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[Michael Anthony](#)*

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Article

Infinitude Condition for Mersenne and Sophie Germain Primes

Michael Mark Anthony

Analytic Foundations of Infinite Class symmetry

Abstract

This paper explores both Mersenne primes of the form $2^p - 1$ where, p is a prime. By extension, the paper also explores Perfect numbers and Sophie primes. Botgh these special primes have a special relationship for “perfectitude”. An insight into these numbers is explored using novel methods that involve the trigonometric functions with integer factorable arguments. Rational functions play a part in the behavior of many functions including regular primes, Mersenne Primes, and Perfect numbers. The paper first determines relationships for primes, and then procedes to show how Perfect number relations can be derived from trigonometric relations. The relationships of trigomentric functions involving the sum of divisors, provide a novel approach to prove that that the analytic structure of $\cot(x)$, when split into Mersenne and non-Mersenne classes through the Bernoulli framework, forces a coupling between the two infinite subsets of integers and the contradiction (negative ratio despite all positive terms) is a proof of necessity for infinite balance between both classes. This paper also explores Sophie Germain primes of the form $2p + 1$ where, p is a prime. By extension, the paper also explores other properties of prime numbers. I derive a trigonometric product identity that isolates the condition for a prime p to form a Sophie Germain prime pair, i.e., such that $2p + 1$ is also prime. The analysis shows that the classical tangent-sine product expansion, when modified by the divisor-sum function, $\sigma(n)$, reproduces a constant equality only under this primality constraint.

Keywords: Mersenne primes; SOPHIE GERMAIN PRIMES; Perfect numbers; abundant numbers; deficient numbers; trigonometric functions; Primes; Cot; trigonometry; sums of divisors; invariance

1. Introduction

The search for a general formula to determine the n^{th} Mersenne prime is an ongoing challenge in mathematics. Mersenne primes are of the form $M_p = 2^p - 1$, where p is a prime number, and M_p is also a prime number. Not all primes p , can generate a Mersenne prime M_p . For example, the primes, 11, 23, 29, are examples that do not generate Mersenne Primes, M_p , they generate what I refer to as Mersenne Numbers M_n , that have the Mersenne form $M_n = 2^p - 1$, where p is a non-generating prime, and M_n is not. It is extremely difficult to find the Mersenne primes, M_p , without tedious factorization, since the known set of Mersenne primes M_p are separated by long distances of non-primes, M_n .

Perfect numbers, N_p , are numbers defined by the product $N_p = (2^p - 1)2^{p-1}$, where, p is a prime that generates a Mersenne prime. They have the Sum of Divisors relation, $\sigma(N_p) = 2N_p$. These numbers are related to Mersenne primes, $M_p = 2^p - 1$, by the relation, $N_p = (2^{p-1} - 1)M_p$. Hence the search for Mersenne primes, M_p , is also the search for Perfect numbers, N_p . It is not known in current art if there are infinitely many Perfect Numbers, N_p and also if there is infinitely many Mersenne primes, M_p . So far, all N_p are even numbers, and it is still not yet determined if there are any odd N_p .The approach used in this paper on Mersenne Primes, M_p and Perfect numbers, N_p is so far as I know, has not yet been used by researchers.

The Gamma-function, denoted as $\Gamma(s)$, was first introduced by Swiss mathematician Leonhard Euler [1] 1729. Euler’s deep insights into Γ -function led to numerous results that provide key insights into many fields of mathematics including Probability theory and Statistics. Other major contributions to the development of the Γ -function used in this paper were developed by Carl

Freidman Gauss [2]. Gauss’s work led to the famous reflection formula of the ζ -function. A key insight into the Γ -function is its multiplicative nature. New results will be presented in this paper resulting from the properties of the Γ -function. So far, there has been little development in the additive representation of the Γ -function as a series of simple terms. The form of the Γ -function [3], p.895:

$$\Gamma(s)\sim z^{s-\frac{1}{2}}e^{-s}\sqrt{2\pi}\left\{1+\frac{1}{12z}+\frac{1}{288s^2}-\frac{139}{51840s^3}-\frac{571}{2488320s^4}+O(s^{-5})\right\},[|\arg s|<\pi] \tag{1.}$$

for s real and positive is well known. Here, the remainder of the series (1) is less than the last term that is retained.

Similar series exists for $\ln \Gamma(s)$. It will be significant if other forms of these series can be found.

The product-form of the Γ -function due to Gauss, provides further insights into many relations that will be developed in this paper. The product form is given by, [4], p. 896:

$$\Gamma(y \cdot n) = (2\pi)^{\frac{1-y}{2}}y^{(n \cdot y)-\frac{1}{2}}\prod_{k=0}^{y-1}\Gamma\left(n+\frac{k}{y}\right) \tag{2.}$$

Certain invariant relations of the product Γ -function will be developed in this paper to show the connections of the Γ -function to other functions, particularly the Riemann-Zeta function, denoted by $\zeta(s)$. The ζ -function, is defined by the additive series:

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{n=1}^{\infty} n^{-s}, \Re(s) > 1 \tag{3.}$$

The importance of the ζ -function is its relation to the distribution of primes and the Riemann hypothesis. There is a one-on-one correspondence between the non-trivial roots of the function and the primes. The ζ -function also has a product relation for primes p , given by [4], p. 1037;

$$\zeta(s) = \prod_p \left(\frac{1}{1-p^{-s}}\right), \Re(s) > 1 \tag{4.}$$

Both the ζ -function, and the Γ -function are factorable. These two functions are related by the ζ -function reflection formula developed by Gauss given by [4], p.1038:

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\pi^{\frac{s-1}{2}}\zeta(1-s) \tag{5.}$$

These relations are well studied, and they provide a wealth of information in Number theory and many disciplines in Mathematics. In this article, I show new relations that govern Mersenne primes and twin primes. All these special integer relations are connected in precious way by powers of 2π .

2. Mersenne Numbers

Mersenne primes were named after the French philosopher and number theorist, Marin Mersenne (1588-1648). Marin Mersenne was also a monk and a theologian, and he had an important influence on many academics such as Fermat, Pascal, Huygens, Descartes and Galileo. He also inspired the invention of the pendulum clock.

Only a few Mersenne primes, M_p are known to exists. It is an arduous task to determine whether a Mersenne number, M_n is either a Mersenne prime, M_p prime or a Mersenne number M_n , since the computation of factors of large Mersenne numbers, M_n is very difficult. When p is a prime, not all $M_n = 2^p - 1$ are Mersenne primes, and it is not known whether there are infinitely many Mersenne primes, M_p . The Great Internet Mersenne Prime Search (GIMPS) has discovered a new Mersenne prime number, $M_p = 282,589,933 - 1$. The first few Mersenne primes are $M_p \in 3, 7, 31, 127, 8191, 131071, 524287, 2147483647, \dots$ (Online Encyclopedia of Integer Sequences, (OEIS) #A000668), corresponding to indices $n \in 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 859433, 1257787, 1398269, 2976221, 3021377, 6972593, 13466917, 20996011, 24036583, 25964951, 30402457, 32582657, 37156667, 42643801, 43112609, 57885161 \dots$ (OEIS A000043).

It is conjectured that there exist an infinite number of Mersenne primes. In Wolfram, we find the best fit line through the origin to the asymptotic number of Mersenne primes M_p with $p \leq \ln x$, for the first 51 known Mersenne primes. The best-fit line gives $C(x) = 2.51763 \ln x$. This fit is illustrated below in Figures 1 and 2. It has been conjectured without any particularly strong evidence, that the constant is given by $e^{\lambda}\sqrt{2} = 2.518..$, where λ is the Euler-Mascheroni constant.

In this paper, I will give strong relations for this constant.

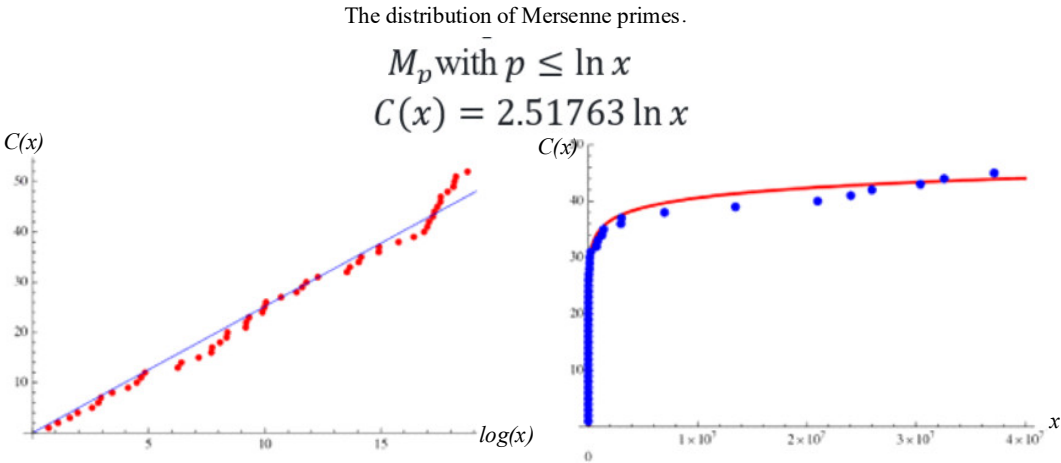


FIGURE 1

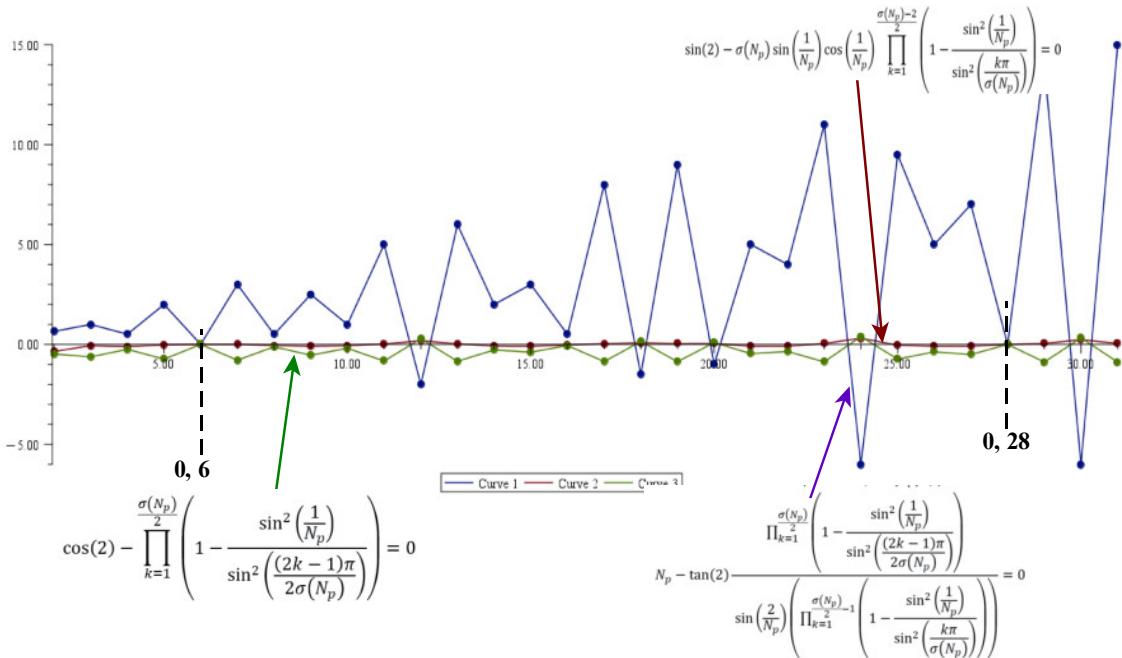


FIGURE 2

Literature on Mersenne primes is mainly dedicated to the search for new Mersenne primes, and very few attempts have made progress on the actual theoretical work. In [8], Zhaodong Cai, Matthew Faust, A.J. Hildebrand, Junxian Li, and Yuan Zhang studied the leading digits of the Mersenne primes. They attempted to show that leading digits of Mersenne numbers behave in many respects more regularly than some sequences of powers of logs of 2. Further information on Mersenne primes can be found in [8–11]. In [12] J. Aust yield bounds on the sums of exponents of Mersenne primes.

Most of this research is related to the present work only in an attempt to categorize properties that Mersenne primes may have found to have, however, the present paper does not rely on any of the current work known on Mersenne primes, but starts a new trend in expoloring the properties of Mersenne primes. To begin, let us explore the concepts that lead to the final proof.

3. The Invariance of the Gamma Function to Substitution $\sigma(m) \rightarrow \sigma(m + j)$

I first want to introduce the curious fact that any function with a relational product $\{n \cdot y\}$, can be represented by the Sums of Divisor function, $\sigma(m)$. Here is a simple example:

$$\log(n \cdot y) = \log n + \log y, (6.)$$

Then, if $n \cdot y = m$, we can put $n = \sigma(m), y = \frac{m}{\sigma(m)}$, and so,

$$\log(m) = \log \sigma(m) + \log \frac{m}{\sigma(m)} (7.)$$

Then, if $n \cdot y = N_p$, we can put $n = \sigma(N_p), y = \frac{N_p}{\sigma(N_p)}$, then, a Perfect number N_p , has the relation:

$$\log(N_p) = \log(\sigma(N_p)) + \log\left(\frac{N_p}{\sigma(N_p)}\right) (8.)$$

$$\log(N_p) = \log(\sigma(N_p)) + \log\left(\frac{1}{2}\right) (9.)$$

Here is another example:

If $n \cdot y = m$, we can put $n = \sigma(m), y = \frac{m}{\sigma(m)}$, and so, applied to the formula [3], p.41:

$$\begin{aligned} \sin(n \cdot x) &= n \sin(x) \cos(x) \prod_{k=1}^{\frac{n-2}{2}} \left(1 - \frac{\sin^2(x)}{\sin^2\left(\frac{k\pi}{n}\right)}\right), [n \text{ is even}] (10.) \\ \cos(n \cdot x) &= \prod_{k=1}^{\frac{n}{2}} \left(1 - \frac{\sin^2(x)}{\sin^2\left(\frac{(2k-1)\pi}{2n}\right)}\right) \end{aligned}$$

$$\begin{aligned} \sin(n \cdot x) &= n \sin(x) \prod_{k=1}^{\frac{n-1}{2}} \left(1 - \frac{\sin^2(x)}{\sin^2\left(\frac{k\pi}{n}\right)}\right), [n \text{ is odd}] (11.) \\ \cos(n \cdot x) &= \cos(x) \prod_{k=1}^{\frac{n-1}{2}} \left(1 - \frac{\sin^2(x)}{\sin^2\left(\frac{(2k-1)\pi}{2n}\right)}\right) \end{aligned}$$

Interestingly, (10) $\in$ even, and (11) $\in$ odd, differentiate between odd and even values of n . Since primes have $\sigma(p) = p + 1$, an even number, and $p + 1$ is always even except for the prime 2, the relations (11) $\in$ odd and does not apply to primes! Since $\sigma(2) = 3$. For example,

$$\cos(2) = \cos\left(\frac{2}{3}\right) \prod_{k=1}^1 \left(1 - \frac{\sin^2\left(\frac{2}{3}\right)}{\sin^2\left(\frac{(2k-1)\pi}{6}\right)}\right), [\sigma(2) \text{ is odd}] (12.)$$

$$-0.4161468365 \dots = 0.7858872608 \dots \left(1 - \frac{0.3823812134 \dots}{0.2500000000}\right) = -0.4161468365 \dots (13.)$$

By using the sum of divisor function, for Perfect numbers, N_p , the even trigonometric relations [(10), (11)] $\in$ even, apply, but the relations, [(12), (13)] $\in$ odd do not apply, so we can put, $\sigma(N_p) =$

$2N_p$. The fact that the sum of divisor function $\sigma(m)$, can be manipulated this way leads to some interesting formulas that can produce significant and unexpected results.

4. Application of the Trigonometric Function to Perfect Numbers

A Perfect Number N_p , is defined as a number for which $\sigma(N_p) = 2N_p$. A list of some known Perfect numbers is

$$N_p \in \{6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128, 2658455991569831744654692615953842176, \dots\}$$

Hence for, example, in (10), putting $n = \sigma(j)$, (n even), $x = \frac{1}{j}$: then, we have

$$\sin\left(\frac{\sigma(j)}{j}\right) = \sigma(j) \sin\left(\frac{1}{j}\right) \cos\left(\frac{1}{j}\right) \prod_{k=1}^{\frac{\sigma(j)-2}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{k\pi}{\sigma(j)}\right)}\right), [\sigma(j) \text{ is even}] \quad (14.)$$

$$\cos\left(\frac{\sigma(j)}{j}\right) = \prod_{k=1}^{\frac{\sigma(j)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)}\right)$$

$$\tan\left(\frac{\sigma(j)}{j}\right) = \frac{\sigma(j) \sin\left(\frac{1}{j}\right) \cos\left(\frac{1}{j}\right) \prod_{k=1}^{\frac{\sigma(j)-2}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{k\pi}{\sigma(j)}\right)}\right)}{\prod_{k=1}^{\frac{\sigma(j)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)}\right)} [\sigma(j) \text{ is even}] \quad (15.)$$

LEMMA 1: The rational trigonometric functions $\sin\left(\frac{\sigma(j)}{j}\right), \cos\left(\frac{\sigma(j)}{j}\right)$ determine Perfect Numbers.

Proof:

$$\sigma(j) = \left[\frac{\tan\left(\frac{\sigma(j)}{j}\right) \prod_{k=1}^{\frac{\sigma(j)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)}\right)}{\sin\left(\frac{1}{j}\right) \cos\left(\frac{1}{j}\right) \prod_{k=1}^{\frac{\sigma(j)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)}\right)} \right] \quad (16.)$$

$$\sigma(j) = 2 \left[\frac{\tan\left(\frac{\sigma(j)}{j}\right) \prod_{k=1}^{\frac{\sigma(j)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)}\right)}{2 \sin\left(\frac{1}{j}\right) \cos\left(\frac{1}{j}\right) \prod_{k=1}^{\frac{\sigma(j)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)}\right)} \right] \quad (17.)$$

$$\sigma(j) = 2 \left[\frac{\tan\left(\frac{\sigma(j)}{j}\right) \prod_{k=1}^{\frac{\sigma(j)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)}\right)}{\sin\left(\frac{2}{j}\right) \prod_{k=1}^{\frac{\sigma(j)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)}\right)} \right] \quad (18.)$$

If $j = N_p$ is a Perfect number, then, the equality applies only when.

$$N_p = \frac{\tan\left(\frac{\sigma(N_p)}{N_p}\right) \prod_{k=1}^{\frac{\sigma(N_p)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(N_p)}\right)}\right)}{\sin\left(\frac{2}{N_p}\right) \left(\prod_{k=1}^{\frac{\sigma(N_p)}{2}-1} \left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{\sigma(N_p)}\right)}\right)\right)} \quad (19.)$$

Taking the limits:

$$\lim_{N_p \rightarrow \infty} N_p = \lim_{N_p \rightarrow \infty} \left(\tan(2) \left\{ \frac{\prod_{k=1}^{\frac{\sigma(N_p)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(N_p)}\right)}\right)}{\sin\left(\frac{2}{N_p}\right) \left(\prod_{k=1}^{\frac{\sigma(N_p)}{2}-1} \left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{\sigma(N_p)}\right)}\right)\right)} \right\} \right) \quad (20.)$$

Now, for large values of y , $\sin\left(\frac{1}{y}\right) \rightarrow \frac{1}{y}$, and so we can approximate the product for large values of N_p as follows:

$$\lim_{N_p \rightarrow \infty} N_p = \lim_{N_p \rightarrow \infty} \left(\frac{\tan(2)}{\sin\left(\frac{2}{N_p}\right)} \left(1 - \left(\frac{2\sigma(N_p)}{N_p(\sigma(N_p)-1)\pi}\right)^2\right) \left\{ \prod_{k=1}^{\frac{\sigma(N_p)}{2}-1} \frac{\left(1 - \left(\frac{2\sigma(N_p)}{N_p(2k-1)\pi}\right)^2\right)}{\left(1 - \left(\frac{\sigma(N_p)}{N_p k\pi}\right)^2\right)} \right\} \right) \quad (21.)$$

$$\lim_{N_p \rightarrow \infty} N_p = \lim_{N_p \rightarrow \infty} \left(N_p \frac{\tan(2)}{2} \left\{ \prod_{k=1}^{\frac{\sigma(N_p)}{2}-1} \frac{\left(1 - \left(\frac{2\sigma(N_p)}{N_p(2k-1)\pi}\right)^2\right)}{\left(1 - \left(\frac{\sigma(N_p)}{N_p k\pi}\right)^2\right)} \right\} \right) \quad (22.)$$

Put $\frac{\sigma(N_p)}{N_p} = x = 2$,

$$1 = \frac{\tan(2)}{2} \left\{ \prod_{k=1}^{\infty} \frac{\left(1 - \frac{4(x)^2}{(2k-1)^2\pi^2}\right)}{\left(1 - \frac{(x)^2}{k^2\pi^2}\right)} \right\} \quad (23.)$$

For the infinite product we have,

$$\frac{\sin(x)}{x} = \prod_{k=1}^{\infty} \left(1 - \left(\frac{x}{k\pi}\right)^2\right), \quad \cos(x) = \prod_{k=0}^{\infty} \left(1 - \frac{4(x)^2}{(2k-1)^2\pi^2}\right) \quad (24.)$$

$$1 = \frac{\tan(2)}{2} \left\{ \frac{2 \cos(2)}{\sin(2)} \right\} = 1 \quad (25.)$$

$$\sin\left(\frac{\sigma(N_p)}{N_p}\right) = \sigma(N_p) \sin\left(\frac{1}{N_p}\right) \cos\left(\frac{1}{N_p}\right) \prod_{k=1}^{\frac{\sigma(N_p)-2}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{\sigma(N_p)}\right)}\right), [\sigma(N_p) \text{ is even}] \quad (26.)$$

$$\cos\left(\frac{\sigma(N_p)}{N_p}\right) = \prod_{k=1}^{\frac{\sigma(N_p)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(N_p)}\right)}\right)$$

It is clear that there if there exists a continued set of infinitely large Perfect Numbers then,

$$\left. \begin{aligned} &\sin(2) - \sigma(N_p) \sin\left(\frac{1}{N_p}\right) \cos\left(\frac{1}{N_p}\right) \prod_{k=1}^{\frac{\sigma(N_p)-2}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{\sigma(N_p)}\right)}\right) \\ &\cos(2) - \prod_{k=1}^{\frac{\sigma(N_p)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(N_p)}\right)}\right) \\ &\prod_{k=1}^{\frac{\sigma(N_p)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(N_p)}\right)}\right) \\ &N_p - \tan(2) \frac{\prod_{k=1}^{\frac{\sigma(N_p)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(N_p)}\right)}\right)}{\sin\left(\frac{2}{N_p}\right) \prod_{k=1}^{\frac{\sigma(N_p)}{2}-1} \left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{\sigma(N_p)}\right)}\right)} \end{aligned} \right\} = 0, \begin{cases} [\sigma(N_p) \text{ is even}], (=) \text{ for } n \in N_p \\ \text{otherwise for } n \notin N_p \end{cases} \quad (27.)$$

Each of these three relations is only true when N_p is a Perfect number.

Figure 2 shows the correlation of the relation (27) with Perfect Numbers.

From symmetry, and considering the form for the divisor function:

$$N_p = \frac{\tan(2) \prod_{k=1}^{N_p} \left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{4N_p}\right)}\right)}{\sin\left(\frac{2}{N_p}\right) \left(\prod_{k=1}^{N_p-1} \left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{2N_p}\right)}\right)\right)} \quad (28.)$$

Since $N_p = (2^p - 1)2^{p-1}$, where p is a prime, we can factor the perfect number N_p , as follows:

$N_p = (2^p - 1)2^{p-1} = (2P - 1)P$, where $P = 2^{p-1}$. This factorization leads to the following results:

$$\begin{aligned}
 F(P) &= P - \frac{\tan\left(\frac{\sigma(P)}{P}\right) \prod_{k=1}^{\frac{\sigma(P)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{P}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(P)}\right)}\right)}{\sin\left(\frac{2}{P}\right) \left(\prod_{k=1}^{\frac{\sigma(P)}{2}-1} \left(1 - \frac{\sin^2\left(\frac{1}{P}\right)}{\sin^2\left(\frac{k\pi}{\sigma(P)}\right)}\right) \right)} \\
 G(P) &= 2P - 1 - \frac{\tan\left(\frac{\sigma(2P-1)}{2P-1}\right) \prod_{k=1}^{\frac{\sigma(2P-1)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{2P-1}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(2P-1)}\right)}\right)}{\sin\left(\frac{2}{2P-1}\right) \left(\prod_{k=1}^{\frac{\sigma(2P-1)}{2}-1} \left(1 - \frac{\sin^2\left(\frac{1}{P}\right)}{\sin^2\left(\frac{k\pi}{\sigma(2P-1)}\right)}\right) \right)}
 \end{aligned} \quad (29.)$$

It is clear that there is a direct correspondence between the Perfect Number N_p , and P . The graphs of the two functions is shown in Figure 3.

$$F(P) = P - \frac{\tan\left(\frac{\sigma(P)}{P}\right) \prod_{k=1}^{\frac{\sigma(P)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{P}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(P)}\right)}\right)}{\sin\left(\frac{2}{P}\right) \left(\prod_{k=1}^{\frac{\sigma(P)}{2}-1} \left(1 - \frac{\sin^2\left(\frac{1}{P}\right)}{\sin^2\left(\frac{k\pi}{\sigma(P)}\right)}\right) \right)} \quad (30.)$$

Graphs of the two functions, showing that the zeroes are strictly on the correspondence *Perfect Numbers* $\rightarrow P$.

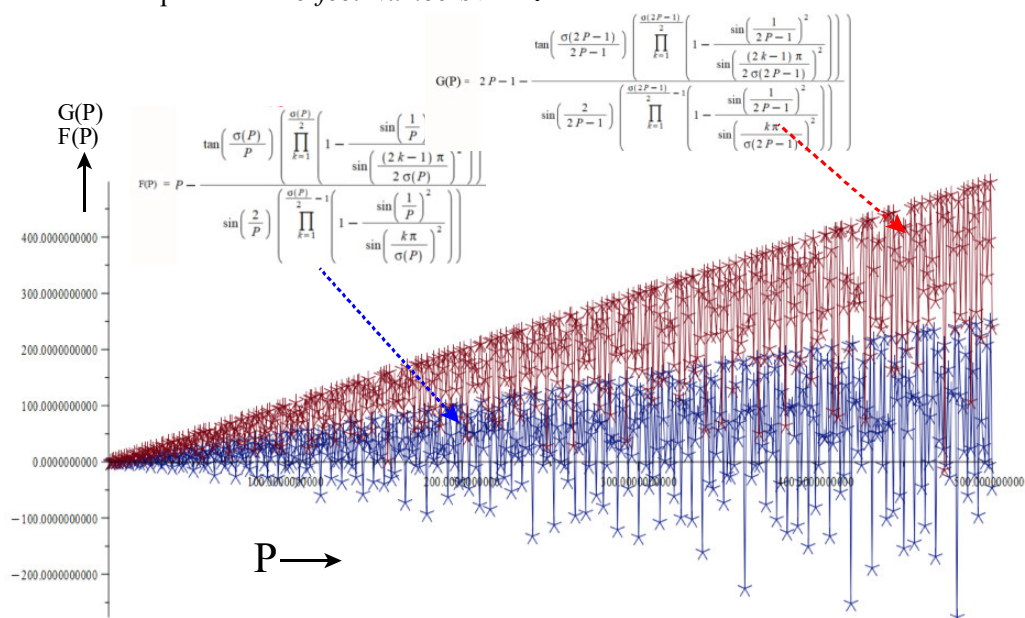


FIGURE 3

Figure 4 shows the correspondence $F(P) \rightarrow N_p$.

Graphs of the two functions, showing that the zeroes of $F(P)$ are strictly on the correspondence *Perfect Numbers* P .

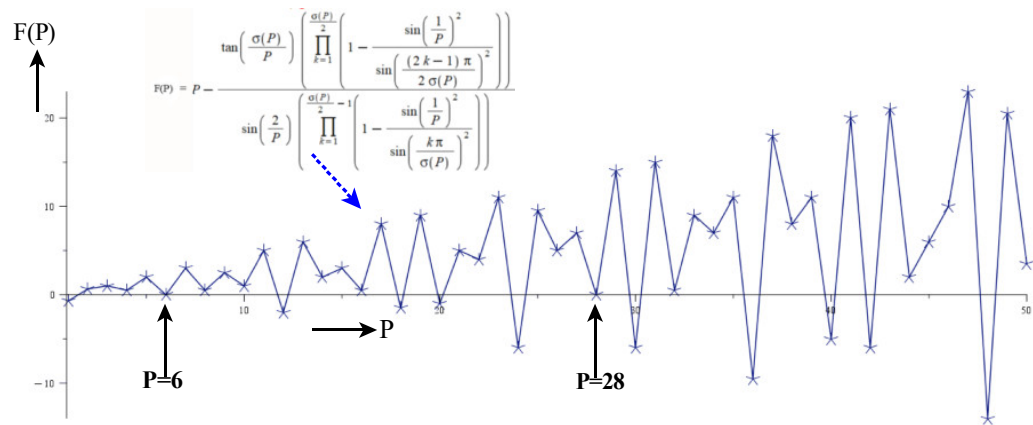


FIGURE 4

FIGURE 5 shows the symmetry of the odd and even product expressions.

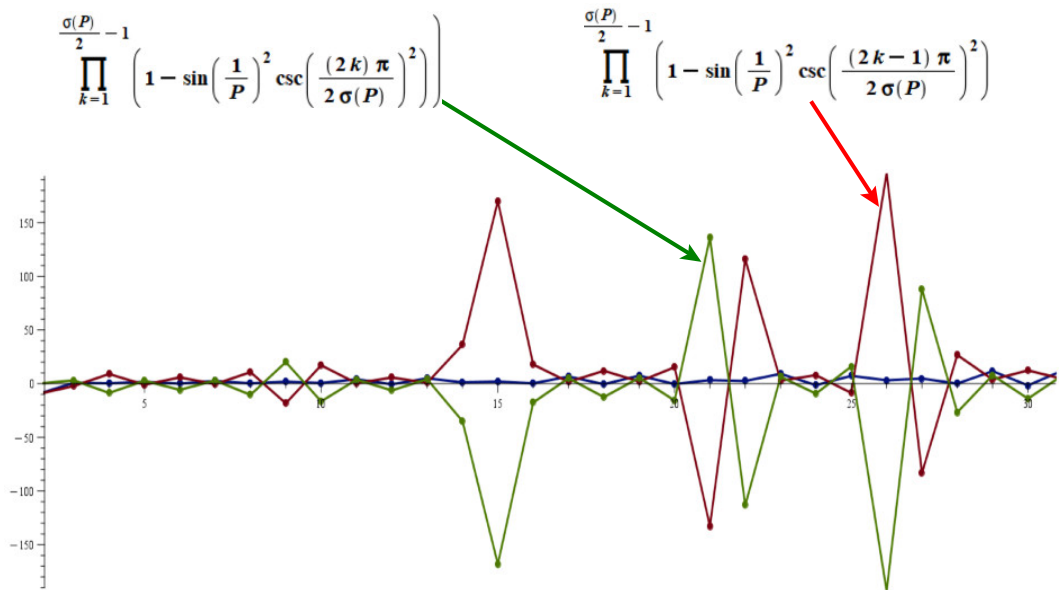


FIGURE 5

The relations (19) hold for all Perfect Numbers. The right hand side of (19) does not depend on implicit rational relationships between $\sigma(N_p)$ and N_p . It is clear that the basic rational trigonometric functions capture the properties of integers. We now explore the general forms of trinometric and exponential forms that capture Perfect numbers, Abundant numbers and deficient numbers in one relation.

5. The General Relation That Captures the Behavior of Abundant Numbers, Perfect Numbers and Deficient Numbers

Definition 1: An **Abundant number** is a positive integer for which the sum of its proper divisors excluding itself is greater than the number itself.

Definition 2: A **Perfect number** is a number for which the sums of all divisors is equal to twice the number.

Definition 3: A **Deficient number** is a number for which the sums of all divisors is less than twice the number.

Lemma 2: If n is a Perfect number, then,

$$\frac{\cos\left(\frac{2\pi n}{\sigma(n)}\right)}{\sin\left(\frac{\pi n}{\sigma(n)}\right)} = -1 \text{ (31.)}$$

Proof: for a Perfect number, $\sigma(n) = 2n$. Hence,

$$\frac{\cos(\pi)}{\sin\left(\frac{\pi}{2}\right)} = -1 \text{ (32.)}$$

The distribution of **perfect numbers**, **abondant numbers** and **deficient numbers** is captured by the general relation:

$$\cos\left(\frac{2n\pi}{\sigma(n)}\right) + \sin\left(\frac{n\pi}{\sigma(n)}\right) = 0 \text{ (33.)}$$

- a. For perfect numbers, $\frac{2n}{\sigma(n)} = 1$, and the relation (33) vanishes.
- b. For **abondant numbers**, $\frac{2n}{\sigma(n)} < 1$, and the relation does not vanish but generates negetaive imaginary values for $n \in$ **abondant numbers**.
- c. For **deficient numbers**, $\frac{2n}{\sigma(n)} < 1$, and the relation does not vanish but generates positive imaginary values for $n \in$ **deficient numbers**.

To see this, put the relation (33) in the form:

$$\frac{\cos\left(\frac{2n\pi}{\sigma(n)}\right)}{\sin\left(\frac{n\pi}{\sigma(n)}\right)} = -1 \sin\left(\frac{n\pi}{\sigma(n)}\right) \neq 0, \text{ (34.)}$$

Obviously, the zeros of the function (34) occur at the Perfect numbers. However, for clarity we convert this relation to the exponential form:

$$F(n) = 1 - ie^{-\frac{in\pi}{\sigma(n)}} - e^{-\frac{2inn\pi}{\sigma(n)}} - ie^{\frac{3inn\pi}{\sigma(n)}} \text{ (35.)}$$

Figure 6 shows the complex map of the function $F(n)$, over the range $n = 0..20,000$.

The map of the first 20,000 integers showing the origin as the point for which n is a Perfect Number.
Using the function:

$$F(n) = 1 - ie^{-\frac{in\pi}{\sigma(n)}} - e^{-\frac{2in\pi}{\sigma(n)}} - ie^{\frac{3in\pi}{\sigma(n)}}$$

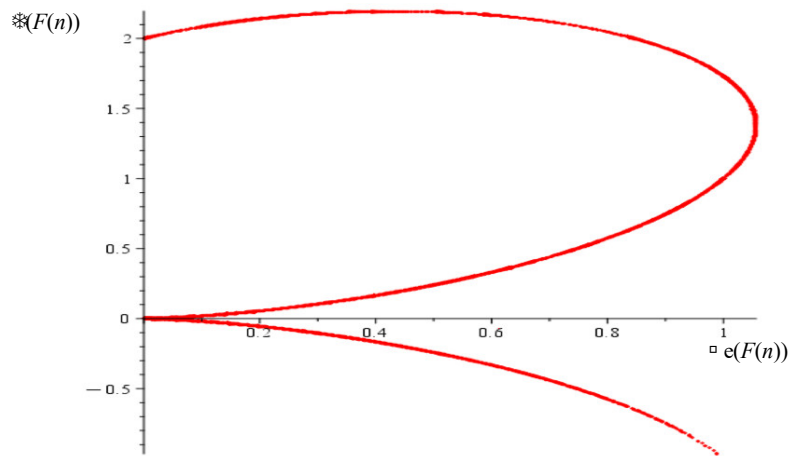


FIGURE 6

The zeros of the function $F(n)$, occur at the values 6, 28, 496, 8124....

NOTE*: The **Mersenne primes** and the perfect numbers can only exist on the upper right quadrant corresponding to **deficient numbers**. **Perfect numbers** are the **zeros** of the function $F(n)$.

The general locations of primes and Mersenne primes are shown in Figure 7. As can be seen, the oprimes do not generate negative imaginary values, and are located on the top-right quadrant of the complex plane.

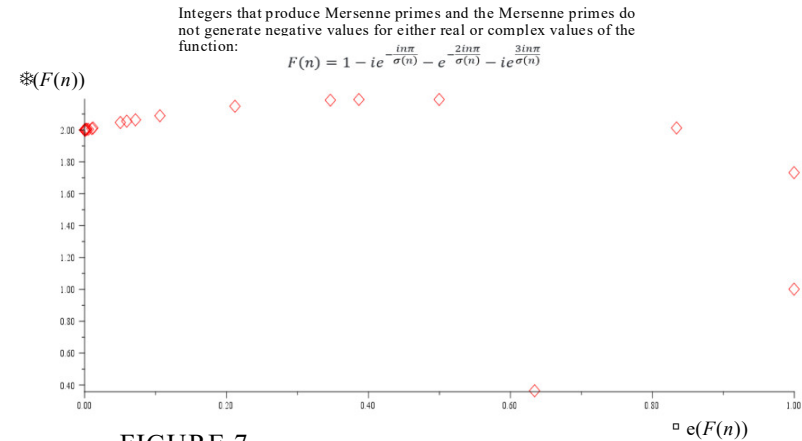


FIGURE 7

Hence, $\sigma(n) > 2n$. It is clear that the sequence of **abundant numbers**, [12, 18, 20, 24, 30, 36, 40, 42, 48, 54, 56, 60, 66, 70, 72, 78, 80, 84, 88, 90, 96, 100, 102, 104, 108, 112, 114, 120, 126, 132, 138, 140, 144, 150, 156, 160, 162, 168, 174, 176, 180, 186, 192, 196, 198, 200, 204, 208, 210, 216, 220, 222, 224, 228, 234, 240, 246, 252, 258, 260, 264, 270, 272, 276, 280, 282, 288, 294, 300, 304, 306, 308, 312, 318, 320, 324, 330, 336, 340, 342, 348, 350, 352, 354, 360, 364, 366, 368, 372, 378, 380, 384, 390, 392, 396, 400, 402, 408, 414, 416, 420, 426, 432, 438, 440, 444, 448, 450, 456, 460, 462, 464, 468, 474, 476, 480, 486, 490, 492, 498, 500], produce values of $F(n)$ in (35) that lie on the lower right quadrant of the complex plane. This distinct observation for the first 500, **abundant numbers** provides a clue as to their distribution.

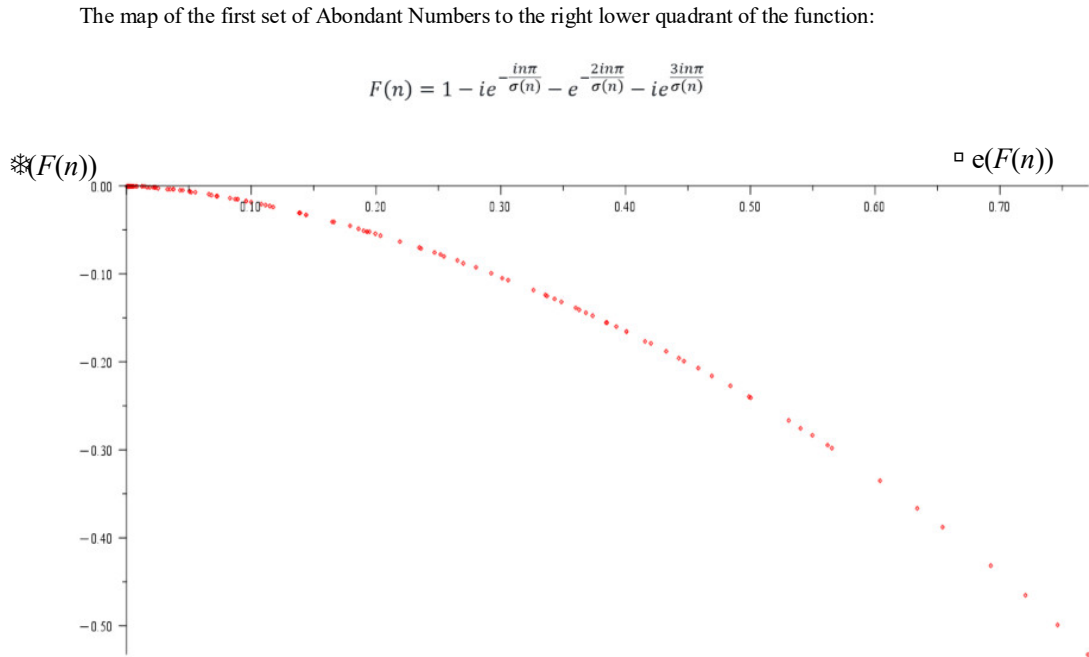


FIGURE 8

It is clear the first numbers between 0 and 500 that generate a sequence of **deficient numbers**:
[2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 21, 22, 23, 25, 26, 27, 29, 31, 32, 33, 34, 35, 37, 38, 39, 41, 43, 44, 45, 46, 47, 49, 50, 51, 52, 53, 55, 57, 58, 59, 61, 62, 63, 64, 65, 67, 68, 69, 71, 73, 74, 75, 76, 77, 79, 81, 82, 83, 85, 86, 87, 89, 91, 92, 93, 94, 95, 97, 98, 99, 101, 103, 105, 106, 107, 109, 110, 111, 113, 115, 116, 117, 118, 119, 121, 122, 123, 124, 125, 127, 128, 129, 130, 131, 133, 134, 135, 136, 137, 139, 141, 142, 143, 145, 146, 147, 148, 149, 151, 152, 153, 154, 155, 157, 158, 159, 161, 163, 164, 165, 166, 167, 169, 170, 171, 172, 173, 175, 177, 178, 179, 181, 182, 183, 184, 185, 187, 188, 189, 190, 191, 193, 194, 195, 197, 199, 201, 202, 203, 205, 206, 207, 209, 211, 212, 213, 214, 215, 217, 218, 219, 221, 223, 225, 226, 227, 229, 230, 231, 232, 233, 235, 236, 237, 238, 239, 241, 242, 243, 244, 245, 247, 248, 249, 250, 251, 253, 254, 255, 256, 257, 259, 261, 262, 263, 265, 266, 267, 268, 269, 271, 273, 274, 275, 277, 278, 279, 281, 283, 284, 285, 286, 287, 289, 290, 291, 292, 293, 295, 296, 297, 298, 299, 301, 302, 303, 305, 307, 309, 310, 311, 313, 314, 315, 316, 317, 319, 321, 322, 323, 325, 326, 327, 328, 329, 331, 332, 333, 334, 335, 337, 338, 339, 341, 343, 344, 345, 346, 347, 349, 351, 353, 355, 356, 357, 358, 359, 361, 362, 363, 365, 367, 369, 370, 371, 373, 374, 375, 376, 377, 379, 381, 382, 383, 385, 386, 387, 388, 389, 391, 393, 394, 395, 397, 398, 399, 401, 403, 404, 405, 406, 407, 409, 410, 411, 412, 413, 415, 417, 418, 419, 421, 422, 423, 424, 425, 427, 428, 429, 430, 431, 433, 434, 435, 436, 437, 439, 441, 442, 443, 445, 446, 447, 449, 451, 452, 453, 454, 455, 457, 458, 459, 461, 463, 465, 466, 467, 469, 470, 471, 472, 473, 475, 477, 478, 479, 481, 482, 483, 484, 485, 487, 488, 489, 491, 493, 494, 495, 497, 499], produce values of $F(n)$ that lie on the **upper right quadrant** of the complex plane. This distinct observation for the first 500, **defficient numbers** and **abondant numbers** provides a clue as to their distributions.

The map of the first set of **Dificient Numbers** , from 0-500, to the right upper quadrant of the function:

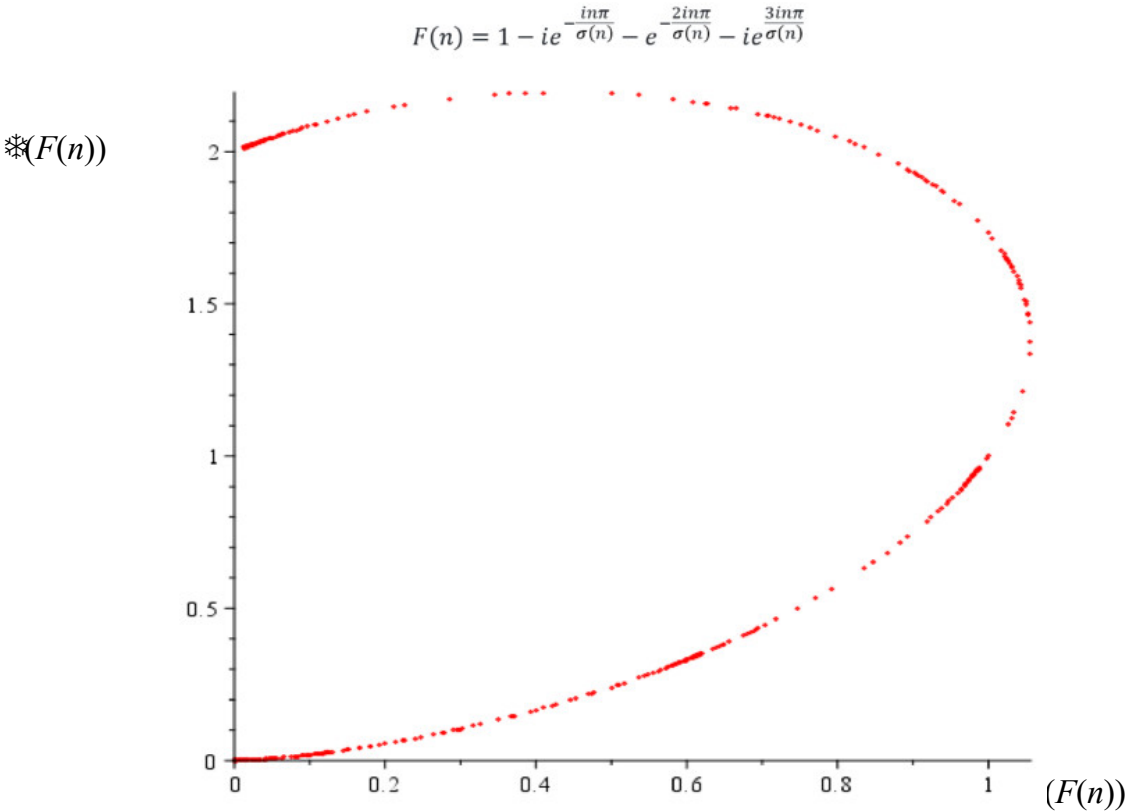


FIGURE 9

Between the **abondant numbers** and the deficient numbers, are the **Perfect Numbers**, [6, 7, 28, 496, 8128, 33550336,...], that generate the zeros of the function:

$$F(n) = 1 - ie^{-\frac{in\pi}{\sigma(n)}} - e^{-\frac{2in\pi}{\sigma(n)}} - ie^{\frac{3in\pi}{\sigma(n)}} = 0. \tag{36.}$$

Hence, the imaginary part of the function $F(n)$ determines if a number is an **abondant number**, a **perfect number** or a **deficient number**.

$$\Im \left(1 - ie^{-\frac{in\pi}{\sigma(n)}} - e^{-\frac{2in\pi}{\sigma(n)}} - ie^{\frac{3in\pi}{\sigma(n)}} \right) \begin{cases} < 0, n \in \text{abondant numbers} \\ = 0, n \in \text{perfect numbers} \\ > 0, n \in \text{deficient numbers} \end{cases} \tag{37.}$$

The first set of even numbers from 0..500 that lie on the **defient number** curve but are not **abondant numbers** are:

[2, 4, 6, 8, 10, 14, 16, 22, 26, 28, 32, 34, 38, 44, 46, 50, 52, 58, 62, 64, 68, 72, 74, 76, 82, 86, 92, 94, 98, 106, 110, 116, 118, 122, 124, 128, 130, 134, 136, 142, 146, 148, 152, 154, 158, 164, 166, 170, 172, 178, 182, 184, 188, 190, 194, 202, 206, 212, 214, 218, 226, 230, 232, 236, 238, 242, 244, 248, 250, 254, 256, 262, 266, 268, 274, 278, 284, 286, 290, 292, 296, 298, 302, 304, 310, 314, 316, 322, 326, 328, 332, 334, 338, 344, 346, 356, 358, 362, 370, 374, 376, 382, 386, 388, 394, 398, 404, 406, 410, 412, 418, 422, 424, 428, 430, 434, 436, 442, 446, 452, 454, 458, 466, 470, 472, 478, 482, 484, 488, 494, 496].

These numbers are clearly defined by (37).

Figure 10 shows the 2D plot of the function covering both odd and even numbers in the range $n = 0..500$.

The map of the first set of **odd numbers** (blue) and even numbers (red). The even numbers that generate points that fall on both the **deficient numbers** and the **abundant numbers**. Hence the graphs are dense with no gaps.

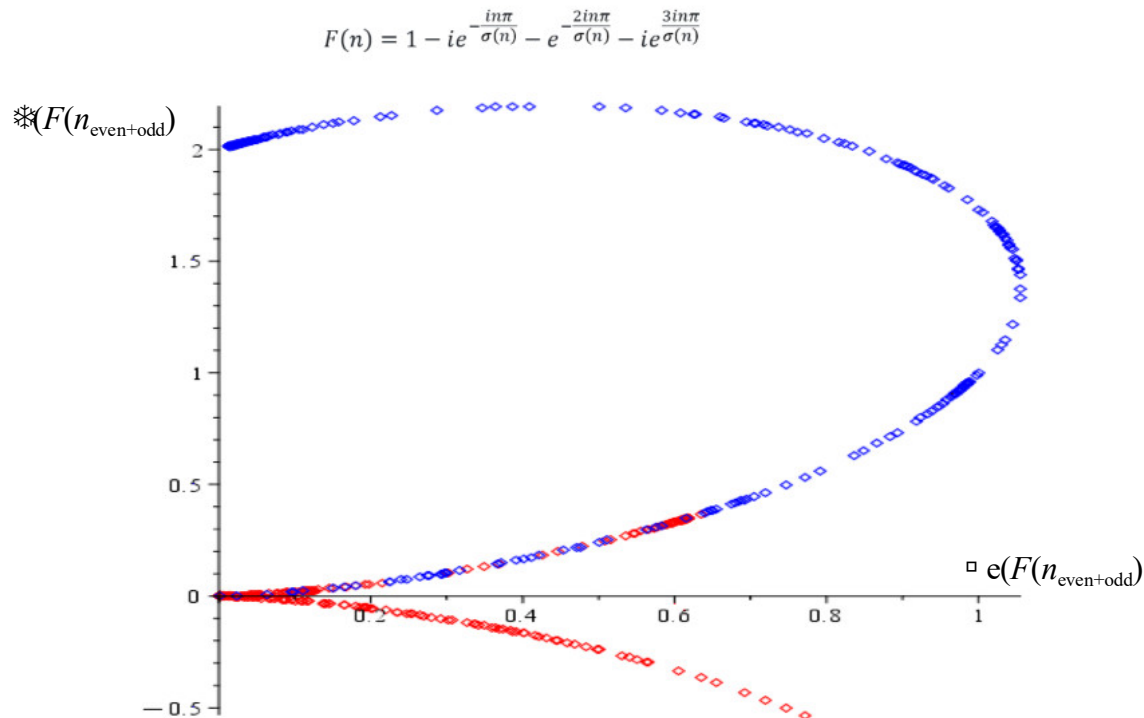


FIGURE 10

It is clear that the even numbers (red points) can fall on both the **deficient number** curve and the **abundant number** curve. The deficient numbers seem to be bounded by the line $1.05629905839783049963 + 1.37659573355141432857i$ and a maximum imaginary value of $0.43293432010231995809 + 2.19494797760015472936i$.

Definition 4: An **Deficient disturbing number**, (DDN), is a **deficient number** which:

$$\Im \left(1 - ie^{-\frac{i n \pi}{\sigma(n)}} - e^{-\frac{2 i n \pi}{\sigma(n)}} - ie^{\frac{3 i n \pi}{\sigma(n)}} \right) > 0 \in \mathbf{DDN} \quad (38.)$$

These are the red points on Figure 10 that intermingle with the blue odd number points.

$DDN \in$

[2,4,8,10,14,16,22,26,32,34,38,44,46,50,52,58,62,64,68,74,76,82,86,92,94,98,106,110,116,118,122,124,128,130,134,136,142,146,148,152,154,158,164,166,170,172,178,182,184,188,190,194,202,206,212,214,218,226,230,232,236,238,242,244,248,250,254,256,262,266,268,274,278,284,286,290,292,296,298,302,310,314,316,322,326,328,332,334,338,344,346,356,358,362,370,374,376,382,386,388,394,398,404,406,410,412,418,422,424,428,430,434,436,442,446,452,454,458,466,470,472,478,482,484,488,494.....].

The map of the first set of **even numbers** (red) . The even numbers generate points that fall on both the **deficient numbers** and the **abundant numbers**.

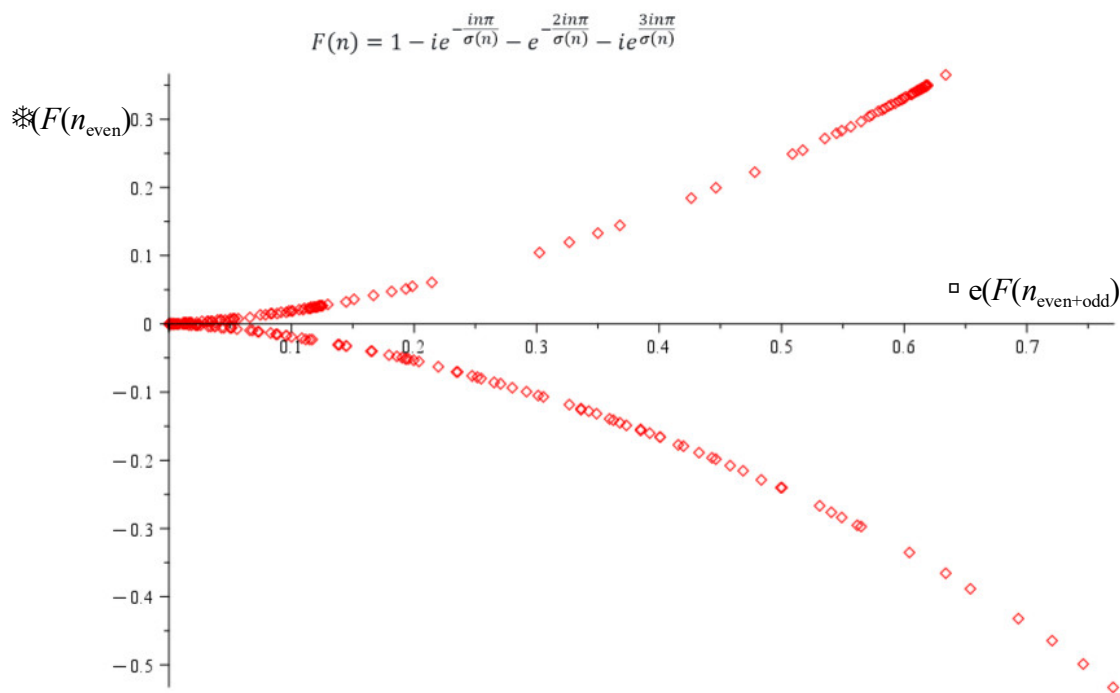


FIGURE 11

The extent to which the even numbers infiltrate the deficient number space for up to $n = 150000$ seems to be confined to the approximate range,

$$0 \leq \left(1 - ie^{-\frac{in\pi}{\sigma(n)}} - e^{-\frac{2in\pi}{\sigma(n)}} - ie^{\frac{3in\pi}{\sigma(n)}}\right) \leq 0.98575151303581662431 + 0.36599952081502975396i, \text{DDN} \leq 150000 \text{ (39.)}$$

The extent to which the even numbers penetrate the abundant number space is unknown. However it is known that there exists in infinite number of abundant numbers. It has been shown that every multiple $6(n \geq 6)$ is either an abundant number, or taking more multiples of 6 of such numbers leads to an bondant number. Since there is an infinite number of multiples of 6, then there are an infinite number of abundant numbers. Erdos &Graham, 1980, [], showed that even numbers greater than 46 are either abundant numbers or the sum of two abundant numbers.

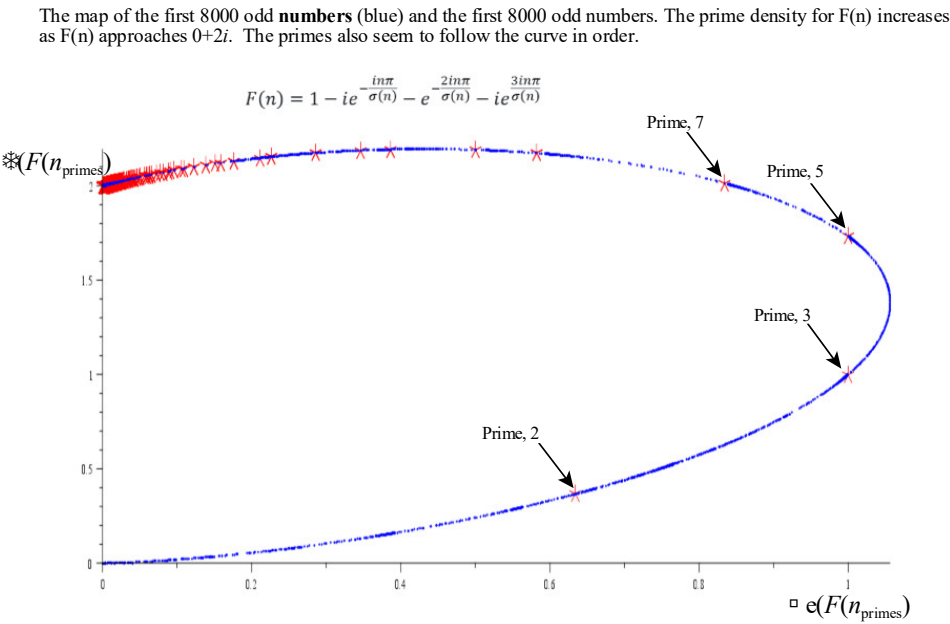


FIGURE 12

Figure 13 shows the distribution of the Mersenne primes with the regular primes.

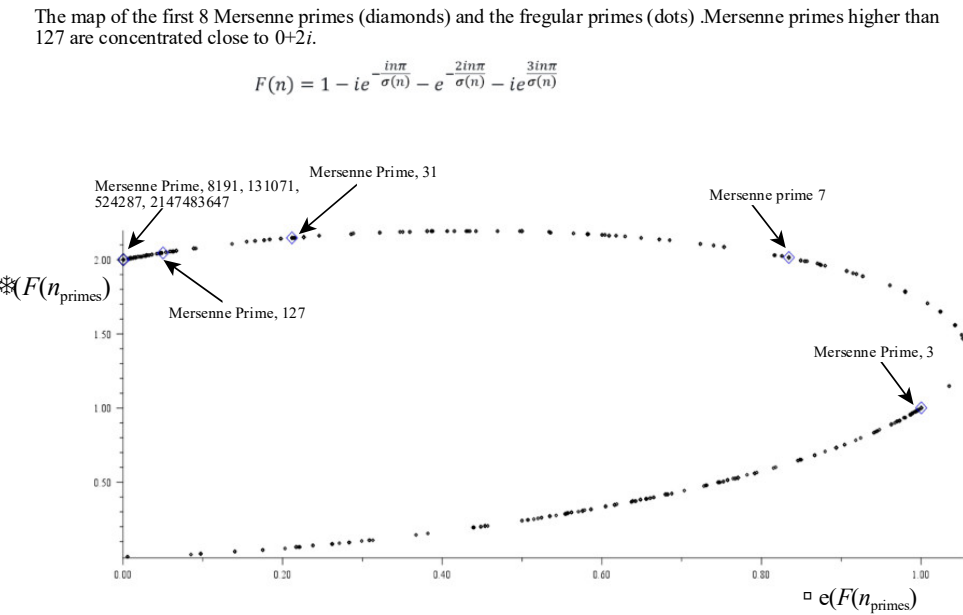


FIGURE 13

6. The Extension of TH Function F(n) to a General Series Form

The function

$$F(n) = 1 - ie^{-\frac{in\pi}{\sigma(n)}} - e^{-\frac{2in\pi}{\sigma(n)}} - ie^{\frac{3in\pi}{\sigma(n)}} \quad (40.)$$

behaves like a CyclotomicPolynomial. CyclotomicPolynomialare the minimal polynomials of primitive roots of unity with rational coefficients. The first few CyclotomicPolynomial are shown below:

$$\Phi_1(x)=x-1$$

(5)

$$\Phi_2(x)=x+1$$

(6)

$$\Phi_3(x)=x^2+x+1$$

(7)

$$\Phi_4(x)=x^2+1$$

(8)

$$\Phi_5(x)=x^4+x^3+x^2+x+1$$

(9)

$$\Phi_6(x)=x^2-x+1$$

(10)

$$\Phi_7(x)=x^6+x^5+x^4+x^3+x^2+x+1$$

(11)

$$\Phi_8(x)=x^4+1$$

(12)

$$\Phi_9(x)=x^6+x^3+1$$

(13)

$$\Phi_{10}(x)=x^4-x^3+x^2-x+1.$$

(14)

An example of a Cyclotomic polynomials are the distribution of the roots of unity on the circle, for $x=50$

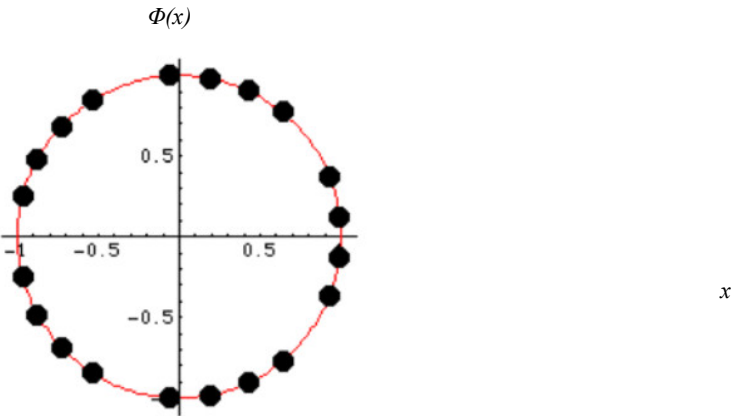


FIGURE 14

A cyclotomic polynomial is of the product form:

$$\Phi_m(x)=\prod_{k=1}^m(x-\zeta_m) \quad (41.)$$

where, ζ_m , are the roots of unity in the complex plane, $\mathbb{C}$. In general, the circle, $\zeta_m = e^{\pi i(\omega(x))}$ where $\omega(x) = \frac{k}{m}$, and k is taken over integers relative prime to m . It is clear that the function

$$F(n)=1-ie^{\frac{in\pi}{\sigma(n)}}-e^{\frac{2in\pi}{\sigma(n)}}-ie^{\frac{3in\pi}{\sigma(n)}} \quad (42.)$$

is composed of functions of cyclotomic polynomials for the the special case of an expansion of some function over the function $\frac{n}{\sigma(n)}$.

Looking at exponential terms with the sequence, $0, -i, 2i, 3i$, we determine the first difference in the powers to be

$$\delta_1 \rightarrow -i, 3i, i, (43.)$$

The second difference gives,

$$\delta_2 \rightarrow 4i, -2i, (43.)$$

The second difference points to the function $F(n)$, following a sequence of powers that is purely linear, but quadratic or alternating in some manner. We assume a quadratic relation, of the form, $Ak^2 + Bk + C$. However, the second differences are not the same constants, and so a recurrence relation of the form, $b_k = f(b_{k-1}, b_{k-2})$ must be used to expand $F(n, m)$ as a series of higher powers for m recurrences. The sequence of powers in $F(n, m)$, follows the recurrence, with initial conditions,

$$b_k = -b_{k-1} - 7b_{k-2}, b_0 = 0, b_2 = -i (44.)$$

The characteristic equation for the recurrence then yields,

$$r^2 + 2r + 7 = 0 (45.)$$

This yields, the two solutions,

$$\begin{aligned} r_1 &= -1 - i\sqrt{6} \\ r_2 &= -1 + i\sqrt{6} \end{aligned} (46.)$$

Since the recurrence (46) follows a second order linear form, the general solution of the recurrence is

$$b_k = C_1 r_1^k + C_2 r_2^k, C_1, C_2 \text{ are constants.} (47.)$$

Solving for C_1 , and C_2 , we get:

$$C_1 = -\frac{\sqrt{6}}{84} + \frac{i}{14}, C_2 = \frac{\sqrt{6}}{84} + \frac{i}{14} (48.)$$

Hence we get

$$b_k = \left(-\frac{\sqrt{6}}{84} + \frac{i}{14}\right)(-1 - \sqrt{6})^k + \left(\frac{\sqrt{6}}{84} + \frac{i}{14}\right)(-1 + \sqrt{6})^k (49.)$$

Hence we have the general form for m terms:

$$F(n, m) = \sum_{k=1}^m \left(\frac{\left(-\frac{\sqrt{6}}{84} + \frac{i}{14}\right)(-1 - \sqrt{6})^k + \left(\frac{\sqrt{6}}{84} + \frac{i}{14}\right)(-1 + \sqrt{6})^k}{k-1} \right)^{k-1} e^{\left(\frac{\pi i n}{\sigma(n)}\right) \left(-\frac{\sqrt{6}}{84} + \frac{i}{14}\right)(-1 - \sqrt{6})^k + \left(\frac{\sqrt{6}}{84} + \frac{i}{14}\right)(-1 + \sqrt{6})^k} (51.)$$

This sum produces the first four terms giving the same function:

$$F(n, 4) = 1 - ie^{-\frac{i n \pi}{\sigma(n)}} - e^{-\frac{2 i n \pi}{\sigma(n)}} - ie^{\frac{3 i n \pi}{\sigma(n)}} (52.)$$

The function,

$$F(n, m) = \sum_{k=1}^m \left(\frac{b_k}{k-1} \right)^{k-1} e^{\left(\frac{\pi i n b_k}{\sigma(n)}\right)} (53.)$$

will only have coefficients that are ± 1 , or $-i$, for the first 4 terms, $m = 4$. The remaining terms $m > 4$ have large coefficients that blow up quickly. For example for $m=7$,

$$F(n, 7) = 1 - ie^{-\frac{i n \pi}{\sigma(n)}} - e^{-\frac{2 i n \pi}{\sigma(n)}} - ie^{\frac{3 i n \pi}{\sigma(n)}} + 625e^{-\frac{20 i n \pi}{\sigma(n)}} + \frac{2476099}{3125}e^{\frac{19 i n \pi}{\sigma(n)}} - 24137569e^{\frac{102 i n \pi}{\sigma(n)}} (54.)$$

In general, for Perfect numbers,

$$F(n,4)=1- ie^{-\frac{in\pi}{\sigma(n)}}- e^{-\frac{2in\pi}{\sigma(n)}}- ie^{\frac{3in\pi}{\sigma(n)}}=0,$$

$$F(n=5..\infty)=625e^{-\frac{20in\pi}{\sigma(n)}}+\frac{2476099}{3125}e^{\frac{19in\pi}{\sigma(n)}}-24137569e^{\frac{102in\pi}{\sigma(n)}}+\dots\left(\frac{b_k}{k-1}\right)^{k-1}e^{\left(\frac{\pi inb_k}{\sigma(n)}\right)}\quad (55.)$$

In general, we have:

$$F(n,\infty)=\sum_{k=1}^{\infty}e^{i\pi\beta_k},\quad (56.)$$

k	β_k
1	0
2	$-\frac{1}{2}-\frac{n}{\sigma(n)}$
3	$1+\frac{2n}{\sigma(n)}$
4	$-\frac{1}{2}+\frac{3n}{\sigma(n)}$
5	$-\frac{20n}{\sigma(n)}-\frac{i\log 5^5}{\pi}$
6	$\frac{1}{2}+\frac{19n}{\sigma(n)}-i\left(\frac{\log 19^5+\log 5^5}{\pi}\right)$
7	$1+\frac{102\,n}{\sigma(n)}-i\frac{\log 17^6}{\pi}$
8	$\frac{1}{2}-\frac{337\,n}{\sigma(n)}-\frac{i}{\pi}\log\left(\frac{337}{7}\right)^7$
9	$-\frac{40\,n}{\sigma(n)}-\frac{i}{\pi}\log(5^8)$
10	$\frac{1}{2}+\frac{2439\,n}{\sigma(n)}-\frac{i}{\pi}\log(271)^9$

A 3-d plot of the function, shows that the function $F(n,4)=0$, is the axis of an infinite cylinder, where the rest of the terms $m>4$ lie.

Distribution of $F(n)$ showing distinct regions for the first 4 terms, (red axis), and the cylindral orbit of the remaining terms on the cylinder (blue). **Perfect numbers occur when $F(n,4)=0$ at the exact axial symmetry of remaining terms higher than the fourth term.**

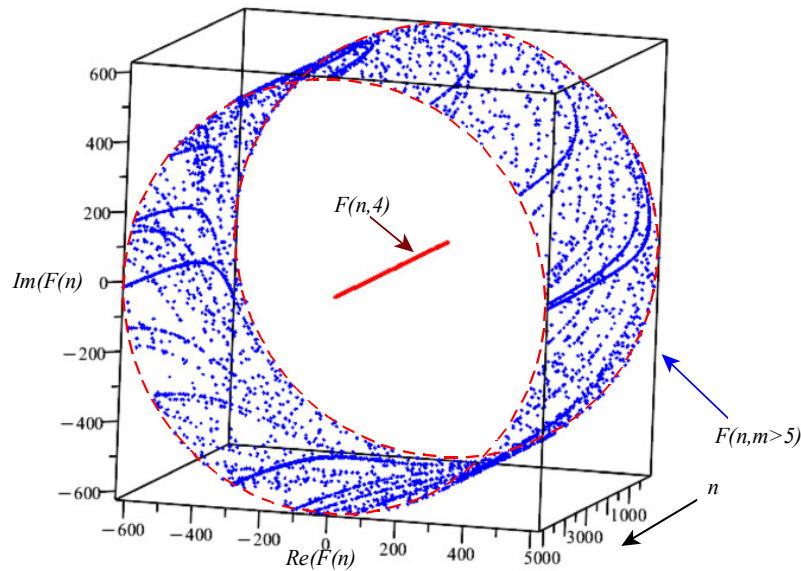


FIGURE 15

Figure 15 shows the cylindrical form with the axis approaching a line when the cylinder radius approaches infinity. The axis of the cylinder becomes the solutions for Perfect numbers,

$$1 - ie^{\frac{in\pi}{\sigma(n)}} - e^{\frac{2in\pi}{\sigma(n)}} - ie^{\frac{3in\pi}{\sigma(n)}} = 0, \quad (57.)$$

Now, from (19), for some integer N ,

$$N_p = \frac{\tan\left(\frac{\sigma(N_p)}{N_p}\right) \prod_{k=1}^{\frac{\sigma(N_p)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(N_p)}\right)}\right)}{\sin\left(\frac{2}{N_p}\right) \left(\prod_{k=1}^{\frac{\sigma(N_p)}{2}-1} \left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{\sigma(N_p)}\right)}\right)\right)} \quad (58.)$$

7. Analytic Mersenne Density and Infinitude

Setup:

Let

$$\cot(x) = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k-1}, \quad [2^2 < \pi^2] \quad (59.)$$

Fix $x_0 \in (0, \pi)$ with $\cot(x_0) \in \mathbb{R} \setminus \{0\}$.

Partition $\mathbb{N}$, into disjoint classes $\wp$, and $\aleph = \mathbb{N} \setminus \wp$, where $\wp$ is the set of Mersenne exponents p , with $(2^{2p} - 1)$ a prime.

Define

$$S_{all(x)} = \sum_{n \geq 1} \frac{2^{2n} |B_{2n}|}{(2n)!} x^{2n-1}, S_{M(x)} = \sum_{p \in \wp} \frac{2^{2p} |B_{2p}|}{(2p)!} x^{2p-1}, S_{N(x)} = S_{all(x)} - S_{M(x)} \quad (60.)$$

Note $S_{all(x)} = \frac{1}{x} - \cot(x)$ and $S_{N(x)}, S_{M(x)} > 0$ for $x \in (0, \pi)$.

Definition (analytic Mersenne density).

$$\rho_M = \frac{S_{M(x)}}{S_{all(x)}} = \frac{\sum_{p \in \wp} \zeta(2p) \left(\frac{x}{\pi}\right)^{2p}}{\sum_{p \in \wp} \zeta(2n) \left(\frac{x}{\pi}\right)^{2n}} \quad [0 < \rho_M < 1] \quad (61.)$$

With the set up above, at a fixed $x_0 \in (0, \pi)$, suppose the following holds true:

(H1) (Regularity/positivity of coefficients).

Each summand is positive and satisfies the classical Bernoulli-Zeta representation:

$$|B_{2n}| = \frac{2(2n)!}{(2\pi)^{2n}} \zeta(2n), \quad (62.)$$

Hence, $S_{all(x)} \in (0, \infty)$.

(H2): (Analytic density at x_0). The decomposition of $\cot(x_0)$ through $S_{N(x)}, S_{M(x)}$ yields a normalized quadratic identity in the $\tan(x_0)$ as shown in **LEMMA 2**, only if **LEMMA1** holds.

(H2): (Single valuedness/discriminat collapse). Since $\tan(x_0)$ is single valued, the discriminant of the quadratic in **LEMMA 2** vanishes.

Proof Sketch:

Absolute positivity and conditional subtraction.

By (H1), $S_{all(x)} = \sum_{n \geq 1} \frac{2^{2n} |B_{2n}|}{(2n)!} x^{2n-1}$, and $S_{M(x)} = \sum_{p \in \wp} \frac{2^{2p} |B_{2p}|}{(2p)!} x^{2p-1}$. The analytic value $\cot(x_0)$ may be negative (e.g., $\cot(x_0 = 2) < 0$), which arises from subtracting the strictly positive $S_{all(x)}$ from $1/x_0$.

Quadratic normalization.

(H2) encodes the partition into a quadratic in $X = \tan(x_0)$:

$$AX^2 + BX + 1 = 0, X = \frac{-B \pm \sqrt{B^2 - 4A}}{2A}.$$

Since $\cot(x_0) \neq 0$ and $S_{M(x)} > 0$, we have A and B finite and nonzero.

Discriminant collapse and consistency.

By **(H3)**, $B^2 - 4A = 0$. Solve for X : the two roots coincide, so the quadratic exactly reproduces $X = \tan(x_0)$.

Contradiction from finiteness.

Assume $\wp$, is finite. Then $S_{M(x)} > 0$ is a fixed positive constant, hence A is fixed. Meanwhile $S_{N(x)} = S_{all(x)} - S_{M(x)} > 0$ is also fixed. The identity $B^2 = 4A$ becomes a rational equality among strictly positive finite constants. But this equality must be compatible with the sign of $\cot(x_0)$ (e.g., negative at $X = 2$); when the decomposition is realized by finite sets, the resulting rational combination cannot produce the required analytic sign/phase (it stays on the “algebraic” positive side). This contradicts the actual value of $\cot(x_0)$.

A symmetric argument applies if $\aleph$, is finite: then S_N is fixed and $S_{M(x)} = S_{all(x)} - S_{N(x)}$ must bear the entire analytic burden; again the finite rational identity cannot reproduce the analytic sign at x_0 . Therefore, both classes must be infinite.

Interpretation via classical pillars.

Pringsheim (nonnegative coefficients $\Rightarrow$ real singular control): Positivity of coefficients yields rigid real-axis behavior of generating series; finite truncations cannot emulate the required analytic sign at x_0 .

Gap/lacunary theorems (Fabry/Hadamard): Attempting to realize the analytic function from a set with “large gaps” (finite or too-sparse) obstructs continuation/phase needed at x_0 ; an infinite contribution from both parts is necessary.

Tauberian philosophy (Wiener–Ikehara): Analytic constraints (here, the discriminant identity at a real point) force “density/infinity-type” conclusions for the underlying index sets. Thus both $\mathbb{N}$, and $\emptyset$ must be infinite.

Corollary A (Intrinsic analytic density)

Under the hypotheses of the intrinsic analytic Mersenne density

$$\rho_M(x) = \frac{S_M(x)}{S_{all}(x)} = \frac{\sum_{p \in \emptyset}^{\infty} \zeta(2p) \left(\frac{x}{\pi}\right)^{2p}}{\sum_{p \in \emptyset}^{\infty} \zeta(2n) \left(\frac{x}{\pi}\right)^{2n}} [0 < \rho_M < 1]$$

is well-defined with $0 < \rho_M < 1$. In particular, $\rho_M(x_0)$ cannot be realized by a finite index set on either side.

Theorem 1: *There exists an infinite number of Mersenne Primes.*

Proof:

I start with the relationship between Perfect numbers and their sums of divisors. Let p be a prime number such that $P_p = (2^p - 1)2^{p-1}$ is a perfect number $N_{p \in P_p}$. Then the following applies.

Lemma 3: *If $N_{p \in P_p}$ is a Perfect number, then,*

$$N_p = \frac{\tan\left(\frac{\sigma(N_p)}{N_p}\right) \prod_{k=1}^{\frac{\sigma(N_p)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(N_p)}\right)}\right)}{\sin\left(\frac{2}{N_p}\right) \left(\prod_{k=1}^{\frac{\sigma(N_p)}{2}-1} \left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{\sigma(N_p)}\right)}\right)\right)} \quad (63.)$$

Proof of LEMMA 3: See equation (19) for Perfect numbers.

Lemma 4: *Let p be a prime that generates a Mersenne prime and a Perfect Number N , then, there exists a unique decomposition of $\cot(x_0)$ into a quadratic identity*

$$A(x_0) \tan^2(x_0) + B(x_0) \tan(x_0) + 1 = 0 \quad (64.)$$

Proof (LEMMA 4):

Now, from [4], p.42, 1.411 (7) we find an expressions for $\cot(x)$:

$$\cot(x) = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k} |B_{2k}|}{(2k)!} x^{2k-1}, [x^2 < \pi^2] \quad (65.)$$

Factoring this form into

$$\cot(x) = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{x^{k-1} (2^k - 1) 2^{2k} |B_{2k}|}{(2^k - 1) (2k)!} x^k, [x^2 < \pi^2] \quad (66.)$$

We find that by choosing $x_0 = 2$, since (66) holds for $\{2^2 < \pi^2\}$, the expression can be modified and separated into two class, one over the sum over Mersenne primes to include Perfect numbers, $N_p =$

$2^{p-1}(2^p - 1)$, when $p \in P_p$, a prime for which $P_p = 2^p - 1$ is a Mersenne prime, and the class of non-Mersenne primes, for $k \notin P_p$

$$\cot(2) = \frac{1}{2} - \sum_{p \in P_p} \frac{2^{p-1}(2^p - 1)2^{2p}|B_{2p}|}{(2^p - 1)(2p)!} 2^p - \sum_{k \notin P_p} \frac{2^{4k-1}|B_{2k}|}{(2k)!}, [2^2 < \pi^2] \quad (67.)$$

Put $N_p = 2^{p-1}(2^p - 1)$, $p \in P_p$ in (67), then,

$$\cot(2) = \frac{1}{2} - \sum_{p \in P_p} N_p \frac{2^{3p}|B_{2p}|}{(2^p - 1)(2p)!} - \sum_{k \notin P_p} \frac{2^{4k-1}|B_{2k}|}{(2k)!}, [2^2 < \pi^2] \quad (68.)$$

From (19), LEMMA 1, for some Perfect number N_q chosen among the set of P_p

$$1 = \frac{\tan(2) \prod_{k=1}^{\frac{\sigma(N_q)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{N_q}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(N_q)}\right)} \right)}{N_q \sin\left(\frac{2}{N_q}\right) \left(\prod_{k=1}^{\frac{\sigma(N_q)}{2}-1} \left(1 - \frac{\sin^2\left(\frac{1}{N_q}\right)}{\sin^2\left(\frac{k\pi}{\sigma(N_q)}\right)} \right) \right)} \quad (69.)$$

Hence we can multiply by 1,

$$\cot(2) = \frac{1}{2} - \tan(2) \sum_{p \in P_p} \left[\left(\frac{2^{3p}|B_{2p}|}{(2^p - 1)(2p)!} \frac{N_p}{N_q} \right) \frac{\prod_{k=1}^{\frac{\sigma(N_q)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{N_q}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(N_q)}\right)} \right)}{\sin\left(\frac{2}{N_q}\right) \left(\prod_{k=1}^{\frac{\sigma(N_q)}{2}-1} \left(1 - \frac{\sin^2\left(\frac{1}{N_q}\right)}{\sin^2\left(\frac{k\pi}{\sigma(N_q)}\right)} \right) \right)} \right] - \sum_{k \notin P_p} \frac{2^{4k-1}|B_{2k}|}{(2k)!} \quad (70.)$$

Divide by $\cot(2)$.

$$1 = \left(\frac{1}{2} - \sum_{k \notin P_p} \frac{2^{4k-1}|B_{2k}|}{(2k)!} \right) \tan(2) - \tan^2(2) \sum_{p \in P_p} \left[\left(\frac{2^{3p}|B_{2p}|}{(2^p - 1)(2p)!} \frac{N_p}{N_q} \right) \frac{\prod_{k=1}^{\frac{\sigma(N_q)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{N_q}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(N_q)}\right)} \right)}{\sin\left(\frac{2}{N_q}\right) \left(\prod_{k=1}^{\frac{\sigma(N_q)}{2}-1} \left(1 - \frac{\sin^2\left(\frac{1}{N_q}\right)}{\sin^2\left(\frac{k\pi}{\sigma(N_q)}\right)} \right) \right)} \right] \quad (71.)$$

$$\tan^2(2) \sum_{p \in P_p} \left[\left(\frac{2^{3p}|B_{2p}|}{(2^p - 1)(2p)!} \frac{N_p}{N_q} \right) \frac{\prod_{k=1}^{\frac{\sigma(N_q)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{N_q}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(N_q)}\right)} \right)}{\sin\left(\frac{2}{N_q}\right) \left(\prod_{k=1}^{\frac{\sigma(N_q)}{2}-1} \left(1 - \frac{\sin^2\left(\frac{1}{N_q}\right)}{\sin^2\left(\frac{k\pi}{\sigma(N_q)}\right)} \right) \right)} \right] + \left(\sum_{k \notin P_p} \frac{2^{4k-1}|B_{2k}|}{(2k)!} - \frac{1}{2} \right) \tan(2) + 1 = 0 \quad (72.)$$

Now, we reduce (73) further with the following identities [[4],page 41]:

$$\sin(n \cdot x) = n \sin(x) \cos(x) \prod_{k=1}^{\frac{n-2}{2}} \left(1 - \frac{\sin^2(x)}{\sin^2\left(\frac{k\pi}{n}\right)} \right), [n \text{ is even}] \quad (73.)$$

$$\cos(n \cdot x) = \prod_{k=1}^{\frac{n}{2}} \left(1 - \frac{\sin^2(x)}{\sin^2\left(\frac{(2k-1)\pi}{2n}\right)} \right)$$

Putting $n = \sigma(N_q)$, $x = \frac{1}{N_q}$, $q \in P_p$, where Q is a constant associated with q .

$$\cos\left(\frac{\sigma(N_q)}{N_q}\right) = \prod_{k=1}^{\frac{\sigma(N_q)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{N_q}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(N_q)}\right)} \right), \frac{\sin\left(\frac{\sigma(N_q)}{N_q}\right)}{N_p} = \sin\left(\frac{2}{N_q}\right) \left(\prod_{k=1}^{\frac{\sigma(N_q)}{2}-1} \left(1 - \frac{\sin^2\left(\frac{1}{N_q}\right)}{\sin^2\left(\frac{k\pi}{\sigma(N_q)}\right)} \right) \right) \quad (74.)$$

Substitute expressions (74) into (73):

$$\tan^2(2) \sum_{p \in P_p} \left[\left(\frac{2^{3p} |B_{2p}|}{(2^p - 1)(2p)!} N_p \right) \cot\left(\frac{\sigma(N_q)}{N_q}\right) \right] + \left(\sum_{k \notin P_p} \frac{2^{4k-1} |B_{2k}|}{(2k)!} - \frac{1}{2} \right) \tan(2) + 1 = 0 \quad (75.)$$

$$\tan^2(2) \sum_{p \in P_p} \left[\left(\frac{2^{3p} |B_{2p}|}{(2^p - 1)(2p)!} (2^p - 1) 2^{p-1} \right) \cot\left(\frac{\sigma(N_q)}{N_q}\right) \right] + \left(\sum_{k \notin P_p} \frac{2^{4k-1} |B_{2k}|}{(2k)!} - \frac{1}{2} \right) \tan(2) + 1 = 0 \quad (76.)$$

$$\tan^2(2) \sum_{p \in P_p} \left[\left(\frac{2^{4p-1} |B_{2p}|}{(2p)!} \right) \cot\left(\frac{\sigma(N_q)}{N_q}\right) \right] + \left(\sum_{k \notin P_p} \frac{2^{4k-1} |B_{2k}|}{(2k)!} - \frac{1}{2} \right) \tan(2) + 1 = 0 \quad (77.)$$

Note that the sum for the Special primes expressed in Mersenne Primes require a modification with the factor $\cot\left(\frac{\sigma(N_q)}{N_q}\right)$ for the original sum definition for $p \in P_p$.

Let

$$A_N = \left\{ \sum_{p \in P_p} \left[\left(\frac{2^{4p} |B_{2p}|}{(2p)!} \right) \cot\left(\frac{\sigma(N_q)}{N_q}\right) \right] \right\} \quad (78.)$$

$$B_N = \left\{ \sum_{k \notin P_p} \frac{2^{4k-1} |B_{2k}|}{(2k)!} - \frac{1}{2} \right\}$$

$$A_N X^2 + B_N X + 1 = 0 \quad (79.)$$

Then,

$$X = -\frac{B_N \pm \sqrt{B_N^2 - 4A_N}}{2A_N} \quad (80.)$$

However, by (H3), $\tan(2)$ can only have one value, and since A_N is positive, hence, we get:

$$X = -\frac{B_N}{2A_N}, B_N = \pm 2\sqrt{A_N} \rightarrow X = \frac{1}{\sqrt{A_N}} = \frac{1}{\sqrt{\cot\left(\frac{\sigma(N_q)}{N_q}\right) \sum_{p \in P_S} \left(\frac{2^{4p-1} |B_{2p}|}{(2k)!} \right)}} \quad (81.)$$

However, if q is a Special prime, (a Perfect number, then, $\cot\left(\frac{\sigma(N_q)}{N_q}\right) = \cot(2)$, and thus, using the negative value of the square root,

$$A_N = - \sum_{p \in P_p} \left[\left(\frac{2^{4p} |B_{2p}|}{(2p)!} \right) \cot\left(\frac{\sigma(N_q)}{N_q}\right) \right] = \cot(2) \sum_{p \in P_S} \left(\frac{2^{4p-1} |B_{2p}|}{(2k)!} \right)$$

$$X = \tan(2) = \frac{1}{\sqrt{-\cot(2) \sum_{p \in P_S} \left(\frac{2^{4p-1} |B_{2p}|}{(2k)!} \right)}} \quad (82.)$$

$$\tan^2(2) = \frac{1}{-\cot(2) \sum_{p \in P_S} \left(\frac{2^{4p-1} |B_{2p}|}{(2k)!} \right)} \quad (83.)$$

$$\tan(2) = - \frac{1}{\sum_{p \in P_S} \left(\frac{2^{4p-1} |B_{2p}|}{(2k)!} \right)} \quad (84.)$$

$$\cot(2) = - \sum_{p \in P_S} \left(\frac{2^{4p-1} |B_{2p}|}{(2k)!} \right) \quad (85.)$$

A similar analysis for Sophie primes gives the following.

8. Application of the Trigonometric Function to Sophie Germain Numbers

A Sophie Germain primes S_p , is defined as a number for which $S_p = 2p + 1$. Consequently, since both numbers are primes, we have

$$\frac{\sigma(2p + 1)}{\sigma(p)} = \frac{2p + 2}{p + 1} = 2 \quad (86.)$$

This is very intimately related to Perfect numbers $\frac{\sigma(N_p)}{N_p} = 2$, were,
 $N_p \in \{6, 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128, 2658455991569831744654692615953842176, \dots\}$

Hence for, example, in (8), putting $n = \sigma(j)$, (n even), $x = \frac{1}{j}$: then, we have

$$\begin{aligned} \sin\left(\frac{\sigma(j)}{j}\right) &= \sigma(j) \sin\left(\frac{1}{j}\right) \cos\left(\frac{1}{j}\right) \prod_{k=1}^{\frac{\sigma(j)-2}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{k\pi}{\sigma(j)}\right)} \right), \text{ } [\sigma(j) \text{ is even}] \quad (87.) \\ \cos\left(\frac{\sigma(j)}{j}\right) &= \prod_{k=1}^{\frac{\sigma(j)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)} \right) \end{aligned}$$

$$\tan\left(\frac{\sigma(j)}{j}\right) = \frac{\sigma(j) \sin\left(\frac{1}{j}\right) \cos\left(\frac{1}{j}\right) \prod_{k=1}^{\frac{\sigma(j)-2}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{k\pi}{\sigma(j)}\right)}\right)}{\prod_{k=1}^{\frac{\sigma(j)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)}\right)} [\sigma(j) \text{ is even}] \quad (88.)$$

$$\sigma(j) = \frac{\tan\left(\frac{\sigma(j)}{j}\right) \prod_{k=1}^{\frac{\sigma(j)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)}\right)}{\sin\left(\frac{1}{j}\right) \cos\left(\frac{1}{j}\right) \prod_{k=1}^{\frac{\sigma(j)-2}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{k\pi}{\sigma(j)}\right)}\right)} [\sigma(j) \text{ is even}] \quad (89.)$$

Lemma 5: The rational trigonometric functions $\sin\left(\frac{\sigma(j)}{j}\right), \cos\left(\frac{\sigma(j)}{j}\right)$ determine *Perfect Numbers, Sophie Germain primes and Twin Primes*.

Proof: From (15),

$$\sigma(j) = 2 \left[\frac{\tan\left(\frac{\sigma(j)}{j}\right) \prod_{k=1}^{\frac{\sigma(j)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)}\right)}{2 \sin\left(\frac{1}{j}\right) \cos\left(\frac{1}{j}\right) \prod_{k=1}^{\frac{\sigma(j)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)}\right)} \right] \quad (90.)$$

$$\sigma(j) = 2 \left[\frac{\tan\left(\frac{\sigma(j)}{j}\right) \prod_{k=1}^{\frac{\sigma(j)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)}\right)}{\sin\left(\frac{2}{j}\right) \prod_{k=1}^{\frac{\sigma(j)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)}\right)} \right] \quad (91.)$$

This is the relation for Perfect Number, and so we arrive at: If $j = N_p$ is a Perfect number, then, the equality applies only when. Since $\frac{\sigma(N_p)}{N_p}$, we have,

$$N_p = \frac{\tan(2) \prod_{k=1}^{\frac{\sigma(N_p)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(N_p)}\right)}\right)}{\sin\left(\frac{2}{N_p}\right) \left(\prod_{k=1}^{\frac{\sigma(N_p)}{2}-1} \left(1 - \frac{\sin^2\left(\frac{1}{N_p}\right)}{\sin^2\left(\frac{k\pi}{\sigma(N_p)}\right)}\right) \right)} \quad (92.)$$

Hence the rational functions of the σ function in the trigonometric functions encodes this behavior of various types of numbers classes.

For Sophie Germain primes, p we have:

$$\sigma(p) = \frac{\tan\left(\frac{\sigma(2p+1)}{\sigma(p)}\right) \prod_{k=1}^{\frac{\sigma(2p+1)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{\sigma(p)}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(2p+1)}\right)}\right)}{\sin\left(\frac{2}{\sigma(p)}\right) \left(\prod_{k=1}^{\frac{\sigma(2p+1)}{2}-1} \left(1 - \frac{\sin^2\left(\frac{1}{\sigma(p)}\right)}{\sin^2\left(\frac{k\pi}{\sigma(2p+1)}\right)}\right)\right)} \quad (93.)$$

Then, since $\frac{\sigma(2p+1)}{\sigma(p)} = 2$,

$$\sigma(p) = \frac{\tan(2) \prod_{k=1}^{\frac{\sigma(2p+1)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{\sigma(p)}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(2p+1)}\right)}\right)}{\sin\left(\frac{2}{\sigma(p)}\right) \left(\prod_{k=1}^{\frac{\sigma(2p+1)}{2}-1} \left(1 - \frac{\sin^2\left(\frac{1}{\sigma(p)}\right)}{\sin^2\left(\frac{k\pi}{\sigma(2p+1)}\right)}\right)\right)}, p \in S_p \quad (94.)$$

Equation (94) only holds for Sophie Germain Primes $p \in S_p$. Hence we find that (18) behave distinctly for the sets of Mersenne primes $p \in M_p$ that yield Perfect numbers N_p , while (94) behaves distinctly for Sophie Germain primes $p \in S_p$. The connection between the two suggest that the infinitum condition applies equally to both sets of numbers iff it applies to one or the other. Perfect numbers are dealt with in a yet unpublished paper "There are infinitely many Mersenne Primes", MDPI: Manuscript ID:mathematics-3942588.

The relations (94) hold exclusively for all Sophie Germain Primes. The right hand side of (20) depends on implicit rational relationships between $\sigma(p)$ and $\sigma(2p+1)$. It is clear that the basic rational trigonometric functions capture the properties of special prime integers. We now explore the general forms of trinometric and exponential forms that capture Sophie Germain Primes.

$$1 = \frac{\left(\tan(2) \prod_{k=1}^{\frac{\sigma(2q+1)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{\sigma(q)}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(2q+1)}\right)}\right) \right)}{\left(\sigma(q) \sin\left(\frac{2}{\sigma(q)}\right) \left(\prod_{k=1}^{\frac{\sigma(2q+1)}{2}-1} \left(1 - \frac{\sin^2\left(\frac{1}{\sigma(q)}\right)}{\sin^2\left(\frac{k\pi}{\sigma(2q+1)}\right)}\right) \right) \right)}, q \in P_s \quad (95.)$$

$$\cot(2) = \frac{1}{2} - \tan(2) \sum_{k \in P_s} \frac{2^{4p-1}|B_{2p}|}{(2k)!} \left(\frac{\prod_{k=1}^{\frac{\sigma(2q+1)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{\sigma(q)}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(2q+1)}\right)}\right)}{\sigma(q) \sin\left(\frac{2}{\sigma(q)}\right) \left(\prod_{k=1}^{\frac{\sigma(2q+1)}{2}-1} \left(1 - \frac{\sin^2\left(\frac{1}{\sigma(q)}\right)}{\sin^2\left(\frac{k\pi}{\sigma(2q+1)}\right)}\right) \right)} \right) - \sum_{k \notin P_s} \frac{2^{4k-1}|B_{2k}|}{(2k)!} \quad (96.)$$

Multiply by $\tan(2)$,

$$\tan^2(2) \sum_{p \in P_s} \left(\frac{\prod_{k=1}^{\frac{\sigma(2q+1)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{\sigma(q)}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(2q+1)}\right)}\right)}{\sigma(q) \sin\left(\frac{2}{\sigma(q)}\right) \left(\prod_{k=1}^{\frac{\sigma(2q+1)}{2}-1} \left(1 - \frac{\sin^2\left(\frac{1}{\sigma(q)}\right)}{\sin^2\left(\frac{k\pi}{\sigma(2q+1)}\right)}\right) \right)} \frac{2^{4p-1}|B_{2p}|}{(2k)!} \right) + \tan(2) \left(\sum_{k \notin P_s} \frac{2^{4k-1}|B_{2k}|}{(2k)!} + \frac{1}{2} \right) + 1 = 0, \quad (97.)$$

Putting $\cos\left(\frac{\sigma(2q+1)}{\sigma(q)}\right) = \prod_{k=1}^{\frac{\sigma(2q+1)}{2}} \left(1 - \frac{\sin^2\left(\frac{1}{\sigma(q)}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(2q+1)}\right)}\right), \frac{\sin\left(\frac{\sigma(2q+1)}{\sigma(q)}\right)}{\sigma(q)} = \sin\left(\frac{2}{\sigma(q)}\right) \left(\prod_{k=1}^{\frac{\sigma(2q+1)}{2}-1} \left(1 - \frac{\sin^2\left(\frac{1}{\sigma(q)}\right)}{\sin^2\left(\frac{k\pi}{\sigma(2q+1)}\right)}\right)\right)$, in (51), then,

$$\tan^2(2) \cot\left(\frac{\sigma(2q+1)}{\sigma(q)}\right) \sum_{p \in P_S} \left(\frac{2^{4p-1}|B_{2p}|}{(2k)!}\right) + \tan(2) \left(\sum_{k \notin P_S} \frac{2^{4k-1}|B_{2k}|}{(2k)!} + \frac{1}{2}\right) + 1 = 0, \tag{98.}$$

$$X = \tan(2), A_N = \cot\left(\frac{\sigma(2q+1)}{\sigma(q)}\right) \sum_{p \in P_S} \left(\frac{2^{4p-1}|B_{2p}|}{(2k)!}\right), B_N = \left(\sum_{k \notin P_S} \frac{2^{4k-1}|B_{2k}|}{(2k)!} - \frac{1}{2}\right), C_N = 1, \tag{99.}$$

$$A_N X^2 + B_N X + 1 = 0 \tag{100.}$$

Then,

$$X = -\frac{B_N \pm \sqrt{B_N^2 - 4A_N}}{2A_N} \tag{100.}$$

However, by (H3), $\tan(2)$ can only have one value, and since A_N is positive, hence, we get:

$$X = -\frac{B_N}{2A_N}, B_N = \pm 2\sqrt{A_N} \rightarrow X = \frac{1}{\sqrt{A_N}} = \frac{1}{\sqrt{\cot\left(\frac{\sigma(2q+1)}{\sigma(q)}\right) \sum_{p \in P_S} \left(\frac{2^{4p-1}|B_{2p}|}{(2k)!}\right)}} \tag{101.}$$

However, if q is a Special prime, then, $\cot\left(\frac{\sigma(2q+1)}{\sigma(q)}\right) = \cot(2)$, and thus, using the negative value of the square root,

$$\cot(2) = -\sum_{p \in P_S} \left(\frac{2^{4p-1}|B_{2p}|}{(2k)!}\right) = \frac{1}{2} - \sum_{p \in P_S} \frac{2^{4p-1}|B_{2p}|}{(2k)!} - \sum_{k \notin P_S} \frac{2^{4k-1}|B_{2k}|}{(2k)!} \tag{102.}$$

$$\sum_{k \notin P_S} \frac{2^{4k-1}|B_{2k}|}{(2k)!} = \frac{1}{2}, \sum_{p \in P_S} \left(\frac{2^{4p-1}|B_{2p}|}{(2k)!}\right) = -\cot(2) = -0.45765755440 \dots \tag{103.}$$

$\cot(2)$ is transcendental, so any finite algebraic or rational sum (like a finite Bernoulli sum) can never equal $\cot(2)$. This conclusion is supported by the following theorems.

The symmetry of both Perfect numbers and Sophie primes is a general result of the cot-function for these special primes. Now $\cot(2)$ is transcendental, so any finite algebraic or rational sum (like a finite Bernoulli sum) can never equal $\cot(2)$. This conclusion is supported by the following theorems.

a. Fabry/Hadamard (sparsity ↔ analytic behavior) [5]: The Fabry and Hadamard theorems, particularly the gap theorems, are central results in complex analysis concerning the analytic continuation of power series with “lacunary” or gapped coefficients. Both theorems establish conditions under which a power series cannot be analytically extended beyond its circle of convergence, which then becomes a “natural boundary” for the function.

b. Lindemann–Weierstrass Theorem (1885) [6]:

If $\alpha_1 \dots \dots \alpha_n$ are distinct algebraic numbers, then the numbers $e^{\alpha_1} \dots \dots e^{\alpha_n}$ are linearly independent over the algebraic numbers.

These are classical results implying that for any nonzero algebraic a ,
 $\sin(a), \cos(a), \tan(a), \cot(a)$

are all transcendental. This guarantees that $\cot(2)$ (with 2 algebraic) is transcendental, so any finite algebraic or rational sum (like a finite Bernoulli sum) can never equal $\cot(2)$. Hence, equality must involve an infinite series and provides a perfect analytical foundation for the infinitude of Special primes.

c. Siegel–Shidlovsky Theorem (1956) [7]

If $f_1(z) \dots f_n(z)$ satisfy a linear differential system with algebraic coefficients and z_0 is algebraic, then the set of values $f_n(z_0)$ that are algebraic is “exceptionally small.”

For most analytic functions, values at algebraic points are transcendental. Bernoulli numbers B_{2p} arise from expansions of $\frac{z}{e^z-1}$, which satisfies such a differential equation. Then, $\sum_{p \in P_S} \left(\frac{2^{4p-1}|B_{2p}|}{(2k)!}\right)$ is an evaluation of a linear combination of such special-function values at $z = 2$. By Siegel–Shidlovsky, its transcendence cannot arise from a finite truncation and only the infinite series can reproduce a transcendental constant. Thus, finite truncations are algebraic, but the limit equals $\cot(2)$ (transcendental), forcing infinitely many contributing terms.

d. Baker’s Theorem (1966) on Linear Forms in Logarithms [8].

Any nontrivial linear combination of logarithms of algebraic numbers with algebraic coefficients is transcendental.

The Bernoulli numbers and trigonometric expansions can both be expressed via logarithmic integrals (e.g., Euler–Maclaurin, zeta relations). This means any equality of the form $\sum_{p \in \text{finite}} \left(\frac{2^{4p-1}|B_{2p}|}{(2k)!}\right) = -\cot(2)$, would imply a linear relation between logarithms of algebraic numbers. This is impossible by Baker’s theorem [8]. Therefore, the equality holds only as an infinite sum.

e. Nesterenko’s Theorem (1996) [9] on the algebraic independence of $e^x, \cot x, \sin(1)$, etc

Certain combinations of transcendental constants (including trigonometric values at algebraic arguments) are algebraically independent over $\mathbb{Q}$.

This determines that $\cot(2)$ cannot be algebraically dependent on any rational or Bernoulli-type term. Thus, no finite algebraic structure The Bernoulli-weighted Special-prime sum structurally resembles these zeta-type series. By analogy, the equality with $\cot(2)$ is consistent with the class of transcendental series equalities known to require infinite index sets.

Lemma 6: (Transcendental Consistency Condition).

By the Lindemann–Weierstrass theorem [6], $\cot(2)$ is transcendental. Since every partial Bernoulli sum in primes such as

$$\sum_N \left(\frac{2^{4p-1}|B_{2p}|}{(2k)!}\right)$$

is algebraic, and equality with a transcendental constant is only possible in the limit $N \rightarrow \infty$.

Define the Special Prime indicator:

$$A(k) = (\mu(k))^2 \frac{\Lambda(k)}{\log(k)} (\mu(2k+1))^2 \frac{\Lambda(2k+1)}{\log(2k+1)}, k \geq 2 \text{ (105.)}$$

where $\Lambda(n)$ is the von Mangoldt function:

$$\Lambda(n) = \begin{cases} \log p, n = p^k \text{ for a prime } p \\ 0 & \text{otherwise} \end{cases} \text{ (106.)}$$

and $\mu(n)$ is the Möbius function. This indicator equals 1 exactly when n and $2n+1$ are both prime.

9. The Quadratic Discriminant Lemma for Special Infinitude

Let P_S be the set of all Special indices p where $A(p) = 1$.
From the cotangent decomposition, define partial sums:

$$A_N = \sum_{p \in P_S} a(p), B_N = \sum_{k \in P_S} b(k), C_N = \sum_{k \notin P_S} c(k) > 0, p \geq 2 \quad (107.)$$

These satisfy (53) the truncated field equation:

$$A_N \cdot \tan^2(2) + B_N \cdot \tan(2) + C_N = R_N \text{ with } R_N \rightarrow 0. \quad (1.)$$

Lemma 7: (Transcendental Consistency Condition)

Let $S = \sum_{p \in P_S} \left(\frac{2^{4p-1} |B_{2p}|}{(2p)!} \right)$ denote the Bernoulli-weighted series over the Special-prime set P_S . Then, if $S = \cot(2)$, the set P_S must be infinite.

Proof.

1. By the **Lindemann–Weierstrass Theorem** [6] (1885), if a is a non-zero algebraic number, then $\sin(a)$ and $\cos(a)$ are transcendental. Hence, $\cot(x)$ is transcendental for any algebraic $x \neq 0$; in particular $\cot(2)$ is transcendental.
2. Each Bernoulli number B_{2p} is rational, and $(2p)!, 2^{4p-1}$ are integers.
Therefore every partial sum $S_N = \sum_{\substack{p \in P_S \\ p \leq N}} \left(\frac{2^{4p-1} |B_{2p}|}{(2p)!} \right)$ is an algebraic number.
3. If P_S were finite, S_N would stabilize at some algebraic value S_r .
Since a finite algebraic sum cannot equal a transcendental constant, equality $S = \cot(2)$ is impossible for finite P_S .
4. Consequently the equality can hold only in the limit of an infinite series, implying that P_S is infinite.

Further support can be gleaned from the following.

10. Conditional Quadratic Discriminant Theorem for Special Infinitude

Theorem 2: (General theory of special primes): Assume the analytic cotangent identity converges as a real equality: $\lim_{N \rightarrow \infty} S_N = \cot(2)$, $\tan(2) \in \mathbb{R}$. Then, if $\Delta_N = 0$, and if a pair of primes have the property $\frac{f(p)}{g(f)} = 2$, then, the set of Special prime indices P_S must be infinite.

Proof:

From the prior analyses, it is clear that property $\frac{f(p)}{g(f)} = 2$, is true for both Perfect number Mersenne primes and Sophie primes and Safe primes. Suppose P_S is finite. Then there exists N such that A_N, B_N, C_N remain stabilized for $N \geq N_0$. The truncated equation (43) reduces to a fixed quadratic in $\tan(2)$ with $\Delta_N < 0$, hence no real solutions occur. However, the analytic identity requires a real solution, producing a contradiction. Therefore, the cotangent field can remain real only if the Special set is infinite and $\Delta_N = 0$.

Corollary. Either the cotangent field identity fails to hold for real arguments, or the Special prime set $\{n : A(n) = 1\}$ is infinite.

Remarks.

1. The theorem provides a conditional consistency proof: finite Special sets will render the analytic system non-real.
2. A full unconditional proof would require establishing the cotangent identity and $\Delta_N = 0$ directly from number-theoretic first principles.

3. This framework connects the σ -perfection field $\frac{\sigma(2p+1)}{\sigma(p)} = 2, \frac{\sigma(p)}{p} = 2$, with the primality condition encoded by (48) and (59), showing that real analytic balance implies infinite continuation of Special primes

a) **Discriminant condition.**

For a single-valued analytic function $\tan(2)$, both roots of (54) must coincide, giving the constraint (55), i.e., $B^2 = 4A$

b) Finite-set contradiction.

Suppose either $\wp$ or $\aleph$ is finite. If $\wp$ is finite, then A is bounded and $\frac{B^2}{4A}$ is strictly positive; hence $\cot(2) > 0$, contradicting the analytic value $\cot(2) \approx -0.4576\dots$

If $\aleph$ is finite, A diverges, destroying convergence and violating the finite analytic value of $\cot(2)$.

Therefore, both subsets must extend infinitely.

c) Analytic necessity.

The negative finite value of $\cot(2)$ arises from the conditional convergence of the full series. Only infinite, interleaved contributions from both classes can reproduce the correct analytic continuation through the real axis.

Finite truncations cannot yield the required sign reversal because all partial sums are positive.

d) The analytic identity demands a real balance.

The equality (59) can hold with a transcendental number $\cot(2)$ only if $|\wp| = |\aleph| = \infty$. Therefore, both the Special-primes and non-Special classes are infinite classes.

$$A_N \cdot \tan^2(2) + B_N \cdot \tan(2) + C_N = R_N \text{ with } R_N \rightarrow 0. \{A_N, B_N, C_N\} > 0, (109.)$$

This is a quadratic equation (63) in $\tan(2)$. To have a real analytic solution, the discriminant must be non-negative:

$$\Delta_N = B_N^2 - 4A_N, (108.)$$

If $\Delta_N < 0$, there is no real number $\tan(2)$ that satisfies this finite equation. That means the analytic equality cannot hold in **real numbers** for any finite truncation of the sums. For a given N , finite, the partial sums $\{A_N, B_N, C_N\} > 0$ will include only finitely many terms of which only finitely many Special and non-Special contributions. If the Special set P_S were finite, there would exist some N_0 beyond which no new Special terms appear:

$$A_N = A_{N_0}, B_N, C_N \text{ stabilize for } N > N_0$$

Then, the quadratic becomes a fixed, finite relation. Because it is shown that for any such finite $N = N_0$, the discriminant $\Delta_N < 0$, this stabilized equation has no real solution i.e., it cannot represent a real-valued field balance. The cotangent identity (from σ and the product expansion) is real for all its parameters. If this real analytic equality holds in the limit, then truncating the series must approach a real number and it cannot “jump” from complex to real unless something is changing as N grows. The only way to recover a real limit from a sequence of non-real finite partials is if the system never stabilizes and this means new terms keep entering forever. That “never stabilizing” is precisely infinitude of Special contributions. If one stops adding Special terms (finite P_S), the balance equation becomes over-constrained and the geometry folds into the complex plane (negative discriminant) and so the discriminant cannot have negative values. To stay real, the balance must keep being adjusted for every $q \in P_S$ which means more Special terms keep entering. In short: Finite Special set $\Rightarrow$ quadratic has no real solution (complex balance). Analytic identity is real $\Rightarrow$ real solution must exist. Therefore, the only way to reconcile them is for the Special set to be infinite.

11. Interpretative Remark

Suppose Equation (62) and (63) both represents an analytic continuation and equilibrium between a sparse harmonic lattice (the Special indices) and the complementary dense continuum (non-Special integers). The finiteness of either subset would destroy the analytic balance and invert the sign of $\cot(2)$. Thus, the very existence of a finite negative cotangent value enforces the infinitude of both classes -a remarkable intersection of trigonometric analysis and arithmetic structure.

12. Remarks and Positioning

- a) Novelty. The Main Theorem, and Theorem 1 are not a re-statement of any single classical result; it's a combination of positivity, analytic identity, discriminant collapse $\Rightarrow$ infinitude of each class. The closest analogues are
- b) Fabry/Hadamard (sparsity $\leftrightarrow$ analytic behavior) [5]: The Fabry and Hadamard theorems, particularly the gap theorems, are central results in complex analysis concerning the analytic continuation of power series with "lacunary" or gapped coefficients. Both theorems establish conditions under which a power series cannot be analytically extended beyond its circle of convergence, which then becomes a "natural boundary" for the function.
- c) Tauberian methods (analytic facts $\Rightarrow$ density/infinitude): Tauberian methods use analytic properties of a function to deduce properties of its underlying sequence of coefficients. In analytic number theory, this approach often uses a Dirichlet series and facts about its analytic continuation to determine the density or infinitude of an arithmetic sequence.

b. The normalization that produces a quadratic in $\tan(2)$ encapsulates the single-valuedness of the trigonometric function at $x_0 = 2$; the vanishing discriminant is precisely the statement that the two algebraic branches coincide with the analytic branch. For finite partitions, that coincidence cannot match the true sign/phase unless both classes are infinite.

Robustness. The argument isn't tied to $x_0 = 2$; any $x_0 \in (0, \pi)$ with $\cot(x_0) \neq 0$, yields the same conclusion under (H1), (H2) and (H3).

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