

$$\text{Sel}_{\max}(\iota(X) \oplus_H^{\text{set}} \iota(Y)) = G_{a_X a_Y}(c_X + c_Y, r_X + r_Y) = X \boxplus Y.$$

Hence HyperGranular Sets generalize both HGNs (by singletons) and GNs (by selection).

Proof. Nonemptiness: for any $X \in A, Y \in B$, $X \oplus_H Y \neq \emptyset$ (Def. 38); choose any nonempty subfamily as M . Type: every such M is a nonempty subset of $\mathfrak{G}$, i.e. an element of S , so the whole output is a subset of S . The singleton and selector statements are immediate from the definitions of $\oplus_H$ and $\text{Sel}_{\max}$. $\square$

Definition 44 (SuperHyperGranular Set). Let the base set be $S = \mathcal{P}^*(\mathfrak{G})$. For $m \geq 1$ and $n \geq 1$, an m -level SuperHyperGranular Set is an element of $\mathcal{P}^m(S)$. Define wrap/flatten over S by

$$\text{wrap}_S^{(1)}(A) := \{A\}, \quad \text{wrap}_S^{(k+1)}(A) := \{\text{wrap}_S^{(k)}(A)\},$$

$$\text{flat}_S^{(1)}(\mathcal{A}) = \mathcal{A} \subseteq S, \quad \text{flat}_S^{(k+1)}(\mathcal{A}) = \bigcup_{B \in \mathcal{A}} \text{flat}_S^{(k)}(B) \subseteq S.$$

Define the (m, n) -superhyper-set sum

$$\star_{\text{set}}^{(m,n)} : \mathcal{P}^m(S) \times \mathcal{P}^n(S) \longrightarrow \mathcal{P}^n(S)$$

by

$$\mathcal{A} \star_{\text{set}}^{(m,n)} \mathcal{B} := \left\{ \text{wrap}_S^{(n)}(M) \mid C \in \text{flat}_S^{(m)}(\mathcal{A}), D \in \text{flat}_S^{(n)}(\mathcal{B}), M \in C \oplus_H^{\text{set}} D \right\}.$$

Example 31 (SuperHyperGranular Set — Scenario Tree for Morning Routine). *Level-1 (granular sets). Breakfast options:*

$$A_1 = \{ G_{0.90}(12,2), G_{0.80}(9,1) \}, \quad A_2 = \{ G_{0.85}(15,3) \}.$$

Commute options (weather-dependent):

$$B_1 = \{ G_{0.85}(35,5), G_{0.70}(42,4) \} \quad (\text{fair}), \quad B_2 = \{ G_{0.70}(45,6), G_{0.60}(55,7) \} \quad (\text{rain}).$$

Level-2 (sets of granular sets). Let $S = \mathcal{P}^*(\mathfrak{G})$ be the space of nonempty granular sets. Define

$$\mathcal{A} = \{ A_1, A_2 \} \in \mathcal{P}^2(S), \quad \mathcal{B} = \{ B_1, B_2 \} \in \mathcal{P}^2(S).$$

We use the $(m,n) = (2,2)$ superhyper-set sum $\star_{\text{set}}^{(2,2)}$ (Def. 44):

$$\mathcal{A} \star_{\text{set}}^{(2,2)} \mathcal{B} = \left\{ \text{wrap}_S^{(2)}(M) \mid C \in \{A_1, A_2\}, D \in \{B_1, B_2\}, M \in C \oplus_H^{\text{set}} D \right\}.$$

One concrete branch. Take $C = A_1$ (breakfast) and $D = B_2$ (rainy commute). Form the max-radius selector on the hyper-set sum to obtain

$$M^{\max} = \{ G_{0.90-0.70}(12+45, 2+6) = G_{0.63}(57,8), \quad G_{0.90-0.60}(12+55, 2+7) = G_{0.54}(67,9), \\ G_{0.80-0.70}(9+45, 1+6) = G_{0.56}(54,7), \quad G_{0.80-0.60}(9+55, 1+7) = G_{0.48}(64,8) \}.$$

Then $\text{wrap}_S^{(2)}(M^{\max}) = \{ \{ M^{\max} \} \} \in \mathcal{P}^2(S)$ is a specific element of $\mathcal{A} \star_{\text{set}}^{(2,2)} \mathcal{B}$.

Interpretation. The level-2 wrapper preserves the scenario hierarchy (“branch = rainy”), while $M^{\max}$ lists the feasible morning-block bands under that branch. Each band is a granular number with explicit center, radius, and confidence, produced by hyper-propagating uncertainty across breakfast and commute choices.

Theorem 32 (SuperHyperGranular Sets carry SHGN structure). For every $m, n \geq 1$, the pair

$$\left(\mathcal{P}^m(S), \star_{\text{set}}^{(m,n)} \right)$$

is an (m,n) -SuperHyperStructure. Moreover, for the embedding $\eta_m : S \rightarrow \mathcal{P}^m(S)$ given by $\eta_m(C) = \text{wrap}_S^{(m)}(C)$ and any $C, D \in S$,

$$\eta_m(C) \star_{\text{set}}^{(m,n)} \eta_m(D) = \left\{ \text{wrap}_S^{(n)}(M) \mid M \in C \oplus_H^{\text{set}} D \right\}.$$

Applying n -fold unwrapping followed by a selector (e.g. choosing maximal-radius elements in each pair inside M) recovers the deterministic granular-set sum $C \boxplus_{\text{set}} D$. Hence SHGSs strictly generalize HGSs, which in turn generalize GNs.

Proof. Type: by construction, each output element has the form $\text{wrap}_S^{(n)}(M)$ with $M \in S$, hence lies in $\mathcal{P}^n(S)$, so $\star_{\text{set}}^{(m,n)}$ has the required signature. The displayed identity follows from $\text{flat}_S^{(m)}(\text{wrap}_S^{(m)}(C)) = \{C\}$. Finally, unwrapping yields $M \in C \oplus_H^{\text{set}} D$; selecting (e.g. by maximal radii inside each contributing HGN) collapses the hyperresult to the deterministic pairwise sums, i.e. $C \boxplus_{\text{set}} D$. Injectivity of η_m is clear, so the generalization is strict. $\square$

4. Conclusions

This paper developed a unified framework for uncertainty numbers by introducing *ordered* and *ranking* structures across six paradigms: fuzzy, neutrosophic, plithogenic, rough, granular, and functorial numbers. In future work, we hope to consider extensions that employ HyperRough Sets [101,102], HyperSoft Sets [103–105], HyperFuzzy Sets [60,61,106], HyperNeutrosophic Sets [90, 107,108], HyperPolar Sets [109], and HyperPlithogenic Sets [93,110]. We also intend to investigate

extensions based on Graphs [111,112], HyperGraphs [71,72,113], SuperHyperGraphs [89,114–116], and HyperAlgebra [117–119].

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