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Square-Difference Factor Absorbing Primary Ideals of Commutative Rings †

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Abstract: Let R be a commutative ring with identity. A proper ideal I of a ring R is called a square-difference factor absorbing primary ideal of R if for $a, b \in R$, whenever $a^2 - b^2 \in I$, then $a + b \in \sqrt{I}$ or $a - b \in I$. Several characterizations and properties of this class of ideals are presented.Various examples are provided to illustrate the obtained results and demonstrate the applicability of our findings. Furthermore, the properties of this class of ideals are investigated in extensions of rings.

Keywords: prime ideal; primary ideal; square-difference factor absorbing ideal; square-difference factor absorbing primary ideal

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1. Introduction

Throughout this paper, we assume that all rings are commutative with nonzero identity. Generalizing prime ideals has been widely explored in the field of commutative algebra, both in theory and practical applications such as graph theory (see for example, [3–6,11–13,15]). As a new generalization of prime ideals, very recently, Anderson, Badawi and Coykendall introduced and studied in [2], the notion of square-difference factor absorbing ideals. According to [2], a proper ideal I of a ring R is called a square-difference factor absorbing ideal (sdf-absorbing ideal) of R if for $0 \neq a, b \in R$, whenever $a^2 - b^2 \in I$, then $a + b \in I$ or $a - b \in I$. Various characterizations of rings in terms of this type of ideals and the relationship with the role of $char(R)$ are given. For example, it is shown that sdf-absorbing ideals are radical and the converse holds when $char(R) = 2$. Thus, every nonzero proper ideal of a commutative ring R is an sdf-absorbing ideal if and only if $R/\sqrt{0}$ is a von Neumann regular ring. Moreover, several additional results about sdf-absorbing ideals are given.

In this paper, expanding the structure of sdf-absorbing ideals, we introduce the concept of square-difference factor absorbing primary ideals. Let R be a ring and I be a proper ideal of R. We call I a square-difference factor absorbing primary ideal (sdf-absorbing primary ideal) of R if for $a, b \in R$, whenever $a^2 - b^2 \in I$, then $a + b \in \sqrt{I}$ or $a - b \in I$. It is clear that any sdf-absorbing ideal is an sdf-absorbing primary. However, the converse implication is not true. This fact can be demonstrated by the Example 1(2).

For the sake of integrity, we list the notations which are commonly used in the sequel. Let R be a ring. As usual, by $char(R)$, $dim(R)$, $U(R)$, $Z(R)$, we denote the characteristic, the Krull dimension, the group of units and the set of zero-divisors of R. For a proper ideal I of R, by $\sqrt{I}$, we denote the radical of I, that is $\{r \in R : r^n \in I \text{ for some } n \in \mathbb{N}\}$. For more comprehensive information about the above notations and terminologies, refer to the book [10].

One of the objectives of this article is to study and investigate characterizations of sdf-absorbing primary ideals in commutative rings. In Section 2, the relationships among primary, sdf-absorbing and sdf-absorbing primary ideals are clarified. It is shown that if $2 \in U(R)$, then sdf-absorbing primary ideals and primary ideals coincide, (Proposition 2). However, these concepts are different in general,

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see Example 1(1). It is proved that in rings of characteristic 2, every proper ideal is sdf-absorbing primary,(see Proposition 1).

Section 3 is devoted to study the behavior of this class of ideals in some extensions of rings such as localization rings, Cartesian product of rings, polynomial rings, idealizations of modules and amalgamations of rings along an ideal, (see Propositions 3, 4 and Theorems 5-8). Furthermore, in ZPI-rings and particularly in principal ideal domains, the class of sdf-absorbing primary ideals is totally characterized, (see Theorem 3 and Corollary 3). Moreover, in terms of this new class of ideals, we present a new characterization for Dedekind domains whose the element 2 is prime, (see Theorem 4). By investigating the structure of homomorphic images of sdf-absorbing primary ideals, we determine all sdf-absorbing primary ideals in the ring of integers modulo n, (see Proposition 5). We present many examples to demonstrate the necessity of the various restrictions imposed in the hypotheses of our theorems.

2. Sdf-Absorbing Primary Ideals

In this section, we give some examples and basic properties of Sdf-absorbing primary ideals and investigate them in several classes of rings.

Definition 1. *A proper ideal I of a ring R is called a square-difference factor absorbing primary ideal (sdf-absorbing primary ideal) of R if for $a, b \in R$, whenever $a^2 - b^2 \in I$, then $a + b \in \sqrt{I}$ or $a - b \in I$.*

We note that by replacing b by $-b$ in the above definition, we conclude that I is an sdf-absorbing primary ideal in R if and only if for $a, b \in R$, whenever $a^2 - b^2 \in I$, then $a - b \in \sqrt{I}$ or $a + b \in I$.

If I is a primary or an sdf-absorbing ideal of a ring R, then clearly I is sdf-absorbing primary. The converses are not true in general as we can see in the following example:

Example 1.

(1) Following ([2] Example 2.8), for any odd prime integer q, the ideal $2q\mathbb{Z}$ is an sdf-absorbing ideal of $\mathbb{Z}$ and so it is sdf-absorbing primary in $\mathbb{Z}$. But, $2q\mathbb{Z}$ is not a primary ideal of $\mathbb{Z}$.

(2) For any positive integer m, the ideal $I = 2q^m\mathbb{Z}$ is sdf-absorbing primary in $\mathbb{Z}$. Indeed, let $a, b \in \mathbb{Z}$ such that $(a+b)(a-b) = a^2 - b^2 \in I \subseteq q^m\mathbb{Z}$. Then $a + b \in \sqrt{q^m\mathbb{Z}} = q\mathbb{Z}$ or $a - b \in q^m\mathbb{Z}$ as $q^m\mathbb{Z}$ is primary in $\mathbb{Z}$. Since the sum of $(a+b)$ and $(a-b)$ is even, then clearly, we must have both $(a+b)$ and $(a-b)$ are even. Thus, $a + b \in q\mathbb{Z} \cap 2\mathbb{Z} = 2q\mathbb{Z} = \sqrt{I}$ or $a - b \in q^m\mathbb{Z} \cap 2\mathbb{Z} = I$. On the other hand, by ([2] Example 2.8), $I = 2q^m\mathbb{Z}$ is not sdf-absorbing in $\mathbb{Z}$ for any $m \geq 2$.

Remark 1. *In the definition of sdf-absorbing ideals [2], the hypothesis "$a, b \neq 0$" is needed when $I = \{0\}$ since otherwise, I is not sdf-absorbing in any non-reduced ring. However, in the case of sdf-absorbing primary ideals, we do not need this hypothesis. Indeed, the zero ideal may or may not sdf-absorbing primary in non-reduced rings. For example, while clearly the zero ideal is sdf-absorbing primary in $\mathbb{Z}_4$, $\{\bar{0}\}$ is not sdf-absorbing primary in $\mathbb{Z}_{12}$. Indeed, $\bar{4}^2 - \overline{10}^2 = \bar{0}$ but $\bar{2} = \bar{4} + \overline{10} \notin \sqrt{\bar{0}}$ and $\bar{6} = \bar{4} - \overline{10} \neq \bar{0}$.*

Next, we present a result which enables us to built some examples of sdf-absorbing primary ideals which are not sdf-absorbing.

Proposition 1. *If R is a ring of characteristic 2, then every proper ideal of R is sdf-absorbing primary.*

Proof. Suppose that $char(R) = 2$ and I is an ideal of R. Let $a, b \in R$ and $a^2 - b^2 \in I$. Then $(a+b)^2 = a^2 - b^2 \in I$ which implies $a + b \in \sqrt{I}$ and I is an sdf-absorbing primary ideal of R. □

Remark 2. *If a ring R is Laskerian, then every proper ideal of R has a primary decomposition, so it has an sdf-absorbing primary decomposition. While the primary decomposition is unique, note that the sdf-absorbing*

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primary decomposition need not be so. For example, by using Example 5, the ideal $I = 120\mathbb{Z}$ *of* $\mathbb{Z}$ *can be decomposed by distinct decompositions such as* $I = (3) \cap (5) \cap (8) = (4) \cap (5) \cap (6) = (3) \cap (4) \cap (10)$. *On the other hand, it is well-known that the ring* $R = \mathbb{Z}_2[X_1, X_2, \cdots]$ *is not Laskerian. However, as* $char(R) = 2$, *every proper ideal of R is sdf-absorbing primary by Proposition 1.*

The next result presents the closed relationship between sdf-absorbing primary and sdf-absorbing ideals.

Theorem 1. *If I is an sdf-absorbing primary ideal of a ring R, then* $\sqrt{I}$ *is an sdf-absorbing ideal of R.*

Proof. Let $a, b \in R$ such that $a^2 - b^2 \in \sqrt{I}$. Then $(a+b)^n(a-b)^n \in I$ for some positive integer n. If n is odd, let $t = a^n + \binom{n}{2} a^{n-2}b^2 + \cdots + \binom{n}{n-1} ab^{n-1}$ and $s = \binom{n}{1} a^{n-1}b + \binom{n}{3} a^{n-3}b^3 + \cdots + b^n$. If n is even, let $t = a^n + \binom{n}{2} a^{n-2}b^2 + \cdots + b^n$ and $s = \binom{n}{1} a^{n-1}b + \binom{n}{3} a^{n-3}b^3 + \cdots + \binom{n}{n-1} ab^{n-1}$. In both cases, we have $t, s \in R$ with $(a+b)^n(a-b)^n = (t+s)(t-s) \in I$. Since I is sdf-absorbing primary in R, then $t+s \in \sqrt{I}$ or $t-s \in I$. It follows that $a+b \in \sqrt{I}$ or $a-b \in \sqrt{I}$ as needed. □

If I is an sdf-absorbing primary ideal of a ring R, then $Q = \sqrt{I}$ is an sdf-absorbing ideal of R by Theorem 1. In this case, we call I a Q-sdf-absorbing primary ideal of R. Via this definition, one can make the following observation: If $I_1, I_2, ..., I_n$ are Q-sdf-absorbing primary ideals of R , then $I = \cap_{j=1}^{n} I_j$ is a Q-sdf-absorbing primary ideal of R. However, the intersection of two sdf-absorbing primary ideals of R is not sdf-absorbing primary. For example, $I = 6\mathbb{Z}$ and $J = 10\mathbb{Z}$ are sdf-absorbing primary ideals of $\mathbb{Z}$ but, $I \cap J = 60\mathbb{Z}$ is not so by Example 5.

We note that the converse of Theorem 1 is not true in general. For example, for any odd prime integer q and a positive integer m, $\sqrt{2^2 q^m \mathbb{Z}} = 2q\mathbb{Z}$ is an sdf-absorbing ideal of $\mathbb{Z}$ by ([2] Example 2.8). On the other hand, $2^2 q^m \mathbb{Z}$ is not sdf-absorbing primary in $\mathbb{Z}$ by Example 5.

In the following proposition, we give a condition under which primary and sdf-absorbing primary ideals coincide.

Proposition 2. *Let I be an ideal of a ring R with* $2 \in U(R)$. *If I is an sdf-absorbing primary ideal of R, then it is a primary ideal of R.*

Proof. Let $x, y \in R$ such that $xy \in I$ and $x \notin \sqrt{I}$. Let $a = \frac{x+y}{2}$, $b = \frac{x-y}{2}$. Then $a, b \in R$ with $a^2 - b^2 = xy \in I$ and $a + b = x \notin \sqrt{I}$. Thus, $y = a - b \in I$ as I is an sdf-absorbing primary ideal of R. Therefore, I is a primary ideal of R. □

The converse of Proposition 1 may not be true. For example, consider $R = \mathbb{Z}_2 \times \mathbb{Z}_p$ where $p \neq 2$ a prime integer. Then every proper ideal of R is sdf-absorbing primary ($(0,0)$ is sdf-absorbing primary by (2) of Theorem 6) but $char(R) = 2p \neq 2$.

By ([2] Example 2.8 (e)), and Proposition 1, we have the following:

Example 2. *Let K be a field of characteristic 2 and* $R = K[X]$. *Then* $J = (X+1)(X-1)R$ *is an sdf-absorbing primary ideal which is not sdf-absorbing.*

3. Extensions of Sdf-Absorbing Primary Ideals

In this section, we discuss the behavior of sdf-absorbing primary ideals under homomorphisms, localizations, Cartesian products, idealizations and amalgamations. Additionally, we characterize this class of ideals in some special rings such as ZPI-rings, Dedekind domains and principal ideal domains.

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Moreover, by using these properties, we completely determine sdf-absorbing primary ideals of the rings $\mathbb{Z}$ and $\mathbb{Z}_n$.

In the following, by $Z_I(R)$, we mean the set of all $r \in R$ such that $ra \in I$ for some $a \in R \backslash I$.

Proposition 3. *Let S be a multiplicatively closed subset of a ring R and I be an ideal of R disjoint with S.*

(1) If I is an sdf-absorbing primary ideal of R, then $S^{-1}I$ is an sdf-absorbing primary ideal of $S^{-1}R$.
(2) If $S \cap Z_I(R) = S \cap Z_{\sqrt{I}}(R) = \emptyset$ and $S^{-1}I$ is an sdf-absorbing primary ideal of $S^{-1}R$, then I is an sdf-absorbing primary ideal of R.

Proof. (1) Let $\left(\frac{a}{s}\right)^2 - \left(\frac{b}{t}\right)^2 \in S^{-1}I$ for some $\frac{a}{s}, \frac{b}{t} \in S^{-1}R$. Then there exists $u \in S$ such that $ut^2a^2 - us^2b^2 \in I$. Hence $(uta)^2 - (usb)^2 \in I$ and since I is sdf-absorbing primary, we have $uta + usb \in \sqrt{I}$ or $uta - usb \in I$. Thus, $\left(\frac{a}{s} + \frac{b}{t}\right) \in S^{-1}\sqrt{I} = \sqrt{S^{-1}I}$ or $\left(\frac{a}{s} - \frac{b}{t}\right) \in S^{-1}I$.

(2) Let $a, b \in R$ with $a^2 - b^2 \in I$. Then $\left(\frac{a}{1}\right)^2 - \left(\frac{b}{1}\right)^2 \in S^{-1}I$ which yields that $\left(\frac{a}{1} + \frac{b}{1}\right) \in \sqrt{S^{-1}I}$ or $\left(\frac{a}{1} - \frac{b}{1}\right) \in S^{-1}I$. Hence, $u(a+b) \in \sqrt{I}$ for some $u \in S$ or $v(a-b) \in I$ for some $v \in S$. By our assumption $S \cap Z_I(R) = S \cap Z_{\sqrt{I}}(R) = \emptyset$, we conclude either $a + b \in \sqrt{I}$ or $a - b \in I$, as required. □

Proposition 4. *Let $f : R \to R'$ be a ring homomorphism.*

(1) If I' is an sdf-absorbing primary ideal of R', then $f^{-1}(I')$ is an sdf-absorbing primary ideal of R.
(2) Suppose f is an epimorphism. If $I \supseteq Ker(f)$ is an sdf-absorbing primary ideal of R, then $f(I)$ is an sdf-absorbing primary ideal of R'.

Proof. (1) Let $x, y \in R'$ with $x^2 - y^2 \in f^{-1}(I')$. Then $f(x)^2 - f(y)^2 \in I'$ which implies $f(x) + f(y) = f(x+y) \in \sqrt{I'}$ or $f(x) - f(y) = f(x-y) \in I$. Thus, $(x+y) \in f^{-1}(\sqrt{I'}) = \sqrt{f^{-1}(I')}$ or $(x-y) \in f^{-1}(I')$, as needed.

(2) Let $a, b \in R$ with $a^2 - b^2 \in f(I)$, say, $a = f(x)$ and $b = f(y)$ for some $x, y \in R$. Since $Ker(f) \subseteq I$, we have clearly, $x^2 - y^2 \in I$ which yields either $x - y \in \sqrt{I}$ or $x - y \in I$. Hence, $a + b \in f(\sqrt{I}) = \sqrt{f(I)}$ or $a - b \in I$ and $f(I)$ is an sdf-absorbing primary ideal of R. □

As a direct consequence of Proposition 4, we conclude the following result.

Corollary 1. *Let R be a ring.*

(1) If $R' \supseteq R$ is a ring and I' is an sdf-absorbing primary ideal of R', then $I' \cap R$ is an sdf-absorbing primary ideal of R.
(2) If I is an ideal of R, then I is an sdf-absorbing primary ideal of R if and only if I/K is an sdf-absorbing primary ideal of R/K where K is an ideal of R with $K \subseteq I$.

The following example shows that the hypothesis "$I \supseteq Ker(f)$" in (2) of Proposition 4 is needed.

Example 3. *Consider the epimorphism $f : \mathbb{Z}[x] \to \mathbb{Z}$ defined by $f(p(x)) = p(0)$. Then $I = (x + 12)$ is clearly an sdf-absorbing primary ideal of $\mathbb{Z}[x]$ but $f(I) = (12)$ is not sdf-absorbing primary in $\mathbb{Z}$ by Theorem 5. Note that $Ker(f) = (x)$ is not contained in I.*

Theorem 2. *Let $P_1, \cdots, P_n$ be distinct comaximal prime ideals of R and $I = P_1 \cap \cdots \cap P_n$. The following are equivalent:*

(1) I is an sdf-absorbing ideal of R.
(2) I is an sdf-absorbing primary ideal of R.
(3) At most one of the P_i's has $char(R/P_i) \neq 2$.

Proof. (1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (3) We note that by the Chinese Remainder Theorem, $R/I \cong R/P_1 \times \cdots \times R/P_n$. Since I is an sdf-absorbing primary ideal of R, then $0_{R/I} = I/I$ is sdf-absorbing primary in R/I by (2) of Corollary 1. Since also $\sqrt{0_{R/I}} = 0_{R/I}$, the proof can be achieved in a similar way to ([2] Theorem 4.1).

(3) $\Rightarrow$ (1) ([2] Theorem 4.1). □

The following is an immediate consequence of Theorems 1 and 2.

Corollary 2. *Let $I = Q_1 \cap \cdots \cap Q_n$ where Q_i's are P_i-primary ideals of R and P_i's are distinct comaximal prime ideals of R. If I is an sdf-absorbing primary ideal of R, then at most one of the P_i's has the property $char(R/P_i) \neq 2$.*

Proof. Suppose I is an sdf-absorbing primary ideal of R. Then $\sqrt{I} = P_1 \cap \cdots \cap P_n$ is an sdf-absorbing ideal of R by Theorem 1. Therefore, $char(R/P_i) \neq 2$ for at most one of the P_i's by Theorem 2. □

The converse of Corollary 2 is not true in general as we can see in the following example:

Example 4. *Consider the ring of Gaussian integers $\mathbb{Z}[i]$ and let $Q_1 = (2) = (1+i)^2$ and $Q_2 = (1+2i)$. Then Q_1 and Q_2 are primary ideals of R where $P_1 = \sqrt{Q_1} = (1+i)$ and $P_2 = \sqrt{Q_2} = (1+2i)$ are distinct comaximal prime ideals of R. Moreover, $char(R/P_1) = 2$ since $2 = (1+i)(1-i) \in P_1$. Therefore, at most one of the P_i's has the property $char(R/P_i) \neq 2$. On the other hand, $I = Q_1 \cap Q_2$ is not an sdf-absorbing primary ideal of R. Indeed, let $x = 2+i$ and $y = -1$. Then $x^2 - y^2 = 2+4i \in I$ but simple computations show that $x+y = 1+i \notin (1+2i) \supseteq \sqrt{I}$ and $x-y = 3+i \notin (2) \supseteq I$.*

An integral domain R is a Dedekind domain if every nonzero proper ideal factors into a product of prime ideals. It can be shown that such a factorization is then necessarily unique up to the order of the factors. More general, a ring R is said to be a ZPI-ring (Zerlegungsatz in Primideale) if each nonzero ideal I of R is uniquely expressible as a product of prime ideals of R. The ring R is said to be a general ZPI-ring if each ideal of R can be expressed as a finite product of prime ideals of R. Dedekind domains and special primary rings are particular examples for general ZPI-rings. A general ZPI-ring R is Noetherian and each primary ideal of R is a prime power. For more details, the reader can refer to [14].

The following theorem plays a crucial role in our subsequent discussion as it provides a comprehensive classification of all sdf-absorbing primary ideals in ZPI-rings.

Theorem 3. *Let R be a ZPI-ring.*

(1) If $char(R) = 2$, then every proper ideal of R is sdf-absorbing primary.
(2) If $2 \in U(R)$, then a proper ideal I of R is sdf-absorbing primary if and only if $I = P^k$ for some prime ideal P of R and $k \in \mathbb{N}$.
(3) If $2 \notin U(R)$ and I is an sdf-absorbing primary ideal of R, then $I = P_1^{k_1} P_2^{k_2} ... P_n^{k_n}$ where the P_i's are comaximal ideals of R such that $2 \notin P_i$ for at most one i.
(4) If (2) is maximal in R, then a proper ideal I of R is sdf-absorbing primary if and only if I is a primary ideal of R or $I = (2)P^k$ for a prime ideal P of R not equal to (2).

Proof. (1) Proposition 1.

(2) Suppose that $2 \in U(R)$ and I is an sdf-absorbing primary ideal of R. Then I is primary by Proposition 2. It is shown in ([14] Result 1) that if Q is a P-primary ideal in a ZPI-ring R, then Q is a power of P. Thus, $I = P^k$ for some prime ideal P of R and $k \in \mathbb{N}$. The converse part is clear.

(3) Suppose that $2 \notin U(R)$ and I is an sdf-absorbing primary ideal of R. Since R is a ZPI-ring, we have $I = P_1^{k_1} P_2^{k_2} ... P_n^{k_n}$ for some comaximal prime ideals $P_1, P_2, ..., P_n$ of R. If $n = 1$, the result is obvious. If $n \gneqq 1$, then $2 \notin P_i$ for at most one $1 \leq i \leq n$ by Corollary 2 and we are done.

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(4) Suppose (2) is maximal in R. Let $I = (2)P^k$ for a prime ideal P of R such that P is not equal to (2). Let $a, b \in R$ such that $a^2 - b^2 \in I$. Since $a^2 - b^2 \in P^k$, then $a + b \in \sqrt{P^k} = P$ or $a - b \in P^k$ as P^k is primary in R. Since $a^2 - b^2 \in (2)$ and $char(R/(2)) = 2$, then $a + b, a - b \in (2)$ by [2, Theorem 2.5]. Hence, $a + b \in (2)P = \sqrt{I}$ or $a - b \in (2)P^k = I$ and I is an sdf-absorbing primary ideal of R. Conversely, suppose I an sdf-absorbing primary ideal of R. Then by (3), $I = P_1^{k_1}P_2^{k_2}...P_n^{k_n}$ where P_i's are comaximal ideals of R such that $2 \notin P_i$ for at most one i, say $i = 1$. Since (2) is maximal in R, then $(2) = P_2...P_n$ and so $I = P_1^{k_1}P_2^{k_2}...P_n^{k_n} = (2^m)P^k$ for some $m, k \in \mathbb{N}$ and prime ideal P which is comaximal with (2). Suppose on contrary that $m \geq 2$ and choose $p \in P$ that is not associate to 2. Then $(1 + 2^{m-2}p^k)^2 - (1 - 2^{m-2}p^k)^2 = 2.2^{m-1}p^k \in I$ but $2 \notin \sqrt{I} = (2)P$ (as $2 \notin P$) and $2^{m-1}p^k \notin I$, a contradiction. Therefore, $m \leq 1$ and so either I is primary in R or $I = (2)P^k$ for a prime ideal P of R which is not equal to (2). □

Recall that a ring R is a Dedekind domain if and only if R is a Noetherian integrally closed integral domain of Krull dimension 1, so that every nonzero prime ideal is maximal. Moreover, if R is Noetherian, then R is a Dedekind domain if and only if whenever M is a maximal ideal of R, then there are no ideals of R strictly between M and M^2 ([10] Theorem 39.2).

In terms of sdf-absorbing primary ideals of a ring, we characterize Dedekind domains for which the ideal (2) is prime:

Theorem 4. *LET R be a Noetherian domain that is not a field such that 2 is a prime element in R. Then, the following are equivalent.*

(1) R is a Dedekind domain.
(2) For any nonzero proper ideal I of R, I is an sdf-absorbing primary ideal if and only if either $I = M^n$ or $I = (2)T^m$ for some maximal ideals M and T of R and $n, m \in \mathbb{N}$.

Proof. (1) $\Longrightarrow$ (2) Let R be a Dedekind domain that is not a field. Suppose that I is a nonzero proper ideal of R. Since every nonzero prime ideal of R is maximal, then $I = M_1^{n_1}M_2^{n_2}\cdots M_k^{nk}$ for some distinct maximal ideals $M_1, \cdots, M_k$ of R and some positive integers $n_1, \cdots, n_k$. Suppose that I is an sdf-absorbing primary ideal of R. Then by (4) of Theorem 3, $I = M^n$ or $I = (2)T^m$ for some maximal ideals M and T of R and $n, m \in \mathbb{N}$. The converse part follows by (4) of Theorem 3.

(2) $\Longrightarrow$ (1) Let M be a maximal ideal of R. We show that there is no ideal properly between M^2 and M. Assume that J is an ideal of R such that $M^2 \subseteq J \subseteq M$. Then clearly J is an M-primary ideal, so it is an sdf-absorbing primary ideal of R. By assumption, we yield that $J = L^n$ or $J = (2)T^m$ for some maximal ideals L and T of R and $n, m \in \mathbb{N}$. If $J = (2)T^m$, then $(2)T = M$ and so $(2) = M = T$ as (2) is also maximal in R. Since also clearly $M = L$, then J is a power of M and so $J = M^2$ or $J = M$, as required. Therefore, R is a Dedekind domain. □

The following corollary establishes a powerful tool in characterizing sdf-absorbing primary ideals in principal ideal domains based just on prime generators.

Corollary 3. *Let R be a principal ideal domain and I a proper ideal of R.*

(1) If $char(R) = 2$, then every proper ideal of R is sdf-absorbing primary.
(2) If $2 \in U(R)$, then I is an sdf-absorbing primary ideal of R if and only if $I = (p^k)$ for some prime element p of R and $k \in \mathbb{N}$.
(3) If $2 \notin U(R)$ and I is sdf-absorbing primary in R, then $I = (p_1^{k_1}p_2^{k_2}...p_n^{k_n})$ where p_i's are non-associate prime elements in R such that $p_i \nmid 2$ for at most one i.
(4) If $2 \in R$ is prime, then I is an sdf-absorbing primary ideal of R if and only if I is a primary ideal of R or $I = (2p^k)$ for a prime element p of R which is not associate to 2.

Note that if 2 is not prime in a ring R, then the converse of (3) in Corollary 3 need not be true as we have seen in Example 4.

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As conclusion, we have the following characterization of sdf-absorbing primary ideals in the ring of integers $\mathbb{Z}$.

Example 5. *A proper ideal I of $\mathbb{Z}$ is an sdf-absorbing primary ideal if and only if I is a primary ideal of $\mathbb{Z}$ or $I = 2q^m\mathbb{Z}$ for some odd prime integer q and positive integer m.*

By using Example 5 and Corollary 1, we determine in the following proposition the sdf-absorbing primary ideal of $\mathbb{Z}_n$ for any positive integer n.

Proposition 5. *Let n be a positive integer. A proper ideal I of $\mathbb{Z}_n$ is an sdf-absorbing primary ideal if and only if I is a primary ideal of $\mathbb{Z}_n$ or $I = \bar{2}\bar{q}^m\mathbb{Z}_n$ for some odd prime integer q and positive integer m.*

Proof. If I is a primary ideal of $\mathbb{Z}_n$, then clearly it is sdf-absorbing primary. Suppose $I = \bar{2}\bar{q}^m\mathbb{Z}_n$ for some odd prime integer $q \mid n$ and positive integer m. Since by using Theorem 5, $J = 2q^m\mathbb{Z}$ is sdf-absorbing primary in $\mathbb{Z}$, then $J/n\mathbb{Z} \cong I$ is an sdf-absorbing primary ideal of $\mathbb{Z}_n$ by (2) of Corollary 1. Conversely, let I be an sdf-absorbing primary ideal of $\mathbb{Z}_n$. With no loss of generality, let us suppose that $I = (\bar{p})^k(\bar{q})^m\mathbb{Z}_n \cong p^kq^m\mathbb{Z}/n\mathbb{Z}$ where p, q are distinct prime integers dividing n and $k, m \geq 1$. Then $J = p^kq^m\mathbb{Z}$ is an sdf-absorbing primary ideal of $\mathbb{Z}$ again by (2) of Corollary 1. Therefore, $p = 2$ and $k = 1$ by Example 5 and we are done. □

As a conclusion, all proper ideals of $\mathbb{Z}_{18}$ are sdf-absorbing primary and all nonzero proper ideals of $\mathbb{Z}_{12}$ or $\mathbb{Z}_{15}$ are sdf-absorbing primary.

The following corollary can be followed directly by Theorem 2 and ([2] Theorem 4.5).

Corollary 4. *Let $P_1, ..., P_n$ be distinct comaximal prime ideals of R and $I = P_1 \cap \cdots \cap P_n$. The following are equivalent:*

(1) I is an sdf-absorbing primary ideal of R.
(2) $I[X]$ is an sdf-absorbing primary ideal of $R[X]$.
(3) At most one of the P_i 's has $char(R/P_i) \neq 2$.

Theorem 5. *Let R be a ring and let I be a proper ideal of R. Then (I, X) is an sdf-absorbing primary ideal of $R[X]$ if and only if I is an sdf-absorbing primary ideal of R. In particular, (X) is an sdf-absorbing primary ideal of $R[X]$ if and only if $\{0\}$ is an sdf-absorbing primary ideal of R.*

Proof. Suppose I is an sdf-absorbing primary ideal of R. Let $f = a + XT(X), g = b + XL(X) \in R[X]$ such that $f^2 - g^2 \in (I, X)$. Then clearly, $a^2 - b^2 \in I$ and by assumption, $a + b \in \sqrt{I}$ or $a - b \in I$. Therefore, $f + g \in (\sqrt{I}, X) = \sqrt{(I, X)}$ or $f - g \in (I, X)$ and (I, X) is an sdf-absorbing primary ideal of $R[X]$. The converse is clear by using (2) of Proposition 4. the "in particular" statement is clear by considering $I = 0$. □

In the following theorem, we determine when the direct product of two ideals is sdf-absorbing primary.

Theorem 6. *Let I_1 and I_2 be proper ideals of rings R_1 and R_2, respectively.*

(1) If $I = I_1 \times I_2$ is an sdf-absorbing primary ideal of $R_1 \times R_2$, then I_1 and I_2 are sdf-absorbing primary ideals of R_1 and R_2, respectively. Moreover, we have $2 \in \sqrt{I_1}$ or $2 \in \sqrt{I_2}$.
(2) If I_1 is an sdf-absorbing ideal of R_1 with $2 \in I_1$ and I_2 is an sdf-absorbing primary ideal of R_2, then $I = I_1 \times I_2$ is an sdf-absorbing primary ideal of $R_1 \times R_2$.
(3) If I_1 is an sdf-absorbing primary ideal of R_1 and I_2 is an sdf-absorbing ideal of R_2 with $2 \in I_2$, then $I = I_1 \times I_2$ is an sdf-absorbing primary ideal of $R_1 \times R_2$.

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(4) $I_1 \times R_2$ is an sdf-absorbing primary ideal of $R_1 \times R_2$ if and only if I_1 is an sdf-absorbing primary ideal of R_1.
(5) $R_1 \times I_2$ is an sdf-absorbing primary ideal of $R_1 \times R_2$ if and only if I_2 is an sdf-absorbing primary ideal of R_2.

Proof. (1) By considering the projection homomorphism in Proposition 4, we conclude that I_1 and I_2 are sdf-absorbing primary ideals of R_1 and R_2, respectively. Now, consider $a = (1,1)$ and $b = (1,-1)$ in $R_1 \times R_2$. Then $a^2 - b^2 = 0 \in I$ and so $(2,0) = a + b \in \sqrt{I}$ or $(0,2) = a - b \in I$. Hence, $2 \in I_2$ or $2 \in \sqrt{I_1}$. Similarly, if we consider $a = (1,1)$ and $b = (-1,1)$, we get $2 \in I_1$ or $2 \in \sqrt{I_2}$. Therefore, we have either $2 \in \sqrt{I_1}$ or $2 \in \sqrt{I_2}$.

(2) Suppose I_1 is an sdf-absorbing ideal of R_1 with $2 \in I_1$. Let $(a_1,a_2),(b_1,b_2) \in R_1 \times R_2$ such that $(a_1,a_2)^2 - (b_1,b_2)^2 \in I$. Then $a_1^2 - b_1{}^2 \in I_1$ and $a_2^2 - b_2{}^2 \in I_2$. Since I_1 is sdf-absorbing in R_1 and $2 \in I_1$, then $a_1 + b_1, a_1 - b_1 \in I_1$ by ([2] Theorem 2.5). Also, $a_2 + b_2 \in \sqrt{I_2}$ or $a_2 - b_2 \in I_2$ as I_2 is an sdf-absorbing primary ideal of R_2. If $a_2 + b_2 \in \sqrt{I_2}$, then $(a_1,a_2) + (b_1,b_2) \in \sqrt{I_1} \times \sqrt{I_2} = \sqrt{I}$. If $a_2 - b_2 \in I_2$, then $(a_1,a_2) - (b_1,b_2) \in I$. Therefore, $I = I_1 \times I_2$ is an sdf-absorbing primary ideal of $R_1 \times R_2$.

(3) Similar to (2).

(4) If $I_1 \times R_2$ is an sdf-absorbing primary ideal of $R_1 \times R_2$, then I_1 is an sdf-absorbing primary ideal of R_1 by Proposition 4. Conversely, suppose I_1 is an sdf-absorbing primary ideal of R_1 and let $(a_1,a_2),(b_1,b_2) \in R_1 \times R_2$ such that $(a_1,a_2)^2 - (b_1,b_2)^2 \in I_1 \times R_2$. Then $a_1^2 - b_1{}^2 \in I_1$ and so $a_1 + b_1 \in \sqrt{I_1}$ or $a_1 - b_1 \in I_1$. Thus, $(a_1,a_2) + (b_1,b_2) \in \sqrt{I_1} \times R_2 = \sqrt{I_1 \times R_2}$ or $(a_1,a_2) - (b_1,b_2) \in I_1 \times R_2$ as required.

(5) Similar to (4). □

Remark 3.

(1) The converse of (1) of Theorem 6 need not be true in general. For example, $4\mathbb{Z}$ and $18\mathbb{Z}$ are sdf-absorbing primary in $\mathbb{Z}$ and $2 \in \sqrt{4\mathbb{Z}}$ but $4\mathbb{Z} \times 18\mathbb{Z}$ is not sdf-absorbing primary in $\mathbb{Z} \times \mathbb{Z}$. Indeed, $(2,2)^2 - (0,2)^2 \in 4\mathbb{Z} \times 18\mathbb{Z}$ but $(2,4) \notin \sqrt{4\mathbb{Z} \times 18\mathbb{Z}} = 2\mathbb{Z} \times 6\mathbb{Z}$ and $(2,0) \notin 4\mathbb{Z} \times 18\mathbb{Z}$.
(2) The condition "$2 \in I_1$" (resp. "$2 \in I_2$") in (3) and (4) of Theorem 6 can not be discarded. For example, consider the ideals $I_1 = 6\mathbb{Z}$ and $I_2 = 18\mathbb{Z}$ of $\mathbb{Z}$. Then I_1 is sdf-absorbing and I_2 is sdf-absorbing primary in $\mathbb{Z}$. But $I_1 \times I_2$ is not sdf-absorbing primary in $\mathbb{Z} \times \mathbb{Z}$ since $(1,1)^2 - (1,-1)^2 \in I_1 \times I_2$ but $(2,0) \notin \sqrt{I_1 \times I_2}$ and $(0,2) \notin I_1 \times I_2$.

Let R be a ring and M be an R-module. We recall that $R(+)M = \{(r,b) : r \in R, b \in M\}$ with coordinate-wise addition and multiplication defined as $(r_1,b_1)(r_2,b_2) = (r_1r_2, r_1b_2 + r_2b_1)$ is a commutative ring with identity $(1,0)$. This ring is called the idealization of M. For an ideal I of R and a submodule N of M, $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$. Moreover, we have $\sqrt{I(+)N} = \sqrt{I}(+)M$, [1].

Next, we justify the relation between sdf-absorbing primary ideals of R and sdf-absorbing primary ideals of $R(+)M$.

Theorem 7. *Let I be a proper ideal of a ring R and N be a submodule of an R-module M. Then*

(1) If $I(+)N$ is an sdf-absorbing primary ideal of $R(+)M$, then I is an sdf-absorbing primary ideal of R.
(2) If I is an sdf-absorbing primary ideal of R, then $I(+)M$ is an sdf-absorbing primary ideal of $R(+)M$.

Proof. (1) Suppose $I(+)N$ is sdf-absorbing primary in $R(+)M$. Then $I \cong I(+)N/0(+)N$ is sdf-absorbing primary in R by (2) of Corollary 1.

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(2) Suppose I is an sdf-absorbing primary ideal of R. Let $(a_1, m_1), (a_2, m_2) \in R(+)M$ such that $(a_1, m_1)^2 - (a_2, m_1)^2 \in I(+)M$. Then $a_1^2 - a_2{}^2 \in I$ and so $a_1 + a_2 \in \sqrt{I}$ or $a_1 - a_2 \in I$. Thus, $(a_1, m_1) + (a_2, m_1) \in \sqrt{I}(+)M = \sqrt{I(+)M}$ or $(a_1, m_1) - (a_2, m_1) \in I(+)M$ as needed. □

As a direct consequence of the theorem above, we conclude the following result.

Corollary 5. *Let I be an ideal of a ring R and M be an R-module. Then $I(+)M$ is an sdf-absorbing primary ideal of $R(+)M$ if and only if I is an sdf-absorbing primary ideal of R.*

If I is an sdf-absorbing primary ideal of R and N is a proper submodule of an R-module M, then $I(+)N$ need not be an sdf-absorbing primary ideal of $R(+)M$. For example, the ideal $18\mathbb{Z}$ is sdf-absorbing primary in $\mathbb{Z}$ but $18\mathbb{Z}(+)6\mathbb{Z}$ is not sdf-absorbing primary in $\mathbb{Z}(+)\mathbb{Z}$. Indeed, $(1,2)^2 - (1,-1)^2 = (1,4) - (1,-2) = (0,6) \in 18\mathbb{Z}(+)6\mathbb{Z}$ but $(2,1) \notin 6\mathbb{Z}(+)\mathbb{Z} = \sqrt{18\mathbb{Z}(+)6\mathbb{Z}}$ and $(0,3) \notin 18\mathbb{Z}(+)6\mathbb{Z}$.

Let R and S be two rings, J be an ideal of S and $f : R \to S$ be a ring homomorphism. As a subring of $R \times S$, the amalgamation of R and S along J with respect to f is defined by $R \bowtie^f J = (a, f(a) + j) : a \in R, j \in J\}$. If f is the identity homomorphism on R, then we get the amalgamated duplication of R along an ideal J, $R \bowtie J = \{(a, a + j) : a \in R, j \in J\}$. For more related definitions and several properties of this kind of rings, one can see [7,8]. If I is an ideal of R and K is an ideal of $f(R) + J$, then $I \bowtie^f J = \{(i, f(i) + j) : i \in I, j \in J\}$ and $\bar{K}^f = \{(a, f(a) + j) : a \in R, j \in J, f(a) + j \in K\}$ are ideals of $R \bowtie^f J$, [9]. By simple computations, one can verify that $\sqrt{I \bowtie^f J} = \sqrt{I} \bowtie^f J$ and $\sqrt{\bar{K}^f} = \overline{\sqrt{K}}^f$.

In the following theorem, we determine the conditions under which the ideals $I \bowtie^f J$ and $\bar{K}^f$ are sdf-absorbing primary ideals of $R \bowtie^f J$.

Theorem 8. *Let R, S, f, J, I and K be as above.*

(1) $I \bowtie^f J$ is an sdf-absorbing primary ideal of $R \bowtie^f J$ if and only if I is an sdf-absorbing primary ideal of R.
(2) $\bar{K}^f$ is an sdf-absorbing primary ideal of $R \bowtie^f J$ if and only if K is an sdf-absorbing primary ideal of $f(R) + J$.

Proof. (1) We firstly note that I is proper in R if and only if $I \bowtie^f J$ is proper in $R \bowtie^f J$. Suppose $I \bowtie^f J$ is an sdf-absorbing primary ideal of $R \bowtie^f J$. Let $a, b \in R$ such that $a^2 - b^2 \in I$. Then $(a, f(a))^2 - (b, f(b))^2 \in I \bowtie^f J$ and so $(a + b, f(a + b)) \in \sqrt{I \bowtie^f J} = \sqrt{I} \bowtie^f J$ or $(a - b, f(a - b)) \in I \bowtie^f J$. Therefore, $a + b \in \sqrt{I}$ or $a - b \in I$ and so I is sdf-absorbing primary in R. Conversely, suppose I is an sdf-absorbing primary ideal of R. Let $(a, f(a) + j_1), (b, f(b) + j_2) \in R \bowtie^f J$ such that $(a, f(a) + j_1)^2 - (b, f(b) + j_2)^2 \in I \bowtie^f J$. Then $a^2 - b^2 \in I$ and so $a + b \in \sqrt{I}$ or $a - b \in I$. Thus, $(a + b, f(a + b) + j_1 + j_2) \in \sqrt{I} \bowtie^f J = \sqrt{I \bowtie^f J}$ or $(a - b, f(a - b) + j_1 - j_2) \in I \bowtie^f J$ as needed.

(2) We can easily check that K is proper in $f(R) + J$ if and only if $\bar{K}^f$ is proper in $R \bowtie^f J$. Suppose $\bar{K}^f$ is an sdf-absorbing primary ideal of $R \bowtie^f J$ and let $f(a) + j_1, f(b) + j_2 \in f(R) + J$ such that $(f(a) + j_1)^2 - (f(b) + j_2)^2 \in K$. Then $(a, f(a) + j_1)^2 - (b, f(b) + j_2)^2 \in \bar{K}^f$ and so $(a + b, f(a + b) + j_1 + j_2) \in \sqrt{\bar{K}^f} = \overline{\sqrt{K}}^f$ or $(a - b, f(a - b) + j_1 - j_2) \in \bar{K}^f$. It follows that $f(a + b) + j_1 + j_2 \in \sqrt{K}$ or $f(a - b) + j_1 - j_2 \in K$ and K is an sdf-absorbing primary ideal of $f(R) + J$. Conversely, suppose K is sdf-absorbing primary in $f(R) + J$. Suppose $(a, f(a) + j_1)^2 - (b, f(b) + j_2)^2 \in \bar{K}^f$ for $(a, f(a) + j_1)$, $(b, f(b) + j_2) \in R \bowtie^f J$. Then $(f(a) + j_1)^2 - (f(b) + j_2)^2 \in K$ and so $f(a + b) + j_1 + j_2 \in \sqrt{K}$ or $f(a - b) + j_1 - j_2 \in K$. Hence, $(a + b, f(a + b) + j_1 + j_2) \in \overline{\sqrt{K}}^f = \sqrt{\bar{K}^f}$ or $(a - b, f(a - b) + j_1 - j_2) \in \bar{K}^f$ and the result follows. □

In particular, we have the following result for amalgamated duplication of R along an ideal J.

Corollary 6. *Let I and J be ideals of a ring R. Then $I \bowtie J$ is an sdf-absorbing primary ideal of $R \bowtie J$ if and only if I is an sdf-absorbing primary ideal of R.*

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It is shown in [2] that every nonzero proper ideal of a commutative ring R is an sdf-absorbing ideal if and only if $R/\sqrt{0}$ is a von Neumann regular ring. If $2 \in U(R)$, then primary and sdf-absorbing primary ideals coincide. Thus, rings in which every proper ideal is an sdf-absorbing primary are the same as rings in which every proper ideal is primary. It is well-known that these rings are characterized as rings that are either with exactly one prime ideal or one dimensional domains with a unique maximal ideal.

Problem 1. *Let R be a commutative ring with identity in which every proper ideal is an sdf-absorbing primary. If $2 \notin U(R)$. Then, what is a characterization for such a ring R?*

References

1. D. D. Anderson, M. Winders, Idealization of a module, J. Commut. Algebra, 1(1) (2009), 3-56.
2. D. F. Anderson, A. Badawi, J. Coykendall, Square-difference factor absorbing ideals of a commutative rings, arXiv:2402.18704, https://doi.org/10.48550/arXiv.2402.18704
3. D. F. Anderson and A. Badawi, On (m,n)-closed ideals of commutative rings, J. Algebra Appl., 16(1) (2017), 1750013 (21 pp).
4. A. Badawi, On 2-absorbing ideals of commutative rings, Bull. Austral. Math. Soc., **75** (2007), 417–429.
5. A. Badawi, Ü. Tekir and E. Yetkin, On 2-absorbing primary ideals in commutative rings, Bull. Korean Math. Soc., **51** (4) (2014), 1163–1173.
6. A. Badawi, M. Issoual, N. Mahdou, On n-absorbing ideals and (m,n)-closed ideals in trivial ring extensions of commutative rings. Journal of Algebra and Its Applications, 18(07) (2019) 1950123.
7. E. M. Bouba, N. Mahdou, and M. Tamekkante, Duplication of a module along an ideal, Acta Math. Hungar. 154(1) (2018), 29-42.
8. M. D'Anna and M. Fontana, An amalgamated duplication of a ring along an ideal: the basic properties, J. Algebra Appl., **6**(**3**) (2007), 443–459.
9. M. D'Anna, C.A. Finocchiaro, and M. Fontana, Properties of chains of prime ideals in an amalgamated algebra along an ideal, J. Pure Appl. Algebra, **214** (2010), 1633-1641.
10. R. Gilmer, Multiplicative Ideal Theory, Queen Papers Pure Appl. Math. 90, Queen'sUniversity, Kingston, 1992
11. H. A. Khashan, E. Yetkin Celikel, (m,n)-prime ideals of commutative rings. Preprints 2024, 2024010472. https://doi.org/10.20944/preprints202401.0472.v1.
12. H. A. Khashan, E. Yetkin Celikel, On weakly (m,n)-prime ideals of commutative rings, Bulletin of the Korean Mathematical Society, In press.
13. H. A. Khashan, E. Yetkin Celikel, A new generalization of (m, n)-closed ideals, Journal of Mathematical Sciences. (2023), https://doi.org/10.1007/s10958-023-06814-2.
14. C. Wood, On general ZPI-rings, Pacific Journal of Mathematics, 30(3) (1969), 837-846.
15. E. Yetkin Celikel, The triple zero graph of a commutative ring. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 70(2) (2021), 653-663.