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Article

Existence and Uniqueness of the Viscous Burgers' Equation with the p-Laplace Operator

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Abstract: In this paper, we investigate the existence and uniqueness of solutions for the viscous Burgers' equation for the isothermal flow of power-law non-Newtonian fluids $\rho(\partial_t u + u\partial_x u) = \mu\partial_x\left(|\partial_x u|^{p-2}\partial_x u\right)$, augmented with the initial condition $u(0,x) = u_0$, $0 < x < L$ and the boundary condition $u(t,0) = u(t,L) = 0$, where ρ is the density, μ the viscosity, u the velocity of the fluid and p, $1 < p < 2$, $L, T > 0$. Moreover, numerical solutions to the problem are constructed by applying the high-level modeling and simulation package COMSOL Multiphysics at small and large Reynold's numbers.

Keywords: p-Laplacian; power-law model; existence and uniqueness; burgers' equation; reynold's number; sobolev space; COMSOL multiphysics

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1. Introduction

The Burgers' equation, first introduced by J.M. Burgers, is a fundamental model for various physical processes, including shock wave propagation and turbulence [1]. Originally formulated for Newtonian fluids, the Burgers' equation has been extensively analyzed; however, its generalization to non-Newtonian fluids—specifically those described by the power-law model—has received comparatively less attention for the case when $1 < p < 2$ [2]. We employ a particular case of the generalized Burgers' equation formulated by Wei and Jordan [3], which was used to analyze acoustic propagation in power-law fluids. Traveling wave solutions of this equation were derived by Wei and Borden [4].

This paper's main focus is to investigate the existence and uniqueness of weak solutions to the generalized viscous Burgers' equation. The study is framed within the context of Sobolev spaces, which provide a robust mathematical environment for addressing weak solutions and ensuring appropriate regularity conditions [5].

Establishing the existence of weak solutions to nonlinear partial differential equations (PDEs) like the generalized Burgers' equation has been well-studied in the context of Newtonian fluids [6], [7]. However, extending these results to the power-law non-Newtonian fluids involves additional complexities due to the nonlinearity introduced by the fluid's viscosity [8]. The unique solutions for the generalized Burgers' equation based on power-law fluids remain largely unexplored for the case when $p \neq 2$, creating a gap in the current literature that this paper aims to address.

Wei and Jordan [3] established the following Burgers' equation

$$\rho(\partial_t u + u\partial_x u) = -P_x + \partial_x\left[\left(\mu_{\mathrm{B}} + \frac{4}{3}\mu k|\partial_x u|^{m-1}\right)\partial_x u\right]$$

as a model to study acoustic traveling waves in Power-law fluids with
$m = p - 1$, $0 < m < 1$ is the power-law index.

In this paper, we study the well-posedness and numerical solutions of a special case of the above equation with initial and boundary conditions

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$$\begin{cases} \rho(\partial_t u + u\partial_x u) = \mu\partial_x\left(|\partial_x u|^{p-2}\partial_x u\right), & \\ u(0,x) = u_0(x), & x \in [0,L], \\ u(t,0) = u(t,L) = 0, & t \in [0,T]. \end{cases}$$

By using the dimensionless variables $x^* = \frac{1}{L}x$, $u^* = \frac{1}{u_0}u$, $t^* = \frac{u_0}{L}t$, to the last equation, we obtain the following problem, in which x^*, u^*, t^* are still denoted by x, u, t

$$\begin{cases} \partial_t u + u\partial_x u = \frac{1}{Re}\partial_x\left(|\partial_x u|^{p-2}\partial_x u\right), & \\ u(0,x) = u_0(x), & x \in [0,1], \\ u(t,0) = u(t,1) = 0, & t \in [0,T] \end{cases} \tag{1}$$

with $1 < p < 2$ and $m = p - 1$. In equation (1), Re stands for the Reynolds number, a dimensionless quantity determining the flow behavior based on the balance of inertial and viscous forces. The Reynolds number for power-law fluids is given by

$$Re = \frac{\rho L^{p-1}}{\mu u_0^{p-3}},$$

where L and u_0 represent the characteristic length and velocity, respectively (see, for example, [9]). The Reynolds number significantly influences the stability of the flow and the type of flow regime observed. Our numerical simulations demonstrate this. When in case $p = 2$, equation (1) becomes a well-known Burgers' equation for a Newtonian fluid

$$\begin{cases} \partial_t u + u\partial_x u = \frac{1}{Re}\partial_x^2 u, & \\ u(0,x) = u_0(x), & x \in [0,1], \\ u(t,0) = u(t,1) = 0, & t \in [0,T]. \end{cases}$$

The well-posedness of the viscous Burgers' equation for Newtonian fluid has been studied intensively in the literature (see, for example, [6], [10], [11]). To our knowledge, the well-posedness of the problem (1) has not been studied before. In our work, we prove the result concerning the existence, uniqueness, and regularity of a solution to the Burgers equation with p-Laplasian right-hand side (1).

Multiplying the equation of (1) by a test function $w \in W_0^{1,p}(0,1)$, and integrating by parts from 0 to 1, we convert the initial boundary value problem (1) to the following integral equation

$$\int_0^1 \partial_t uw dx + \int_0^1 u\partial_x uw dx + \frac{1}{Re}\int_0^1 \left(|\partial_x u|^{p-2}\partial_x u\right)\partial_x w dx = 0 \tag{2}$$

for all $w \in W_0^{1,p}(0,1)$, $t \in (0,T)$ and $1 < p < 2$.

Definition 1.1. *For $1 < p < 2$, a function $u \in L^2\left(0,T; W_0^{1,p}(0,1)\right)$, for which $\partial_t u \in L^2(0,T; L^2(0,1))$ is said to be a weak solution of problem (1) if it satisfies equation (2) for each $w \in W_0^{1,p}(0,1)$ and a.e. $t \in (0,T)$ and $u(0,x) = u_0$ for all $x \in (0,1)$.*

Now, we present the main result of the work.

Theorem 1.2. *Let $u_0 \in W_0^{1,p}(0,1)$. Then there exists a unique solution $u \in L^2\left(0,T; W_0^{1,p}(0,1)\right)$ of (1) such that $\partial_t u \in L^2(0,T; L^2(0,1))$, in the sense of Definition 1.1.*

In Section 2, we state some preliminary information needed to study the well-posedness of the problem (1), formulated in the weak form (2). Section 3 consists of two subsections 3.1 and 3.2. We obtain the existence result in 3.1 and establish the uniqueness of the solution in 3.2. Moreover, in

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Section 4, we construct the numerical solutions for the various values of the Reynolds number Re, by using the COMSOL PDE solver with some numerical manipulations to avoid singularity in $\triangle_p u$ at points where $\partial_x u = 0$.

2. Preliminary

We present well-known facts that we will use to prove Theorem 1.2. The Gronwall's inequality, some Sobolev inequalities, and the monotonicity of the p-Laplacian operator defined by $\triangle_p u = \partial_x \left(|\partial_x u|^{p-2} \partial_x u \right)$ are required to establish the main result.

Recall that $L^p(0,1)$ and $W^{k,p}(0,1)$ are the standard spaces of Lebesgue and Sobolev, respectively, for $1 < p < \infty$ and $k \in \mathbb{Z}$. For any Banach space X, we define $L^p(0,T;X)$ to be the space of measurable functions $u : (0,T) \to X$ such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u\|_X^p \, \mathrm{d}t \right)^{1/p} < \infty$$

for $1 < p < \infty$ and $\|u\|_{L^\infty(0,T;X)} = \operatorname{ess\,sup}_{0<t<T} \|u\|_X < \infty$ if $p = \infty$. $L^p(0,T;X)$ is a Banach space [12].

Proposition 2.1 (The differential form of Gronwall's inequality [13]).

(i) Let $\eta(\cdot)$ be a nonnegative, absolutely continuous function on $[0,T]$, which satisfies for a.e. t the differential inequality

$$\eta'(t) \le \phi(t)\eta(t) + \psi(t),$$

where $\phi(t)$ and $\psi(t)$ are nonnegative, summable functions on $[0,T]$. Then

$$\eta(t) \le e^{\int_0^t \phi(s)ds} \left[\eta(0) + \int_0^t \psi(s)ds \right]$$

for all $0 \le t \le T$.

(ii) In particular, if

$$\eta' \le \phi\,\eta \text{ on } [0,T] \text{ and } \eta(0) = 0$$

then

$$\eta \equiv 0 \text{ on } [0,T].$$

Proposition 2.2 (Poincaré's inequality [13]). *Assume U is a bounded, open subset of $\mathbb{R}^n$. Suppose $u \in W_0^{1,p}(U)$ for some $1 \le p < n$. Then we have the estimate*

$$\|u\|_{L^q(U)} \le C\|Du\|_{L^p(U)}$$

for each $q \in [1, \frac{np}{n-p}]$, the constant C depending only on p, q, n and U.

In particular, for all $1 \le p \le \infty$,

$$\|u\|_{L^p(U)} \le C\|Du\|_{L^p(U)}. \tag{3}$$

Next, we recall the Bounded Convergence Theorem.

Proposition 2.3 ([14]). *If f is a Lebesgue measurable function defined on a closed, bounded interval [a,b], with its L^p norm for $1 < p < \infty$ defined by*

$$\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p}$$

and L^∞ norm

$$\|f\|_\infty = \text{ ess. sup. } \{|f(x)| : a \leqq x \leqq b\}.$$

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If $f_1, f_2, f_3, \cdots$ is a sequence of measurable functions and M is a positive constant such that

$$\|f_n\|_\infty \leqq M \quad n = 1, 2, \cdots$$

and $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. $x \in [a, b]$ then

$$\lim_{n\to\infty} \|f_n - f\|_p = 0$$

for any p satisfying $0 < p < \infty$.

Now, we state the well-known Theorem.

Proposition 2.4 ([15]). *The space $L^2(0, T; W_0^{1,p}(0, 1))$ is complete and separable if $1 < p < \infty$.*

Proposition 2.5 (Jensen's inequality [16]). *Let (Ω, Λ, μ) be a measure space with $0 < \mu(\Omega) < \infty$ and let $\phi : I \to \mathbb{R}$ be a convex function defined on an open interval I in $\mathbb{R}$ if $f : \Omega \to I$ is such that $f, \phi \circ f \in L(\Omega, \Lambda, \mu)$, then*

$$\phi\left(\frac{1}{\mu(\Omega)} \int_\Omega f d\mu\right) \leq \frac{1}{\mu(\Omega)} \int_\Omega \phi(f) d\mu.$$

Proposition 2.6 ([17]). *If a sequence (f_n) of continuous functions $f_n : A \to \mathbb{R}$ converges uniformly on $A \subset \mathbb{R}$ to $f : A \to \mathbb{R}$, then f is continuous on A.*

3. Existence and Uniqueness

3.1. Existence

In this subsection, we will prove the existence of solutions to the integral equation (2).

Firstly, we choose the basis $\{e_i\}_{j\in\mathbb{N}} \in L^2(0, 1)$ defined as a subset of the eigenfunction of the Laplacian for the Dirichlet problem.

$$\begin{cases} -\partial_x^2 e_j = \lambda_j e_j, & j \in \mathbb{N}, \\ e_j = 0, & x \in \{0, 1\}, \end{cases} \tag{4}$$

where

$$e_j = \sqrt{2} \sin j\pi x, \quad \lambda_j = (j\pi)^2 \tag{5}$$

for $j \in \mathbb{N}$.
Since $\{e_j\}_{j\in N}$ is an orthonormal basis in $L^2(0, 1)$, then $\{e_j\}_{j\in N}$ is also an orthonormal basis in $W_0^{1,p}(0, 1)$.
Then for $f \in L^2(0, 1)$ we can write

$$f = \sum_{j=1}^{\infty} c_j(t) e_j,$$

with $c_j(t) = \left(\int_0^1 |f e_j|^2 dx\right)^{1/2} < \infty, \quad j = 1, 2, \ldots$ and the series $\{c_j\}_1^\infty$ is converges in $L^2(0, 1)$.
Hence, the approximate solution $u_n(t, x)$ can be represented as follows

$$u_n(t, x) = \sum_{j=1}^{n} c_j(t) e_j, \tag{6}$$

$$u_n(0) = u_{0n} = \sum_{j=1}^{n} c_j(0) e_j. \tag{7}$$

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Moreover, we suppose that $u_n(t,x)$ satisfy the following approximating problem

$$\int_0^1 \partial_t u_n e_j dx + \int_0^1 u_n \partial_x u_n e_j dx + \frac{1}{Re}\int_0^1 \left(|\partial_x u|^{p-2}\partial_x u\right)\partial_x e_j dx = 0, \tag{8}$$

$$u_n(0) = u_{0n}$$

for all $j = 1,\dots,n$, $t \in [0,T]$.

Remark 3.1. *The coefficients $c_j(0)$ can be chosen such that $\lim_{n\to\infty} u_{0n} = u(0)$ in $W_0^{1,p}(0,1)$.*

Furthermore, we prove the lemma concerning the solution of approximate problem (8).

Lemma 3.2. *There exists a positive constant $C_0 > 0$ independent on n, that satisfies the following inequality*

$$\|u_n\|^2_{L^2(0,1)} + \frac{2}{Re}\int_0^t \|\partial_x u_n\|^p_{L^p(0,1)} d\tau \leqslant C_0$$

for all $t \in [0,T]$.

Proof. Taking $e_j := u_n$ and substituting into equation (8), we obtain

$$\int_0^1 u_n \partial_t u_n dx + \int_0^1 u_n^2 \partial_x u_n dx + \frac{1}{Re}\int_0^1 |\partial_x u_n|^p dx = 0. \tag{9}$$

The first term of (9) can be written as follows

$$\int_0^1 u_n \partial_t u_n dx = \frac{1}{2}\frac{d}{dt}\int_0^1 u_n^2 dx = \frac{1}{2}\frac{d}{dt}\|u_n\|^2_{L^2(0,1)}. \tag{10}$$

Computing the second integral of (9) by the initial condition in (8) leads us to

$$\int_0^1 u_n^2 \partial_x u_n dx = \int_0^1 \partial_x\left(u_n^3\right) dx = u_n^3\Big|_0^1 = 0. \tag{11}$$

By substituting the integrals (10) and (11) into (9), then integrating the equation (9) from 0 to t with respect to τ, we have

$$\|u_n\|^2_{L^2(0,1)} + \frac{2}{\mathrm{Re}}\int_0^t \|\partial_x u_n\|^p_{L^p(0,1)} d\tau = \|u_{n0}\|^2_{L^2(0,1)}. \tag{12}$$

Application of inequality (3) to the left-hand side of (12) leads us to the assertion of the lemma. The proof of the lemma is complete. □

Lemma 3.3. *There exists at least one solution of the approximating problem (8) such that $u_n \in L^2(0,T;W_0^{1,p}(0,1))$.*

Proof. It is obvious that

$$\int_0^1 \langle e_i, e_j\rangle dx = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases} \tag{13}$$

We note that

$$\begin{aligned}\frac{\partial}{\partial x}\left(\left|\frac{\partial u_n}{\partial x}\right|^{m-2}\frac{\partial u_n}{\partial x}\right)\\ &= \left|\frac{\partial u_n}{\partial x}\right|^{m-1}\frac{\partial^2 u_n}{\partial x^2} + (m-1)\left|\frac{\partial u_n}{\partial x}\right|^{m-2}\frac{\partial^2 u_n}{\partial x^2}\frac{\partial u_n}{\partial x}\\ &= m\left|\frac{\partial u_n}{\partial x}\right|^{m-1}\frac{\partial^2 u_n}{\partial x^2}.\end{aligned} \tag{14}$$

Now, we represent the approximate equation (8) in the following form

$$\int_0^1 \frac{\partial u_n}{\partial t} e_j dx = -\int_0^1 u_n \partial u_n e_j dx - \frac{1}{Re}\int_0^1 \frac{\partial}{\partial x}\left(\left|\frac{\partial u_n}{\partial x}\right|^{m-1}\frac{\partial u_n}{\partial x}\right) e_j dx \tag{15}$$

for $j = 1, \ldots, n$.

The substitution (14) into (15) gives

$$\int_0^1 \frac{\partial u_n}{\partial t} e_j dx = -\int_0^1 u_n \partial u_n e_j dx - \frac{m}{Re}\int_0^1 \left|\frac{\partial u_n}{\partial x}\right|^{m-1}\frac{\partial u_n}{\partial x} e_j dx \tag{16}$$

for $j = 1, \ldots, n$. We find the derivative of (6) with respect to t, then by (13), the integral on the left-hand side of (16) can be written as below

$$\int_0^1 \frac{\partial u_n}{\partial t} e_j dx = \int_0^1 \sum_{i=1}^n \dot{c}_i(t) e_i e_j dx = \dot{c}_j(t), \quad j = 1, \ldots, n. \tag{17}$$

First integral on the right-hand side of (16), we rewrite as follows

$$\begin{aligned}\int_0^1 u_n \frac{\partial u_n}{\partial x} e_j dx &= \int_0^1 \left(\sum_{i=1}^n c_i(t) e_i\right)\left(\sum_{i=1}^n c_i(t) e_i'\right) e_j dx\\ &= \int_0^1 \sum_{i=1}^n c_i(t) e_i' c_j(t) dx\\ &= \left[\sum_{i=1}^n c_i(t)\int_0^1 e_i' dx\right] c_j(t)\end{aligned} \tag{18}$$

for $j = 1, \ldots, n$. Here $\sum_{i=1}^n c_i(t)\int_0^1 e_i' dx$ is bounded on $[0, T]$, since

$$\int_0^1 e_i' dx = a_i, \quad i = 1, \ldots, n \tag{19}$$

with $a_i < \infty$ constant numbers (see (5)).

Equation in (4) implies that

$$\frac{\partial^2 u_n}{\partial x^2} = \sum_{i=1}^n \lambda_i c_i(t) e_i. \tag{20}$$

By equation (20), we transform the second integral on the right-hand side of (16) in the following way

$$\frac{m}{Re}\int_0^1 \left|\frac{\partial u_n}{\partial x}\right|^{m-1}\frac{\partial^2 u_n}{\partial x^2} e_j dx = \frac{m\lambda_j c_j(t)}{Re}\int_0^1 \left|\sum_{i=1}^n c_i(t) e_i'\right|^{m-1} dx \tag{21}$$

for $j = 1, \ldots, n$.

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Thus, substitutions (17), (18), and (21) into (16) lead us to the following homogeneous ordinary system of equations with respect to t

$$\dot{c}_j(t) = \left\{ -\sum_{i=1}^{n} c_i(t) \int_0^1 e_i' dx - \frac{m\lambda_j}{Re} \int_0^1 \left| \sum_{i=1}^{n} c_i(t) e_i' \right|^{m-1} dx \right\} c_j(t) \tag{22}$$

for $j = 1, \ldots, n$.

Furthermore, we introduce the following notations

$$\vec{c}(t) = \begin{pmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_n(t) \end{pmatrix}, \quad \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad \vec{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}$$

with a_i $(i = 1, \ldots, n)$ from (19).

Therefore, by using our notations, the homogeneous system of ODE (22) can be written as follows

$$\dot{\vec{c}} = -(\vec{a} \cdot \vec{c})\vec{c} - \frac{m}{Re} \int_0^1 |\vec{e}\,' \cdot \vec{c}|^{m-1} \vec{\lambda} \odot \vec{c}\, dx. \tag{23}$$

Here $\lambda \odot \vec{c}$ is the Hadamard product of two vectors specified by the following formula

$$\vec{\lambda} \odot \vec{c} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} \odot \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \lambda_1 c_1 \\ \lambda_2 c_2 \\ \vdots \\ \lambda_n c_n \end{pmatrix}.$$

Now we denote the right-hand side of(23) by F

$$F(\vec{c}(t)) = -(\vec{a} \cdot \vec{c})\vec{c} - \frac{m}{Re} \int_0^1 |\vec{e}\,' \cdot \vec{c}|^{m-1} \vec{\lambda} \odot \vec{c}\, dx. \tag{24}$$

The first term on the right-hand side of (24) is the linear. The second term on the right-hand side of (24) is a homogeneous system of ODE.

Now we denote the second term of (24) by $\tilde{F}$.

$$F(\vec{c}) = \begin{cases} -(\vec{a} \cdot \vec{c})\vec{c} - \frac{m}{Re} \int_0^1 (\vec{e}\,' \cdot \vec{c})^{m-1} (\vec{\lambda} \odot \vec{c})\, dx, & \text{for } \vec{e}\,' \cdot \vec{c} > 0, \\ -(\vec{a} \cdot \vec{c})\vec{c} - \frac{m}{Re} \int_0^1 (-\vec{e}\,' \cdot \vec{c})^{m-1} (\vec{\lambda} \odot \vec{c})\, dx & \text{for } \vec{e}\,' \cdot \vec{c} < 0. \end{cases} \tag{25}$$

So, equation (23) can be rewritten as the homogeneous system of equations

$$\dot{c}_j(t) = F_j\big(c_j(t)\big), \quad j = 1, \ldots, n. \tag{26}$$

We show that

$$\nabla \vec{F}(c) = \begin{pmatrix} \frac{\partial F_1}{\partial c_1} & \frac{\partial F_1}{\partial c_2} & \ldots & \frac{\partial F_1}{\partial c_n} \\ \frac{\partial F_2}{\partial c_1} & \frac{\partial F_2}{\partial c_2} & \ldots & \frac{\partial F_2}{\partial c_n} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial F_n}{\partial c_1} & \frac{\partial F_n}{\partial c_2} & \ldots & \frac{\partial F_n}{\partial c_n} \end{pmatrix}$$

is bounded and continuous.

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If $\vec{e}\cdot\vec{c} > 0$, then

$$\frac{\partial F_i}{\partial c_j} = -\left[a_j c_i + (\vec{a}\cdot\vec{c})\cdot\delta_{ij}\right] - \frac{m}{Re}\int_0^1 \Big[(m-1)|\vec{e}\,'\cdot\vec{c}|^{p-2}\vec{e'}_j\lambda_i c_i + |\vec{e}\,'\cdot\vec{c}|^{p-1}\lambda_i\delta_{ij}\Big]dx.$$

Notice that

$$-\left[a_j c_i + (\vec{a}\cdot\vec{c})\delta_{ij} + \frac{m}{Re}\int_0^1 |\vec{e}\,'\cdot\vec{c}|^{p-1}\lambda_i\delta_{ij}dx\right]$$

is bounded and continuous in $\vec{c}$.

We only consider to prove that

$$\int_0^1 |\vec{e}\,'\cdot\vec{c}|^{p-2}\vec{e}_j\,'\lambda_i c_i dx = \lambda_i\int_0^1 |\vec{e}\,'\cdot\vec{c}|^{p-2}\vec{e}_j\,' c_i dx$$

is bounded and continuous at one point where $\vec{e}\,'\cdot\vec{c} = 0$. So, each integrand $|\vec{e}\,'\cdot\vec{c}|^{p-2}|\lambda_i e_i' c_j|$ of the following integrals

$$\int_0^1 |\vec{e}\,'\vec{c}|^{p-2}\begin{pmatrix} \lambda_1 e_1' c_1 & \cdots & \lambda_1 e_n' c_1 \\ \vdots & \ddots & \cdots \\ \lambda_n e_1' c_n & \cdots & \lambda_n e_n' c_n \end{pmatrix} dx \tag{27}$$

is bounded, if $\|\vec{c}\| \to 0$. By Lemma 3.2, we have

$$\|u_n\|^2_{L^2} = \langle u_n, u_n\rangle = \left\langle \sum_{i=1}^n c_i e_i, \sum_{i=1}^n c_i e_i \right\rangle = \sum_{i=1}^n c_i^2 = \|\vec{c}\|^2_{\mathbb{R}^n}.$$

Equation (12) implies that $\|\vec{c}\|^2_{\mathbb{R}^n} \leqslant \|\vec{c}(0)\|^2$. Hence, $\vec{c}$ is bounded in $\mathbb{R}^n$. Then there exists sequence $\{\vec{c}_k(t)\}_1^\infty$ that converges to $\vec{c}(t)$ in $\mathbb{R}^n$ for $0 < t < \infty$. Moreover, this sequence $\{\dot{\vec{c}}_k(t)\}_1^\infty$ satisfies the estimates

$$\|\dot{\vec{c}}_k\| \leq \|(\vec{a}\cdot\vec{c})\vec{c}\| + \left\|\frac{m}{Re}\int_0^1 |\vec{e}\,'\cdot\vec{c}|^{m-1}\vec{\lambda}\odot\vec{c}\,dx\right\| \leqslant \infty. \tag{28}$$

Consequently, there exist functions $\vec{c}(t)$, $\dot{\vec{c}}(t)$ such that $\lim_{n\to\infty}\vec{c}_k(t) = \vec{c}(t)$ and $\lim_{n\to\infty}\dot{\vec{c}}_k(t) = \dot{\vec{c}}(t)$. Functions $\vec{c}_k(t), \dot{\vec{c}}_k(t)$ are continuous and uniformly converge to $\vec{c}(t), \dot{\vec{c}}(t)$ in $\mathbb{R}^n$ for all $0 \leqslant t < +\infty$. Then by the Proposition 2.6 on continuity of the sum of uniformly convergent series, functions $\vec{c}(t)$, $\dot{\vec{c}}(t)$ in $\mathbb{R}^n$ are also continuous in $\mathbb{R}^n$ for all $0 \leqslant t < +\infty$. Thus, $\vec{c}(t) \in C([0,+\infty);\mathbb{R}^n)$ and $\dot{\vec{c}}(t) \in C^\infty([0,+\infty);\mathbb{R}^n)$.

Therefore, a system of equations (26) satisfies the condition of the existence of the homogeneous system of ODE. The right-hand side of equations (26) $F_j\big(c_j(t)\big)$ is continuous for all $j = 1,\ldots,n$ on the bounded domain $[0,T]$. Consequently, there exists a unique solution

$$\vec{c}(t) = (c_1(t), c_2(t), \ldots, c_n(t)) \tag{29}$$

of equation (26) on $[0,T]$ that satisfies the initial condition $\vec{c} = \vec{c}(0)$ or $\vec{c}(0) = (c_1(0), c_2(0), \ldots, c_n(0))$ for $t = 0$.

The proof of the lemma is complete. □

We are now ready to prove the following lemma.

Lemma 3.4. *There exists a weak solution of the equation (2) such that $u \in L^2(0,T;W_0^{1,p}(0,1))$.*

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Proof. Thus, by Lemma 3.3 there exists a weak solution u_n of the approximating problem (8) in $L^2(0,T;W_0^{1,p}(0,1))$. Moreover, by Lemma 3.2 the sequence of functions $\{u_n\}_{n=1}^{\infty}$ and $\{\partial_t u_n\}_{n=1}^{\infty}$ are bounded in $L^2(0,T;W_0^{1,p}(0,1))$. Therefore, by Proposition 2.4, the sequence $\{u_n\}_{n=1}^{\infty}$ has a subsequence that converges to some $u \in L^2(0,T;W_0^{1,p}(0,1))$. We denote this subsequence still by $\{u_n\}_{n=1}^{\infty}$, we obtain

$$\lim_{n\to\infty}\int_0^T \|u_n(t)-u(t)\|_{1,p}dt = 0,$$

which implies

$$\lim_{n\to\infty}\|u_n(t)-u(t)\|_{L,p}dt = 0$$

for a.e. t.

Since $L^2(0,T;W_0^{1,p}(0,1))$ is complete space, then any $\{u_n\}_{n=1}^{\infty} \in L^2(0,T;W_0^{1,p}(0,1))$ has a limit such that $u \in L^2(0,T;W_0^{1,p}(0,1))$. Since $\{u_n\}_{n=1}^{\infty} \in L^2(0,T;W_0^{1,p}(0,1))$, hence $\{\partial_x u_n\}_{n=1}^{\infty} \in L^2(0,T;L^p(0,1))$. Therefore $\lim_{n\to\infty}\partial_x u_n = \partial_x u$. □

3.2. Uniqueness

In the following lemma, we establish some auxiliary statements that we apply in the proof of Lemma 3.6.

Lemma 3.5. *Let $u \in W_0^{1,p}(0,1)$. Then there exists the positive number $C_1 < \infty$ such that $C_1 = \max_{x\in[0,1]}|\partial_x u|$.*

Proof. Since $u \in W_0^{1,p}(0,1)$, then the Embedding theorem

$$W_0^{1,p}(0,1) \subset C^1[0,1]$$

implies that $\partial_x u$ is continuous on $[0,1]$. Hence, by the Extreme Value theorem, if a function is continuous on a closed interval, it attains its maximum and minimum values on this interval. Consequently, there exists the positive number C_1 such that $C_1 = \max_{x\in[0,1]}|\partial_x u|$. □

Now we prove the uniqueness result.

Lemma 3.6. *If $C_1 < \infty$, the integral equation (2) has a unique weak solution.*

Proof. Assume that there exist two solutions of equation (2), $u_1(x,t)$ and $u_2(x,t)$. Taking $w := \hat{u}$ in the equation (2) and denoting by $\hat{u} = u_1 - u_2$, we obtain

$$\int_0^1 \hat{u}\partial_t\hat{u}dx + \int_0^1 (u_1\partial_x u_1 - u_2\partial_x u_2)\hat{u}dx + \frac{1}{\text{Re}}\int_0^1 \Big(|\partial_x u_1|^{p-2}\partial_x u_1 - \left|\partial_x u_2\right|^{p-2}\partial_x u_2\Big)\hat{u}dx = 0 \quad (30)$$

with $1 < p < 2$.

The first integral on the left-hand side of (30) can be written as (10)

$$\int_0^1 \hat{u}\partial_t\hat{u}dx = \frac{1}{2}\frac{d}{dt}\|\hat{u}\|_{L^2(0,1)}. \quad (31)$$

We note that

$$(u_1-u_2)(\partial_x u_1 - \partial_x u_2) = u_1\partial_x u_1 - u_2\partial_x u_1 - u_1\partial_x u_2 + u_2\partial_x u_2.$$

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Therefore,

$$u_1\partial_x u_1 - u_2\partial_x u_2 = (u_1-u_2)(\partial_x u_1 - \partial_x u_2) + u_2\partial_x(u_1-u_2) + (u_1-u_2)\partial_x u_2 = \hat{u}\partial_x\hat{u} + u_2\partial_x\hat{u} + \hat{u}\partial_x u_2.$$

Hence, the second integral of (30) can be represented in the following way

$$\int_0^1 (u_1\partial_x u_1 - u_2\partial_x u_2)\hat{u}dx = \int_0^1 \hat{u}^2\partial_x\hat{u}dx + \int_0^1 u_2\hat{u}\partial_x\hat{u}dx + \int_0^1 \hat{u}^2\partial_x u_2 dx. \tag{32}$$

By (11), the first term on the right-hand side of equation (32) equals zero.

By invoking Lemma 3.5, we bound the third integral on the right-hand side of (32)

$$|\int_0^1 \hat{u}^2\partial_x u_2 dx| \leqslant \max_{x\in[0,1]}|\partial_x u_2| \int_0^1 \hat{u}^2 dx = C_1\|\hat{u}\|^2_{L^2(0,1)}, \tag{33}$$

where $C_1 = \max_{x\in[0,1]}|\partial_x u_2|$.

Integrating by parts the second term on the right-hand side of equation (32), we have the same estimate as (33),

$$\left|\int_0^1 u_2\hat{u}\partial_x\hat{u}dx\right| = \left|-\frac{1}{2}\int_0^1 \hat{u}^2\partial_x u_2 dx\right| \leqslant \frac{C_1}{2}\|\hat{u}\|^2_{L^2(0,1)}.$$

So, we have the following estimate

$$\int_0^1 (u_1\partial_x u_1 - u_2\partial_x u_2)\hat{u}dx \leqslant \frac{3}{2}C_1\|\hat{u}\|^2_{L^2(0,1)} \tag{34}$$

for the second integral of (30).

Furthermore, we bound the third term on the left-hand side of (30). Therefore, we apply the following inequality [18]

$$(p-1)|b-a|^2 \le \langle |b|^{p-2}b - |a|^{p-2}a, b-a\rangle$$

for $b-a>0$, $a,b\in\mathbb{R}^n$ and $1<p<2$, to the last term of the equation (30)

$$\int_0^1 |\partial_x\hat{u}|^2\hat{u}_x dx \leqslant \int_0^1 \Big(|\partial_x u_1|^{p-2}\partial_x u_1 - |\partial_x u_2|^{p-2}\partial_x u_2\Big)\hat{u}_x dx. \tag{35}$$

The left-hand side of (35) can be written as follows

$$\int_0^1 |\partial_x\hat{u}|^2\partial_x\hat{u}dx = \int_0^1 |\partial_x\hat{u}|^3 dx = \|\partial_x u\|^3_{L^3(0,1)}.$$

By Jensen's inequality, we bound the norm in the last equation from below

$$\|\partial_x u\|_{L^2(0,1)} = \Big(\int_0^1 |\partial_x\hat{u}|^2 dx\Big)^{\frac{1}{2}} = \Big[\Big(\int_0^1 |\partial_x\hat{u}|^{3\cdot\frac{2}{3}}dx\Big)^{\frac{2}{3}\cdot\frac{1}{2}}\Big]^{\frac{3}{2}} \leqslant \Big(\int_0^1 |\partial_x\hat{u}|^{3\cdot\frac{2}{3}\cdot\frac{3}{2}}dx\Big)^{\frac{1}{3}} \leqslant \|\partial_x u\|_{L^3(0,1)}.$$

The last inequality and the inequality (35) implies that

$$\|\partial_x u\|^3_{L^2(0,1)} \leqslant \int_0^1 \Big(|\partial_x u_1|^{p-2}\partial_x u_1 - |\partial_x u_2|^{p-2}\partial_x u_2\Big)\hat{u}_x dx. \tag{36}$$

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By combining inequalities (36), (34) and (31), the equation (30) implies that

$$\frac{d}{dt}\|\hat{u}\|^2_{L^2(0,1)} + \frac{2}{Re}\|\partial_x\hat{u}\|^3_{L^2(0,1)} \leqslant 3C_1\|\hat{u}\|^2_{L^2(0,1)}.$$

Using Poincare's inequality (2.2) to the last inequality leads us

$$\frac{d}{dt}\|\hat{u}\|^2_{L^2(0,1)} + \frac{2}{\mathrm{Re}}\|\hat{u}\|^3_{L^2(0,1)} \leqslant 3C_1\|\hat{u}\|^2_{L^2(0,1)}.$$

Combining the similar terms in the last inequality, we conclude that there exists a non-negative constant K such that

$$\frac{d}{dt}\|\hat{u}\|^2_{L^2(0,1)} \leqslant K\|\hat{u}\|^2_{L^2(0,1)} \tag{37}$$

with $K = \left(3C_1 + \frac{2}{Re}\|\hat{u}\|_{L^2(0,1)}\right)$.

To the inequality (37) using Gronwall's Inequality that stated in Proposition 2.1, we have that $\hat{u} = 0$. Thus, the solution of the problem (9) is unique. The proof of the lemma is complete. □

4. Numerical Part

In this section, we first create a data set by solving numerically the Reynolds number with prescribed Power-law rheology parameters. The corresponding parameters and values for the Reynolds number are listed in Table 1. Problem (1) is solved using the finite element software COMSOL Multiphysics 6.0. To apply the software COMSOL, the equation in (1) has been written in a standard form for the solver

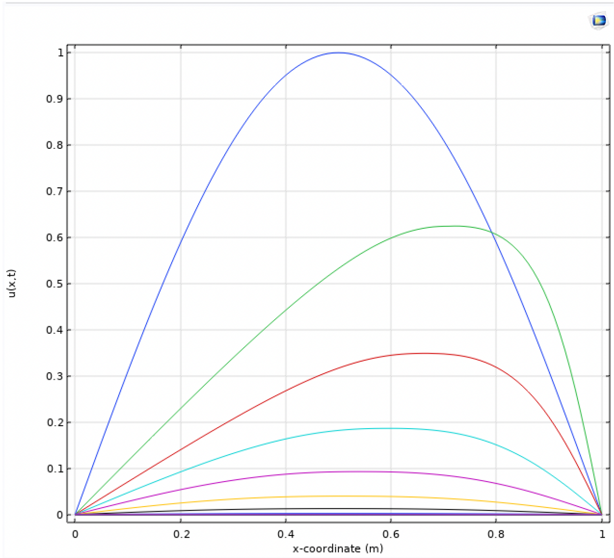
$$e_a\frac{\partial^2 u}{\partial t^2} + d_a\frac{\partial u}{\partial t} + \nabla\cdot(-c\nabla u - \alpha u + \gamma) + \beta\cdot\nabla u + au = f,$$

where $\nabla = \partial_x$ and $u = u(t,x)$. We construct the numerical solutions of problem (1) taking coefficients $e_a = 0$, $d_a = 1$, $a = 0$, $f = 0$, $\alpha = 0$, $\gamma = 0$, $\beta = u$ and $c = \frac{1}{Re}(|\nabla u|)^{m-1}$ in the last equation. The Dirichlet boundary condition is applied on both ends of the computational domain.

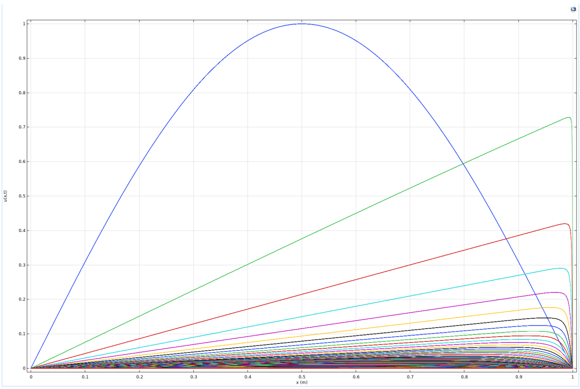
Table 1. Parameter values for Power-law fluids.

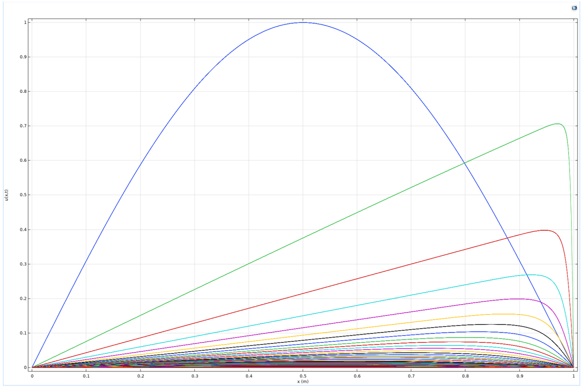
Parameters	Case 1	Case 2	Case 3
$\rho\ (kg/m^3)$	1000	1100	1200
$u\ (m/s)$	0.10	10.0	2.00
$L\ (m)$	0.05	0.05	0.10
n	0.70	0.70	0.80
$K\ (Pa\cdot s^n)$	1.00	2.00	2.00
$\gamma\ (s^{-1})$	1000	200	500
Re	9.98	537.1	165.4

Figure 1 shows the simulated solutions obtained for the Burgers' Equation based on a Power-law model with three different Reynolds numbers. In each case, the velocity profile has been shown for several instances of time. It can be observed that within a finite time T, the solution u and its derivative u_x are bounded in $[0,1]$, as predicted by our theoretical results.



(a) Reynolds number $Re = 9.98$





(b) Reynolds number $Re = 537.1$

(c) Reynolds number $Re = 165.4$

Figure 1. Velocity components for the flow of Power-law fluid.

5. Conclusions

This work proves the existence and uniqueness of weak solutions of the viscous Burgers equation for the isothermal flow of power-law non-Newtonian fluids with initial and boundary conditions. A numerical experiment on COMSOL Multiphysics was conducted, confirming the theoretical part.

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