Article

Not peer-reviewed version

Existence of Positive Solutions for a System of Generalized Laplacian Problems

Chan-Gyun Kim *

Posted Date: 1 September 2025

doi: 10.20944/preprints202508.2208.v1

Keywords: generalized Laplacian system; positive solutions; singular weight function

Article

Existence of Positive Solutions for a System of Generalized Laplacian Problems

Chan-Gyun Kim *

Department of Mathematics Education, Chinju National Univeristy of Education, Jinju, 52673, Korea
* Correspondence: cgkim75@cue.ac.kr

Abstract

This paper investigates the existence and multiplicity of positive solutions for a system of generalized Laplacian problems. By analyzing the asymptotic behavior of the nonlinearity, we establish conditions for the existence of positive solutions and the presence of multiple positive solutions. Our main results reveal how the norm of the positive solutions behaves as the parameter λ approaches 0 or ∞, specifically that the norm tends to either 0 or ∞.

Keywords: generalized Laplacian system; positive solutions; singular weight function

MSC: 34B08; 34B16; 34B18

---

1. Introduction

In this paper, we investigate the existence of positive solutions to the following system:

$$\begin{cases} (P(t) \bullet \Phi(u'(t)))' + \lambda h(t) \bullet f(u(t)) = \theta, \;\; t \in (0,1), \\ u(0) = u(1) = \theta, \end{cases} \tag{1.1}$$

where $\lambda \in \mathbb{R}_+ := [0, \infty)$ is a parameter and θ denotes the origin of $\mathbb{R}^N$ with $N \geq 2$. The vector-valued function $P := (p_1, \cdots, p_N)$ is defined with $p_i \in C([0,1], (0,\infty))$ for $i \in \{1, \cdots, N\}$. The vector-valued function $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ is defined by

$$\Phi(s) = (\varphi_1(s_1), \cdots, \varphi_N(s_N)) \text{ for } s = (s_1, \cdots, s_N) \in \mathbb{R}^N,$$

where each $\varphi_i : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism for $i \in \{1, \cdots, N\}$. In addition, $h := (h_1, \cdots, h_N)$ is a vector-valued function with $h_i \in C((0,1), (0,\infty))$ for $i \in \{1, \cdots, N\}$, and $f := (f_1, \cdots, f_N)$ is a vector-valued function with $f_i \in C(\mathbb{R}_+^N, \mathbb{R}_+)$ such that $f_i(y) > 0$ for $y \in \mathbb{R}_+^N \setminus \{\theta\}$ and $i \in \{1, \cdots, N\}$. Here $\mathbb{R}_+^N := \{(x_1, \cdots, x_N) : x_j \in \mathbb{R}_+ \text{ for } 1 \leq j \leq N\}$. The symbol $\bullet$ denotes the entry-wise product, i.e., $(a_1, \cdots, a_N) \bullet (b_1, \cdots, b_N) := (a_1 b_1, \cdots, a_N b_N)$.

By a solution $u = (u_1, \cdots, u_N)$ to problem (1.1), we mean a vector-valued function $u = (u_1, \cdots, u_N) \in C^1((0,1), \mathbb{R}^N)$ that satisfies (1.1). A solution $u = (u_1, \cdots, u_N)$ to problem (1.1) is positive if, for each $i \in \{1, \cdots, N\}$, $u_i(t) \geq 0$ for all $t \in [0,1]$ and at least one component of u is nontrivial. We note that, according to Lemma 2.1(2), any nontrivial component of u is positive on $(0,1)$.

Let $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing homeomorphism. Then we denote by $\mathcal{H}_\vartheta$ the set

$$\left\{ k \in C((0,1), \mathbb{R}_+) : \int_0^1 \vartheta^{-1}\left(\left| \int_s^{\frac{1}{2}} k(\tau) d\tau \right| \right) ds < \infty \right\}.$$

In the present work, we adopt the following hypotheses. Hypothesis (K_1) will be assumed in all subsequent sections, whereas (K_2) will be invoked only in Section 3, where the main results are established:

(K_1) For each $i \in \{1, \cdots, N\}$, there exist increasing homeomorphisms $\psi_i^1, \psi_i^2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi_i(x)\psi_i^1(y) \le \varphi_i(xy) \le \varphi_i(x)\psi_i^2(y) \text{ for all } x, y \in \mathbb{R}_+. \tag{1.2}$$

(K_2) $h_i \in \mathcal{H}_{\psi_i^1} \setminus \{0\}$ for all $i \in \{1, \cdots, N\}$.

As is well known, it follows from (K_1) that, for $i \in \{1, \cdots, , N\}$,

$$\varphi_i^{-1}(x)(\psi_i^2)^{-1}(y) \le \varphi_i^{-1}(xy) \le \varphi_i^{-1}(x)(\psi_i^1)^{-1}(y) \text{ for all } x, y \in \mathbb{R}_+ \tag{1.3}$$

and

$$L^1(0,1) \cap C(0,1) \subsetneq \mathcal{H}_{\psi_i^1} \subseteq \mathcal{H}_{\varphi_i} \subseteq \mathcal{H}_{\psi_i^2}.$$

(see, e.g., Remark 1 in [1]).

In recent decades, considerable attention has been devoted to the study of positive solutions for boundary value problems involving generalized Laplacian operators. Early contributions mainly focused on the scalar case (see, e.g., [2–5]), where topological methods were used to establish existence and multiplicity results. Building upon the scalar results, subsequent works have shifted the system case ([6–10]), providing conditions for the existence and multiplicity of positive solutions. For example, Wang [6] laid the foundation for generalized elliptic systems by proving the existence and multiplicity of positive solutions under specific structural hypotheses on the nonlinearity, while Medekhel et al. [10] investigated a parabolic $(p(x), q(x))$-Laplacian system, establishing its existence and asymptotic behavior using sub- and super-solution methods.

More recently, Li et al. [11] investigated a system of fractional q-difference equations with generalized p-Laplacian operators, establishing existence results of positive solutions under various superlinear and sublinear conditions. Furthermore, Yang and Zhang [12] studied a (p, q)-Laplacian coupled system with perturbations and two parameters on locally finite graphs, proving the existence and multiplicity of nontrivial solutions by means of variational methods.

This paper extends previous research by analyzing a system of generalized Laplacian problems with a parameter λ. While earlier studies ([6–9]) focused on cases where the operator's characteristics were uniform across all equations (i.e., φ_i was the same for all i) and required the function p_i to be non-decreasing on $[0, 1]$, our work considers a more general setting where these conditions are relaxed. Our approach accommodates a variable operator φ_i and makes no monotonicity assumption on p_i. Moreover, unlike the aforementioned papers that primarily focused on establishing the existence and multiplicity of positive solutions, our study provides a more detailed analysis of the asymptotic behavior of the solution norm with respect to the parameter λ.

Our main contributions are twofold. First, depending on the behavior of the nonlinearity at zero and ∞, we establish the existence of a positive solution for every $\lambda > 0$ and precisely determine the asymptotic behavior of the solution norm as λ tends to 0 or ∞. These results provide a global existence result along with precise asymptotic information. Second, under different structural assumptions on the nonlinearity, we prove the existence of two distinct positive solutions for certain ranges of the parameter λ. We also determine the asymptotic behavior of these solutions as λ tends to 0 or ∞. Unlike the first case, however, this multiplicity result leaves open the question of global existence for all $\lambda > 0$.

The remainder of this paper is organized as follows. Section 2 presents preliminary results and their immediate consequences. Section 3 establishes auxiliary lemmas and our main results, illustrating them with examples. Section 4 concludes with a summary of the main results and a discussion of open problems, particularly concerning the global existence of positive solutions under the assumptions in Theorem 3.6.

2. Preliminaries

Throughout this section, we assume that (K_1) and $h_i \in \mathcal{H}_{\varphi_i}$ for all $i \in \{1, \cdots, N\}$ hold.

Let

$$X := \{(v_1, \cdots, v_N) : v_i \in C[0,1] \text{ for } 1 \le i \le N\}$$

and

$$\|v\|_X := \sum_{i=1}^{N} \|v_i\|_\infty \text{ for } v = (v_1, \cdots, v_N) \in X.$$

Here,

$$\|w\|_\infty := \max\{|w(t)| : t \in [0,1]\} \text{ for } w \in C[0,1].$$

Then $(X, \|\cdot\|_X)$ be a Banach space. For $s = (s_1, \cdots, s_N) \in \mathbb{R}^N$, let $\|s\|_N := \sum_{i=1}^{N} |s_i|$.

Let $\mathcal{K} := \mathcal{P}_1 \times \cdots \times \mathcal{P}_N$. Here, for $i \in \{1, \cdots, N\}$, $\mathcal{P}_i$ is the set of all nonnegative continuous functions v_i satisfying, for $t \in [\frac{1}{4}, \frac{3}{4}]$,

$$v_i(t) \ge \frac{1}{4} p_i^1 \|v_i\|_\infty.$$

Here $p_i^1 := \psi_2^{-1}\left(\frac{1}{\|p_i\|_\infty}\right)\left[\psi_1^{-1}\left(\frac{1}{p_i^0}\right)\right]^{-1} \in (0,1]$ and $p_i^0 := \min_{t\in[0,1]} p_i(t) > 0$. Then $\mathcal{K}$ is a cone in X. For $v = (v_1, \cdots, v_N) \in \mathcal{K}$ and $t \in [\frac{1}{4}, \frac{3}{4}]$,

$$\|v(t)\|_N = \sum_{i=1}^{N} v_i(t) \ge \sum_{i=1}^{N} \frac{1}{4} p_i^1 \|v_i\|_\infty \ge \rho \|v\|_X. \tag{2.1}$$

Here $\rho := \min\{\frac{1}{4} p_i^1 : 1 \le i \le N\}$.

For $l > 0$, let

$$\mathcal{K}_l := \{v \in \mathcal{K} \ : \ \|v\|_X < l\}, \partial\mathcal{K}_l := \{v \in \mathcal{K} \ : \ \|v\|_X = l\} \text{ and } \overline{\mathcal{K}_l} := \mathcal{K}_l \cup \partial\mathcal{K}_l.$$

Let $i \in \{1, \cdots, N\}$ be given. For $g_i \in \mathcal{H}_{\varphi_i}$, consider the following problem

$$\begin{cases} (p_i(t)\varphi_i(u_i'(t)))' + g_i(t) = 0, \ \ t \in (0,1), \\ u_i(0) = u_i(1) = 0. \end{cases} \tag{2.2}$$

Define a function $S_i : \mathcal{H}_{\varphi_i} \to C[0,1]$ by, for $g_i \in \mathcal{H}_{\varphi_i}$,

$$S_i(g)(t) = \begin{cases} \int_0^t \varphi_i^{-1}\left(\frac{1}{p_i(s)} \int_s^{\sigma_i} g_i(\varrho) d\varrho\right) ds, & \text{if } 0 \le t \le \sigma_i, \\ \int_t^1 \varphi_i^{-1}\left(\frac{1}{p_i(s)} \int_{\sigma_i}^{s} g_i(\varrho) d\varrho\right) ds, & \text{if } \sigma_i \le t \le 1. \end{cases} \tag{2.3}$$

Here $\sigma_i = \sigma_i(g_i)$ is a constant satisfying

$$\int_0^{\sigma_i} \varphi_i^{-1}\left(\frac{1}{p_i(s)} \int_s^{\sigma_i} g_i(\varrho) d\varrho\right) ds = \int_{\sigma_i}^{1} \varphi_i^{-1}\left(\frac{1}{p_i(s)} \int_{\sigma_i}^{s} g_i(\varrho) d\varrho\right) ds. \tag{2.4}$$

For any $g_i \in \mathcal{H}_{\varphi_i}$ and any σ_i satisfying (2.4), $S_i(g_i)$ is monotone increasing on $[0, \sigma_i)$ and monotone decreasing on $(\sigma_i, 1]$. Note that $\sigma_i = \sigma(g_i)$ is not necessarily unique, but $S_i(g_i)$ is independent of the choice of σ_i satisfying (2.4) (see [13, Remark 2]).

Based on Lemma 1 and Lemma 2 in [1], we have the following lemma.

Lemma 2.1. *Let $i \in \{1, \cdots, N\}$ be given, and assume that (K_1) and $g_i \in \mathcal{H}_{\varphi_i}$ hold. Then*

(1) $S_i(g_i)$ is a unique solution to problem (2.2) with the following property

$$S_i(g_i)(t) \ge \min\{t, 1-t\} p_i^1 \|S_i(g_i)\|_\infty$$

for $t \in [0,1]$, and thus $S_i(g_i) \in \mathcal{P}_i$.

(2) If $g_i \not\equiv 0$, then there exists a subinterval $[\sigma_i^1, \sigma_i^2]$ of $(0,1)$ such that

$$(S_i(g_i))'(t) > 0,\ t \in (0, \sigma_i^1),\ (S_i(g_i))'(t) = 0,\ t \in [\sigma_i^1, \sigma_i^2] \text{ and } (S_i(g_i))'(t) < 0,\ t \in (\sigma_i^2, 1).$$

Define a function $G_i : \mathbb{R}_+ \times \mathcal{K} \to C(0,1)$ by, for $t \in (0,1)$,

$$G_i(\lambda, u)(t) := \lambda h_i(t) f_i(u_1(t), \cdots, u_N(t))$$

for $\lambda \in \mathbb{R}_+$ and $u = (u_1, \cdots, u_N) \in \mathcal{K}$. From the facts that $h_i \in \mathcal{H}_\varphi$ and the continuity of f_i, it clearly follows that

$$G_i(\lambda, u) \in \mathcal{H}_\varphi$$

for any $(\lambda, u) \in \mathbb{R}_+ \times \mathcal{K}$.

Next, we define an operator $H : \mathbb{R}_+ \times \mathcal{K} \to \mathcal{K}$ by

$$H(\lambda, u) := (T_1(\lambda, u), \cdots, T_N(\lambda, u))$$

for $(\lambda, u) \in \mathbb{R}_+ \times \mathcal{K}$. Here,

$$T_i(\lambda, u) := S_i(G_i(\lambda, u))$$

for $(\lambda, u) \in \mathbb{R}_+ \times \mathcal{K}$ and $i \in \{1, \cdots, N\}$. More precisely, for $(\lambda, u) \in \mathbb{R}_+ \times \mathcal{K}$,

$$T_i(\lambda, u)(t) = \begin{cases} \int_0^t \varphi^{-1}(\frac{1}{p_i(s)} \int_s^\sigma G_i(\lambda, u)(\tau) d\tau) ds, & \text{if } 0 \le t \le \sigma_i, \\ \int_t^1 \varphi^{-1}(\frac{1}{p_i(s)} \int_\sigma^s G_i(\lambda, u)(\tau) d\tau) ds, & \text{if } \sigma_i \le t \le 1. \end{cases}$$

Here, for $i \in \{1, \cdots, N\}$, $\sigma_i = \sigma_i(\lambda, u)$ is a number that satisfies

$$\int_0^{\sigma_i} \varphi^{-1}\left(\frac{1}{p_i(s)} \int_s^{\sigma_i} G_i(\lambda, u)(\tau) d\tau\right) ds = \int_{\sigma_i}^1 \varphi^{-1}\left(\frac{1}{p_i(s)} \int_{\sigma_i}^s G_i(\lambda, u)(\tau) d\tau\right) ds. \tag{2.5}$$

By Lemma 2.1(1),

$$T_i(\lambda, u) \in \mathcal{P}_i$$

for all $(\lambda, u) \in \mathbb{R}_+ \times \mathcal{K}$ and $i \in \{1, \cdots, N\}$, which implies

$$H(\mathbb{R}_+ \times \mathcal{K}) \subseteq \mathcal{K}.$$

Remark 2.2. (1) *It is evident that* (1.1) *has a solution if and only if* $H(\lambda, \cdot)$ *has a fixed point in* $\mathcal{K}$.

(2) *From* $H(0, u) = 0$ *for any* $u \in \mathcal{K}$, *it follows that* θ *is a unique solution to problem* (1.1) *with* $\lambda = 0$.

(3) *By Lemma* 2.1(2), u *is a positive solution, provided that* u *is a nontrivial solution to problem* (1.1).

Using (1.3), by the similar arguments as in the proof of Lemma 3 in [2] (or Lemma 3.3 in [14]), one can show that, for $i \in \{1, \cdots, N\}$, $T_i : \mathbb{R}_+ \times \mathcal{K} \to \mathcal{P}_i$ is completely continuous. Thus the complete continuity of H can be obtained as follows.

Lemma 2.3. *Assume that* (K_1) *and* $h_i \in \mathcal{H}_{\varphi_i}$ *for all* $i \in \{1, \cdots, N\}$ *hold. Then the operator* $H : \mathbb{R}_+ \times \mathcal{K} \to \mathcal{K}$ *is completely continuous.*

For $i \in \{1, \cdots, N\}$ and $m \in \mathbb{R}_+$, let

$$(f_i)_*(m) := \min\{f_i(y) : y \in \mathbb{R}_+^N \text{ with } \rho m \le \|y\|_N \le m\}$$

and

$$(f_i)^*(m) := \max\{f_i(y) : y \in \mathbb{R}_+^N \text{ with } \|y\|_N \le m\}.$$

For $a \in \{0, \infty\}$ and $i \in \{1, \cdots, N\}$, let

$$f_i^a := \lim_{\|y\|_N \to a} \frac{f_i(y)}{\varphi_i(\|y\|_N)}.$$

Remark 2.4. *It is straightforward to observe that, for $a \in \{0, \infty\}$ and $i \in \{1, \cdots, N\}$,*

$$\lim_{m \to a} \frac{(f_i)_*(m)}{\varphi_i(m)} = \lim_{m \to a} \frac{(f_i)^*(m)}{\varphi_i(m)} = 0 \text{ if } f_i^a = 0$$

and

$$\lim_{m \to a} \frac{(f_i)_*(m)}{\varphi_i(m)} = \lim_{m \to a} \frac{(f_i)^*(m)}{\varphi_i(m)} = \infty \text{ if } f_i^a = \infty.$$

For the reader's convenience, we provide the proofs. First, we show that $f_i^0 = 0$ implies

$$\lim_{m \to 0} \frac{(f_i)_*(m)}{\varphi_i(m)} = \lim_{m \to 0} \frac{(f_i)^*(m)}{\varphi_i(m)} = 0.$$

Let $\epsilon > 0$ be given and $f_i^0 = 0$ be assumed. Then there exists $\delta > 0$ such that for any s with $\|s\|_N \in (0, \delta)$,

$$0 < \frac{f_i(s)}{\varphi_i(\|s\|_N)} < \epsilon.$$

Since $f_i \in C(\mathbb{R}_+^N, \mathbb{R}_+)$, by the extreme value theorem, for any $m \in (0, \delta)$, $(f_i)^*(m) = f_i(x_m)$ for some $x_m \in \mathbb{R}_+^N \setminus \mathbf{0}$ with $\|x_m\|_N \le m$. Then

$$0 \le \frac{(f_i)_*(m)}{\varphi_i(m)} \le \frac{(f_i)^*(m)}{\varphi_i(m)} = \frac{f_i(x_m)}{\varphi_i(m)} \le \frac{f_i(x_m)}{\varphi_i(\|x_m\|_N)} < \epsilon$$

for any $m \in (0, \delta)$, which implies

$$\lim_{m \to 0} \frac{(f_i)_*(m)}{\varphi_i(m)} = \lim_{m \to 0} \frac{(f_i)^*(m)}{\varphi_i(m)} = 0.$$

Next, we prove that $f_i^\infty = 0$ implies

$$\lim_{m \to \infty} \frac{(f_i)_*(m)}{\varphi_i(m)} = \lim_{m \to \infty} \frac{(f_i)^*(m)}{\varphi_i(m)} = 0.$$

Indeed, let $\epsilon > 0$ be given and $f_i^\infty = 0$ be assumed. Then there exists $N > 0$ such that $y \in \mathbb{R}_+^N$ with $\|y\|_N \ge N$,

$$\frac{f_i(y)}{\varphi_i(\|y\|_N)} < \epsilon.$$

For any m with $m \ge N$,

$$(f_i)^*(m) \le (f_i)^*(N) + f_i(y_{N,m}),$$

where $y_{N,m} \in \mathbb{R}_+^N$ satisfying

$$N \le \|y_{N,m}\|_N \le m \text{ and } f_i(y_{N,m}) = \max\{f_i(y) : N \le \|y\|_N \le m\}.$$

Then

$$0 \le \frac{(f_i)_*(m)}{\varphi_i(m)} \le \frac{(f_i)^*(m)}{\varphi_i(m)} \le \frac{(f_i)^*(N)}{\varphi_i(m)} + \frac{(f_i)^*(y_{N,m})}{\varphi_i(\|y_{N,m}\|_N)} \le \frac{(f_i)^*(N)}{\varphi_i(m)} + \epsilon.$$

Consequently,

$$0 \le \limsup_{m \to \infty} \frac{f_*(s)}{\varphi_i(s)} \le \limsup_{m \to \infty} \frac{f^*(m)}{\varphi_i(m)} \le \epsilon,$$

which is true for all $\epsilon > 0$. Thus

$$\lim_{m \to \infty} \frac{(f_i)_*(m)}{\varphi_i(m)} = \lim_{m \to \infty} \frac{(f_i)^*(m)}{\varphi_i(m)} = 0.$$

Finally, we establish that, for $a \in \{0, \infty\}$, $f_i^a = \infty$ implies

 doi:10.20944/preprints202508.2208.v1

$$\lim_{m\to a}\frac{(f_i)_*(m)}{\varphi_i(m)}=\lim_{m\to a}\frac{(f_i)^*(m)}{\varphi_i(m)}=\infty.$$

For each $m\in(0,\infty)$, the extreme value theorem ensures the existence of $y_m\in\mathbb{R}_+^N$ with $\rho m\le\|y_m\|_N\le m$ satisfying $(f_i)_*(m)=f_i(y_m)$. Hence

$$\frac{(f_i)^*(m)}{\varphi_i(m)}\ge\frac{(f_i)_*(m)}{\varphi_i(m)}=\frac{f_i(y_m)}{\varphi_i(m)}\ge\frac{f_i(y_m)}{\varphi_i(\|y_m\|_N)\psi_i^2(\frac{m}{\|y_m\|_N})}\ge\frac{f(y_m)}{\varphi_i(\|y_m\|_N)\psi_i^2(\rho^{-1})}.$$

As $m\to a\in\{0,\infty\}$, $\|y_m\|_N\to a$, and thus $f_i^a=\infty$ implies

$$\lim_{m\to a}\frac{(f_i)_*(m)}{\varphi_i(m)}=\lim_{m\to a}\frac{(f_i)^*(m)}{\varphi_i(m)}=\infty.$$

Finally, we recall a well-known theorem of the fixed point index theory.

Theorem 2.5. (see, e.g., [15,16]) *Assume that, for some $r>0$, $T:\overline{\mathcal{K}}_r\to\mathcal{K}$ is completely continuous. Then*

(i) if $\|T(v)\|_X>\|v\|_X$ for $v\in\partial\mathcal{K}_r$, then $i(T,\mathcal{K}_r,\mathcal{K})=0$;

(ii) if $\|T(v)\|_X<\|v\|_X$ for $v\in\partial\mathcal{K}_r$, then $i(T,\mathcal{K}_r,\mathcal{K})=1$.

3. Main Results

In this section, we assume that (K_1) and (K_2) hold. For $i\in\{1,\cdots,N\}$, define continuous functions $R_i^1:(0,\infty)\to(0,\infty)$ and $R_i^2:(0,\infty)\to(0,\infty)$ by

$$R_i^1(m):=C_1p_i^0\frac{\varphi_i(m)}{(f_i)^*(m)}\text{ and }R_i^2(m):=C_2\|p_i\|_\infty\frac{\varphi_i(m)}{(f_i)_*(m)}\text{ for }m\in(0,\infty).$$

Here,

$$C_1:=\min_{1\le i\le N}\left\{\psi_i^1\left(\frac{1}{2}\left[\int_0^{\frac{1}{2}}(\psi_i^1)^{-1}\left(\int_s^{\frac{1}{2}}h_i(\tau)d\tau\right)ds\right]^{-1}\right),\psi_i^1\left(\frac{1}{2}\left[\int_{\frac{1}{2}}^1(\psi_i^1)^{-1}\left(\int_{\frac{1}{2}}^s h_i(\tau)d\tau\right)ds\right]^{-1}\right)\right\}$$

and

$$C_2:=\max_{1\le i\le N}\left\{\psi_i^2\left(\left[\int_{\frac{1}{4}}^{\frac{1}{2}}(\psi_i^2)^{-1}\left(\int_s^{\frac{1}{2}}h_i(\tau)d\tau\right)ds\right]^{-1}\right),\psi_i^2\left(\left[\int_{\frac{1}{2}}^{\frac{3}{4}}(\psi_i^2)^{-1}\left(\int_{\frac{1}{2}}^s h_i(\tau)d\tau\right)ds\right]^{-1}\right)\right\}.$$

By (1.2) and (1.3), $\psi_i^1(z)\le\psi_i^2(z)$ and $(\psi_i^2)^{-1}(z)\le(\psi_i^1)^{-1}(z)$ for $z\in\mathbb{R}_+$. Thus

$$0<C_1<C_2\text{ and }0<R_i^1(m)<R_i^2(m)$$

for all $i\in\{1,\cdots,N\}$ and all $m\in(0,\infty)$.

Let, for $m\in(0,\infty)$,

$$R_*^1(m):=\min\{R_i^1(m):1\le i\le N\}\text{ and }R_*^2(m):=\min\{R_i^2(m):1\le i\le N\}.$$

Then, for all $m\in(0,\infty)$,

$$0<R_*^1(m)<R_*^2(m).\tag{3.1}$$

Lemma 3.1. *Assume that (K_1) and (K_2) hold. Let $m\in(0,\infty)$ be fixed. Then, for any $\lambda\in(0,R_*^1(m))$,*

$$\|H(\lambda,v)\|_X<\|v\|_X\text{ for all }v\in\partial\mathcal{K}_m$$

and

$$i(H(\lambda,\cdot),\mathcal{K}_m,\mathcal{K})=1.\tag{3.2}$$

Proof. Let $\lambda\in(0,R_*^1(m))$ and $v\in\partial\mathcal{K}_m$ be fixed. Then $\|v\|_X=m$ and

 doi:10.20944/preprints202508.2208.v1

$$\|v(t)\|_N \leq m \text{ for } t \in [0,1].$$

Let $i \in \{1, \cdots, N\}$ be given. Since $\lambda \in (0, R_*^1(m))$,

$$\lambda f_i(v(t)) \leq \lambda (f_i)^*(m) = \frac{\lambda \varphi_i(m)}{R_i^1(m)} C_1 p_i^0 < \varphi_i(m) C_1 p_i^0 \text{ for } t \in [0,1]. \tag{3.3}$$

Let σ_i be a number satisfying $T_i(\lambda, v)(\sigma_i) = \|T_i(\lambda, v)\|_\infty$. We have two cases: either (i) $\sigma_i \in (0, \frac{1}{2})$or (ii) $\sigma_i \in [\frac{1}{2}, 1)$. We only consider the case (i), since the case (ii) can be dealt in a similar manner. From (1.3), (3.3) and the definition of C_1, it follows that

$$\begin{aligned}
\|T_i(\lambda, v)\|_\infty &= \int_0^{\sigma_i} \varphi_i^{-1}\left(\frac{1}{p_i(s)} \int_s^{\sigma_i} \lambda h_i(\tau) f_i(v(\tau)) d\tau\right) ds \\
&< \int_0^{\frac{1}{2}} \varphi_i^{-1}\left(\int_s^{\frac{1}{2}} h(\tau) d\tau \varphi_i(m) C_1\right) ds \\
&\leq \int_0^{\frac{1}{2}} (\psi_i^1)^{-1}\left(\int_s^{\frac{1}{2}} h(\tau) d\tau\right) ds \varphi_i^{-1}(\varphi_i(m) C_1)) \\
&\leq \int_0^{\frac{1}{2}} (\psi_i^1)^{-1}\left(\int_s^{\frac{1}{2}} h(\tau) d\tau\right) ds (\psi_i^1)^{-1}(C_1) m \leq \frac{1}{2}\|v\|_X.
\end{aligned}$$

Thus $\|H(\lambda, v)\|_X < \|v\|_X$ for all $v \in \partial\mathcal{K}_m$, and by Theorem 2.5, (3.2) holds for any $\lambda \in (0, R_*^1(m))$. $\square$

Lemma 3.2. *Assume that* (K_1) *and* (K_2) *hold. Let* $m \in (0, \infty)$ *be fixed. Then, for any* $\lambda \in (R_*^2(m), \infty)$,

$$\|H(\lambda, v)\|_X > \|v\|_X \text{ for all } v \in \partial\mathcal{K}_m$$

and

$$i(H(\lambda, \cdot), \mathcal{K}_m, \mathcal{K}) = 0. \tag{3.4}$$

Proof. Let $\lambda \in (R_*^2(m), \infty)$ and $v \in \partial\mathcal{K}_m$ be fixed. Then $\|v\|_X = m$ and, by (2.1),

$$\rho m \leq \|v(t)||_N \leq m \text{ for } t \in [\tfrac{1}{4}, \tfrac{3}{4}].$$

For fixed i satisfying $R_i^2(m) = R_*^2(m)$,

$$\lambda f_i(v(t)) \geq \lambda (f_i)_*(m) = \frac{\lambda \varphi_i(m)}{R_i^2(m)} C_2 \|p_i\|_\infty > \varphi_i(m) C_2 \|p_i\|_\infty \text{ for } t \in [\frac{1}{4}, \frac{3}{4}]. \tag{3.5}$$

Let σ_i be a number satisfying $T_i(\lambda, v)(\sigma_i) = \|T_i(\lambda, v)\|_\infty$. We have two cases: either (i) $\sigma_i \in [\frac{1}{2}, 1)$ or (ii) $\sigma_i \in (0, \frac{1}{2})$. We only consider the case (i), since the case (ii) can be dealt in a similar manner. Since $\lambda > R_i^2(m)$, it follows from (1.3), (3.5) and the definition of C_2 that

$$\begin{aligned}
\|T_i(\lambda, v)\|_\infty &= \int_0^{\sigma_i} \varphi_i^{-1}\left(\frac{1}{p_i(s)} \int_s^{\sigma} \lambda h_i(\tau) f_i(v(\tau)) d\tau\right) ds \\
&> \int_{\frac{1}{4}}^{\frac{1}{2}} \varphi_i^{-1}\left(\int_s^{\frac{1}{2}} h_i(\tau) d\tau \varphi_i(m) C_2\right) ds \\
&\geq \int_{\frac{1}{4}}^{\frac{1}{2}} (\psi_i^2)^{-1}\left(\int_s^{\frac{1}{2}} h_i(\tau) d\tau\right) ds \varphi_i^{-1}(\varphi_i(m) C_2) \\
&\geq \int_{\frac{1}{4}}^{\frac{1}{2}} (\psi_i^2)^{-1}\left(\int_s^{\frac{1}{2}} h_i(\tau) d\tau\right) ds (\psi_i^2)^{-1}(C_2) m \geq m = \|v\|_X.
\end{aligned}$$

Thus $\|H(\lambda, v)\|_X > \|v\|_X$ for all $v \in \partial\mathcal{K}_m$, and by Theorem 2.5, (3.4) holds for any $\lambda \in (R_*^2(m), \infty)$. $\square$

By Lemma 3.1 and Lemma 3.2, we give the result for the existence of positive solutions to problem (1.1).

Theorem 3.3. *Assume that* (K_1) *and* (K_2) *hold, and that there exist positive constants* m_1 *and* m_2 *such that* $m_1 < m_2$ *(resp.,* $m_2 < m_1$*) and* $R_*^2(m_2) < R_*^1(m_1)$*. Then problem* (1.1) *has a positive solution* $u = u(\lambda)$ *satisfying* $m_1 < \|u\|_\infty < m_2$ *(resp.,* $m_2 < \|u\|_\infty < m_1$*) for any* $\lambda \in (R_*^2(m_2), R_*^1(m_1))$*.*

Proof. We prove only the case $0 < m_1 < m_2$, since the other case $m_2 < m_1$ is analogous. Let $\lambda \in (R_*^2(m_2), R_*^1(m_1))$ be given. From Lemma 3.1 and Lemma 3.2, it follows that

$$i(H(\lambda,\cdot), \mathcal{K}_{m_1}, \mathcal{K}) = 1 \text{ and } i(H(\lambda,\cdot), \mathcal{K}_{m_2}, \mathcal{K}) = 0.$$

Since $H(\lambda, v) \neq v$ for all $v \in \partial\mathcal{K}_{m_1}$, the additivity property implies

$$i(H(\lambda,\cdot), \mathcal{K}_{m_2} \setminus \overline{\mathcal{K}}_{m_1}, \mathcal{K}) = -1.$$

Hence, by the solution property, there exists $u \in \mathcal{K}_{m_2} \setminus \overline{\mathcal{K}}_{m_1}$ such that $H(\lambda, u) = u$. This completes the proof. □

For $a \in \{0, \infty\}$, let $f^a := \sum_{i=1}^{N} f_i^a$. Recall that

$$f_i^a = \lim_{\|y\|_N \to a} \frac{f_i(y)}{\varphi_1(\|y\|_N)} \text{ for } i \in \{1, \cdots, N\}.$$

Remark 3.4. *For* $a \in \{0, \infty\}$*,*

$$f^a = 0 \textit{ if and only if } f_i^a = 0 \textit{ for all } i \in \{1, \cdots, N\}$$

and

$$f^a = \infty \textit{ if and only if } f_i^a = \infty \textit{ for some } i \in \{1, \cdots, N\}.$$

By Remark 2.4*, for* $a \in \{0, \infty\}$*,*

$$\lim_{m \to a} R_*^1(m) = \lim_{m \to a} R_*^2(m) = \infty \textit{ if } f^a = 0 \tag{3.6}$$

and

$$\lim_{m \to a} R_*^1(m) = \lim_{m \to a} R_*^2(m) = 0 \textit{ if } f^a = \infty. \tag{3.7}$$

Theorem 3.5. *Assume that* (K_1) *and* (K_2) *hold.*

(1) *If* $f^0 = 0$ *and* $f^\infty = \infty$*, then problem* (1.1) *has a positive solution* $u(\lambda)$ *for any* $\lambda \in (0, \infty)$ *satisfying*

$$\|u_\lambda\|_X \to \infty \textit{ as } \lambda \to 0 \textit{ and } \|u_\lambda\|_X \to 0 \textit{ as } \lambda \to \infty.$$

(2) *If* $f^0 = \infty$ *and* $f^\infty = 0$*, then problem* (1.1) *has a positive solution* $u(\lambda)$ *for any* $\lambda \in (0, \infty)$ *satisfying*

$$\|u_\lambda\|_X \to 0 \textit{ as } \lambda \to 0 \textit{ and } \|u_\lambda\|_X \to \infty \textit{ as } \lambda \to \infty.$$

Proof. We prove only the case $f^0 = 0$ and $f^\infty = \infty$, as the case $f^0 = \infty$ and $f^\infty = 0$ can be treated analogously. Since $f^0 = 0$ and $f^\infty = \infty$, it follows from Remark 3.4 that

$$R_*^j(m) \to \infty \text{ as } m \to 0 \text{ and } R_*^j(m) \to 0 \text{ as } m \to \infty \text{ for all } j \in \{1, 2\}.$$

For any $\lambda \in (0, \infty)$, by (3.1), there exist $m_1(\lambda)$ and $m_2(\lambda)$ such that

$$0 < m_1(\lambda) < m_2(\lambda) \text{ and } R_*^2(m_2(\lambda)) < \lambda < R_*^1(m_1(\lambda)).$$

By Theorem 3.3, there exists a positive solution u_λ to problem (1.1) such that $m_1(\lambda) < \|u_\lambda\|_X < m_2(\lambda)$. Since $R_*^j(m) \to \infty$ as $m \to 0$ for all $j \in \{1, 2\}$, we can choose $m_1(\lambda)$ and $m_2(\lambda)$ such that

 doi:10.20944/preprints202508.2208.v1

$$0 < m_1(\lambda) < m_2(\lambda) \text{ and } m_2(\lambda) \to 0 \text{ as } \lambda \to \infty.$$

Consequently, we can choose a positive solution u_λ to problem (1.1) for large $\lambda > 0$ so that $\|u_\lambda\|_X \to 0$ as $\lambda \to \infty$. Similarly, since $R_*^j(m) \to 0$ as $m \to \infty$ for all $j \in \{1, 2\}$, we can choose a positive solution u_λ to problem (1.1) for small $\lambda > 0$ so that $\|u_\lambda\|_X \to \infty$ as $\lambda \to 0$. □

Theorem 3.6. *Assume that* (K_1) *and* (K_2) *hold.*

(1) *If* $f^0 = f^\infty = 0$, *then there exists* $\lambda_* \in (0, \infty)$ *such that problem* (1.1) *has two positive solutions* $u^1(\lambda)$ *and* $u^2(\lambda)$ *for any* $\lambda \in (\lambda_*, \infty)$ *and it has a positive solution* $u(\lambda_*)$ *for* $\lambda = \lambda_*$. *Moreover,* $u^1(\lambda)$ *and* $u^2(\lambda)$ *can be chosen so that*

$$\lim_{\lambda\to\infty} \|u^1(\lambda)\|_X = 0 \text{ and } \lim_{\lambda\to\infty} \|u^2(\lambda)\|_X = \infty.$$

(2) *If* $f^0 = f^\infty = \infty$, *then there exists* $\lambda^* \in (0, \infty)$ *such that problem* (1.1) *has two positive solutions* $u^1(\lambda)$ *and* $u^2(\lambda)$ *for any* $\lambda \in (0, \lambda^*)$ *and it has a positive solution* $u(\lambda^*)$ *for* $\lambda = \lambda^*$. *Moreover,* $u^1(\lambda)$ *and* $u^2(\lambda)$ *can be chosen so that*

$$\lim_{\lambda\to 0} \|u^1(\lambda)\|_X = 0 \text{ and } \lim_{\lambda\to 0} \|u^2(\lambda)\|_X = \infty.$$

Proof. (1) Since $f^0 = f^\infty = 0$, it follows that

$$\lim_{m\to 0} R_*^j(m) = \lim_{m\to\infty} R_*^j(m) = \infty \text{ for all } j \in \{1, 2\}.$$

Then there exists $m_* \in (0, \infty)$ satisfying $R_*^2(m_*) = \min\{R_*^2(m) : m \in \mathbb{R}_+\} \in (0, \infty)$. Let $\lambda_* = R_*^2(m_*)$. For any $\lambda \in (\lambda_*, \infty)$, by (3.1), there exist $m_1(\lambda), m_2(\lambda), M_1(\lambda)$ and $M_2(\lambda)$ such that

$$0 < m_1(\lambda) < m_2(\lambda) < m_* < M_2(\lambda) < M_1(\lambda)$$

and

$$R_*^2(m_2(\lambda)) = R_*^2(M_2(\lambda)) < \lambda < R_*^1(m_1(\lambda)) = R_*^1(M_1(\lambda)).$$

By Theorem 3.3, positive solutions $u^1(\lambda)$ and $u^2(\lambda)$ to problem (1.1) exist that satisfy

$$m_1(\lambda) < \|u^1(\lambda)\|_X < m_2(\lambda) \text{ and } M_2(\lambda) < \|u^2(\lambda)\|_X < M_1(\lambda).$$

Since $\lim_{m\to 0} R_*^j(m) = \lim_{m\to\infty} R_*^j(m) = \infty$ for all $j \in \{1, 2\}$, we can choose $m_2(\lambda)$ and $M_2(\lambda)$ satisfying

$$m_2(\lambda) \to 0 \text{ and } M_2(\lambda) \to \infty \text{ as } \lambda \to \infty.$$

Consequently, we can choose positive solutions $u^1(\lambda)$ and $u^2(\lambda)$ to problem (1.1) for large $\lambda > 0$ so that

$$\|u^1(\lambda)\|_X \to 0 \text{ and } \|u^2(\lambda)\|_X \to \infty \text{ as } \lambda \to \infty.$$

For each $n \in \mathbb{N}$, let $\lambda_n := \lambda_* + \frac{1}{n}$. We can then choose m_1^n and m_2^n such that

$$0 < \epsilon < m_1^n < m_2^n < m_* \text{ and } R_*^2(m_2^n) < \lambda_n < R_*^1(m_1^n) \text{ for all } n.$$

Consequently, for each n, there exists a positive solution $u_n \in \mathcal{K}$ to problem (1.1) such that

$$H(\lambda_n, u_n) = u_n \text{ and } \epsilon < \|u_n\|_X < m_*.$$

Since $\{(\lambda_n, u_n)\}$ is bounded in $\mathbb{R}_+ \times \mathcal{K}$ and $H : \mathbb{R}_+ \times \mathcal{K} \to \mathcal{K}$ is compact, there exist a subsequence $\{(\lambda_{n_k}, u_{n_k})\}$ of $\{(\lambda_n, u_n)\}$ and $u_* \in \mathcal{K}$ such that

$$H(\lambda_{n_k}, u_{n_k}) = u_{n_k} \to u_* \text{ in } \mathcal{K} \text{ as } k \to \infty.$$

Given that $\lambda_n \to \lambda_*$ as $n \to \infty$ and H is continuous,

$$H(\lambda_*, u_*) = u_* \text{ and } \|u_*\|_X \geq \epsilon > 0.$$

Therefore, problem (1.1) has a positive solution u_* for $\lambda = \lambda_*$. The proof is now complete.

(2) Since $f^0 = f^\infty = \infty$, it follows that

$$\lim_{m\to 0} R_*^j(m) = \lim_{m\to\infty} R_*^j(m) = 0 \text{ for all } j \in \{1,2\}.$$

Let $\lambda^* = \max\{R_*^1(m) : m \in \mathbb{R}_+\} \in (0,\infty)$ and $m^* \in (0,\infty)$ satisfying $R_*^1(m^*) = \lambda^*$. Then the proof is complete by the argument similar to those in the proof of Theorem 3.6(1). □

Finally, we conclude by providing some examples that illustrate the assumptions of Theorem 3.5 and Theorem 3.6.

Example 3.7. *Consider*

$$\begin{cases} ((1+t^2)\varphi_1(u_1'(t)))' + \lambda h_1(t) f_1(u_1(t), u_2(t)) = 0, & t \in (0,1), \\ ((1+t)^{-1}\varphi_2(u_2'(t)))' + \lambda h_2(t) f_2(u_1(t), u_2(t)) = 0, & t \in (0,1), \\ u_1(0) = u_1(1) = u_2(0) = u_2(1) = 0. \end{cases} \tag{3.8}$$

Let

$$\varphi_1(x) = \frac{x|x|}{1+|x|} \text{ and } \varphi_2(x) = x + x|x| \text{ for } x \in \mathbb{R}.$$

It is easy to verify that condition (K_1) holds for

$$\psi_1^1(y) = \psi_2^1(y) = \min\{y, y^2\} \text{ and } \psi_1^2(y) = \psi_2^2(y) = \max\{y, y^2\} \text{ for } y \in \mathbb{R}_+.$$

Define $h_1 : (0,1) \to (0,\infty)$ and $h_2 : (0,1) \to (0,\infty)$ by

$$h_1(t) = t^{-a} \text{ and } h_2(t) = (1-t)^{-b} \text{ for } t \in (0,1),$$

where $a, b \in [1,2)$ are fixed. Then, for $i \in \{1,2\}$, $(\psi_i^1)^{-1}(s) = s$ for all $s \geq 1$. Consequently, condition (K_2) is also satisfied.

In order to illustrate our results more concretely, we now discuss the four cases determined by f^0 and f^∞: First, for the case where $f^0 = 0$ and $f^\infty = \infty$, one may take

$$f_1(s_1, s_2) = (s_1 + s_2)^3 = \|s\|_2^3,\ f_2(s_1, s_2) = (s_1 + s_2)^2 = \|s\|_2^2 \text{ for } s = (s_1, s_2) \in \mathbb{R}_+^2.$$

By Theorem 3.5(1), problem (3.8) has a positive solution $u(\lambda)$ for any $\lambda \in (0,\infty)$ satisfying

$$\|u_\lambda\|_X \to \infty \text{ as } \lambda \to 0 \text{ and } \|u_\lambda\|_X \to 0 \text{ as } \lambda \to \infty.$$

Next, if $f^0 = \infty$ and $f^\infty = 0$, for instance with

$$f_1(s_1, s_2) = (s_1 + s_2)^{\frac{1}{3}} = \|s\|_2^{\frac{1}{3}},\ f_2(s_1, s_2) = (s_1 + s_2)^{\frac{3}{2}} = \|s\|_2^{\frac{3}{2}} \text{ for } s = (s_1, s_2) \in \mathbb{R}_+^2,$$

then by Theorem 3.5(2), problem (3.8) has a positive solution $u(\lambda)$ for any $\lambda \in (0,\infty)$ satisfying

$$\|u_\lambda\|_X \to 0 \text{ as } \lambda \to 0 \text{ and } \|u_\lambda\|_X \to \infty \text{ as } \lambda \to \infty.$$

Moreover, in the case where $f^0 = f^\infty = 0$, by choosing

$$f_1(s_1, s_2) = \begin{cases} (s_1 + s_2)^3 = \|s\|_2^3, & \text{if } \|s\|_2 \leq 1, \\ (s_1 + s_2)^{\frac{1}{2}} = \|s\|_2^{\frac{1}{2}}, & \text{if } \|s\|_2 > 1 \end{cases}$$

and

$$f_2(s_1, s_2) = (s_1 + s_2)^{\frac{3}{2}} = \|s\|_2^{\frac{3}{2}} \text{ for } s = (s_1, s_2) \in \mathbb{R}_+^2,$$

Theorem 3.6(1) shows that there exists $\lambda_* \in (0,\infty)$ such that (3.8) has two positive solutions $u^1(\lambda)$ and $u^2(\lambda)$ for any $\lambda \in (\lambda_*, \infty)$ and it has a positive solution $u(\lambda_*)$ for $\lambda = \lambda_*$. Moreover, $u^1(\lambda)$ and $u^2(\lambda)$ can be chosen so that

$$\lim_{\lambda\to\infty}\|u^1(\lambda)\|_X = 0 \textit{ and } \lim_{\lambda\to\infty}\|u^2(\lambda)\|_X = \infty.$$

Finally, when $f^0 = f^\infty = \infty$, an example is given by

$$f_1(s_1,s_2) = (s_1+s_2)^{\frac{3}{2}} = \|s\|_2^{\frac{3}{2}},\ f_2(s_1,s_2) = (s_1+s_2)^{\frac{1}{2}} = \|s\|_2^{\frac{1}{2}} \textit{ for } s=(s_1,s_2)\in\mathbb{R}_+^2.$$

According to Theorem 3.6(2), there exists $\lambda^*\in(0,\infty)$ such that problem (3.8) has two positive solutions $u^1(\lambda)$ and $u^2(\lambda)$ for any $\lambda\in(0,\lambda^*)$ and it has a positive solution $u(\lambda^*)$ for $\lambda=\lambda^*$. Moreover, $u^1(\lambda)$ and $u^2(\lambda)$ can be chosen so that

$$\lim_{\lambda\to 0}\|u^1(\lambda)\|_X = 0 \textit{ and } \lim_{\lambda\to 0}\|u^2(\lambda)\|_X = \infty.$$

4. Conclusions

In this work, we have investigated the existence of positive solutions to problem (1.1) under various growth conditions on the nonlinearity. Our main results are summarized in Theorem 3.5 and Theorem 3.6.

Theorem 3.5 establishes that, when the hypotheses (K_1) and (K_2) hold and the nonlinear term exhibits suitable asymptotic behavior, problem (1.1) admits a positive solution for every parameter $\lambda\in(0,\infty)$. Moreover, the theorem provides detailed information on the asymptotic behavior of these solutions as $\lambda\to 0$ or $\lambda\to\infty$, thereby providing a more comprehensive understanding of the solution structure in this case.

In contrast, Theorem 3.6 shows that, under different assumptions on the nonlinearity, one can guarantee the existence of two distinct positive solutions for problem (1.1), but only for certain ranges of the parameter λ. While this multiplicity result is significant, the theorem does not yield any information about the existence of positive solutions for the remaining values of λ. Hence, unlike Theorem 3.5, the global existence of positive solutions with respect to λ remains unclear in this setting.

This limitation suggests a natural direction for future research. A central open problem is to obtain further information on the existence or nonexistence of positive solutions for all values of $\lambda>0$ under the assumptions of Theorem 3.6, possibly by introducing additional conditions on the nonlinearity. Achieving such a characterization, possibly by combining variational techniques, upper and lower solution methods, and bifurcation methods, would provide a more complete understanding of the solution set.

Funding: This research received no external funding.

Conflicts of Interest: The author declares no conflicts of interest.

References

1. Jeong, J.; Kim, C.G. Existence of Positive Solutions to Singular Boundary Value Problems Involving φ-Laplacian. *Mathematics* **2019**, *7(7)*, 654. doi:10.3390/math7070654.
2. Agarwal, R.P.; Lü, H.; O'Regan, D. Eigenvalues and the one-dimensional p-Laplacian. *J. Math. Anal. Appl.* **2002**, *266*, 383–400. doi:10.1006/jmaa.2001.7742.
3. Wang, H. On the structure of positive radial solutions for quasilinear equations in annular domains. *Adv. Differential Equations* **2003**, *8*, 111–128. doi:10.57262/ade/1355926870.
4. Lee, Y.H.; Xu, X. Existence and multiplicity results for generalized Laplacian problems with a parameter. *Bull. Malays. Math. Sci. Soc.* **2020**, *43*, 403–424. doi:10.1007/s40840-018-0691-0.
5. Jeong, J.; Kim, C.G. Existence, Nonexistence and Multiplicity of Positive Solutions for Generalized Laplacian Problems with a Parameter. *Mathematics* **2024**, *12*, 3668. doi:10.3390/math12233668.
6. Wang, H. On the number of positive solutions of nonlinear systems. *Journal of Mathematical Analysis and Applications* **2003**, *281*, 287–306. doi:10.1016/S0022-247X(03)00100-8.
7. Chu, J.; O'Regan, D.; Zhang, M. Positive solutions and eigenvalue intervals for nonlinear systems. Proc. Math. Sci., **2007**, *117*, 85–95. doi:10.1007/s12044-007-0007-z.
8. Lee, Y.H.; Xu, X. Global existence structure of parameters for positive solutions of a singular (p_1,p_2)-Laplacian system. *Bull. Malays. Math. Sci. Soc.* **2019**, *42*, 1143–1159. doi:10.1007/s40840-017-0539-z.

9. Lee, Y.H.; Xu, X. Multiplicity results of positive solutions for singular generalized Laplacian systems. *J. Korean Math. Soc.* **2019**, *56*, 1309–1331. doi:10.4134/JKMS.j180658.
10. Medekhel, H.; Boulaaras, S.; Zennir, K.; Allahem, A. Existence of Positive Solutions and Its Asymptotic Behavior of (p(x), q(x))-Laplacian Parabolic System. *Symmetry* **2019**, *11*. doi:10.3390/sym11030332.
11. Li, H.; Wang, L.; Cui, Y. Positive solutions for a system of fractional q-difference equations with generalized p-Laplacian operators. *Electronic Research Archive* **2024**, *32*, 1044–1066. doi:10.3934/era.2024051.
12. Yang, P.; Zhang, X. Existence and multiplicity of nontrivial solutions for a (p, q)-Laplacian system on locally finite graphs. *Taiwanese Journal of Mathematics* **2024**. doi:10.11650/tjm/240201.
13. Jeong, J.; Kim, C.G. Existence of Positive Solutions to Singular φ-Laplacian Nonlocal Boundary Value Problems when φ is a Sup-multiplicative-like Function. *Mathematics* **2020**, *8(3)*, 420. doi:10.3390/math8030420.
14. Kim, C.G.; Lee, Y.H. Existence of multiple positive solutions for p-Laplacian problems with a general indefinite weight. *Commun. Contemp. Math.* **2008**, *10*, 337–362. doi:10.1142/S021919970800279X.
15. Deimling, K. *Nonlinear Functional Analysis*; Springer: Berlin/Heidelberg, Germany, 1985. https://doi.org/10.1007/978-3-662-00547-7.
16. Guo, D.J.; Lakshmikantham, V. *Nonlinear Problems in Abstract Cones*; Academic Press, Inc.: Boston, MA, USA, 1988.