Article

Not peer-reviewed version

Nonlinear Hybrid Second-Order Neutral Delay Differential Equation with Mixed Sign Terms:Oscillation via Symmetric Transform

Ganesh Purushothaman , Pandarinathan Nandakumar , George E. Chatzarakis * , Ethiraju Thandapani

Posted Date: 2 September 2025

doi: 10.20944/preprints202509.0256.v1

Keywords: second-order; nonlinear neutral hybrid equation; symmetric transform; oscillation

Article

Nonlinear Hybrid Second-Order Neutral Delay Differential Equation with Mixed Sign Terms:Oscillation via Symmetric Transform

Ganesh Purushothaman [1], Pandarinathan Nandakumar[2], George E. Chatzarakis [3,*] and Ethiraju Thandapani [4]

[1] Department of Mathematics, St. Joseph's College of Engineering, Chennai 600119, India
[2] Department of Mathematics, Perunthalaivar Kamarajar Institute of Engineering and Technology,(Constituent College of Puducherry Technological University), Karaikal 609603, India
[3] Department of Electrical and Electronic Engineering Educators, School of Pedagogical and Technological Education, Marousi, 15122 Athens, Greece
[4] Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai 600005, India
* Correspondence: gea.xatz@aspete.gr

Abstract

This paper investigates the oscillatory behavior of solutions to a class of second-order nonlinear neutral delay differential equations with both positive and negative terms.To facilitate the analysis, the equation is transformed into an equivalent binomial form using a positive solution of an auxiliary second-order ordinary differential equation. The comparison method and integral averaging technique combined with the arithmetic-geometry mean inequality are employed to establish new sufficient conditions for the oscillation of all solutions. The obtained results extend and improve several existing criteria in the literature. Finally, illustrative examples are provided to demonstrate the effectiveness, novelty, and applicability of the proposed oscillation criteria.

Keywords: second-order; nonlinear neutral hybrid equation; symmetric transform; oscillation

MSC: 34C10; 34K11

.

1. Introduction

Second-order neutral differential equations with positive and negative terms are important for modeling complex systems in engineering and science, particularly where delays occur in the system dynamics, such as in high-speed electrical networks, biological population models, and mechanical systems. The inclusion of both positive and negative coefficients allows for more realistic and intricate descriptions of these phenomena, leading to diverse behaviors such as oscillations or stability, which are crucial for understanding and predicting the system's long-term behavior. The applicability of these equations extends to diverse fields, including control theory, population dynamics, neuroscience, and, more recently, the study of complex networks and cyber-physical systems. Their ability to capture intricate temporal dynamics makes them a powerful tool for both theoretical analysis and practical applications (see [1,2]).

Motivated by these facts, in this paper we focus on the oscillatory properties of solutions of the second-order nonlinear hybrid-type neutral delay differential equation of the form

$$\big(a(t)z'(t)\big)' - p(t)x(t) + q(t)x^{\alpha}(\sigma(t)) = 0, \quad t \geq t_0, \tag{E}$$

where $z(t) = x(t) + b(t)x(\tau(t))$, subject to the following conditions:

(H1) $p(t), q(t), b(t) \in C([t_0, \infty), \mathbb{R})$, with $0 \leq b(t) < 1$, $p(t) > 0$, and $q(t) \geq 0$;

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 September 2025 doi:10.20944/preprints202509.0256.v1

(H2) $a(t) \in C'([t_0, \infty), (0, \infty))$ and α is a ratio of odd positive integers;
(H3) $\sigma(t), \tau(t) \in C([t_0, \infty), \mathbb{R})$, with $\sigma(t) \leq t, \tau(t) \leq t$,

$$\lim_{t\to\infty} \sigma(t) = \lim_{t\to\infty} \tau(t) = \infty,$$

and let $\delta(t) = \min\{\tau(t), \sigma(t)\}$.
As usual, by a proper solution of equation (E) we mean a function that satisfies (E) for all sufficiently large t and for which $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq t_0$. A proper solution is called oscillatory if it has infinitely many zeros; otherwise, it is called nonoscillatory.

Note that if $p(t) = 0$, then (E) reduces to the second-order nonlinear neutral delay differential equation

$$\left(a(t)z'(t)\right)' + q(t)x^{\alpha}(\sigma(t)) = 0, \quad t \geq t_0 \tag{E_1}$$

and if $q(t) = 0$, then we obtain the linear neutral differential equation

$$\left(a(t)z'(t)\right)' - p(t)x(t) = 0, \quad t \geq t_0. \tag{E_2}$$

Therefore, we may refer to equation (E) as a hybrid-type nonlinear neutral differential equation.

These equations model systems in which the state at a given time depends not only on its current behavior but also on its past states. The "neutral" term, where the derivative also depends on delayed versions of the unknown function, captures these memory effects. The presence of both positive and negative coefficients allows for the modeling of feedback mechanisms within the system. This can lead to a richer array of behaviors, including stable, unstable, or oscillatory dynamics observed in many natural and engineered systems (see [3,4] and the references therein).

A significant focus in the study of these equations is the analysis of oscillation of their solutions. The presence of positive and negative terms can lead to solutions that oscillate around an equilibrium point, a critical characteristic in many physical and biological phenomena. Oscillatory solutions are particularly important as they model repeating behaviors found in natural and engineered systems, such as waves, vibrations, and population dynamics. Studying these solutions provides insight into system stability and behavior, allowing scientists and engineers to predict and understand phenomena across physics, biology, and engineering. By analyzing oscillatory solutions, one can determine the conditions under which systems oscillate, thereby aiding in the design and control of such systems.

Because of their practical importance, many researchers have studied oscillation criteria for second-order neutral differential equations of different types. Numerous results have appeared in the literature concerning oscillation and asymptotic behavior of second-order differential equations with linear, sub-linear, and super-linear neutral terms (see the monographs [5,6] and the papers [7–24]).

Oscillation criteria for second-order differential equations involving both positive and negative terms remain a topic of ongoing research. The presence of both positive and negative terms complicates the analysis, as the structure of the set of nonoscillatory solutions of the hybrid neutral equation (E) is not well understood. Hence, only a limited number of criteria have been developed to study the oscillatory behavior of such equations.

In [3,25,26], the authors examined the oscillation and nonoscillation of the first-order neutral differential equation

$$(x(t) - R(t)x(t-r))' + P(t)x(t-\tau) - Q(t)x(t-\sigma) = 0, \tag{E_3}$$

using the integral averaging method and fixed point theorems. Furthermore, second-order neutral differential equations of the form

$$(r(t)(x(t) \pm C(t)h(x(t-\tau)))')' + P(t)f(x(t-\delta)) - Q(t)g(x(t-\sigma)) = 0 \tag{E_4}$$

and their generalizations were studied in [3,27–32] via the comparison method and fixed point theorems.

More recently, in [33], the authors investigated oscillation criteria for the second-order linear neutral differential equation

$$\left(a(t)z'(t)\right)' + p(t)z(t) + q(t)x(\sigma(t)) = 0, \quad t \geq t_0, \tag{E_5}$$

where $z(t) = x(t) + b(t)x(\tau(t))$, using the Riccati transformation technique and the integral averaging method.

Observe that the equations considered in [3,27–32] involve deviating arguments in both positive and negative terms, while the equation considered in [33] is linear and contains only positive terms. However, our equation (E) is nonlinear and contains a positive term with delay and a negative term without delay. Therefore, equation (E) is fundamentally and structurally different from those previously examined in the literature.

Moreover, we employ a different method to transform the trinomial hybrid equation into a binomial form, which allows a clear identification of the structure of nonoscillatory solutions of the transformed equation. This symmetry transformation is crucial for establishing oscillation criteria for equation (E) since it preserves the oscillation of original equation and the transformed equation.

By employing the comparison method, integral averaging technique and arithmetic-geometric mean inequality, we derive new sufficient conditions ensuring the oscillation of all solutions of the transformed binomial-type equation, which in turn guarantees the oscillation of all solutions of equation (E). Three illustrative examples are provided to demonstrate the applicability and novelty of the main results. Notably, for one illustrative example, an explicit solution is obtained and shown to be oscillatory.

2. Preliminary Results

In this section, we present some preliminary results that will be needed to establish our main theorems. First, we transform the hybrid equation (E) into a binomial-type equation using a positive solution of the related auxiliary second-order linear ordinary differential equation

$$(a(t)u'(t))' = p(t)u(t), \quad t \geq t_0. \tag{1}$$

Lemma 1. *Let (H_1) and (H_2) hold. Then, equation (1) has a positive decreasing solution on $[t_0, \infty)$.*

Proof. The proof is similar to Theorem 2.46 of [34], and hence the details are omitted. □

Next, note that equation (E) can be rewritten in the equivalent form

$$\left(a(t)z'(t)\right)' - p(t)z(t) + p(t)b(t)x(\tau(t)) + q(t)x^{\alpha}(\sigma(t)) = 0. \tag{2}$$

Our next result is based on an equivalent representation of the differential operator

$$L(z(t)) = \left(a(t)z'(t)\right)' - p(t)z(t) \tag{3}$$

in terms of a positive solution $u(t)$ of (1).

Lemma 2. *Let $u(t)$ be a positive decreasing solution of (1) on $[t_0, \infty)$. Then the operator (3) can be represented as*

$$L(z(t)) = \frac{1}{u(t)}\left(a(t)u^2(t)\left(\frac{z(t)}{u(t)}\right)'\right)'. \tag{4}$$

Proof. By direct computation, we obtain

$$\begin{aligned} L(z(t)) &= \frac{1}{u(t)}\big(a(t)u(t)z'(t) - a(t)z(t)u'(t)\big)' \\ &= \big(a(t)z'(t)\big)' - \big(a(t)u'(t)\big)'\frac{z(t)}{u(t)} \\ &= \big(a(t)z'(t)\big)' - p(t)z(t), \end{aligned}$$

where we have used the fact that $u(t)$ is a positive solution of (1). The proof of the lemma is complete. □

Lemma 3. *Let $u(t)$ be a positive decreasing solution of (1) on $[t_0, \infty)$. Then equation (E) can be written in the form*

$$\big(\eta(t)\mu'(t)\big)' + u(t)p(t)b(t)x(\tau(t)) + u(t)q(t)x^{\alpha}(\sigma(t)) = 0, \tag{5}$$

where

$$\eta(t) = a(t)u^2(t), \quad \mu(t) = \frac{z(t)}{u(t)}.$$

Proof. The result follows immediately by combining (2) with (4). □

For our investigation, it is convenient to assume that (5) is in canonical form, i.e.,

$$\int_{t_0}^{\infty} \frac{1}{\eta(t)}\,dt = \int_{t_0}^{\infty} \frac{1}{a(t)u^2(t)}\,dt = \infty. \tag{6}$$

Next, we establish the structure of nonoscillatory solutions of (5) when condition (6) holds.

Lemma 4. *Let $u(t)$ be a positive decreasing solution of (1) on $[t_0, \infty)$ satisfying (6). If $x(t)$ is a positive solution of (E), then the corresponding function $\mu(t)$ satisfies*

$$\mu(t) > 0, \quad \eta(t)\mu'(t) > 0, \quad \big(\eta(t)\mu'(t)\big)' \le 0 \tag{7}$$

for all $t \ge t_1 \ge t_0$.

Proof. Let $x(t)$ be an eventually positive solution of (E), i.e., $x(t) > 0$, $x(\sigma(t)) > 0$, and $x(\tau(t)) > 0$ for all $t \ge t_1$ for some $t_1 \ge t_0$. Then, by definition, $z(t) > 0$ for all $t \ge t_1$. Since $u(t) > 0$ for all $t \ge t_0$, it follows that $\mu(t) = \frac{z(t)}{u(t)} > 0$ for all $t \ge t_1$. From (5), we have $\big(\eta(t)\mu'(t)\big)' \le 0$ for all $t \ge t_1$. Moreover, by condition (6), it follows that $\eta(t)\mu'(t) > 0$ for all $t \ge t_1$. a positive solution of (1). The proof of the lemma is complete. □

Lemma 5. *Let $u(t)$ be a positive decreasing solution of (1) on $[t_0, \infty)$ satisfying (6). Then*

$$x(t) \ge u(t)F(t)\mu(t), \quad t \ge t_1 \ge t_0, \tag{8}$$

where

$$F(t) = 1 - b(t)\frac{u(\tau(t))}{u(t)} > 0.$$

Proof. From the definition of $\mu(t)$ and its monotonicity, we have

$$u(t)\mu(t) = z(t) = x(t) + b(t)x(\tau(t)).$$

Thus,

$$x(t) \ge u(t)\mu(t) - b(t)u(\tau(t))\mu(\tau(t))$$

 doi:10.20944/preprints202509.0256.v1

and therefore,

$$x(t) \geq u(t)\left(1 - b(t)\frac{u(\tau(t))}{u(t)}\right)\mu(t),$$

which proves the result. □

Before stating and proving the next result, let us define

$$Q_1(t) = u(t)b(t)p(t)u(\tau(t))F(\tau(t)), \quad Q_2(t) = u(t)q(t)u^{\alpha}(\sigma(t))F^{\alpha}(\sigma(t)),$$
$$Q_3(t) = Q_1(t) + Q_2(t), \quad \Omega(t) = \int_{t_1}^{t} \frac{1}{\eta(s)}\,ds, \quad F(t) > 0,$$

where $t_1 \geq t_0$.

Lemma 6. *Let $u(t)$ be a positive decreasing solution of (1) on $[t_0, \infty)$ satisfying (6). Then, equation (E) is oscillatory provided that*

$$(\eta(t)\mu'(t))' + Q_1(t)\mu(\tau(t)) + Q_2(t)\mu^{\alpha}(\sigma(t)) = 0, \quad t \geq t_0, \tag{9}$$

is oscillatory.

Proof. For the sake of contradiction, assume that $x(t)$ is an eventually positive solution of (E), i.e., $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all $t \geq t_1 \geq t_0$. Then the corresponding function $z(t) > 0$ for all $t \geq t_1$. By Lemma 4, we have $\mu(t) = \frac{z(t)}{u(t)} > 0$, and it satisfies condition (7) for all $t \geq t_1$. Using (8) in (5), we obtain that $\mu(t)$ is a positive and increasing solution of

$$(\eta(t)\mu'(t))' + Q_1(t)\mu(\tau(t)) + Q_2(t)\mu^{\alpha}(\sigma(t)) \leq 0.$$

However, by Theorem 2 of [35], the corresponding equation (9) also has a positive solution, which yields a contradiction. This completes the proof. □

3. Oscillation Results

In this section, we derive oscillation criteria for equation (E) with the help of equation (9). We begin with the following theorem.

Theorem 1. *Let $u(t)$ be a positive decreasing solution of (1) on $[t_0, \infty)$ satisfying (6). If*

$$\int_{t_0}^{\infty} Q_1(t)\,dt = \infty \quad \text{or} \quad \int_{t_0}^{\infty} Q_2(t)\,dt = \infty, \tag{10}$$

then equation (E) is oscillatory.

Proof. Assume, for the sake of contradiction, that $x(t)$ is an eventually positive solution of (E), i.e., $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all $t \geqslant t_1$ for some $t_1 \geqslant t_0$. Then the corresponding function satisfies $z(t) > 0$, $z(\tau(t)) > 0$, and $z(\sigma(t)) > 0$ for all $t \geqslant t_1$.

From Lemma 4, the function $\mu(t) = \dfrac{z(t)}{u(t)} > 0$ satisfies condition (7). Since $\mu(t)$ is increasing, there exists a constant $M > 0$ such that $\mu(t) \geqslant M > 0$ for all $t \geqslant t_2 \geqslant t_1$.

Using this in (9) and integrating from t_2 to t, we obtain

$$\int_{t_2}^{t}(MQ_1(s) + M^{\alpha}Q_2(s))ds \leq \eta(t_2)\mu'(t_2) - \eta(t)\mu'(t).$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 September 2025 doi:10.20944/preprints202509.0256.v1

As $t \to \infty$, it follows that

$$\int_{t_2}^{\infty} (MQ_1(t) + M^{\alpha} Q_2(t))dt \leq \eta(t_2)\mu'(t_2) < \infty,$$

which contradicts (10).The proof of the theorem is complete. □

Remark 1. *Theorem 1 is independent of α and the delay argument. Hence, it is applicable to linear, sub-linear, and super-linear equations, as well as to both delay and advanced type differential equations.*

In the following, we present several oscillation criteria for equation (E) when condition (10) fails to hold.

Theorem 2. *Assume $\alpha = 1$ and let $u(t)$ be a positive decreasing solution of (1) on $[t_0, \infty)$ satisfying (6). If*

$$\liminf_{t\to\infty} \int_{\delta(t)}^{t} Q_3(s)\Omega(\delta(s))\, ds > \frac{1}{e}, \tag{11}$$

then equation (E) is oscillatory.

Proof. Assume, for the sake of contradiction, that $x(t)$ is an eventually positive solution of (E), i.e., $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all $t \geqslant t_1 \geqslant t_0$. Then the corresponding function satisfies $z(t) > 0$, $z(\sigma(t)) > 0$, and $z(\tau(t)) > 0$ for all $t \geqslant t_1$.

From Lemma 4, the function $\mu(t) = \dfrac{z(t)}{u(t)} > 0$ satisfies condition (7). From the monotonicity of $\eta(t)\mu'(t)$, we obtain

$$\mu(t) \geqslant \int_{t_1}^{t} \frac{\eta(s)\mu'(s)}{\eta(s)}\, ds \geqslant \Omega(t)\eta(t)\mu'(t) \tag{12}$$

and hence

$$\left(\frac{\mu(t)}{\Omega(t)}\right)' = \frac{\Omega(t)\eta(t)\mu'(t) - \mu(t)}{\eta(t)\Omega^2(t)} \leqslant 0,$$

which implies that $\dfrac{\mu(t)}{\Omega(t)}$ is decreasing.

Now, from (9), we have

$$(\eta(t)\mu'(t))' + Q_1(t)\mu(\tau(t)) + Q_2(t)\mu(\sigma(t)) = 0. \tag{13}$$

Since $\delta(t) = \min\{\tau(t), \sigma(t)\}$ and $\mu(t)$ is increasing, it follows from (13) that

$$(\eta(t)\mu'(t))' + Q_3(t)\mu(\delta(t)) \leqslant 0. \tag{14}$$

Combining (12) with (14) and letting $\omega(t) = \eta(t)\mu'(t)$, we see that $\omega(t)$ is a positive solution of the inequality

$$\omega'(t) + Q_3(t)\Omega(\delta(t))\omega(\delta(t)) \leqslant 0. \tag{15}$$

However, by Theorem 2.2.6(i) in [14], condition (11) implies that inequality (15) has no eventually positive solutions, a contradiction. The proof of the theorem is complete. □

Theorem 3. *Assume that $\alpha = 1$ and let $u(t)$ be a positive decreasing solution of (1) on $[t_0, \infty)$ satisfying (6). If*

$$\liminf_{t\to\infty} \Omega(\delta(t)) \int_{t}^{\infty} Q_3(s)\, ds > \frac{1}{4}, \tag{16}$$

then equation (E) is oscillatory.

 doi:10.20944/preprints202509.0256.v1

Proof. Suppose, on the contrary, that $x(t)$ is an eventually positive solution of (E), that is, $x(t) > 0$, $x(e(t)) > 0$, and $x(\sigma(t)) > 0$ for all $t \geqslant t_1 \geqslant t_0$. Proceeding as in the proof of Theorem 2, we see that

$$\mu(t) = \frac{z(t)}{u(t)} > 0$$

satisfies condition (1) and the inequality

$$\left(\eta(t)\mu'(t)\right)' + Q_3(t)\,\mu(\delta(t)) \leq 0. \tag{17}$$

However, by Theorem 3 of [36], condition (16) implies that $\mu(t)$ is oscillatory, which is a contradiction. The proof of the theorem is complete. □

Theorem 4. *Assume that $\alpha = 1$ and let $u(t)$ be a positive decreasing solution of (1) on $[t_0, \infty)$ satisfying (6). If*

$$\limsup_{t\to\infty} \left\{ \frac{1}{\Omega(\delta(t))} \int_{t_1}^{\delta(t)} Q_3(s)\,\Omega(s)\,\Omega(\delta(s))\,ds + \int_{\delta(t)}^{t} Q_3(s)\,\Omega(\delta(s))\,ds + \Omega(\delta(t)) \int_t^{\infty} Q_3(s)\,ds \right\} > 1, \tag{18}$$

for some $t_1 \geqslant t_0$, then equation (E) is oscillatory.

Proof. Suppose, on the contrary, that $x(t)$ is an eventually positive solution of (E), that is, $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all $t \geqslant t_1$, for some $t_1 \geqslant t_0$. Proceeding as in the proof of Theorem 3, we see that

$$\mu(t) = \frac{z(t)}{u(t)} > 0$$

satisfies condition (7) and inequality (14) for all $t \geqslant t_1$. Integrating (14) gives

$$\mu'(t) \geqslant \frac{1}{\eta(t)} \int_t^{\infty} Q_3(s)\,\mu(\delta(s))\,ds.$$

Integrating once more yields

$$\begin{aligned} \mu(t) &\geqslant \int_{t_1}^{t} \frac{1}{\eta(s)} \int_s^{\infty} Q_3(s_1)\,\mu(\delta(s_1))\,ds_1\,ds \\ &= \int_{t_1}^{t} \frac{1}{\eta(s)} \int_s^{t} Q_3(s_1)\,\mu(\delta(s_1))\,ds_1\,ds + \int_{t_1}^{t} \frac{1}{\eta(s)} \int_t^{\infty} Q_3(s_1)\,\mu(\delta(s_1))\,ds_1\,ds. \end{aligned}$$

Applying integration by parts, we obtain

$$\mu(t) \geqslant \int_{t_1}^{t} Q_3(s)\,\Omega(s)\,\mu(\delta(s))\,ds + \Omega(t) \int_t^{\infty} Q_3(s)\,\mu(\delta(s))\,ds.$$

Hence,

$$\begin{aligned} \mu(\delta(t)) \geqslant{}& \int_{t_1}^{\delta(t)} Q_3(s)\,\Omega(s)\,\mu(\delta(s))\,ds + \Omega(\delta(t)) \int_{\delta(t)}^{t} Q_3(s)\,\mu(\delta(s))\,ds \\ &+ \Omega(\delta(t)) \int_t^{\infty} Q_3(s)\,\mu(\delta(s))\,ds. \end{aligned}$$

Since $\mu(t)$ is increasing and $\mu(t)/\Omega(t)$ is decreasing, the previous inequality becomes

$$\mu(\delta(t)) \geqslant \frac{\mu(\delta(t))}{\Omega(\delta(t))} \int_{t_1}^{\delta(t)} Q_3(s)\,\Omega(s)\,\Omega(\delta(s))\,ds + \mu(\delta(t)) \int_{\delta(t)}^{t} Q_3(s)\,\Omega(\delta(s))\,ds + \Omega(\delta(t))\,\mu(\delta(t)) \int_t^\infty Q_3(s)\,ds.$$

Dividing through by $\mu(\delta(t))$, we obtain

$$1 \geqslant \frac{1}{\Omega(\delta(t))} \int_{t_1}^{\delta(t)} Q_3(s)\,\Omega(s)\,\Omega(\delta(s))\,ds + \int_{\delta(t)}^{t} Q_3(s)\,\Omega(\delta(s))\,ds + \Omega(\delta(t)) \int_t^\infty Q_3(s)\,ds,$$

which contradicts (18). The proof of the theorem is complete. □

Next, we derive an oscillation condition for equation (E) in the case $\alpha > 1$.

Theorem 5. *Assume that β is a ratio of odd positive integers such that $\alpha > \beta > 1$, and let $u(t)$ be a positive decreasing solution of (1) on $[t_0, \infty)$ satisfying (6). If*

$$\int_{t_0}^\infty \frac{1}{\eta(t)} \int_t^\infty Q_4(s) \frac{\Omega^\beta(\delta(s))}{\Omega^\beta(s)}\,ds\,dt = \infty, \tag{19}$$

where

$$Q_4(t) = \left(\frac{Q_1(t)}{\eta_1}\right)^{\eta_1} \left(\frac{Q_2(t)}{\eta_2}\right)^{\eta_2}, \quad \eta_1 = \frac{\alpha-\beta}{\alpha-1}, \quad \eta_2 = \frac{\beta-1}{\alpha-1}, \tag{20}$$

then equation (E) is oscillatory.

Proof. Assume, on the contrary, that $x(t)$ is an eventually positive solution of (E), i.e., $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all $t \geqslant t_1$, for some $t_1 \geqslant t_0$. Proceeding as in the proof of Lemma 7, we see that $\mu(t) = \frac{z(t)}{w(t)} > 0$ satisfies condition (7) and the inequality

$$(\eta(t)\mu'(t))' + Q_1(t)\mu(\tau(t)) + Q_2(t)\mu^\alpha(\sigma(t)) \leqslant 0.$$

Since μ is increasing and $\delta(t) = \min\{\tau(t), \sigma(t)\}$, we obtain

$$(\eta(t)\mu'(t))' + Q_1(t)\mu(\delta(t)) + Q_2(t)\mu^\alpha(\delta(t)) \leqslant 0. \tag{21}$$

Recall the well-known arithmetic–geometric mean inequality

$$\eta_1 y_1 + \eta_2 y_2 \geqslant y_1^{\eta_1} y_2^{\eta_2}, \quad y_1, y_2 > 0, \tag{22}$$

where η_1 and η_2 are chosen to satisfy (20). Now, taking $y_1 = \eta_1^{-1} Q_1(t)\mu^{1-\beta}(\delta(t))$ and $y_2 = \eta_2^{-1} Q_2(t)\mu^{\alpha-\beta}(\delta(t))$ in (22), we get

$$Q_1(t)\mu^{1-\beta}(\delta(t)) + Q_2(t)\mu^{\alpha-\beta}(\delta(t)) \geqslant \left(\frac{Q_1(t)}{\eta_1}\right)^{\eta_1} \left(\frac{Q_2(t)}{\eta_2}\right)^{\eta_2}. \tag{23}$$

Substituting (23) into (21), we obtain

$$(\eta(t)\mu'(t))' + Q_4(t)\mu^\beta(\delta(t)) \leqslant 0. \tag{24}$$

Since $\mu(t)/\Omega(t)$ is decreasing, using this property in (24), we arrive at

$$(\eta(t)\mu'(t))' + Q_4(t)\frac{\Omega^\beta(\delta(t))}{\Omega^\beta(t)}\mu^\beta(t) \leqslant 0, \quad t \geqslant t_1. \tag{25}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 September 2025 doi:10.20944/preprints202509.0256.v1

Integrating (25) from t to ∞ yields

$$\int_t^\infty Q_4(s)\frac{\Omega^\beta(\delta(s))}{\Omega^\beta(s)}\mu^\beta(s)\,ds \leqslant \eta(t)\mu'(t),$$

or equivalently,

$$\frac{1}{\eta(t)}\int_t^\infty Q_4(s)\frac{\Omega^\beta(\delta(s))}{\Omega^\beta(s)}\,ds \leqslant \frac{\mu'(t)}{\mu^\beta(t)}.$$

Integrating this inequality from t_1 to t, we obtain

$$\begin{aligned}\int_{t_1}^t \frac{1}{\mu(s)}\int_s^\infty Q_4(s_1)\frac{\Omega^\beta(\delta(s_1))}{\Omega^\beta(s_1)}\,ds_1\,ds &\leqslant \int_{t_1}^t \frac{\mu'(s)}{\mu^\beta(s)}\,ds \\ &\leqslant \frac{\mu^{1-\beta}(t_1)}{\beta-1} - \frac{\mu^{1-\beta}(t)}{\beta-1}.\end{aligned}$$

As $t \to \infty$, we obtain

$$\int_{t_1}^\infty \frac{1}{\mu(t)}\int_t^\infty Q_4(s)\frac{\Omega^\beta(\delta(s))}{\Omega^\beta(s)}\,ds\,dt \leq \frac{\mu^{1-\beta}(t_1)}{\beta-1} < \infty,$$

which contradicts (19). The proof of the theorem is complete. □

We conclude this section with the following theorem for the case $\alpha < 1$.

Theorem 6. *Assume that γ is a ratio of odd positive integers such that $1 > \gamma > \alpha$, and let $u(t)$ be a positive decreasing solution of (1) on $[t_0, \infty)$ satisfying (6). If*

$$\int_{t_0}^\infty \Omega^\gamma(\delta(t))Q_5(t)\,dt = \infty, \tag{26}$$

where

$$Q_5(t) = \left(\frac{Q_1(t)}{\eta_1}\right)^{\eta_1}\left(\frac{Q_2(t)}{\eta_2}\right)^{\eta_2}, \quad \eta_1 = \frac{\gamma-\alpha}{1-\alpha}, \quad \eta_2 = \frac{1-\gamma}{1-\alpha}, \tag{27}$$

then equation (E) is oscillatory.

Proof. Assume, on the contrary, that $x(t)$ is an eventually positive solution of (E), i.e., $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all $t \geqslant t_1$, for some $t_1 \geqslant t_0$. Proceeding as in the proof of Theorem 5, we arrive at (21) for all $t \geqslant t_1$. Recall the well-known arithmetic–geometric mean inequality

$$\eta_1 y_1 + \eta_2 y_2 \geqslant y_1^{\eta_1} y_2^{\eta_2}, \quad y_1, y_2 > 0, \tag{28}$$

where η_1 and η_2 are chosen to satisfy (27).
Now, setting $y_1 = \eta_1^{-1}Q_1(t)\mu^{1-\gamma}(\delta(t))$ and $y_2 = \eta_2^{-1}Q_2(t)\mu^{\alpha-\gamma}(\delta(t))$ in (28), we obtain

$$Q_1(t)\mu^{1-\gamma}(\delta(t)) + Q_2(t)\mu^{\alpha-\gamma}(\delta(t)) \geqslant \left(\frac{Q_1(t)}{\eta_1}\right)^{\eta_1}\left(\frac{Q_2(t)}{\eta_2}\right)^{\eta_2}. \tag{29}$$

Using (29) in (21), we obtain

$$\left(\eta(t)\mu'(t)\right)' + Q_5(t)\mu'(\delta(t)) \leq 0. \tag{30}$$

Now, using (12) in (30), we find

$$\left(\eta(t)\mu'(t)\right)' + Q_5(t)\Omega^\gamma(\delta(t))\left(\eta(\delta(t))\mu'(\delta(t))\right)^\gamma \leq 0, \quad t \geqslant t_1.$$

 doi:10.20944/preprints202509.0256.v1

Since $\delta(t) \leq t$ and $\eta(t)\mu'(t)$ is decreasing, we further obtain

$$\left(\eta(t)\mu'(t)\right)' + Q_5(t)\Omega^\gamma(\delta(t))\left(\eta(t)\mu'(t)\right)^\gamma \leq 0, \quad t \geqslant t_1.$$

Dividing by $\left(\eta(t)\mu'(t)\right)'$ and integrating from t_1 to t, we get

$$\begin{aligned}\int_{t_1}^{t} Q_5(s)\Omega^\gamma(\delta(s))\,ds &\leqslant -\int_{t_1}^{t} \frac{\left(\eta(s)\mu'(s)\right)'}{\left(\eta(s)\mu'(s)\right)^\gamma}\,ds \\ &\leqslant \frac{\left(\eta(t_1)\mu'(t_1)\right)^{1-\gamma}}{1-\gamma} - \frac{\left(\eta(t)\mu'(t)\right)^{1-\gamma}}{1-\gamma}. \end{aligned} \tag{31}$$

Letting $t \to \infty$ in (31) gives a contradiction with (26). This completes the proof. □

Remark 2. *Theorems 1–6 extend and complement several known oscillation criteria for second-order nonlinear neutral delay differential equations. In particular, they unify oscillation results for both linear and nonlinear equations with mixed arguments.*

4. Examples

In this section, we present three examples to demonstrate the importance and novelty of the main results. In the first example, we provide an explicit solution for the considered equation.

Example 1. *Consider the second-order hybrid neutral delay differential equation*

$$z''(t) - \frac{2}{t^2}x(t) + \left(\frac{t^2+4}{2t^2}\right)x(t-2\pi) = 0, \quad t \geqslant 3\pi, \tag{32}$$

where $z(t) = x(t) + \frac{1}{2}x(t-\pi)$.
The corresponding auxiliary equation (1) *takes the form*

$$u''(t) - \frac{2}{t^2}u(t) = 0, \quad t \geqslant 3\pi. \tag{33}$$

The function $u(t) = t^{-1}$ *is a positive decreasing solution of* (33), *and* $\eta(t) = \frac{1}{t^2}$ *satisfies condition* (6). *Further calculations show that*

$$F(t) = \frac{t-2\pi}{2(t-\pi)}, \quad \delta(t) = t - 2\pi, \quad Q_1(t) \approx \frac{1}{t^4}, \quad Q_2(t) \approx \frac{1}{4t^2} + \frac{1}{t^4}, \quad \Omega(t) = \frac{t^3}{3}.$$

It is easy to verify that condition (16) *becomes*

$$\liminf_{t\to\infty} \frac{(t-2\pi)^3}{3}\int_t^\infty \left(\frac{1}{4s^2} + \frac{2}{s^4}\right)ds = \infty > \frac{1}{4},$$

which implies that condition (16) *holds. Therefore, by Theorem 3, equation* (32) *is oscillatory. In fact,* $x(t) = \sin t$ *is one such oscillatory solution.*

Example 2. *Consider the second-order hybrid neutral delay differential equation*

$$\left(t^{3/2}z'(t)\right)' - \frac{1}{2\sqrt{t}}x(t) + q_0 t\, x^{5/3}(t/3) = 0, \quad t \geqslant 1, \tag{34}$$

where $z(t) = x(t) + \frac{1}{3}x(t/2)$ *and* $q_0 > 0$ *is a constant.*
The corresponding auxiliary equation (1) *takes the form*

$$\left(t^{3/2}u'(t)\right)' - \frac{1}{2\sqrt{t}}u(t) = 0, \quad t \geqslant 1. \tag{35}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 September 2025 doi:10.20944/preprints202509.0256.v1

The function $u(t) = 1/t$ is a positive decreasing solution of (35), and $\eta(t) = \frac{1}{\sqrt{t}}$ satisfies condition (6). Further calculations show that

$$F(t) = \tfrac{1}{3}, \quad \delta(t) = t/3, \quad Q_1(t) = \tfrac{1}{6}t^{-5/2}, \quad Q_2(t) = q_0 t^{-5/3}.$$

Choose $\beta = 7/5$ so that $5/3 > 7/5 > 1$. Then $\eta_1 = 2/5,\ \eta_2 = 3/5$, and

$$Q_4(t) = q_0^{3/5}\left(\frac{5}{12}\right)\left(\frac{1}{4}\right)^{1/5}\frac{1}{t^2}, \quad \Omega(t) \approx \tfrac{2}{3}t^{3/2}.$$

Condition (19) becomes

$$q_0^{3/5}\int_1^\infty \sqrt{t}\int_t^\infty \frac{5}{54\sqrt{3}}(1/4)^{1/5}\frac{1}{s^2}ds\,dt = \frac{5}{54\sqrt{3}}(1/4)^{1/5}q_0^{3/5}\int_1^\infty \frac{1}{\sqrt{t}}dt = \infty,$$

which shows that (19) holds for all $q_0 > 0$. Hence, by Theorem 5, equation (34) is oscillatory for all $q_0 > 0$.

Example 3. *Consider the second-order hybrid neutral delay differential equation*

$$(tz'(t))' - \frac{1}{t}x(t) + q_0 x^{1/3}(t/2) = 0, \quad t \geqslant 1, \tag{36}$$

where $z(t) = x(t) + \frac{1}{4}x(t/3)$ and $q_0 > 0$ is a constant.
The corresponding auxiliary equation (1) takes the form

$$(tu'(t))' - \frac{1}{t}u(t) = 0, \quad t \geqslant 1. \tag{37}$$

The function $u(t) = \frac{1}{t}$ is a positive decreasing solution of (37), and $\eta(t) = \frac{1}{t}$ satisfies condition (6). Further calculations show that

$$F(t) = \tfrac{1}{4}, \quad \delta(t) = t/3, \quad Q_1(t) = \tfrac{3}{16t^3}, \quad Q_2(t) = \tfrac{q_0}{2^{1/3}}t^{-1/3}.$$

Choose $\gamma = \frac{3}{5}$ so that $1 > \frac{3}{5} > \frac{1}{3}$. Then $\eta_1 = \frac{2}{5},\ \eta_2 = \frac{3}{5}$, and

$$Q_5(t) = q_0^{3/5}\left(\tfrac{1}{2}\right)^{1/5}\left(\tfrac{15}{32}\right)^{2/5}\left(\tfrac{5}{3}\right)^{3/5}t^{-7/5}, \quad \Omega(t) \approx \tfrac{t^2}{2}.$$

Condition (26) becomes

$$\int_1^\infty q_0^{3/5}\left(\tfrac{1}{2}\right)^{4/5}\left(\tfrac{15}{32}\right)^{2/5}\left(\tfrac{5}{3}\right)^{3/5}t^{-1/5}\,dt = \infty,$$

which shows that condition (26) holds whenever $q_0 > 0$. Therefore, by Theorem 6, equation (36) is oscillatory.

5. Conclusions

In this paper, we investigated the oscillatory properties of solutions to a hybrid second-order nonlinear neutral delay differential equation (E). The studied equation was first transformed into a binomial form. By applying the integral averaging method, comparison technique and the arithmetic-geometric mean inequality, we derived new sufficient conditions ensuring that all solutions of (E) oscillate. The obtained results are novel and complement the existing oscillation theory of functional differential equations. Furthermore, three illustrative examples were provided to demonstrate the significance and novelty of the main findings, since none of the known results are directly applicable to equations (32), (34), and (36).

It would be interesting to extend the results of this work to the case when condition (6) fails to hold. Another promising direction for future research is to study the oscillatory behavior of the following hybrid nonlinear neutral equation:

$$\left(a(t)z'(t)\right)' + p(t)x(t) - q(t)x^{\alpha}(\sigma(t)) = 0,$$

where $z(t) = x(t) + b(t)x(\tau(t))$.

Author Contributions: All of the authors contributed to the preparation of this paper and all are in agreement with submitting the paper to this journal. In particular: Methodology: G.P., P.N., G.E.C. and E.T.; Investigation, G.P., P.N., G.E.C. and E.T.; Writing—Original Draft: G.P.,P.N. and E.T.; Writing review and editing, G.E.C. and E.T. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in this study are included in the article.Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Cooke, K.L. Differential, Difference Equations. Academic Press: New York, USA, **1963**.
2. Goyri,I.; Ladas,G. Oscillation Theory of Delay Differential Equations with Applications.Clarendan Press: Oxford, UK, **1991**.
3. Bainov, D. D.; Mishev, D.P. Oscillation Theory for Neutral Differential Equations with Delay, Adam Hilger, Bristol, **1991**.
4. Hale, J.K. Theory of Functional Differential Equations. Springer, New York, **1977**.
5. Agarwal, R.P.; Bohner, M.; Li, W.T. Nonoscillation and Oscillation: Theory of Functional Differential Equations, Marcel Dekker, New York, **2004**.
6. Agarwal, R.P.; Grace, S.R.; Regan, D.O. Oscillation Theory for Second-order Linear, Halflinear, Superlinear and Sublinear Dynamic Equations. Kluwer Acad. publ. Dordrecht, The Netherlands, **2002**.
7. Bohner M. ; Grace, S.R. ; Jadlovska. Oscillation criteria for second order neutral delay differential equations. *Electron. J. Qual. Theory Diff. Equ.* **2017**, 60, 1-12.
8. Karpuz, B.; Manojlovic, J.; Ocalan, O.; Shoukaku, Y. Oscillation criteria for a class of second-order neutral delay differential equations. *Appl. Math. Comput.* **2009**, 210, 303-312.
9. Baculikova, B. Oscillatory criteria for second order differential equations with several sublinear neutral terms. *Opus. Mat.* **2019**, 39, 753-763.
10. Baculikova, B.; Li, T.; Dzurina, J. Oscillation theorems for second order superlinear neutral differential equations. *Math.Slovaca.* **2013**, 63, 123-134.
11. Shi, S.; Han, Z. Oscillation of second order mixed functional differential equations with sub linear and superlinear neutral terms. *Turk. J. Math.* **2022**, 46, 3045-3056.
12. Abdelnasen, A.; Moaz. O.; Cesarano, C.; Askar, S.; Elbbasy, E.M. Oscillation test for second-order differential equations with several delays. *Symmetry.* **2023**, 15, 452-462.
13. Dzurina, J.; Thandapani, E.; Baculikova, B.; Dharuman, C.; Prabh, N. Oscillation of second order nonlinear differential equations with several sublinear neutral terms. *Nonlinear Dyn. Syst.Theo.* **2019**, 19, 124-132.
14. Dzurina, J.; Grace S.R.; Jadlovska, I.; Li. T. Oscillation criteria for second order Emden-Fowler delay differential equations with a sublinear neutral term, *Math. Nach.* **2020**, 1-13.
15. Alomain, A.M.; Muhib A. Some new oscillation results for second order differential equations with neutral term. *AIMS Mathematics.* **2025**, 10. 694-704.
16. Bohner, M.; Srinivasan, R.; Thandapani, E. Oscillation of second order damped noncanonical differential equations with superlinear neutral term. *J. Inequ. Spl. Funct.* **2021**, 12, 44-53.
17. Nabih, A.; Al-Jaser, A.; Moaaz, O. Neutral Emden-Fowler differential equations of second order: oscillation criteria of coles type. *Symmetry.* **2024**, 16, 931.
18. Moaaz, O.; Muhib, A.; Owyed, S.; Mahmoud, E.E. Second-order neutral differential equations: improved criteria for testing the oscillation. *J. Math.* **2021**, 6665103, 1-7.

19. Ozdemir, O.; Kilic, A. Oscillation of second order functional differential equations with superlinear neutral terms. *Bull. Malays. math. Sci. Soc.* **2022**, 45, 83-99.
20. Ruggieri, M.; Santra, S.S.; Scapellato, A. Oscillatory behaviour of second-order neutral differential equations, *Bull. Braz. Math. Soc.* **2022**, 53, 665-675.
21. Bohner, M.; Sudha, B.; Thangavelu, K.; Thandapani, E. Oscillation criteria for second-order differential equations with superlinear neutral term. *Nonlinear Stud.* **2019**, 26, 425-434.
22. Dzurina, J. Oscillation of second order trinomial differential equations with retarded and advanced arguments. *Appl. Math. Lett.* **2024**, 153, 109058.
23. Baculikova, B. Property (A) and oscillation of higher-order trinomial differential equations with retarded and advanced arguments. *Mathematics.* **2024**, 12, 910.
24. Li, T.; Rogovchenko, Y.V. Oscillation criteria for second order superlinear Emden-Fowler neutral differential equations. *Monatsh. Math.* **2017**, 184, 489-500.
25. Ocalan, O. Oscillation of neutral differential equations with positive and negative coefficients. *J. Math. Anal. Appl.* **2007**, 331, 644-654.
26. Tao, Y; Gao, G. Oscillation of forced neutral differential equations with positive and negative coefficients. *Appl. Math.* **1998**, 13, 271-280.
27. Shoukaku, Y. Oscillation criteria of second-order differential equation with positive and negative coefficients. *Hacette.J.Math. Stat*. **2022**, 51, 970-980.
28. Deng, X.; Wang. Q.; Zhou, Z. Oscillation criteria for second-order neutral dynamic equations of Emden-Fowler type with positive and negative coefficients on time scales. *Sci.China Math.* **2017**, 60, 113-132.
29. Bai, Y.; Liu. L. New oscillation criteria for second-order neutral delay differential equations with positive and the negative coefficients. *Abstr. Appl. Anal.* 2010, 564068, 1-11.
30. Guo, R.; Huang, Q.; Tian, H. Nonoscillation and oscillation criteria for a class of second-order nonlinear neutral delay differential equations with positive and negative coefficients. *Axioms.* **2022**, 281, 1-11.
31. Thandapani, E.; Muthulakshmi, V.; Graef, J.R. Oscillation criteria for second-order nonlinear neutral delay differential equations with positive and negative coefficients. *Int.J.Pure Appl.Math.* **2011**, 70, 261-274.
32. Shoukaku, Y. Oscillation of solutions of second order neutral differential equations with positive and negative coefficients. *J. Appl. Anal.* **2009**, 15, 281-298.
33. Zafer,A.; Candan,T.; Gurkan,Z.N. Equivalence transformation for neutral differential equations: oscillation of solutions. *Mathematics.* **2025**, 13, 2243.
34. Swansan, C.A. Comparison and Oscillation Theory of Linear Differential Equations. Academic Press, New York, **1968**.
35. Kusano, T.; Naito, M. Comparison theorems for functional differential equations with deviating arguments, *J. math. Soc. Japan.* **1981**, 33, 509-533.
36. Kusano, T; Wang, J. Oscillation properties of half linear functional differential equations of the second order, *Hiroshima Math. J.* **1995**, 25, 371-385.