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Article

Ex Post Nash Equilibrium in Games for Decision Making in Multi-environments

Abbas Edalat [1], Samira Hossein Ghorban [2]* and Ali Ghoroghi[1] †

1 Department of Computing, Imperial College London; a.edalat@imperial.ac.uk
2 School of Computer Science, Institute for Research in Fundamental Sciences (IPM); s.hosseinghorban@ipm.ir
* Correspondence: s.hosseinghorban@ipm.ir; Tel.: +98 21 24509404
† Current address: Department of Computer Engineering, University of Science and Culture; ghoroghi@usc.ac.ir

Abstract: We employ the solution concept of ex post Nash equilibrium to predict the interaction of a finite number of agents competing in a finite number of basic games simultaneously. The competition is called a multi-game. For each agent, a specific weight, considered as private information, is allocated to each basic game representing its investment in that game and the utility of each agent for any strategy profile is the weighted sum, i.e., convex combination, of its utilities in the basic games. Multi-games can model decision making in multi-environments in a variety of circumstances, including decision making in multi-markets and decision making when there are both material and social utilities for agents as, we propose, in the Prisoner's Dilemma and the Trust Game. Given a set of pure Nash equilibria, one for each basic game in a multi-game, we construct a pure Bayesian Nash equilibrium for the multi-game. We then focus on the class of so-called uniform multi-games in which each agent is constrained to play in all games the same strategy from an action set consisting of a best response per game. Uniform multi-games are equivalent to multi-dimensional Bayesian games where the type of each agent is a finite dimensional vector with non-negative components. A notion of pure type-regularity for uniform multi-games is developed and it is shown that a multi-game that is pure type-regular on the boundary of its type space has a pure ex post Nash equilibrium which is computed in constant time with respect to the number of the types and is independent of prior probability distributions. We then develop an algorithm, linear in the number of types of the agents, which tests if a multi-game is pure type-regular on the boundary of its type space in which case it returns a pure ex post Nash equilibrium for the multi-game.

Keywords: Bayesian game, Ex post Nash equilibrium, Prisoner's Dilemma, Trust Game

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1. Introduction

The key concept of *Parato efficiency* can be used to evaluate the performance of various economic systems. In an economy with incomplete information, a mechanism or decision rule is efficient if and only if there is no other feasible mechanism which may make some agents better off without ever making any other individuals worse off [1]. An agent's welfare evaluates the terms "better off" and "worse off". Computing the agents' welfare depends on what information the agents possess at the time. Holmstrom and Myerson suggested three evaluation stages: Ex ante, before the agents have received any private information; interim, when each agent has received its private information, but does not know the other's information; and ex post, when the information is public knowledge [1]. This classification leads to three different notions of domination, called ex ant domination, interim domination and ex post domination.

The revelation principle for ex post optimality states that there is a direct mechanism in which no agent will have an incentive to change its strategy after the private information of the other agents are revealed [3]. This principle has been discussed under various names in different contexts by several authors: Harris and Townsend have examined a large class of allocation problems with

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asymmetric information based on a notion of "full-information" optimality which is equivalent to "ex post" optimality [2].

Since the problem of mechanism design can be examined by games with incomplete information, the three notions of ex ante, interim and ex post domination are interpreted in the relevant Bayesian games with incomplete information. In [3], ex post Nash equilibrium (ex post NE) is employed to deal with the optimal design of mechanisms of resource allocations. Bergemann and Morris justify employing ex post NE as a solution concept in which no agent would have an incentive to change its strategy even if it were to be informed of the true type profile of the other agents; a related justification for ex post NE, discussed by the same authors, is the distinguished feature of lack of regret in this type of equilibrium, which does not hold for Bayesian Nash equilibrium (BNE) in general [4].

In this paper, we consider a new context in which the solution concept of ex post NE is similarly useful. Multi-Games, as introduced here, model the common situation in which a number of agents allocate their resources to several independent environments or games and for each of these environments each agent chooses a particular action from a set given for that environment. The amount of resources each agent allocates to each environment is regarded as private information or type and is determined by Nature according to a joint probability distribution on the set of types of all agents. The utilities in a multi-game depend on the types of the agents as well as the actions chosen. Thus, MGs form a particular class of Bayesian games.

Multi-games have some similarities and yet some basic differences with polymatrix games [5], one of several well-studied classes of compactly represented games, which also include graphical games [6], hypergraphical game [7], and graphical multi-hypermatrix games [8]. Recall that a 2-agent Bayesian game with a finite number of types can be represented by a polymatrix game [9]. In a polymatrix game, every agent plays the same strategy in every 2-agent subgame, and its utility is the sum of its subgame utilities. In a MG, however, the utility of each agent for any strategy profile in any local game, i.e, for any type profile, is a weighted sum of its n-agent basic game utilities where the weights, considered as private information, are given by the components of the agent's type.

Here, we stipulate the simplest but non-trivial condition for the dependence of the utilities on the types, namely that, for a given set of types of all agents, the total pay-off of each agent only depends linearly on its own investment profile or type in the different environments. i.e., it is independent of the types of the other agents. If the utilities are normalised, the amount of investment of agents in each environment would be replaced with their investment rate in that environment. In the normalised setting, the total utility is reduced to the convex combination of the individual utility for each environment weighted by the rate of investment for that environment.

We actually show the simple result that a pure ex post NE can be obtained in constant time if a set of pure NE for all the environments is already given. In fact, the ex post NE in this context is simply provided by the strategy profile where each agent for any given type takes, in each environment, the action specified for that agent by the NE strategy profile for that environment.

1.1. Pure type-regular multi-games

In the bulk of the paper, we keep the assumption on the linear dependence of utilities on the agents' own types but focus on a class of MGs, called uniform MGs, where agents play uniformly in all games by choosing one action from an action set consisting of a best response per game. In [10], it is shown that uniform MGs are equivalent to multi-dimensional Bayesian games, where the type of each agent is a finite dimensional vector with non-negative components, in which the utility of each agent is a linear function of its type components.

In general, due to the exponential size of the type profile set in the expanded game, the computation of an ex post NE of a uniform multi-game is intractable even if the number of agents and basic games are small. We develop a notion of *pure type-regularity* for uniform MGs and show that for a pure type-regular MG an ex post NE can be computed in constant time. Intuitively, a uniform multi-game is pure type-regular if, for each type of each agent, there is a strategy for the agent such that

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the resulting strategy profile is a NE for the local game induced by the type profile. Based on several technical results, it turns out that the type space of the multi-game is partitioned into different regions in each of which all local games have the same NE. This property is used for a pure type-regular MG to construct an ex post and consequently a pure BNE, which is shown to be independent of the beliefs of the agents.

We establish that a multi-game is pure type-regular if its restriction to the boundary of its type space is pure type-regular. In practical applications, as in classical game theory, the number of agents and that of basic games in a MG are both small so that computing pure NEs in its basic games is feasible. When the type space is discrete, consisting of a finite set, this leads to an algorithm, which is linear in the number of types, to check if a multi-game is pure type-regular, in which case an ex post NE can be efficiently computed. By employing this linear algorithm, we can efficiently establish if a multi-game is pure type-regular or not. Thus, the solution concept of ex post NEs is a reasonable and useful equilibrium concept for multi-games and therefore for modelling simultaneous decision making of agents with private information in multi-environments when the number of agents and that of basic games are both small.

1.2. Applications of multi-games

In economics, MGs would, for example, arise when several companies divide up and allocate their funds to invest in different independent markets where each market has its own rate of return or utility. Companies generally are unaware of the amount of investment their rivals make and thus these can be considered as private information, yet the rate of return or utility in each market would depend on the amount the companies invest therein. Uniform MGs can be used to model economic behaviour in multi-environments in different contexts. They commonly arise, for example, when companies working in different markets consider whether or not to adopt a new technology. They can also be employed to model the situation in which several companies compete in m markets for the production and sale of $k \leq m$ products where in each market one of these k products is optimal.

Similarly, multi-games can arise in multi-agent systems. When agents interact in a network, an agent can divide its resources such as time and allocate different portions of it to engage with other agents as in friendship networks [11]. If their resources are actually allocated to engage in different network games, we will have a multi-game.

In a completely different setting, uniform MGs can model decision making when both material and social utilities are involved. In recent years, a growing number of leading researchers in different disciplines have highlighted that decision-making has also a social and an emotional component. In many situations, agents do not seem to behave in their self-interest, but rather behave pro-socially. In the past quarter of a century, various findings in neuroscience have also established that decision making has a significant and substantial emotional component which has to be taken into account together with its more cognitive and rational component (see [14–16] for a more in-depth discussion of this issue).

Gintis [13, page xiv] has argued that while the assumption that humans are rational is an excellent first approximation, "the Bayesian rational actors favored by contemporary game theory live in a universe of subjectivity and instead of constructing a truly social epistemology, game theorists have developed a variety of subterfuges that make it appear that rational agents may enjoy a commonality of belief (common priors, common knowledge), but all are failures". Humans have a *social epistemology*, meaning that we have reasoning processes that afford us forms of knowledge and understanding, especially the understanding and sharing of other minds, that are unavailable to merely "rational" creatures. This social epistemology characterizes our species. The bounds of reason are thus not the irrational, but the social." Gintis proposes a preference ordering in order to reflect individual beliefs and perceptions [13, chapter 1].

A classic benchmark for modeling human decision making when self-interests are at stake is provided by the Prisoner's Dilemma (PD) (see [17,18]). As it is well-known, Axelrod organised

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two international round-robin tournaments in which strategies for the iterated PD competed with each other [19,20]. The strategy Tit-for-Tat, i.e., cooperate on the first move and then reciprocate the opponent's last move, proved to be robust and became the overall winner.

However, when confronted with the choice to cooperate or defect, human beings not only consider their material score, but also the social and moral implications of their decisions, and the consequent social and moral utilities. In [12], Housman discusses the influence of social norms on decision making of individuals in the PD and observes that agents wouldn't play the behavior expected from the NE since the game theoretic model simplifies drastically the complexities of human mental life: "The Prisoner's Dilemma shows that there are some institutional frameworks in which the self-interested choices of rational individuals are not socially beneficial."

In recent years, there have been some empirical evidence on the PD with real people that corroborate this argument. Khadjavi and Lange present an experiment to compare female prison inmates and students in a simultaneous and an iterated PD [21]. In the simultaneous PD, the cooperation rate among inmates exceeded the rate of cooperating students. The authors have concluded that a similar and significant fraction of inmates and students hold social preferences. Brosig provides findings from a face-to-face experiment that used the PD to analyse whether individuals who possess a willingness to cooperate can credibly signal it and whether it is recognisable by the partner [22]. Results revealed that both capabilities, signaling and recognising, depend upon the individual's propensity to cooperate.

In the past two decades, the so-called Trust Game with two agents and an experimenter has been proposed to measure trust in human economic behaviour [23]. Initially the two agents are given an equal amount of money. Then in stage one, the first agent is asked to send some of its money to the experimenter who triples it and sends the tripled amount to the second agent. In stage two, the second agents is asked to send some of the money it has received by the experimenter to the first agent. The NE in the Trust Game stipulates that the first agent sends no money to the experimenter and the second agent also sends no money back to the first agent; see Section 7. However, in practice, human agents deviate from the NE as reported in experiments in [23] and also in a meta-analysis of 162 replications of the Trust Game involving more than 23,000 participants [25]. An explanation for this deviation has been proposed by evolutionary psychology: "Evolutionary models predict the emergence of trust because it maximises genetic fitness" [23].

Chaudhuri and Gangadharant model the role of reciprocation in the decision making with a Bayesian game with two types, a "reciprocator" or a "non-reciprocator". Based on the expectation of reciprocation, they show that there is no significant difference in the level of reciprocity between two groups of men and women. They have examined gender differences in the decision to send money and show that the level of trust in men is higher than women and hypothesise that this is due to a higher degree of risk aversion in women [24].

We propose that in games such as PD and the Trust Game there are two independent utilities: a material utility and a social one. This motivates us to use multi-games to combine material and social utilities by allowing one environment, called the material game, to specify the material utilities and another environment, called the social game, to represent the social utilities. The weights the agents allocate as private information to the material and the social game indicate their materialistic and altruistic inclinations respectively, giving rise to a Bayesian game. This in particular gives an alternative approach to that of Gintis [13] and that of Chaudhuri and Gangadharant [24] referred to above.

We will use uniform multi-games to model decision making in the PD and in the Trust Game when, as well as the standard game with its material utilities, the agents simultaneously take part in a social game where social utilities of their decisions are additionally considered. Since civilised societies generally commend collaboration and trust and censure defection and mistrust, the social game will have a NE when both agents collaborate, which counters the NE for the standard PD in which the NE occurs when both agents defect and the NE in the Trust Game when the two agents send each

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other zero money. We show that uniform multi-games provide a more reasonable model of human behaviour in both the PD and the Trust Game.

1.3. Other related work

In [26], Bulow et al. provide a numerical example of Cournot markets in which two firms sell in one market and one of them is a monopolist in a second market. Thus, they analyse a network of two firms and two markets where one of markets is accessible for both firms and another one is only accessible for one of them. A market where only one supplier is present will have a monopolistic higher price while price in a market which has alternative suppliers will be lower as a result of the competition among the firms. More recently, several authors have examined a network approach to Cournot competition [27,28]. In this approach, a bipartite graph determines which subset of markets a given firm can supply to. In [27], the aim is to derive algorithms that compute the pure strategy Nash equilibria while [28] provide characterisations of equilibrium production quantities and study the impact of changes in the competition structure such as expansion to a new market and merger of companies. In the above studies, the structure of the network is common knowledge. In real life, however, firms may be uncertain about the new markets their competitors would decide to access. Multi-games can be used to model a network Cournot competition in which the subset of markets each firm has access to becomes private information.

As far as the notion of a double game consisting of a material and a social game is concerned, our work has certain parallels with the altruistic extension in [29] which endows each agent with an altruistic level in the unit interval providing the degree of the pro-social attitude of the agent. This is similar to the weights for materialistic and pro-social tendencies in a DG. In fact, the altruistic extension is a special case of a DG in which the utilities of the social game is determined by a sum-bounded cost function. However, such altruistic extensions are not general enough to differentiate between individual strategies in the social utilities and thus cannot model the double PD referred to above, which we will analyse in detail later in the paper.

2. Multi-games and their ex post NEs

A multi-game, as a Bayesian game, models the behavior of a finite number of rational agents who play in a number of different environments simultaneously. Each environment is represented by a basic game and the resources of each agent are allocated with varying proportions, as private information, to these basic games. Here is the formal definition.

Definition 1. *A multi-game G is a game in strategic form with incomplete information which has the following structure:*

$$G = \left\langle I, J, \{w_i\}_{i\in I}, \{G_j\}_{j\in J}, \{\Theta_i\}_{i\in I}, \{A_{ij}\}_{i\in I, j\in J}, \{U_{ij}\}_{i\in I, j\in J}, p(\cdot) \right\rangle$$

1. *The set of agents, $I = \{1, \ldots, n\}$.*
2. *The set of n-agent basic games G_j where $j \in J = \{1, \ldots, m\}$ with action space A_{ij} and utility function U_{ij} for each agent $i \in I$ in the game G_j.*
3. *Agent i's strategy $s_i = (s_{i1}, \ldots, s_{im}) \in S_i = \prod_{j\in J} A_{ij}$ where s_{ij} is agent i's action in G_j.*
4. *Agent i' type $\theta_i = (\theta_{i1}, \ldots, \theta_{im}) \in \Theta_i$ with $\theta_{ij} \geq 0$, $\sum_{j\in J} \theta_{ij} \leq w_i$, and $w_i > 0$ as agent i's total resource.*
5. *Agent i's utility for the strategy profile (s_i, s_{-i}) and type profile (θ_i, θ_{-i}) depends linearly on its types:*

$$U_i(s_i, s_{-i}, \theta_i, \theta_{-i}) = \sum_{j\in J} \theta_{ij} U_{ij}(s_{1j}, \ldots, s_{nj}).$$

6. *The agents' type profile $(\theta_1, \ldots, \theta_n) \in \prod_{i\in I} \Theta_i$ is drawn from a given joint probability distribution $p(\theta_1, \ldots, \theta_n)$. For any $\theta_i \in \Theta_i$, the function $p(\cdot|\theta_i)$ specifies a conditional probability distribution over Θ_{-i} representing what agent i believes about the types of the other agents if its own type were θ_i.*

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Note that w_i is an upper bound for the sum of investments of agent i in the basic games. In the above definition of a multi-game, the utility of each agent only depends linearly on its own type. More general utilities will be considered in future work.

Since the types of each agent $i \in I$ are private information, a MG is a Bayesian game with a (pure) strategy map space $S_i^{\Theta_i} = \{s_i(\cdot) : \Theta_i \to S_i\}$ for agent i so that $\prod_{i\in I} S_i^{\Theta_i}$ represents the space of all strategy map profiles for the expanded game. For a strategy map profile $(s_i(\cdot), s_{-i}(\cdot)) \in S_i^{\Theta_i} \times S_{-i}^{\Theta_{-i}}$, the expected utility of agent $i \in I$ is given by

$$U_i(s_i(\cdot), s_{-i}(\cdot)) = \sum_{\theta_{-i}} \sum_{\theta_i} p_i(\theta_{-i}|\theta_i) U_i\big((s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i, \theta_{-i}\big).$$

Recall that a strategy map profile $(s_1(\cdot), \ldots, s_n(\cdot))$ is a pure Bayesian Nash equilibrium (BNE) if for each agent $i \in I$ and $s_i'(\cdot) \in S_i^{\Theta_i}$, we have $U_i(s_i(\cdot), s_{-i}(\cdot)) \geq U_i(s_i'(\cdot), s_{-i}(\cdot))$ [31, p. 215]. For discrete type spaces, this is equivalent to $s_i(\theta_i) \in \arg\max_{s_i \in S_i} \sum_{\theta_{-i}} p_i(\theta_{-i}|\theta_i) U_i\big((s_i, s_{-i}(\theta_{-i}), \theta_i, \theta_{-i}\big)$, for each $\theta_i \in \Theta_i$ and $i \in I$.

We say that a multi-game is *normalised* if for each agent $i \in I$, we have $\sum_{j\in J} \theta_{ij} = w_i = 1$. By considering an additional game, we can always normalise a multi-game as follows.

Proposition 1. *The multi-game* $G = \langle I, J, \{w_i\}_{i\in I}, \{G_j\}_{j\in J}, \{\Theta_i\}_{i\in I}, \{A_{ij}\}_{i\in I, j\in J}, \{U_{ij}\}_{i\in I, j\in J}, p(\cdot)\rangle$ *is equivalent with* $\hat{G} = \langle I, \hat{J}, \{\hat{w}_i\}_{i\in I}, \{\hat{G}_j\}_{j\in \hat{J}}, \{\hat{\Theta}_i\}_{i\in I}, \{A_{ij}\}_{i\in I, j\in J}, \{\hat{U}_{ij}\}_{i\in I, j\in J}, p(\cdot)\rangle$ *where:*

1. *The set of n-agent basic games* $\hat{G}_j$ *where* $\hat{J} = J \cup \{m+1\}$ *with action space* A_{ij} *and utility function* $\hat{U}_{ij}$ *for each agent* $i \in I$ *in the game* $\hat{G}_j$ *where* $\hat{U}_{ij} = w_i U_{ij}$ *for each* $j \in J$ *and also action set* $A_{i(m+1)} = A_{ij_i}$ *for some* $j_i \in J$ *and utility function* $\hat{U}_{i(m+1)}(s_{i(m+1)}, s_{-i(m+1)}) = 0$ *for each* $s_{i(m+1)} \in A_{i(m+1)}$ *and* $s_{-i(m+1)} \in A_{-i(m+1)}$.
2. *Agent i's type space,*

$$\hat{\Theta}_i = \left\{ \left(\frac{\theta_{i1}}{w_i}, \ldots, \frac{\theta_{im}}{w_i}, 1 - \frac{\sum_{j\in J}\theta_{im}}{w_i} \right) \Big| (\theta_{i1}, \ldots, \theta_{im}) \in \Theta_i \right\}.$$

3. *Agent i's total resource* $\hat{w}_i = 1$.

Proof. Assume $\hat{\theta}_k = \left(\theta_{k1}/w_k, \ldots, \theta_{km}/w_k, 1 - \sum_{j\in J} \theta_{kj}/w_k\right) \in \Theta_k$ where $k \in I$. For each agent $i \in I$ and $(s_i, s_{-i}) \in S_i \times S_{-i}$, we have

$$\hat{U}_i(s_i, s_{-i}, \hat{\theta}_i, \hat{\theta}_{-i}) = \sum_{j\in\hat{J}} \hat{\theta}_{ij} \hat{U}_{ij}(s_{ij}, s_{-ij}) = \sum_{j\in J} \theta_{ij} U_{ij}(s_{ij}, s_{-ij}) = U_i(s_i, s_{-i}, \theta_i, \theta_{-i}).$$

□

By Proposition 1, we will from now on assume that a multi-game is always normalised. Thus, Θ_i for each agent i is a subset of the $(m-1)$-dimensional simplex

$$\Delta^{m-1} = \left\{ x \in \mathbb{R}^m | x_j \geq 0, \sum_{j=1}^m x_j = 1 \right\}.$$

A multi-game is called a *Double Game* (DG) if $m = 2$. In a DG, it is convenient to write the type $(\theta_{i1}, \theta_{i2})$ of agent i as $\theta_i := \theta_{i2}$ with $\theta_{i1} = 1 - \theta_{i2} = 1 - \theta_i$. Thus, for an n-agent DG we have $\Theta_i \subseteq [0,1]$ and $\prod_{i\in I} \Theta_i \subseteq [0,1]^n$.

Definition 2. *A type* $\theta_i = (\theta_{i1}, \ldots, \theta_{im})$ *for agent i with* $\theta_{ij} = 1$ *for some* $j \in J$ *and* $\theta_{it} = 0$ *for* $t \neq j$ *is called an* extreme type, *and is denoted by* p_j, *which is the unit vector along the j axis. If* $\theta_i = p_j$, *we say that agent i*

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only plays in the basic game j. The set of all extreme types for agent $i \in I$ is denoted by $\Theta_i^e = \{p_j : 1 \leq j \leq m\}$. The type profile $(p_{j_1}, \ldots, p_{j_n})$, where $p_{j_i} \in \Theta_i^e$ and $j_i \in J$ for each $i \in I$, is called an extreme type profile.

Although Θ_i^e is independent of i, it is convenient in practice to explicitly use the index i to indicate the agent for which the types are considered. We will assume that the type space Θ_i of each agent i contains all its m extreme types, i.e., $\Theta_i^e \subset \Theta_i$. This implies that Δ^{m-1} is the convex hull of Θ_i for each $i \in I$. It also follows that $\Theta = \prod_{i \in I} \Theta_i$ contains m^n extreme type profiles, i.e., $\prod_{i \in I} \Theta_i^e \subset \Theta$. The *boundary* of the type space Θ is defined as $\Theta^b = \bigcup_{i \in I} \Theta_i \times \Theta_{-i}^e$. For example, in the case of a DG with n agents, the boundary of the type space is the faces of the hyper-cube $[0,1]^n$.

Definition 3. *The restriction of a multi-game G to a given type profile $(\theta_1, \ldots, \theta_n)$ is denoted by $G_{(\theta_1, \ldots, \theta_n)}$ and is called the* local game *for G at $(\theta_1, \ldots, \theta_n)$.*

As a general example, consider a 2-agent DG with basic games G_1 and G_2 such that $A_{11} = \{a_{111}, a_{112}\}$, $A_{12} = \{a_{121}, a_{122}\}$, $A_{21} = \{a_{211}, a_{212}\}$ and $A_{22} = \{a_{221}, a_{222}\}$. Table 1 represents the utilities of the two basic games.

Table 1. Utilities for basic games G_1 and G_2.

	a_{211}	a_{212}
a_{111}	(b_1, b_2)	(d_1, d_2)
a_{112}	(e_1, e_2)	(f_1, f_2)

	a_{221}	a_{222}
a_{121}	(g_1, g_2)	(h_1, h_2)
a_{122}	(k_1, k_2)	(ℓ_1, ℓ_2)

The utilities of agent $i = 1, 2$ for the local game $G_{(\theta_1, \theta_2)}$, for each $(\theta_1, \theta_2) \in \Theta_1 \times \Theta_2$, are respectively depicted in Table 2.

Table 2. Agent i's utility of $G_{(\theta_1, \theta_2)}$ for a given $(\theta_1, \theta_2) \in \Theta$.

	(a_{211}, a_{221})	(a_{211}, a_{222})	(a_{212}, a_{221})	(a_{212}, a_{222})
(a_{111}, a_{121})	$(1-\theta_i)b_i + \theta_i g_i$	$(1-\theta_i)b_i + \theta_i h_i$	$(1-\theta_i)d_i + \theta_i g_i$	$(1-\theta_i)d_i + \theta_i h_i$
(a_{111}, a_{122})	$(1-\theta_i)b_i + \theta_i k_i$	$(1-\theta_i)b_i + \theta_i \ell_i$	$(1-\theta_1)d_i + \theta_i k_i$	$(1-\theta_i)d_i + \theta_i \ell_i$
(a_{112}, a_{121})	$(1-\theta_i)e_i + \theta_i g_i$	$(1-\theta_i)e_i + \theta_i h_i$	$(1-\theta_i)f_i + \theta_i g_i$	$(1-\theta_i)f_i + \theta_i h_i$
(a_{112}, a_{122})	$(1-\theta_i)e_i + \theta_i k_i$	$(1-\theta_i)e_i + \theta_i \ell_i$	$(1-\theta_i)f_i + \theta_i k_i$	$(1-\theta_i)f_i + \theta_i \ell_i$

Example 1. *Suppose G is a 2-agent DG with basic games G_1 and G_2 with $A_{11} = \{a_{111}, a_{112}\}$, $A_{12} = \{a_{121}, a_{122}\}$, $A_{21} = \{a_{211}, a_{212}\}$, $A_{22} = \{a_{221}, a_{222}\}$ and, $\Theta = [0,1]^2$. Assume the utilities of the two basic games G_1 and G_2 are given in Table 3 and that the joint prior probability distribution is the uniform (Lebesgue) measure on $[0,1]^2$. Then, it is easy to check that $(s_1(\cdot), s_2(\cdot))$, where $s_1(\cdot) : \Theta_1 \to S_1$ and $s_2(\cdot) : \Theta_2 \to S_2$ are given by $s_1(\theta_1) = (a_{111}, a_{122})$ and $s_2(\theta_2) = (a_{211}, a_{222})$, is a BNE for G.*

Table 3. Utilities for basic games G_1 and G_2 of Example 1.

	a_{211}	a_{212}
a_{111}	$(3,5)$	$(1,4)$
a_{112}	$(2,8)$	$(6,3)$

	a_{221}	a_{222}
a_{121}	$(10,1)$	$(0,5)$
a_{122}	$(4,1)$	$(2,3)$

In Example 1, (a_{111}, a_{122}) and (a_{211}, a_{222}) are, respectively, NEs for the basic games G_1 and G_2. The following simple result shows that a BNE for a multi-game can be computed in constant time with respect to the number of types and independent of the prior probability distribution if a set of NEs for all its basic games is given.

Proposition 2. *Let G be a multi-game and $(s_{1j}, \ldots, s_{nj})$ a NE for its basic game G_j with $j \in J$. Then $(s_1(\cdot), \ldots, s_n(\cdot))$, where $s_i(\cdot) : \Theta_i \to S_i$ is given by $s_i(\theta_i) = (s_{i1}, \ldots, s_{im})$, is a BNE for all priors.*

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Proof. Consider any strategy $s'_i(\cdot) : \Theta_i \to S_i$. By definition: $U_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) = \sum_{j\in J} \theta_{ij} U_{ij}(s_{ij}, s_{-ij}) + c_i(s_i(\theta_i), s_{-i}(\theta_{-i}))$. Since (s_{ij}, s_{-ij}) is a NE for G_j, $U_{ij}(s_{ij}, s_{-ij}) \geq U_{ij}(s'_{ij}, s_{-ij})$ for $j \in J$ where $s'(\theta_i) = (s'_{ij})_{j\in J}$. Therefore $U_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) \geq U_i(s'_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i, \theta_{-i})$. Hence

$$U_i(s_i(\cdot), s_{-i}(\cdot)) = \sum_{\theta_i\in\Theta_i} \sum_{\theta_{-i}\in\Theta_{-i}} p(\theta_{-i}|\theta'_i) U_i((s(\theta_i), s_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) \geq U_i(s'_i(\cdot), s_{-i}(\cdot)),$$

which completes the proof. □

Definition 4. *[3] Let G be a Bayesian game. A strategy profile $(s_1(\cdot), \ldots, s_n(\cdot)) \in \prod_{i\in I} S_i^{\Theta_i}$ is an ex post (pure) NE if $U_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta) \geq U_i(a_i, s_{-i}(\theta_{-i}), \theta)$ for all $i \in I, \theta \in \Theta, a_i \in S_i$.*

It follows immediately from the definition that if in a Bayesian game the strategy profile $(s_1(\cdot), \ldots, s_n(\cdot))$ is an ex post NE then it is a BNE which is independent of the prior probability distribution. This property therefore justifies ex post NE as a solution concept.

Corollary 1. *Let G be a multi-game and $(s_{1j}, \ldots, s_{nj})$ a NE for its basic game G_j with $j \in J$. Then $(s_1(\cdot), \ldots, s_n(\cdot))$, where $s_i(\cdot) : \Theta_i \to S_i$ is given by $s_i(\theta_i) = (s_{i1}, \ldots, s_{im})$, is an ex post NE.*

Therefore, in a multi-game, the best response of an agent, whatever its private information, is to simply play its NE in each basic game. In the next section, we consider a class of multi-games in which agents are constrained in their strategies for the basic games and the above simple recipe no longer provides the best response.

3. Uniform MGs

In many applications, the action set of each agent consists of one optimal action per basic game and the agents select the same action in all these games. We formalise this class of multi-games in the following definition.

Definition 5. *A multi-game G is* uniform *if the following conditions hold:*

1. *Agent i's action set in game G_j is given by $A_{ij} = A_i$, i.e., A_{ij} is independent of $j \in J$ and contains an action denoted by s_{ij} for each $j \in J$. The actions s_{ij} for each $j \in J$ are not necessarily distinct and thus A_i contains at most m actions.*
2. *The strategy profile $(s_{1j}, \ldots, s_{nj})$ is a NE for the basic game G_j for each $j \in J$.*
3. *Each agent plays the same action in all basic games, i.e., $S_i = \{(s, \ldots, s) | s \in A_i\}$ for each $i \in I$.*

In [10], it is shown that a uniform multi-game is equivalent to an m-dimensional n-agent Bayesian game (where the type of each agent is a vector in $\mathbb{R}^m_+$), in which the utility of each agent only depends linearly on its own type. In the rest of this paper, all multi-games will be uniform games. It is therefore convenient, by an abuse of notation, to denote the strategy $(s, \ldots, s) \in A_i^m$ of agent i simply as $s \in A_i$, which implies that agent i plays action s in all basic games. We will adopt this notation in what follows.

In the rest of this section, we present three different types of uniform games in different contexts. In the first, following the discussions in the introduction, we use a uniform DG to model the PD when social or moral utilities of the agents as well as their material utilities, are also taken into account. This generalises the framework proposed in [29] for altruistic behaviour in the context of a DG. In the second, we show how adoption of a new technology by a number of companies in a multi-market can be captured by a uniform MG. In the third, a model of competition between multi-national companies simultaneously investing in a number of markets is presented.

These problems are briefly explained in this section. In Section 6, after obtaining results on ex post NEs for pure type-regular games, we will revisit these examples to derive some equilibria for them.

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3.1. A double game for Prisoner's Dilemma (PD)

As argued in the Introduction, in many circumstances, human beings consider not only their material score, but also the social utilities of any decision they make. This can be modelled by allowing agents to engage simultaneously in a social game and a standard material game for their utilities. We will show this for the case of the PD.

Consider the standard PD with the utilities as given in Table 4 (left) with $t > r > p > s$ and $r > (t+s)/2$. The first equation specifies the order of the utilities and defines the dilemma, since the best an agent can do is to get t (i.e. the utility for defection when the other agent cooperates). The worst an agent can do is get s (i.e. the sucker's utility for cooperating while the other agent defects), and, in ordering the other two outcomes, r (i.e. the utility for mutual cooperation), is assumed to be better than p (i.e. the utility for mutual defection) [30]. The second equation ensures that, in the repeated game, the agents cannot get out of the dilemma by taking turns in exploiting each other. This means that an even chance of exploiting and being exploited is not as good an outcome for each agent as mutual cooperation. Therefore, it is assumed that r is greater than the average of t and s [30].

The social game (SG) encourages cooperation and discourages defection, as cooperating is usually considered to be the right ethical and moral choice when interacting with others in social dilemmas. This can be done in different ways which correspond to different types of utilities. Here, we will only consider the case in which SG encourages cooperation and discourages defection for each agent, independently of the action chosen by the other agent. We present the normal form and the mathematical formulation of the SG as follows. Assume that the competing participants in the SG are agent 1 and 2. Each of them can select C or D. When they have both made their choice, the utilities assigned to them are calculated according to Table 4(right). Let $y > z$. Then (D, D) and (C, C) are NEs for PD and SG, respectively.

Table 4. Utilities for DP (left) and SG (right)

	C	D
C	(r,r)	(s,t)
D	(t,s)	(p,p)

	C	D
C	(y,y)	(y,z)
D	(z,y)	(z,z)

The type $\theta_i \in [0,1]$ of agent $i = 1,2$ is their *prosocial* coefficient, with $\theta_i = 0$ reflecting complete selfishness while $\theta_i = 1$ indicating maximum pro-sociability.

In order to use the DG to model agent behavior with varying degrees of pro-sociability, we argue, as in [32], that we need to assume three new sets of relations: (i) $r > y > p$, (ii) $y > (r+p)/2$ and (iii) $z = s$. With respect to (i), note that if y is equal to or less than p, then, cooperation is discouraged, since one would have no incentive to select a high pro-social coefficient and choose C. In addition, y should be strictly less than r, as we would like to encourage cooperation in the SG by assigning to it a utility value that is somewhat less than the utility value obtained through mutual cooperation in the PD. As for (ii), we assume that y should be greater than the average of r and p, so that the dilemma of whether to cooperate or defect becomes more intense. Finally, regarding (iii), we argue that z can be taken to be equal to s, so as to discourage defection with a high social coefficient, which would be self-contradictory, and, also, to punish, in a sense, defection, since z is the utility value for defection in the SG, which, by its definition, should not give a high value to defection.

This framework for considering the PD with a SG, we believe, reflects more accurately real-life situations, as, in general, decisions based on pro-social or moral incentives and beliefs do not bring high material benefits.

3.2. Adoption of technology

Consider n companies, competing in a multi-market, which have to adopt a long-term strategy as to whether they should implement a new technology in their production or not (e.g., energy companies choosing between fossil-based or renewable sources of energy). Due to the high cost of shifting to

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the new technology, they need to use the same strategy in all the available markets. We thus have a uniform game in which the agents have two possible strategies.

We will model one such scenario with two firms ($i = 1, 2$), each with the choice either to adopt (a) or to reject (r) the new technology. Suppose the the two firms compete in two markets $j = 1, 2$, where they have different costs for the adoption of the new technology as well as different returns from the adoption.

For instance consider two firms in two markets who have a strategy space $\{a, r\}$ for the adoption (a) and rejection (r) of the new technology. Assume Table 5 shows utilities for firms in market G_1 and market G_2, respectively. Let $b_1 > e_1,\ \ell_1 > h_1$ and $b_2 > d_2,\ \ell_2 > k_2$. Then the strategy profiles (a, a) and (r, r) are NEs for G_1 and G_2 respectively.

Table 5. Utilities for markets G_1 and G_2.

	a	r
a	(b_1, b_2)	(d_1, d_2)
r	(e_1, e_2)	(f_1, f_2)

	a	r
a	(g_1, g_2)	(h_1, h_2)
r	(k_1, k_2)	(ℓ_1, ℓ_2)

3.3. *Multinational companies*

Consider n multinational companies which compete in multi-markets consisting of, say, m different markets each with its own rate of return. Assume that in each market j a given product s_j yields the greatest return but due to the design and manufacturing costs each company has to produce the same product in all the m markets. In this way, we have a uniform MG with $S_i = \{s_j \mid j \in J\}$ for all $i \in I$ where θ_{ij} is the investment fraction of company i in market j.

We give a numerical example with two companies competing in three markets, in which Table 6 provides the utilities.

Table 6. Utilities for basic games G_1, G_2 and G_3.

	s_1	s_2	s_3
s_1	$(3,7)$	$(11,4)$	$(4,3)$
s_2	$(2,8)$	$(10,5)$	$(3,4)$
s_3	$(1,5)$	$(9,2)$	$(2,1)$

	s_1	s_2	s_3
s_1	$(4,1)$	$(5,4)$	$(9,2)$
s_2	$(5,5)$	$(6,8)$	$(10,6)$
s_3	$(3,0)$	$(4,3)$	$(8,1)$

	s_1	s_2	s_3
s_1	$(2,2)$	$(3,6)$	$(7,10)$
s_2	$(4,4)$	$(6,8)$	$(8,12)$
s_3	$(3,6)$	$(5,10)$	$(9,14)$

4. Pure type-regularity

The computation of a NE in classical game theory is a hard problem in relation to the number of agents and strategies, which is why in applications one uses small numbers of agents and strategies (see [33]). Likewise in applications of uniform MGs we always deal with small n and m, but here the number of types can be large. This means that computation of an ex post NE in uniform MGs is a hard problem with respect to the number of types even for small numbers of agents and basic games. To redress this problem, we introduce a notion of pure type-regularity for a subset of the type profile set. We show that if the boundary of the type profile set satisfies the pure type-regularity condition, then an ex post NE can be computed efficiently, as we will show in this section. Let the projection map $\pi_i : \Theta \subseteq (\mathbb{R}^m)^n \to \mathbb{R}^m$, for each $i \in I$, be given by $\pi_i(\theta_1, \ldots, \theta_n) = \theta_i$.

Definition 6. *A uniform multi-game G is* pure type-regular *on* $\Theta' \subseteq \Theta$ *if for each* $i \in I$ *there is a function* $s_i^*(\cdot) : \pi_i(\Theta') \to S_i$ *such that*

(i) $s_i^*(p_j) = s_{ij}$ *if* $p_j \in \pi_i(\Theta')$*, and,*
(ii) the strategy profile $(s_1^*(\theta_1), \ldots, s_n^*(\theta_n))$ *is a NE for the local game* $G_{(\theta_1,\ldots,\theta_n)}$ *whenever* $(\theta_1, \ldots, \theta_n) \in \Theta'$.

If G is pure type-regular on Θ *then G is simply called* pure type-regular.

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In words, item (i) in Definition 6 states that if there is a strategy map profile in Θ' for which agent i plays only in the game G_j, then the expanded game strategy $s_i^*(\cdot)$ assigns p_j, the type for agent i that corresponds to playing only in the game G_j, to action s_{ij}, which is agent i's action in the NE action profile for game G_j. Intuitively, this means that a uniform multi-game G is pure type-regular on Θ' if for each agent and a given type component for it from the set Θ', (i) the agent chooses its associated NE action if it has an extreme type, i.e., if it only plays in one basic game, and (ii) the agent can select an action dependent only on the given type, such that for each type profile of all agents in Θ' the resulting action profile is a NE for the local game specified by that type profile. Note from the definition that if G is pure type-regular on $\Theta' \subseteq \Theta$, then it is pure type-regular on any subset of Θ'. The result below is an immediate consequence of Definition 4.

Proposition 3. *If G is pure type-regular, then the strategy map profile $(s_1^*(\cdot), \ldots, s_n^*(\cdot))$ that witnesses the pure type-regularity of G is an ex post NE for G.*

The following result gives an equivalent characterisation of a pure type-regular MG, which sheds light on the structure of the type space its agents.

Proposition 4. *Let G be a MG. Then G is pure type-regular if and only if, for each agent i, there is a partition of its type set $\Theta_i = \bigcup_{1 \le j \le m} \Theta_{ij}$, where Θ_{ij} are disjoint subsets for $1 \le j \le m$, such that*

(i) $p_j \in \Theta_{ij}$ for $1 \le j \le m$, and,
(ii) for all $\theta \in \Theta$, the local game $G_{(\theta_1,\ldots,\theta_n)}$ has as a NE the action profile $(s_{1j_1}, \ldots, s_{ij_i}, \ldots, s_{nj_n})$, where j_i is given by $\theta_i \in \Theta_{ij_i}$ for $1 \le i \le n$.

Proof. Suppose G is pure type-regular and $s_i^*(\cdot)$ is given as in Definition 6. For each agent i, let $\Theta_{ij} = (s_i^*)^{-1}(s_{ij})$, where, $(s_i^*)^{-1}$ is the inverse map of s_i^* and, recall, s_{ij} is the agent i's action in the action profile $(s_{1j}, \ldots, s_{ij}, \ldots, s_{nj})$ that is a NE for the game G_j. Then Θ_{ij} are disjoint for $1 \le j \le m$ and $\Theta_i = \bigcup_{1 \le j \le m} \Theta_{ij}$, and $p_j \in \Theta_{ij}$ by Definition 6(i). Moreover, by Definition 6(ii), $(s_{1j_1}, \ldots, s_{ij_i}, \ldots, s_{nj_n})$, where j_i is given by $\theta_i \in \Theta_{ij_i}$, is a NE for $G_{(\theta_1,\ldots,\theta_n)}$.

Next, suppose conditions (i) and (ii) hold. Let $s_i^*(\cdot) : \pi_i(\Theta) \to S_i$ be given by $s_i^*(\theta_i) = s_{ij}$ where j is such that $\theta_i \in \Theta_{ij}$. Condition (i) shows that $s_i^*(p_j) = s_{ij}$ for each $i \in I$ and $j \in J$. Also, we have $(s_1^*(\theta_1), \ldots, s_n^*(\theta_n)) = (s_{1j_1}, \ldots, s_{nj_n})$ where $\theta_i \in \Theta_{ij_i}$. Thus, by condition (ii), $(s_1^*(\theta_1), \ldots, s_n^*(\theta_n))$ is a NE for $G_{(\theta_1,\ldots,\theta_n)}$. $\square$

We will develop a test to ascertain if a uniform MG is pure type-regular in the next section. Before embarking on this task, we present two examples of DG: the first is pure type-regular with a partition of its type set according to Proposition 4, while the second is not pure type-regular.

Example 2. *Suppose G is a 2-agent DG with basic games G_1 and G_2 where $A_1 = \{s, t\}$, $A_2 = \{u, v\}$ and $\Theta_1 = \Theta_2 = [0, 1]$. Assume the utilities of the two basic games G_1 and G_2 are given in Table 7.*

Table 7. Utilities for G_1 (left) and G_2 (right) of Example 2.

	u	v
s	$(2,4)$	$(6,2)$
t	$(1,10)$	$(3,8)$

	u	v
s	$(4,7)$	$(9,20)$
t	$(11,4)$	$(30,17)$

Let $s_1^*(\cdot) : \Theta_1 \to S_1$ and $s_2^*(\cdot) : \Theta_2 \to S_2$ be given by

$$s_1^*(\theta_1) = \begin{cases} s & \theta_1 \le 1/8 \\ t & \theta_1 > 1/8 \end{cases} \qquad s_2^*(\theta_2) = \begin{cases} u & \theta_2 \le 2/15 \\ v & \theta_2 > 2/15 \end{cases}$$

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Then $(s_1^*(\theta_1), s_2^*(\theta_2))$ *is a NE of* $G_{(\theta_1,\theta_2)}$ *for each* $(\theta_1, \theta_2) \in \Theta$ *as depicted in Figure 1. Hence, G is pure type-regular. By Proposition 3,* $(s_1^*(\cdot), s_2^*(\cdot))$ *is an ex post NE for G.*

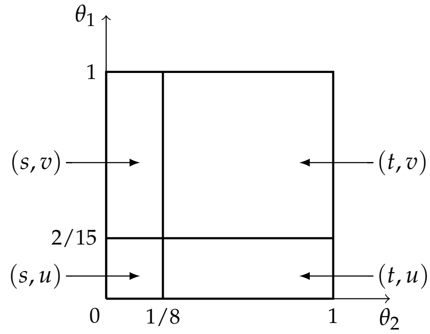

Figure 1. MG of Example 2 is pure type-regular: the type set is partitioned into four regions

Example 3. *Suppose G is a 2-agent DG with basic games* G_1 *and* G_2 *where* $A_1 = \{s,t\}$, $A_2 = \{u,v\}$ *and* $\Theta_1 = \{0, 5/8, 1\}$ *and,* $\Theta_2 = \{0, 2/5, 1\}$. *Assume the utilities of the two basic games* G_1 *and* G_2 *are as in Table 8.*

Table 8. Utilities for G_1 (left) and G_2 (right) of Example 3.

	u	v
s	$(5,3)$	$(9,1)$
t	$(2,14)$	$(7,12)$

	u	v
s	$(-1,10)$	$(4,13)$
t	$(0,8)$	$(6,11)$

For each $(\theta_1, \theta_2) \in \Theta^b$, *the local game* $G_{(\theta_1,\theta_2)}$ *has a unique NE which is given in Figure 2. Hence, G is pure type-regular on* Θ^e. *Since* (s,u) *and* (t,v) *are, respectively, unique NEs for local games* $G_{(5/8,0)}$ *and* $G_{(5/8,1)}$, *respectively, it follows that G is not pure type-regular on* Θ^b, *and, hence, not pure type-regular.*

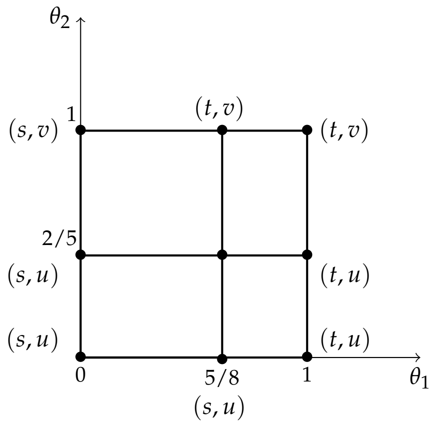

Figure 2. MG of Example 3 is pure type-regular on Θ^e but not pure type-regular on Θ^b.

In the next section, we will derive an algorithm to decide if a MG which is pure type-regular on Θ^e is pure type-regular.

5. Ex post Nash equilibrium analysis

In this section, we first show that if a MG is pure type-regular then the corresponding partition of its type space given by Proposition 4 provides us with an ex post NE. Our aim is then to show that a MG is pure type-regular if it is pure type-regular on the boundary of the type space. In the case of a

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finite number of discrete types, we derive an algorithm, linear in the number of types, to decide if a MG is pure type-regular if it is pure type-regular on its boundary types.

We need a lemma which we first show in the case of DGs with a constructive proof, which provides the intuitive idea behind it.

Lemma 1. *Let G be a 2-agent DG with $A_1 = \{s, t\}$ and $A_2 = \{u, v\}$. Assume that the strategy profiles (s, u) and (s, v) are NEs for $G_{(\theta_1,0)}$ and $G_{(\theta_1,1)}$ for $\theta_1 \in \Theta_1$. Then, there exists $\theta_2^* \in \Theta_2$, independent of $\theta_1 \in \Theta_1$, such that (s, u) is NE for $G_{(\theta_1,\theta_2)}$ if $\theta_2 \leq \theta_2^*$ and (s, v), respectively if $\theta_2 > \theta_2^*$.*

Proof. Assume that the utilities of the two basic games are as in Table 9.

Table 9. Utilities for G_1 (left) and G_2 (right) in Lemma 1.

	u	v
s	(b_1, b_2)	(d_1, d_2)
t	(e_1, e_2)	(f_1, f_2)

	u	v
s	(g_1, g_2)	(h_1, h_2)
t	(k_1, k_2)	(ℓ_1, ℓ_2)

Since (s, u) and (s, v) are NEs for the local games $G_{(\theta_1,0)}$ and $G_{(\theta_1,1)}$ respectively, we have $U_2(s, u, \theta_1, 0) \geq U_2(s, v, \theta_1, 0)$ and also $U_2(s, v, \theta_1, 1) \geq U_2(s, u, \theta_1, 1)$ which imply $b_2 \geq d_2$ and $h_2 \geq g_2$. Let $f(\theta_2) = U_2(s, u, \theta_1, \theta_2) - U_2(s, v, \theta_1, \theta_2)$, i.e., $f(\theta_2) = \theta_2(g_2 - h_2 + d_2 - b_2) + b_2 - d_2$. If $g_2 - h_2 + d_2 - b_2 = 0$ then $f(\theta_2) \geq 0$ for each $\theta_2 \in \Theta_2$ and the strategy profile (s, u) is NE for the local games $G_{(\theta_1,\theta_2)}$ for all $0 \leq \theta_2 \leq 1$. In this case, put $\theta_2^* = 1$. Next, suppose $g_2 - h_2 + d_2 - b_2 \neq 0$. Let

$$\theta_2^* = \frac{b_2 - d_2}{b_2 - d_2 + h_2 - g_2}.$$

Since $b_2 \geq d_2$ and $h_2 \geq g_2$, we have $0 \leq \theta_2^* \leq 1$ and $f(\theta_2) \geq 0$ for $\theta_2 \leq \theta_2^*$ and $f(\theta_2) \leq 0$ for $\theta_2^* < \theta_2$ and the result follows. □

We next obtain a result similar to Lemma 1 for the general case of a MG, which uses a proof by contradiction.

Lemma 2. *Let G be a pure type-regular MG on $\Theta_i^e \times \{\theta_{-i}\}$ for a given agent $i \in I$ and $\theta_{-i} \in \Theta_{-i}$. Then there exists $\Theta_{ij}(\theta_{-i}) \subseteq \Theta_i$, for each $j \in J$, such that*

1. *$\Theta_i = \bigcup_{j \in J} \Theta_{ij}(\theta_{-i})$ where $\Theta_{ij}(\theta_{-i}) \cap \Theta_{ik}(\theta_{-i}) = \emptyset$ for $j, k \in J$ with $j \neq k$, and,*
2. *given $\theta_i \in \Theta_i$, the local game $G_{(\theta_i,\theta_{-i})}$ has a NE for a strategy profile $\left(s_i^*(p_j), s_{-i}^*(\theta_{-i})\right)$, where $j \in J$ is the game for which $\theta_i \in \Theta_{ij}(\theta_{-i})$.*

Proof. Let $(s_i^*(p_j), s_{-i}^*(\theta_{-i})) \in S_i \times S_{-i}$, for agent i and for $j \in J$ be the strategy profile induced from the pure type-regularity of G on $\Theta_i^e \times \{\theta_{-i}\}$. Let $P_j : \Theta_i \to \mathbb{R}$ be the plane

$$P_j(\theta_i) = U_i(s_i^*(p_j), s_{-i}^*(\theta_{-i}), \theta_i, \theta_{-i}),$$

for $j \in J$. For each $k, j \in J$, put

$$T_{jk} := \{\theta_i \in \Theta_i | P_j(\theta_i) - P_k(\theta_i) \geq 0\}.$$

Let $\Theta_{i1}(\theta_{-i}) = \bigcap_{k=1}^m T_{1k}$ and, for $j > 1$,

$$\Theta_{ij}(\theta_{-i}) = \left\{ \theta_i \in \Theta_i \backslash \Theta_{i(j-1)}(\theta_{-i}) \mid \theta_i \in \bigcap_{k=1}^m T_{jk} \right\}.$$

We claim that $\Theta_i = \bigcup_{j \in J} \Theta_{ij}(\theta_{-i})$. Suppose, for a contradiction, that there exists $\theta_i \in \Theta_i \setminus \bigcup_{j \in J} \Theta_{ij}(\theta_{-i})$. Then, there exists $j_1 \in J$ such that $\theta_i \notin T_{1j_1}$ as $\theta_i \notin \Theta_{i1}(\theta_{-i})$. Since $\theta_i \notin \Theta_{ij_1}(\theta_{-i})$, there exists $j_2 \in J$

such that $\theta_i \notin T_{j_1 j_2}$. Inductively, for each integer $r > 2$, there exists $j_r \in J$ such that $\theta_i \notin T_{j_{r-1} j_r}$. Put $j_0 := 1$. Since J is finite, there exist $r, k \geq 0$ with $k < r$ and $j_r = j_k$. Thus,

$$\begin{cases} P_{j_{k+1}}(\theta_i) & - & P_{j_k}(\theta_i) & > & 0 \\ P_{j_{k+2}}(\theta_i) & - & P_{j_{k+1}}(\theta_i) & > & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{j_k}(\theta_i) & - & P_{j_{r-1}}(\theta_i) & > & 0. \end{cases} \tag{1}$$

Adding Inequalities (1) yields: $P_{j_k}(\theta_i) - P_{j_k}(\theta_i) > 0$, which is a contradiction. Therefore, $\Theta_i = \bigcup_{j \in J} \Theta_{ij}(\theta_{-i})$, and by construction, $\Theta_{ij}(\theta_{-i}) \cap \Theta_{ik}(\theta_{-i}) = \emptyset$ for each $j \neq k$. Assume that $\theta_i \in \Theta_{ij}(\theta_{-i})$. Thus, we have

$$U_i(s_i^*(p_j), s_{-i}^*(\theta_{-i}), \theta_i, \theta_{-i}) \geq U_i(s_i^*(p_k), s_{-i}^*(\theta_{-i}), \theta_i, \theta_{-i})$$

for each $k \in J$. Note that, by the definition of pure type-regularity, we have: $A_i = \{s_i^*(p_k) | k \in J\}$. Hence, for each $s_i \in S_i$, we have

$$U_i(s_i^*(p_j), s_{-i}^*(\theta_{-i}), \theta_i, \theta_{-i}) \geq U_i(s_i, s_{-i}^*(\theta_{-i}), \theta_i, \theta_{-i}),$$

i.e., the strategy profile $(s_i^*(p_j), s_{-i}^*(\theta_{-i}))$ is a NE for $G_{(\theta_i, \theta_{-i})}$ where $\theta_i \in \Theta_{ij}(\theta_{-i})$ and $j \in J$. □

Note that the set $\Theta_{ij}(\theta_{-i})$ in the statement of the Lemma 2 can be empty for some $j > 1$. It also follows from the construction of $\Theta_{ij}(\theta_{-i})$ in the proof that if $\Theta_{ij}(\theta_{-i}) = \emptyset$ for some $j > 1$ then $\Theta_{ij'}(\theta_{-i}) = \emptyset$ for $j \leq j' \leq m$. We can now define the notion of a partition of an agent's type set.

Definition 7. *We call a family of subsets $\Theta_{ij}(\theta_{-i}) \subseteq \Theta_i$, for agent i with $j \in J$, satisfying the conditions of Lemma 2 a partition of Θ_i with respect to θ_{-i}.*

Our first theoerm relates to pure type-regularity on the set of extreme types which is immediately obtained from Lemma 2.

Theorem 1. *Let G be a MG which is pure type-regular on Θ^e. Then for each $i \in I$ and $\theta_{-i} \in \Theta^e_{-i}$, there is a partition of Θ_i with*

$$\Theta_i = \bigcup_{j \in J} \Theta_{ij}(\theta_{-i})$$

such that the strategy profile $(s_i^*(p_j), s_{-i}^*(\theta_{-i}))$ is a NE for the local game $G_{(\theta_i, \theta_{-i})}$ where $\theta_i \in \Theta_{ij}(\theta_{-i})$.

Our next theorem is concerned with pure type-regularity on the boundary of the type space. Recall the definition of the strategy function $s_i^*(\cdot)$ for pure type-regularity in Definition 6.

Theorem 2. *Let G be a MG which is pure type-regular on Θ^e. Then the following two conditions are equivalent.*

(i) G is pure type-regular on Θ^b.
(ii) For each agent $i \in I$ and $\theta_{-i} \in \Theta_{-i}$ there exists a partition $\{\Theta_{ij}(\theta_{-i})\}_{j \in J}$ of Θ_i with $\Theta_i = \bigcup_{j \in J} \Theta_{ij}(\theta_{-i})$ such that the set $\Theta_{ij}(\theta_{-i})$ is independent of $\theta_{-i} \in \Theta^e_{-i}$ for each $j \in J$.

Proof. Suppose (ii) holds. Then, $\Theta_i = \bigcup_{j \in J} \Theta_{ij}$ and Theorem 1 implies that the strategy profile $(s_i^*(p_j), s_{-i}^*(\theta_{-i}))$ is a NE for the local game $G_{(\theta_i, \theta_{-i})}$ where $\theta_i \in \Theta_{ij}$ and $\theta_{-i} \in \Theta^e_{-i}$. Hence, G is pure type-regular on Θ^b.

Now, suppose G is pure type-regular on Θ^b. Since G is a uniform MG, $A_i = \{s_i^*(p_k) | k \in J\}$. Thus, by pure type-regularity, for each $\theta_i \in \Theta_i$, there exists $j \in J$ such that the strategy profile $(s_i^*(p_j), s_{-i}^*(\theta_{-i}))$ is a NE for the local game $G_{(\theta_i, \theta_{-i})}$ where $\theta_{-i} \in \Theta^e_{-i}$. Let $\Theta_{i1} \subseteq \Theta_i$ where

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$(s_i^*(p_1), s_{-i}^*(\theta_{-i}))$ is a NE for the local game $G_{(\theta_i,\theta_{-i})}$ for $\theta_i \in \Theta_{i1}$ and $\theta_{-i} \in \Theta_{-i}^e$. Next, for each $1 \le j \le m-1$, iteratively construct $\Theta_{i(j+1)} \subseteq \Theta_i \setminus \bigcup_{1\le k\le j} \Theta_{ik}$ such that for each $\theta_i \in \Theta_{i(j+1)}$, the strategy profile $(s_i^*(p_{j+1}), s_{-i}^*(\theta_{-i}))$ is a NE for the local game $G_{(\theta_i,\theta_{-i})}$ for $\theta_{-i} \in \Theta_{-i}^e$. Since G is pure type-regular on Θ^b it follows that $\Theta_i = \bigcup_{j\in J} \Theta_{ij}$ as required. □

We now present two examples of uniform multi-games, the first is pure type-regular while the second is not.

Example 4. *Suppose G is a 3-agent DG with basic games G_1 and G_2 where $A_i = \{u_i, v_i\}$ for $1 \le i \le 3$ and $\Theta = [0,1]^3$ and assume the utilities of G_1 and G_2 are as in Table 10 and Table 11 respectively.*

Table 10. The utility matrix of Example 4 for the basic game G_1 for $a \in A_3$.

	u_2	v_2
u_1	$(1,10,3)$	$(6,9,9)$
v_1	$(0,11,11)$	$(4,7,10)$

	u_2	v_2
u_1	$(12,3,0)$	$(3,1,6)$
v_1	$(9,13,8)$	$(-1,12,4)$

Table 11. The utility matrix of Example 4 for the basic game G_2 for $a \in A_3$.

	u_2	v_2
u_1	$(1,3,1)$	$(8,6,4)$
v_1	$(2,8,12)$	$(10,20,18)$

	u_2	v_2
u_1	$(5,2,2)$	$(7,8,5)$
v_1	$(8,2,13)$	$(11,5,20)$

Hence, (u_1,u_2,u_3) and (v_1,v_2,v_3) are, respectively, NEs for G_1 and G_2. Figure 3(a) shows the partition of $\Theta = [0,1]^3$ into regions of constant local NE, as described in Lemma 2, by the three planes $\theta_1 = 1/2$ (blue), $\theta_2 = 1/4$ (red) and $\theta_3 = 3/4$ (green) as in Figure 3. The strategy profile (u_1,v_2,u_3) is abbreviated as uvu. The 3-agent DG is pure type-regular and $(s_1^*(\cdot), s_2^*(\cdot), s_3^*(\cdot))$ is an ex post NE for the DG such that $s_1^*(\cdot) : \Theta_1 \to S_1$, $s_2^*(\cdot) : \Theta_2 \to S_2$ and $s_3^*(\cdot) : \Theta_3 \to S_3$ are given by

$$s_1^*(\theta_1) = \begin{cases} u_1 & \theta_1 \le 1/2 \\ v_1 & \theta_1 > 1/2 \end{cases} \quad s_2^*(\theta_2) = \begin{cases} u_2 & \theta_2 \le 1/4 \\ v_2 & \theta_2 > 1/4 \end{cases} \quad s_3^*(\theta_3) = \begin{cases} u_3 & \theta_3 \le 3/4 \\ v_3 & \theta_3 > 3/4 \end{cases}$$

Example 5. *Suppose G is a 2-agent MG with 3 basic games and $A_i = \{u_i, v_i, w_i\}$ for each $i \in I$. The utilities of G_1, G_2 and G_3 have been depicted in Table 12.*

Table 12. Utilities for basic games G_1, G_2 and G_3 of Example 5.

	u_2	v_2	w_2
u_1	$(3,3)$	$(3,2)$	$(3,2)$
v_1	$(2,3)$	$(2,2.5)$	$(2,2)$
w_1	$(2,3)$	$(2,2)$	$(2,1)$

	u_2	v_2	w_2
u_1	$(2,2)$	$(2,3)$	$(2,2)$
v_1	$(3,2)$	$(3,3)$	$(3,1)$
w_1	$(2,2)$	$(2,3)$	$(2,1)$

	u_2	v_2	w_2
u_1	$(2,2)$	$(2,2)$	$(1,3)$
v_1	$(2,2)$	$(2,2)$	$(2,3)$
w_1	$(3,2)$	$(3,1)$	$(3,3)$

The strategy profiles (u_1,u_2), (v_1,v_2) and (w_1,w_2) are respectively NEs for the basic games G_1, G_2 and G_3. Table 13 gives the NE for all local games and shows that G is pure type-regular on Θ^e.

Table 13. NE of $G_{(\theta_1,\theta_2)}$ for all $(\theta_1,\theta_2) \in \Theta^e$

	p_1	p_2	p_3
p_1	(u_1,u_2)	(u_1,v_2)	(u_1,w_2)
p_2	(v_1,u_2)	(v_1,v_2)	(v_1,w_2)
p_3	(w_1,u_2)	(w_1,v_2)	(w_1,w_2)

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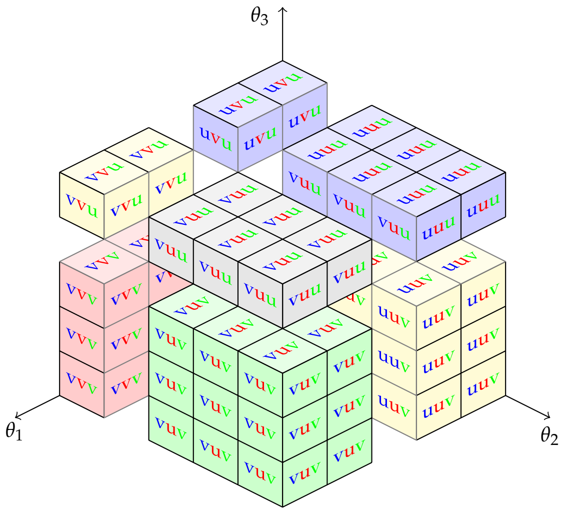
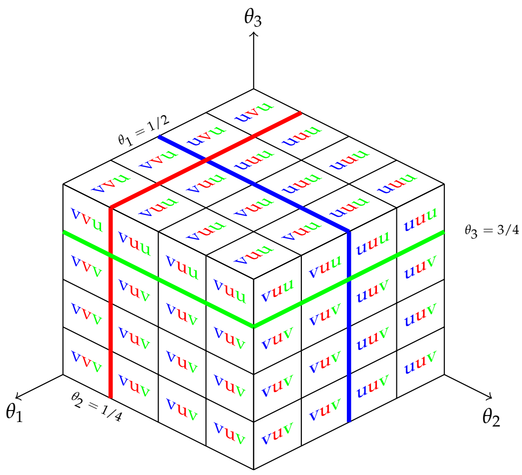


(a)



(b)

Figure 3. The partitioning of Θ in Example 4 into regions of constant local NE for G.

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The partition of Θ_i for each $1 \le i \le 3$, as given by Corollary 1, is illustrated in Figure 4. From these partitions, we can see that G is not pure type-regular. In fact, let $\theta_2 = (1/4, 1/4, 1/2)$. We have: $\theta_2 \in \Theta_{23}(p_1)$ and $\theta_2 \notin \Theta_{23}(p_2) \cup \Theta_{23}(p_3)$. Therefore, $\Theta_{23}(\theta_1)$ is not independent of $\theta_1 \in \Theta^e_1$ and, thus, by Theorem 2, G is not pure type-regular on Θ^b.

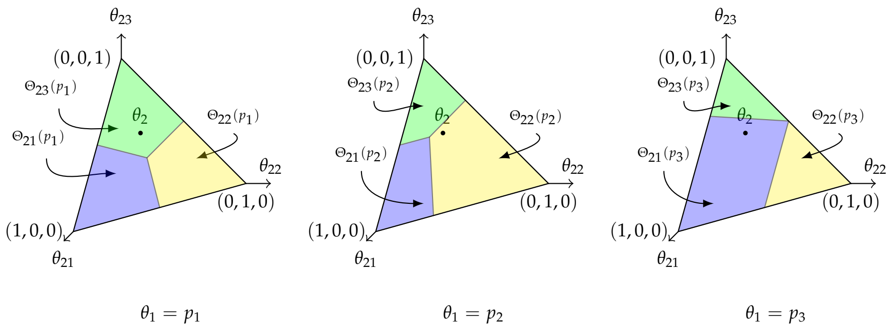

Figure 4. The MG of Example 5 which is not pure type-regular by Theorem 2. The type $\theta_2 = (1/4, 1/4, 1/2)$ shows that the partition of Θ_2 is not independent of $\theta_1 \in \Theta^e_1$.

We can now deduce one of our main results for a MG.

Theorem 3. *A multi-game is pure type-regular if and only if it is pure type-regular on the boundary Θ^b of the type space Θ.*

Proof. The "only if" part follows immediately from the definition of a pure type-regularity in Definition 6. Suppose the multi-game G is pure type-regular on Θ^b. Then, for each $i \in I$, there exists $s^*_i : \Theta_i \to S_i$ and $s^*_{-i} : \Theta^e_{-i} \to S_{-i}$ such that $(s^*_i(\theta_i), s^*_{-i}(\theta_{-i}))$ is a NE for $G_{(\theta_i, \theta_{-i})}$ for each $(\theta_i, \theta_{-i}) \in \Theta_i \times \Theta^e_{-i}$. Since $A_i = \{s^*_i(p_j) | j \in J\}$, there exists $j_i \in J$ such that $s^*_i(\theta_i) = s^*_i(p_{j_i})$. Thus, there exists $j_r \in J$, for each $r \in I$, such that $(s^*_1(\theta_1), \ldots, s^*_n(\theta_n)) = (s^*_1(p_{j_1}), \ldots, s^*_n(p_{j_n}))$, for each $(\theta_1, \ldots, \theta_n) \in \Theta$. We claim $(s^*_1(\theta_1), \ldots, s^*_n(\theta_n))$ is a NE for $G_{(\theta_1, \ldots, \theta_n)}$ for each $(\theta_1, \ldots, \theta_n) \in \Theta$. We have

$$U_i(s^*_i(\theta_i), s^*_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) = U_i(s^*_i(p_{j_i}), s^*_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) = \sum_{j \in J} \theta_{ij} U_{ij}(s^*_i(p_{j_i}), s^*_{-i}(\theta_{-i})).$$

Suppose $i \in I$ and $s'_i \in S_i$. Since $(s^*_i(\theta_i), s^*_{-i}(\theta_{-i}))$ is a NE for the local game $G_{(p_{j_1}, \ldots, p_{j_{i-1}}, \theta_i, p_{j_{i+1}}, \ldots, p_{j_n})}$, we have $\sum_{j \in J} \theta_{ij} U_{ij}(s^*_i(p_{j_i}), s^*_{-i}(\theta_{-i})) \ge \sum_{j \in J} \theta_{ij} U_{ij}(s'_i, s^*_{-i}(\theta_{-i}))$, for each $s'_i \in S_i$. Hence $U_i(s^*_i(\theta_i), s^*_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) \ge U_i(s'_i, s^*_{-i}(\theta_{-i}), \theta_i, \theta_{-i})$.
□

From Theorem 3 and Theorem 2, we obtain:

Corollary 2. *Let G be a MG pure type-regular on Θ^e. Then G is pure type-regular if for each $i \in I$ and $j \in J$ the set $\Theta_{ij}(\theta_{-i})$ in the partition of Θ_i is independent of $\theta_{-i} \in \Theta^e_{-i}$.*

Assume now that the type space Θ_i is finite for each agent $i \in I$. Based on Corollary 2, we can derive an algorithm to check if a MG, which is pure type-regular on Θ^e, is also pure type-regular on Θ^b and is hence pure type-regular by Theorem 3. If the strategy profile function $\{(s^*_i(p_1), \ldots, s^*_i(p_m))\}_{i \in I}$ is a witness for pure type-regularity of G on Θ^e, we define:

$$\mathsf{T}(i; (s^*_i(p_1), \ldots, s^*_i(p_m); \theta_{-i}) = \prod_{j \in J} \Theta_{ij}(\theta_{-i}),$$

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Algorithm 1: Pure type-regularity Test for G pure type-regular on Θ^e

Input : $\{(s_i^*(p_1), \ldots, s_i^*(p_m))\}_{i \in I}$ a witness for pure type-regularity of G on Θ^e

$K = 1$;
for $i \leftarrow 1$ **to** n **do**
  $C = T(i; (s_i^*(p_1), \ldots, s_i^*(p_m)); (p_1, \ldots, p_1))$;
  for $\theta_{-i} \in \Theta^e_{-i} \backslash \{(p_1, \ldots, p_1)\}$ **do**
    if $C \neq T(i; (s_i^*(p_1), \ldots, s_i^*(p_m)); \theta_{-i})$ **then**
      $K = 0$
    end
  end
end
if $K = 1$ **then**
  G is Type-regular.
end

where $\Theta_{ij}(\theta_{-i})$ for $j \in J$ is given in Theorem 1. Consider now Algorithm 1. The number of calls to $\mathsf{T}(i; (s_i^*(p_1), \ldots, s_i^*(p_m)); \theta_{-i})$ is nm^{n-1}. Moreover, the runtime of T for $\theta_{-i} \in \Theta^e_{-i}$ is $O(|\Theta_i|)$. Thus, the computational complexity of Algorithm 1 is $O(knm^{n-1})$ where $k = \max_{i \in I} |\Theta_i|$. Therefore, for small n and m as we have in applications, Algorithm 1 is linear with respect to the size of the type space.

As an example, Figure 5 illustrates a 2-agent multi-game G with 3 basic games and $\Theta_1 = \Theta_2 = \Delta^2$ where the types of each agent are shown by small discs. From the figure, we see that, for each $i \in I$ and $j \in J$, the set $\Theta_{ij}(\theta_{-i})$ is independent of θ_{-i}. Theorem 2 implies that G is pure type-regular on Θ^b. Hence G is pure type-regular by Theorem 3 and its ex post NE can be efficiently computed by Algorithm 1.

6. Ex post NEs in some applications of MGs

In this section, we analyse the three uniform MGs introduced in Section 3 and derive their ex post NE.

6.1. DG for Prisoner's Dilemma

We now analyse the DG for PD, which was described in Subsection 3.1. Given the utilities of the social game in the DG for PD, as well as the standard utilities of the PD, there are two cases for the partition of the type space $[0,1]^2$ depending on the relative value of two parameters

$$\mu := \frac{p - s}{y + p - 2s} \qquad \lambda := \frac{t - r}{t - s + y - r}$$

namely: (i) $\mu < \lambda$ or (ii) $\mu \geq \lambda$. These two cases are broadly similar and we will only consider (i). In this case, the partition of the type space is given in Figure 6. It is clear from the figure that if $\Theta_1, \Theta_2 \subseteq [0, \mu] \cup [\lambda, 1]$ then the DG is pure type-regular. However, pure type-regularity (and thus pure type-regularity on the boundary) is lost if $\Theta_1 \cap (\mu, \lambda) \neq \emptyset$ or $\Theta_2 \cap (\mu, \lambda) \neq \emptyset$.

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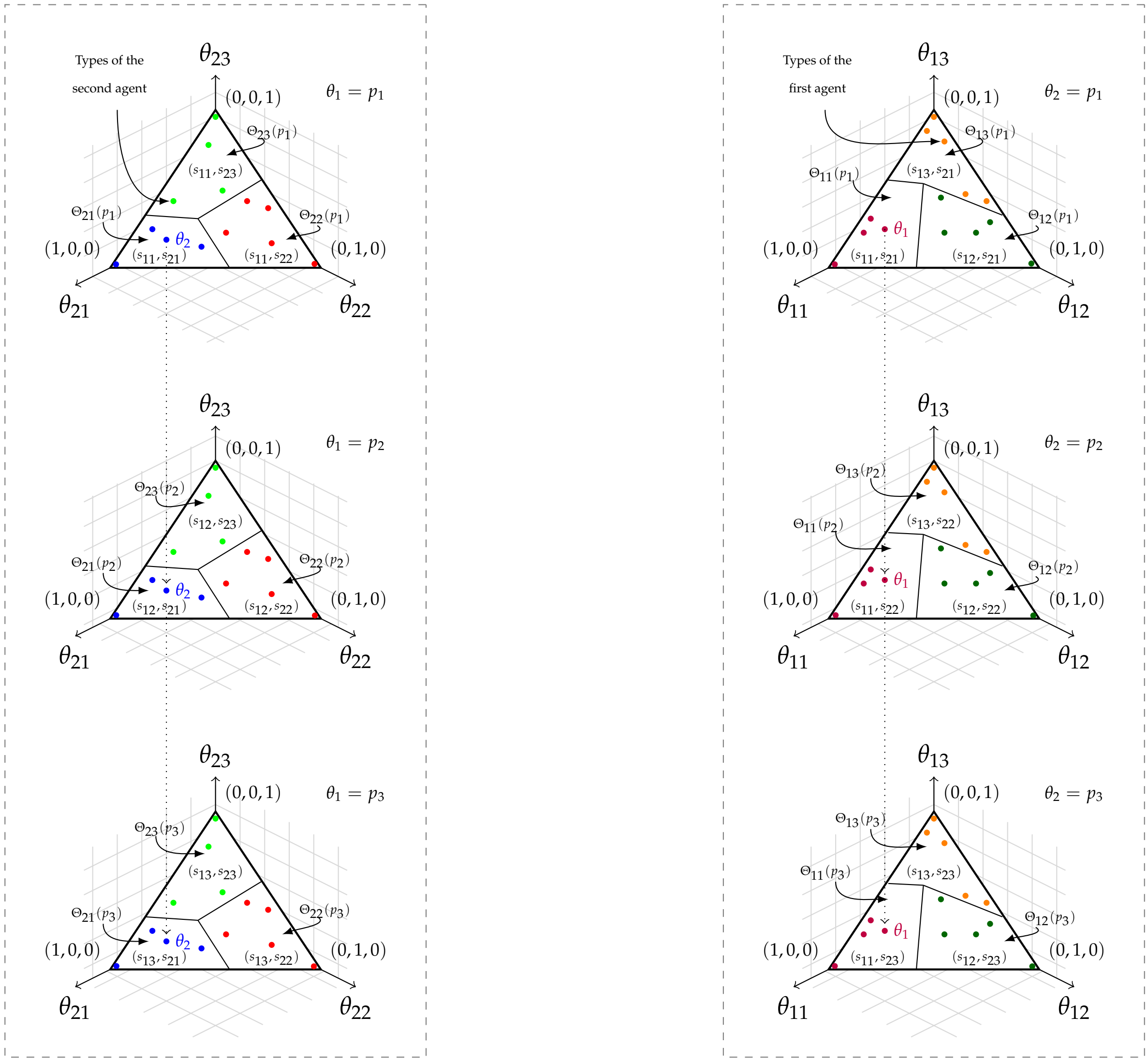

(a) **(b)**

Figure 5. The partitioning of Θ^b in a 2-agent MG with three games which shows that the MG is pure type-regular. The types of each agent in the three partitionings of its type space are shown in three different colours.

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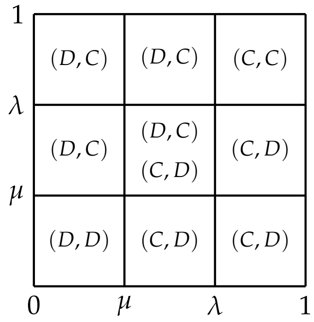

Figure 6. NEs in the type space of DG for PD ($\mu < \lambda$)

6.2. Adoption of technology

Consider a DG with two basic games G_1 and G_2 whose utilities have been depicted in Table 5 in Subsection 6.2 and $\Theta_i = [0,1]$. Assume $k_1 > g_1$, $d_1 > f_1$, $h_2 > g_2$, and $e_2 > f_2$. These inequalities guarantee that the DG is pure type-regular on Θ^e. Furthermore, suppose $(k_1 - g_1)/(b_1 - e_1) = (\ell_1 - h_1)/(d_1 - f_1)$ and $(h_2 - g_2)/(b_2 - d_2) = (\ell_2 - k_2)/(e_2 - f_2)$. Then the DG is pure type-regular on the boundary. Theorem 3 implies that the DG is pure type-regular and has an ex post NE. If $b_1 - e_1 + k_1 - g_1 = 0$ then let $\theta_1^* = 1$ otherwise let $\theta_1^* = (b_1 - e_1)/(b_1 - e_1 + k_1 - g_1)$. Similarly, if $b_2 - d_2 + h_2 - g_2 = 0$ let $\theta_2^* = 1$ otherwise let $\theta_2^* = (b_2 - d_2)/(b_2 - d_2 + h_2 - g_2)$. Hence, $(s_1(\cdot), s_2(\cdot))$ is an ex post NE where $s_i(\cdot) : \Theta_i \to \{a, r\}$ is given by:

$$s_i(\theta_i) = \begin{cases} a & \text{if} \quad \theta_i \le \theta_i^* \\ r & \text{if} \quad \theta_i > \theta_i^*. \end{cases}$$

6.3. Multinational companies

We present a numerical example which models the competition of two firms in three markets with a uniform MG. Assume the three markets are represented by three basic games G_1, G_2 and G_3 whose utilities are shown in Table 6 in Subsection 3.3 with $\Theta_i = \Delta^2$ and $S_i = \{s_j \mid j \in J\}$. Recall that for each $1 \le j \le 3$, the strategy profile (s_j, s_j) is a NE for the basic game G_j. For each agent $i = 1, 2$, let $s_i^*(\cdot) : \Theta_i^e \to S_i$ be given by $s_i^*(p_j) = s_j$ for each $1 \le j \le 3$. For each vector $x \in \mathbb{R}^m$, we say $x = (x_1, \cdots, x_m) \ge 0$ if $x_r \ge 0$ for each $0 \le r \le m$, and we denote the transpose of x by x^t.

Using, step by step, the method of proof in Lemma 2, we will show that the MG has an ex post NE. We start by putting $i = 1$ and $\theta_{-i} = p_1$. Let $P_j : \Delta^2 \to \mathbb{R}$ be the plane $P_j(\theta_1) = U_1(s_1^*(p_j), s_2^*(p_1), \theta_1, p_1)$, for $1 \le j \le 3$. Thus, $P_1(\theta_1) = \theta_{11} + 2\theta_{12} + 2$, $P_2(\theta_1) = -2\theta_{11} + \theta_{12} + 4$ and $P_3(\theta_1) = -2\theta_{11} + 3$ where $\theta_1 = (\theta_{11}, \theta_{12}, 1 - \theta_{11} - \theta_{12})$. Put $T_{jk} := \{\theta_1 \in \Delta^2 | P_j(\theta_1) - P_k(\theta_1) \ge 0\}$, for each $1 \le k, j \le 3$ and let $\Theta_{11}(p_1) = \bigcap_{k=1}^3 T_{1k}$ and $\Theta_{1j}(p_1) = \left\{\theta_1 \in \Delta^2 \backslash \Theta_{1(j-1)}(p_1) \mid \theta_1 \in \bigcap_{k=1}^3 T_{jk}\right\}$ for $j = 2, 3$. Then we obtain:

$$\begin{aligned} \Theta_{11}(p_1) &= \{\theta_1 \in \Delta^2 \mid A_{11}\theta_1^t \ge 0\} \\ \Theta_{12}(p_1) &= \{\theta_1 \in \Delta^2 \mid A_{12}\theta_1^t \ge 0\} \backslash \Theta_{11}(p_1) \\ \Theta_{13}(p_1) &= \{\theta_1 \in \Delta^2 \mid A_{13}\theta_1^t \ge 0\} \backslash \Theta_{11}(p_1) \cup \Theta_{12}(p_1) \end{aligned}$$

where

$$A_{11} = \begin{bmatrix} 1 & -1 & -2 \\ -2 & 2 & 2 \end{bmatrix} \qquad A_{12} = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \qquad A_{13} = \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \end{bmatrix}$$

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We have: $\Theta_1 = \bigcup_{j \in J} \Theta_{1j}(p_1)$. Repeating the above computation with $\theta_{-i} = p_2$ and $\theta_{-i} = p_3$, we find that $\Theta_{1j}(p_1) = \Theta_{1j}(p_2) = \Theta_{1j}(p_3)$ for each $1 \le j \le 3$; thus $\Theta_1 = \bigcup_{1 \le j \le 3} \Theta_{1j}(p_1)$ is independent of p_1. Similarly, there exists a partition $\Theta_2 = \bigcup_{1 \le j \le 3} \Theta_{2j}(p_1)$ such that

$$\begin{aligned}
\Theta_{21}(p_1) &= \{\theta_2 \in \Delta^2 \mid A_{21}\theta_2^t \ge 0\}, \\
\Theta_{22}(p_1) &= \{\theta_2 \in \Delta^2 \mid A_{22}\theta_2^t \ge 0\} \backslash \Theta_{21}(p_1), \\
\Theta_{23}(p_1) &= \{\theta_2 \in \Delta^2 \mid A_{23}\theta_2^t \ge 0\} \backslash \Theta_{21}(p_1) \cup \Theta_{22}(p_1),
\end{aligned}$$

where

$$A_{21} = \begin{bmatrix} 3 & -3 & -8 \\ 4 & -1 & -8 \end{bmatrix} \qquad A_{22} = \begin{bmatrix} -3 & 3 & 8 \\ 1 & 2 & -4 \end{bmatrix} \qquad A_{23} = \begin{bmatrix} -1 & -2 & -4 \\ -4 & 1 & 8 \end{bmatrix}.$$

Moreover, $\Theta_{2j}(p_1)$ is independent of $p_1 \in \Theta_2^e$ for each $1 \le j \le 3$. Hence, G is pure type-regular on the boundary of its type profile space. It now follows from Theorem 2 that this MG is pure type-regular. The strategy map profile $(s_1(\cdot), s_2(\cdot))$ is an ex post NE for G where $s_i(\cdot) : \Theta_i \to \{s_1, s_2, s_3\}$ is given by:

$$s_i(\theta_i) = \begin{cases} s_1 & \text{if} \quad \theta_i \in \Theta_{i1} \\ s_2 & \text{if} \quad \theta_i \in \Theta_{i2} \\ s_3 & \text{if} \quad \theta_i \in \Theta_{i3}. \end{cases}$$

7. Multi-games with multi-stage basic games

In this section, we will show how we can develop a multi-stage MG to provide a more realistic model for the behaviour of human beings when they play the well-known trust game. This approach presents an alternative Bayesian model compared to that of Chaudhuri and Gangadharant [24] as mentioned in the Introduction. We first formally recall the Trust Game.

Berg et al. [23] designed an experiment, called the Trust Game, to measure trust in economic decisions by human agents [23]. The Trust Game is played with two agents, who are both given initially some equal amount of money, and an experimenter. The game is played as follows: In stage one, the first agent is asked to send some of the money it has been given to the second agent even though the amount sent can be zero. When the first agent chooses an amount to send to the second agent, the experimenter will triple the money and send the tripled amount to the second agent. In stage two, the second agent must decide to send some of the money it has received from the experimenter back to the first agent. The NE for the Trust Game is $(0, 0)$. However, experiments reported in [23] show that in actual fact the first agents, on average, do send a proportion of their endowment and that the second agents, on average, send back at least the amount sent by the first agents. We develop a stage DG, which includes the Trust Game and a conscience game with moral and social utilities, to model the actual behaviour of human agents.

Example 6. The Trust Game *Consider a two-agent stage game G_1 in which $A_1 = [0, 1]$, $A_2 = \{x | 3y \ge x, y \in A_1\}$ and $u_1(y, x) = x - y$, $u_2(y, x) = 3y - x$ for $y \in A_1$ and $x \in A_2$. By backward induction, when the first agent plays first, $(0, 0)$ is the Nash equilibrium. If, for the sake of illustration, we restrict agent 1's actions to $A_1' = \{0, 1\}$, then Figure 7 shows the branches of the stage game where the two agents are named a_1 and a_2, respectively. As usual, the label on each edge is the action taken by the agent on the node above and, under each leaf, the first number is the utility of agent 1 for the branch corresponding to the leaf and the second number is agent 2's utility.*

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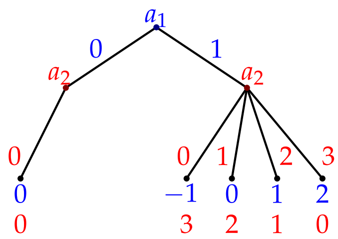

Figure 7. Trust Game

Under the standard economic assumption of rational self-interest, the predicted actions of the first agent in the Trust Game will be to send nothing, and any behaviour that deviates from this self-interest is viewed as irrational. Since in actual experiments, individuals significantly deviate from this Nash equilibrium, we argue that, as well as their material interest, they seek to build or protect their social reputation or their own ethical and pro-social values. We thus propose to develop a more realistic model of trust in economic behaviour by using a DG which includes the Trust Game above and a second social or conscience game a follows.

Example 7. DG for Trust Game *Let G_1 be the Trust Game as in Example 6 and let G_2 be the associated* conscience *game in which $A_1 = [0, 1]$, $A_2 = \{x | 3y \geq x, y \in A_1\}$ and*

$$u_1(y, x) = y \quad u_2 = x - 2y$$

for $y \in A_1$ and $x \in A_2$. By backward induction, $(1, 3)$ is the NE. If again, we restrict agent 1's actions to $A'_1 = \{0, 1\}$, then Figure 8 shows the branches of the stage game.

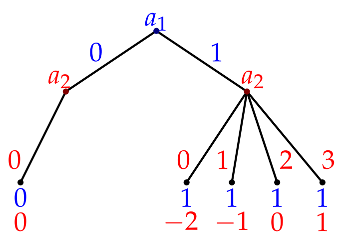

Figure 8. Conscience Game

Consider a double game G with basic games G_1 and G_2 where $A_1 = [0, 1]$, $A_2 = \{x | 3y \geq x, y \in A_1\}$, $\Theta_1 = \{1/4\}$ and $\Theta_2 = \{0, 2/3\}$. We have $U_1(y, x, 1/4, 2/3) = 3/4x - 1/2y$ and $U_2(y, x, 1/4, 2/3) = 1/3x - 1/3y$. Thus, $\arg\max_x U_2(y, x, 1/4, 2/3) = \{3y\}$. Put

$$s_2(\theta_2) = \begin{cases} 0 & \theta_2 = 0 \\ 3y & \theta_2 = 2/3 \end{cases}$$

We have

$$U_1(y, 0, 1/4, \theta_2) = -y/2 \qquad U_1(y, 3y, 1/4, \theta_2) = 7y/4.$$

As a result, $U_1(y, s_2(\theta_2)(y)) = p_0(-9y/4) + 7y/4$. Therefore

$$s_1(1/4) = \arg\max_y U_1(y, s_2(\theta_2)(y)) = \begin{cases} 1 & p_0 < 7/9 \\ y & p_0 = 7/9 \\ 0 & p_0 > 7/9 \end{cases}$$

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Hence $(s_1(1/4), s_2(\theta_2))$ *is a sub-game perfect equilibrium for the DG. We now see that, depending on its belief about agent 2, agent 1 can send any amount of money to agent 2 and agent 2 can return different amounts of money as an optimal solution for the trust DG.*

8. Conclusion

We have proposed multi-games to model human rational-social decision making and, more generally, for decision making by agents investing with their individual weights in multiple environments or markets that are considered as basic games. We have developed the notion of pure type-regularity of a multi-game on a subset of types in uniform multi-games, in which actors play the same strategy in all basic games. We have shown that a pure ex post NE for a multi-game which is pure type-regular on the space of types can be computed efficiently and have constructed an algorithm, linear in the number of types, to check if a multi-game is pure type-regular. We have presented applications in the PD, the Trust Game and in the global economy.

We envisage applications of multi-games for multi-agent systems in a variety of contexts [34–36]. In addition, future work considers extensions of the results in this paper to mixed ex post NE and extensions of multi-games to the case for which the utilities of each agent depend affinely or piecewise affinely on its types and to the case when the utilities of each agent depend linearly or affinely on the types of all agents. A framework for implementing multi-games in networks is also considered.

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