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Abstract: A co-variational inequality problem involving two generalized Yosida approximation operators is considered and solved. We show some characteristics of generalized Yosida approximation operator which are used in our main proof. We apply the concept of nonexpansive sunny retraction to obtain solution of our problem. Convergence analysis is also discussed.

Keywords: Yosida; solution; convergence; inequality; operator

MSC: 65J15; 47J25; 65K15

1. Introduction

Variational inequality theory is an influential unifying methodology for solving many obstacles of pure as well as applied sciences. Hartman and Stampacchia [1] in 1966 initiated the study of variational inequalities while dealing with some problems of mechanics.

The concept of variational inequalities furnish us various devices for modelling many problems existing in variational analysis related to applicable sciences. One can ensure the existence of solution and convergence of iterative sequences using these devices. For applications, see [2–10] and references therein.

Alber and Yao [11] first considered and studied co-variational inequalities using nonexpansive sunny retraction concept. They obtained solution of co-variational inequality problem and discussed the convergence criteria. Their work is extended by Ahmad and Irfan [12].

Yosida approximation operators are useful for obtaining solution of various types of differential equations. Petterson [13] first solved the stochastic differential equation by using Yosida approximation operator approach. For the study of heat equations, problem of couple sound and heat flow in compressible fluids, wave equations, etc., the concept of Yosida approximation operator is applicable. For more details, we refer to [14–18].

After above important discussion, the aim of this work is to introduce a different version of co-variational inequality which involve two generalized Yosida approximation operators. We obtain the solution of our problem and we also discuss convergence criteria for the sequences achieved by iterative method.

2. Preface

Throughout this document, we denote real Banach space by $\widetilde{E}$ and its dual space by $\widetilde{E}^*$. Let $\langle \dot{a}, \dot{b} \rangle$ be the duality pairing between $\dot{a} \in \widetilde{E}$ and $\dot{b} \in \widetilde{E}^*$. The usual norm on $\widetilde{E}$ is denoted by $\|\cdot\|$, the class of nonempty subsets of $\widetilde{E}$ by $2^{\widetilde{E}}$ and the class of nonempty compact subsets of $\widetilde{E}$ by $\widehat{C}(\widetilde{E})$.

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The Normalized duality operator $J : \widetilde{E} \to \widetilde{E}^*$ is defined by

$$J(\dot{a}) = \left\{\dot{b} \in \widetilde{E}^* : \langle \dot{a}, \dot{b}\rangle = \|\dot{a}\|^2 = \|\dot{b}\|^2\right\}, \ \forall \dot{a} \in \widetilde{E}.$$

Some characteristics of normalized duality operator can be discovered in [19].

For the space $\widetilde{E}$, modulus of smoothness is given by the function

$$\rho_{\widetilde{E}}(t) = \sup\left\{\frac{\|\dot{c}+\dot{d}\| + \|\dot{c}-\dot{d}\|}{2} - 1 : \|\dot{c}\| = 1, \|\dot{d}\| = t\right\}.$$

In fact $\widetilde{E}$ is uniformly smooth if and only if

$$\lim_{t\to 0} t^{-1}\rho_{\widetilde{E}}(t) = 0.$$

The following result is instrumental for our main result.

Proposition 1. [11] *Let $\widetilde{E}$ be a uniformly smooth Banach space and J be the normalized duality operator. Then, for any $\dot{a}, \dot{b} \in \widetilde{E}$, we have*

(i) $\|\dot{a} + \dot{b}\|^2 \leq \|\dot{a}\|^2 + 2\langle \dot{b}, J(\dot{a} + \dot{b})\rangle$,
(ii) $\langle \dot{a} - \dot{b}, J(\dot{a}) - J(\dot{b})\rangle \leq 2d^2\rho_{\widetilde{E}}(4\|\dot{a} - \dot{b}\|/d)$, *where* $d = \sqrt{\|\dot{a}\|^2 + \|\dot{b}\|^2/2}$

Definition 1. *The operator $\widehat{h}_1 : \widetilde{E} \to \widetilde{E}$ is called*

(i) *accretive, if*
$$\langle \widehat{h}_1(\dot{a}) - \widehat{h}_1(\dot{b}), J(\dot{a} - \dot{b})\rangle \geq 0, \ \forall \dot{a}, \dot{b} \in \widetilde{E}.$$
(ii) *Strongly accretive, if*
$$\langle \widehat{h}_1(\dot{a}) - \widehat{h}_1(\dot{b}), J(\dot{a} - \dot{b})\rangle \geq r_1\|\dot{a} - \dot{b}\|^2, \ \forall \dot{a}, \dot{b} \in \widetilde{E},$$
where $r_1 > 0$ is a constant.
(iii) *Lipschitz continuous, if*
$$\|\widehat{h}_1(\dot{a}) - \widehat{h}_1(\dot{b})\| \leq \lambda_{\widehat{h}_1}\|\dot{a} - \dot{b}\|, \ \forall \dot{a}, \dot{b} \in \widetilde{E},$$
where $\lambda_{\widehat{h}_1} > 0$ is a constant.
(iv) *expansive, if*
$$\|\widehat{h}_1(\dot{a}) - \widehat{h}_1(\dot{b})\| \geq \beta_{\widehat{h}_1}\|\dot{a} - \dot{b}\|, \ \forall \dot{a}, \dot{b} \in \widetilde{E}$$
where $\beta_{\widehat{h}_1} > 0$ is a constant.

Definition 2. *Let $\widetilde{A} : \widetilde{E} \to \widetilde{E}$ be an operator. The operator $S : \widetilde{E} \times \widetilde{E} \times \widetilde{E} \to \widetilde{E}$ is said to be*

(i) *Lipschitz continuous in the first slot, if*
$$\|S(\widehat{u}_1, \cdot, \cdot) - S(\widehat{u}_2, \cdot, \cdot)\| \leq \delta_{S_1}\|\widehat{u}_1 - \widehat{u}_2\|, \ \forall \dot{a}, \dot{b} \in \widetilde{E} \ \text{and for some } \widehat{u}_1 \in \widetilde{A}(\dot{a}), \widehat{u}_2 \in \widetilde{A}(\dot{b}),$$
where $\delta_{S_1} > 0$ is a constant.
Similarly, we can obtain Lipschitz continuity of S in other slots.
(ii) *Strongly accretive in the first slot with respect to $\widetilde{A}$, if*
$$\langle S(\widehat{u}_1, \cdot, \cdot) - S(\widehat{u}_2, \cdot, \cdot), J(\dot{a} - \dot{b})\rangle \geq \lambda_{S_1}\|\dot{a} - \dot{b}\|^2, \ \forall \dot{a}, \dot{b} \in \widetilde{E} \ \text{and for some } \widehat{u}_1 \in \widetilde{A}(\dot{a}), \widehat{u}_2 \in \widetilde{A}(\dot{b}),$$
where $\lambda_{S_1} > 0$ is a constant.
Similarly strong accretivity of S in other slots and with respect to other operators can be obtained.

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Definition 3. *The operator $\widetilde{A} : \widetilde{E} \to \widehat{C}(\widetilde{E})$ is called D-Lipschitz continuous if*

$$D(\widetilde{A}(\dot{a}), \widetilde{A}(\dot{b})) \le \alpha_{\widetilde{A}} \|\dot{a} - \dot{b}\|, \;\; \forall\, \dot{a}, \dot{b} \in \widetilde{E},$$

where $\alpha_{\widetilde{A}} > 0$ is a constant and $D(\cdot, \cdot)$ denotes the Housdörff metric.

Definition 4. [11] *Suppose $\widetilde{\Omega}$ is the nonempty closed convex subset of $\widetilde{E}$. Then an operator $Q_{\widetilde{\Omega}} : \widetilde{E} \to \widetilde{\Omega}$ is called:*

(i) *retraction on $\widetilde{\Omega}$, if $Q^2_{\widetilde{\Omega}} = Q_{\widetilde{\Omega}}$,*
(ii) *nonexpansive retraction on $\widetilde{\Omega}$, if it satisfies the inequality:*

$$\|Q_{\widetilde{\Omega}}(\dot{a}) - Q_{\widetilde{\Omega}}(\dot{b})\| \le \|\dot{a} - \dot{b}\|, \;\; \forall \dot{a}, \dot{b} \in \widetilde{E},$$

(iii) *nonexpansive sunny retraction on $\widetilde{\Omega}$, if*

$$Q_{\widetilde{\Omega}}(Q_{\widetilde{\Omega}}(\dot{a}) + \hat{t}(\dot{a} - Q_{\widetilde{\Omega}}(\dot{a}))) = Q_{\widetilde{\Omega}}(\dot{a}),$$

for all $\dot{a} \in \widetilde{E}$ and for $0 \le \hat{t} < +\infty$.

Some characteristics of nonexpansive sunny retraction operator are mentioned below, which can be found in [20–22].

Proposition 2. *The operator $Q_{\widetilde{\Omega}}$ is a nonexpansive sunny retraction, if and only if*

$$\langle \dot{a} - Q_{\widetilde{\Omega}}(\dot{a}), J(Q_{\widetilde{\Omega}}(\dot{a}) - \dot{b}) \rangle \ge 0,$$

for all $\dot{a} \in \widetilde{E}$ and $\dot{b} \in \widetilde{\Omega}$.

Proposition 3. *Suppose $\widetilde{m} = \widetilde{m}(\dot{a}) : \widetilde{E} \to \widetilde{E}$ and $Q_{\widetilde{\Omega}} : \widetilde{E} \to \widetilde{\Omega}$ is a nonexpansive sunny retraction. Then for all $\dot{a} \in \widetilde{E}$, we have*

$$Q_{\widetilde{\Omega} + \widetilde{m}(\dot{a})}(\dot{a}) = \widetilde{m}(\dot{a}) + Q_{\widetilde{\Omega}}(\dot{a} - \widetilde{m}(\dot{a})).$$

Definition 5. *The multi-valued operator $\widehat{M} : \widetilde{E} \to 2^{\widetilde{E}}$ is called accretive, if*

$$\langle \widehat{u} - \widehat{v}, J(\dot{a} - \dot{b}) \rangle \ge 0, \;\; \forall\, \dot{a}, \dot{b} \in \widetilde{E} \text{ and for some } \widehat{u} \in \widehat{M}(\dot{a}), \widehat{v} \in \widehat{M}(\dot{b}).$$

Definition 6. *Let $\widehat{h_1} : \widetilde{E} \to \widetilde{E}$ be an operator. The multi-valued operator $\widehat{M} : \widetilde{E} \to 2^{\widetilde{E}}$ is said to be $\widehat{h_1}$-accretive if $\widehat{M}$ is accretive and*

$$[\widehat{h_1} + \lambda \widehat{M}](\widetilde{E}) = \widetilde{E}, \;\; \textit{where } \lambda > 0 \textit{ is a constant.}$$

Definition 7. *We define $R^{\widehat{M}}_{\widehat{h_1},\lambda} : \widetilde{E} \to \widetilde{E}$ such that*

$$R^{\widehat{M}}_{\widehat{h_1},\lambda}(\dot{a}) = [\widehat{h_1} + \lambda \widehat{M}]^{-1}(\dot{a}), \; \forall \dot{a} \in \widetilde{E}, \; \textit{where } \lambda > 0 \textit{ is a constant.}$$

We call it generalized resolvent operator.

Definition 8. *We define $Y^{\widehat{M}}_{\widehat{h_1},\lambda} : \widetilde{E} \to \widetilde{E}$ such that*

$$Y^{\widehat{M}}_{\widehat{h_1},\lambda}(\dot{a}) = \frac{1}{\lambda}[\widehat{h_1} - R^{\widehat{M}}_{\widehat{h_1},\lambda}](\dot{a}), \; \forall \dot{a} \in \widetilde{E}, \; \textit{where } \lambda > 0 \textit{ is a constant.}$$

We call it generalized Yosida approximation operator.

Proposition 4. *[23] Let $\widehat{h}_1 : \widetilde{E} \to \widetilde{E}$ is r_1-strongly accretive and $\widehat{M} : \widetilde{E} \to 2^{\widetilde{E}}$ is $\widehat{h}_1$-accretive multi-valued operator. Then, the operator $R^{\widehat{M}}_{\widehat{h}_1,\lambda} : \widetilde{E} \to \widetilde{E}$ satisfies the following condition.*

$$\left\| R^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}) - R^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{b}) \right\| \leq \frac{1}{r_1} \|\dot{a} - \dot{b}\|, \ \ \forall \dot{a}, \dot{b} \in \widetilde{E}.$$

That is, $R^{\widehat{M}}_{\widehat{h}_1,\lambda}$ is $\dfrac{1}{r_1}$-Lipschitz continuous.

Proposition 5. *If $\widehat{h}_1 : \widetilde{E} \to \widetilde{E}$ is r_1-strongly accretive, $\beta_{\widehat{h}_1}$-expansive, $\lambda_{\widehat{h}_1}$-Lipschitz continuous operator and $R^{\widehat{M}}_{\widehat{h}_1,\lambda} : \widetilde{E} \to \widetilde{E}$ is $\frac{1}{r_1}$-Lipschitz continuous, then the operator $Y^{\widehat{M}}_{\widehat{h}_1,\lambda} : \widetilde{E} \to \widetilde{E}$ satisfy the following condition.*

$$\left\langle Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}) - Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{b}), J(\widehat{h}_1(\dot{a}) - \widehat{h}_1(\dot{b})) \right\rangle \geq \delta_{Y_{\widehat{h}_1}} \|\dot{a} - \dot{b}\|^2, \ \ \forall \dot{a}, \dot{b} \in \widetilde{E},$$

where $\delta_{Y_{\widehat{h}_1}} = \dfrac{\beta^2_{\widehat{h}_1} r_1 - \lambda_{\widehat{h}_1}}{\lambda r_1}$, $\beta^2_{\widehat{h}_1} r_1 > \lambda_{\widehat{h}_1}, \lambda r_1 \neq 0$ and all the constants involved are positive. That is, $Y^{\widehat{M}}_{\widehat{h}_1,\lambda}$ is $\delta_{Y_{\widehat{h}_1}}$-strongly accretive with respect to the operator $\widehat{h}_1$.

Proof. Since $Y^{\widehat{M}}_{\widehat{h}_1,\lambda} = \frac{1}{\lambda}[\widehat{h}_1 - R^{\widehat{M}}_{\widehat{h}_1,\lambda}]$, we evaluate

$$\begin{aligned}
& \left\langle Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}) - Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{b}), J(\widehat{h}_1(\dot{a}) - \widehat{h}_1(\dot{b})) \right\rangle \\
= & \ \frac{1}{\lambda} \left\langle \widehat{h}_1(\dot{a}) - R^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}) - \left[\widehat{h}_1(\dot{b}) - R^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{b}) \right], J(\widehat{h}_1(\dot{a}) - \widehat{h}_1(\dot{b})) \right\rangle \\
= & \ \frac{1}{\lambda} \left\langle \widehat{h}_1(\dot{a}) - \widehat{h}_1(\dot{b}), J(\widehat{h}_1(\dot{a}) - \widehat{h}_1(\dot{b})) \right\rangle \\
& - \frac{1}{\lambda} \left\langle R^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}) - R^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{b}), J(\widehat{h}_1(\dot{a}) - \widehat{h}_1(\dot{b})) \right\rangle.
\end{aligned}$$

As $\widehat{h}_1$ is expansive and Lipschitz continuous and the generalized resolvent operator $R^{\widehat{M}}_{\widehat{h}_1,\lambda}$ is Lipschitz continuous, we obtain

$$\begin{aligned}
& \left\langle Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}) - Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{b}), J(\widehat{h}_1(\dot{a}) - \widehat{h}_1(\dot{b})) \right\rangle \\
\geq & \ \frac{1}{\lambda} \left\| \widehat{h}_1(\dot{a}) - \widehat{h}_1(\dot{b}) \right\|^2 - \frac{1}{\lambda} \left\| R^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}) - R^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{b}) \right\| \left\| \widehat{h}_1(\dot{a}) - \widehat{h}_1(\dot{b}) \right\| \\
\geq & \ \frac{1}{\lambda} \left\| \widehat{h}_1(\dot{a}) - \widehat{h}_1(\dot{b}) \right\|^2 - \frac{1}{\lambda} \frac{1}{r_1} \|\dot{a} - \dot{b}\| \left\| \widehat{h}_1(\dot{a}) - \widehat{h}_1(\dot{b}) \right\| \\
\geq & \ \frac{1}{\lambda} \beta^2_{\widehat{h}_1} \|\dot{a} - \dot{b}\|^2 - \frac{1}{\lambda} \frac{1}{r_1} \lambda_{\widehat{h}_1} \|\dot{a} - \dot{b}\| \|\dot{a} - \dot{b}\| \\
\geq & \ \frac{\beta^2_{\widehat{h}_1}}{\lambda} \|\dot{a} - \dot{b}\|^2 - \frac{\lambda_{\widehat{h}_1}}{\lambda r_1} \|\dot{a} - \dot{b}\|^2 \\
\geq & \ \frac{\beta^2_{\widehat{h}_1} r_1 - \lambda_{\widehat{h}_1}}{\lambda r_1} \|\dot{a} - \dot{b}\|^2 \\
= & \ \delta_{Y_{\widehat{h}_1}} \|\dot{a} - \dot{b}\|^2.
\end{aligned}$$

That is,

$$\left\langle Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}) - Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{b}), J(\widehat{h}_1(\dot{a}) - \widehat{h}_1(\dot{b})) \right\rangle \geq \delta_{Y_{\widehat{h}_1}} \|\dot{a} - \dot{b}\|^2.$$

That is, $Y^{\widehat{M}}_{\widehat{h}_1,\lambda}$ is $\delta_{Y_{\widehat{h}_1}}$-strongly accretive with respect to $\widehat{h}_1$. □

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Proposition 6. *Let $\widehat{h_1} : \widetilde{E} \to \widetilde{E}$ be $\lambda_{\widehat{h_1}}$-Lipschitz continuous, r_1-strongly accretive operator and $R^{\widehat{M}}_{\widehat{h_1},\lambda} : \widetilde{E} \to \widetilde{E}$ is $\frac{1}{r_1}$-Lipschitz continuous, then the operator $Y^{\widehat{M}}_{\widehat{h_1},\lambda} : \widetilde{E} \to \widetilde{E}$ satisfy the following condition.*

$$\left\| Y^{\widehat{M}}_{\widehat{h_1},\lambda}(\dot{a}) - Y^{\widehat{M}}_{\widehat{h_1},\lambda}(\dot{b}) \right\| \leq \lambda_{Y_{\widehat{h_1}}} \|\dot{a} - \dot{b}\|, \ \ \forall \dot{a}, \dot{b} \in \widetilde{E},$$

where $\lambda_{Y_{\widehat{h_1}}} = \dfrac{\lambda_{\widehat{h_1}} r_1 + 1}{\lambda r_1}$, $\lambda r_1 \neq 0$. That is, $Y^{\widehat{M}}_{\widehat{h_1},\lambda}$ is $\lambda_{Y_{\widehat{h_1}}}$-Lipschitz continuous.

Proof. Since $\widehat{h_1}$ and the generalized resolvent operator $R^{\widehat{M}}_{\widehat{h_1},\lambda}$ are Lipschitz continuous, we obtain

$$\begin{aligned}
\left\| Y^{\widehat{M}}_{\widehat{h_1},\lambda}(\dot{a}) - Y^{\widehat{M}}_{\widehat{h_1},\lambda}(\dot{b}) \right\| &= \left\| \frac{1}{\lambda}\left[\widehat{h_1}(\dot{a}) - R^{\widehat{M}}_{\widehat{h_1},\lambda}(\dot{a})\right] - \frac{1}{\lambda}\left[\widehat{h_1}(\dot{b}) - R^{\widehat{M}}_{\widehat{h_1},\lambda}(\dot{b})\right] \right\| \\
&= \frac{1}{\lambda}\left\| \widehat{h_1}(\dot{a}) - \widehat{h_1}(\dot{b}) \right\| + \frac{1}{\lambda}\left\| R^{\widehat{M}}_{\widehat{h_1},\lambda}(\dot{a}) - R^{\widehat{M}}_{\widehat{h_1},\lambda}(\dot{b}) \right\| \\
&\leq \frac{1}{\lambda}\lambda_{\widehat{h_1}} \|\dot{a} - \dot{b}\| + \frac{1}{\lambda}\frac{1}{r_1}\|\dot{a} - \dot{b}\| \\
&= \left(\frac{\lambda_{\widehat{h_1}} r_1 + 1}{\lambda r_1} \right) \|\dot{a} - \dot{b}\| \\
&= \lambda_{Y_{\widehat{h_1}}} \|\dot{a} - \dot{b}\|.
\end{aligned}$$

That is

$$\left\| Y^{\widehat{M}}_{\widehat{h_1},\lambda}(\dot{a}) - Y^{\widehat{M}}_{\widehat{h_1},\lambda}(\dot{b}) \right\| \leq \lambda_{Y_{\widehat{h_1}}} \|\dot{a} - \dot{b}\|, \ \ \forall \dot{a}, \dot{b} \in \widetilde{E}.$$

Thus, the operator $Y^{\widehat{M}}_{\widehat{h_1},\lambda}$ is $\lambda_{Y_{\widehat{h_1}}}$-Lipschitz continuous. □

3. Problem Formation and Iterative Method

Suppose $S : \widetilde{E} \times \widetilde{E} \times \widetilde{E} \to \widetilde{E}$ is a nonlinear operator, $\widetilde{A}, \widetilde{B}, \widetilde{C} : \widetilde{E} \to \widehat{C}(\widetilde{E})$ be multi-valued operators and $\widetilde{K} : \widetilde{E} \to 2^{\widetilde{E}}$ be a multi-valued operator such that $\widetilde{K}(\dot{a})$ is a nonempty, closed and convex set for all $\dot{a} \in \widetilde{E}$. Let $\widehat{h_1}, \widehat{h_2} : \widetilde{E} \to \widetilde{E}$ be the single-valued operators, $\widehat{M} : \widetilde{E} \to 2^{\widetilde{E}}$ be $\widehat{h_1}$-accretive multi-valued operator and $\widehat{N} : \widetilde{E} \to 2^{\widetilde{E}}$ be $\widehat{h_2}$-accretive multi-valued operator, $Y^{\widehat{M}}_{\widehat{h_1},\lambda} : \widetilde{E} \to \widetilde{E}$ and $Y^{\widehat{N}}_{\widehat{h_2},\lambda} : \widetilde{E} \to \widetilde{E}$ be the generalized Yosida approximation operators.

We consider the problem of finding $\dot{a} \in \widetilde{E}, \widehat{u} \in \widetilde{A}(\dot{a}), \widehat{v} \in \widetilde{B}(\dot{a}), \widehat{w} \in \widetilde{C}(\dot{a})$ such that

$$\left\langle Y^{\widehat{M}}_{\widehat{h_1},\lambda}(\dot{a}) - Y^{\widehat{N}}_{\widehat{h_2},\lambda}(\dot{a}), J(S(\widehat{u}, \widehat{v}, \widehat{w})) \right\rangle \geq 0, \ \ \forall\, S(\widehat{u}, \widehat{v}, \widehat{w}) \in \widetilde{K}(\dot{a}), \text{ and } \lambda > 0 \text{ is a constant.} \tag{1}$$

We call problem (1) as co-variational inequality problem involving generalized Yosida approximation operators..

Clearly for problem (1), it is easily accessible to obtain co-variational inequalities studied by Alber and Yao [11] and Ahmad and Irfan [12].

We provide few characterizations of solution of problem (1).

Theorem 1. *Let $\widetilde{A}, \widetilde{B}, \widetilde{C} : \widetilde{E} \to \widehat{C}(\widetilde{E})$ be the multi-valued operators, $S : \widetilde{E} \times \widetilde{E} \times \widetilde{E} \to \widetilde{E}$ be the nonlinear operator and $\widetilde{K} : \widetilde{E} \to 2^{\widetilde{E}}$ be a multi-valued operator such that $\widetilde{K}(\dot{a})$ is a nonempty, closed and convex set for all $\dot{a} \in \widetilde{E}$. Let $\widehat{h_1}, \widehat{h_2} : \widetilde{E} \to \widetilde{E}$ be the single-valued operators, $\widehat{M} : \widetilde{E} \to 2^{\widetilde{E}}$ be $\widehat{h_1}$-accretive multi-valued operator and $\widehat{N} : \widetilde{E} \to 2^{\widetilde{E}}$ be $\widehat{h_2}$-accretive multi-valued operator, $Y^{\widehat{M}}_{\widehat{h_1},\lambda} : \widetilde{E} \to \widetilde{E}$ and $Y^{\widehat{N}}_{\widehat{h_2},\lambda} : \widetilde{E} \to \widetilde{E}$ be the generalized Yosida approximation operators, where $\lambda > 0$ is a constant. Then the following assertions are similar:*

(i) $\dot{a} \in \widetilde{E}, \widehat{u} \in \widetilde{A}(\dot{a}), \widehat{v} \in \widetilde{B}(\dot{a}), \widehat{w} \in \widetilde{C}(\dot{a})$ *constitute the solution of problem* (1).

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(ii) $\dot{a} \in \widetilde{E}, \widehat{u} \in \widetilde{A}(\dot{a}), \widehat{v} \in \widetilde{B}(\dot{a}), \widehat{w} \in \widetilde{C}(\dot{a})$ *such that*

$$S(\widehat{u},\widehat{v},\widehat{w}) = Q_{\widetilde{K}(\dot{a})}[S(\widehat{u},\widehat{v},\widehat{w}) - \lambda\left(Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}) - Y^{\widehat{N}}_{\widehat{h}_2,\lambda}(\dot{a})\right)].$$

Proof. For proof, see [6,19]. □

Combining Proposition 3 and Theorem 1, we obtain the theorem mentioned below.

Theorem 2. *Suppose all the conditions of Theorem 1 are fulfilled and additionally* $\widetilde{K}(\dot{a}) = \widetilde{m}(\dot{a}) + F$, *for all* $\dot{a} \in \widetilde{E}$, *where F is nonempty closed convex subset of* $\widetilde{E}$ *and* $Q_F : \widetilde{E} \to F$ *be a nonexpansive sunny retraction. Then* $\dot{a} \in \widetilde{E}, \widehat{u} \in \widetilde{A}(\dot{a}), \widehat{v} \in \widetilde{B}(\dot{a})$ *and* $\widehat{w} \in \widetilde{C}(\dot{a})$ *form the solution of problem (1), if and only if*

$$\dot{a} = \dot{a} + \widetilde{m}(\dot{a}) - S(\widehat{u},\widehat{v},\widehat{w}) + Q_F[S(\widehat{u},\widehat{v},\widehat{w}) - \lambda\left(Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}) - Y^{\widehat{N}}_{\widehat{h}_2,\lambda}(\dot{a})\right) - \widetilde{m}(\dot{a})], \tag{2}$$

where $\lambda > 0$ *is a constant.*

Using Theorem 2, we construct the following iterative method.

Iterative Method 3.1. *For initial* $\dot{a}_0 \in \widetilde{E}, \widehat{u}_0 \in \widetilde{A}(\dot{a}_0), \widehat{v}_0 \in \widetilde{B}(\dot{a}_0), \widehat{w}_0 \in \widetilde{C}(\dot{a}_0)$, *let*

$$\dot{a}_1 = \dot{a}_0 + \widetilde{m}(\dot{a}_0) - S(\widehat{u}_0,\widehat{v}_0,\widehat{w}_0) + Q_F[S(\widehat{u}_0,\widehat{v}_0,\widehat{w}_0) - \lambda\left(Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_0) - Y^{\widehat{N}}_{\widehat{h}_2,\lambda}(\dot{a}_0)\right) - \widetilde{m}(\dot{a}_0)].$$

Since $\widetilde{A}(\dot{a}_0), \widetilde{B}(\dot{a}_0)$ *and* $\widetilde{C}(\dot{a}_0)$ *are nonempty convex sets, by Nadler [24], there exists* $\widehat{u}_1 \in \widetilde{A}(\dot{a}_1), \widehat{v}_1 \in \widetilde{B}(\dot{a}_1)$ *and* $\widehat{w}_1 \in \widetilde{C}(\dot{a}_1)$ *such that*

$$\begin{aligned} \|\widehat{u}_1 - \widehat{u}_0\| &\leq D(\widetilde{A}(\dot{a}_1), \widetilde{A}(\dot{a}_0)), \\ \|\widehat{v}_1 - \widehat{v}_0\| &\leq D(\widetilde{B}(\dot{a}_1), \widetilde{B}(\dot{a}_0)), \\ \text{and } \|\widehat{w}_1 - \widehat{w}_0\| &\leq D(\widetilde{C}(\dot{a}_1), \widetilde{C}(\dot{a}_0)), \end{aligned}$$

where $D(\cdot,\cdot)$ *denotes the Hausdorff metric.*
Proceeding in a similar manner, we can find the sequences $\{\dot{a}_n\}, \{\widehat{u}_n\}, \{\widehat{v}_n\}$ *and* $\{\widehat{w}_n\}$ *by the following method:*

$$\dot{a}_{n+1} = \dot{a}_n + \widetilde{m}(\dot{a}_n) - S(\widehat{u}_n,\widehat{v}_n,\widehat{w}_n) + Q_F[S(\widehat{u}_n,\widehat{v}_n,\widehat{w}_n) - \lambda\left(Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_n) - Y^{\widehat{N}}_{\widehat{h}_2,\lambda}(\dot{a}_n)\right) - \widetilde{m}(\dot{a}_n)], \tag{3}$$

$$\widehat{u}_n \in \widetilde{A}(\dot{a}_n),\ \|\widehat{u}_{n+1} - \widehat{u}_n\| \leq D(\widetilde{A}(\dot{a}_{n+1}), \widetilde{A}(\dot{a}_n)), \tag{4}$$
$$\widehat{v}_n \in \widetilde{B}(\dot{a}_n),\ \|\widehat{v}_{n+1} - \widehat{v}_n\| \leq D(\widetilde{B}(\dot{a}_{n+1}), \widetilde{B}(\dot{a}_n)), \tag{5}$$
$$\widehat{w}_n \in \widetilde{C}(\dot{a}_n),\ \|\widehat{w}_{n+1} - \widehat{w}_n\| \leq D(\widetilde{C}(\dot{a}_{n+1}), \widetilde{C}(\dot{a}_n)), \tag{6}$$

for $n = 0, 1, 2, 3, \cdots\cdots$, *where* $\lambda > 0$ *is a constant.*

4. Convergence Result

Theorem 3. *Suppose* $\widetilde{E}$ *be real uniformly smooth Banach space and* $\rho_{\widetilde{E}}(t) \leq Ct^2$, *for some* $C > 0$, *is the modulus of smoothness. Suppose F be a closed convex subset of* $\widetilde{E}$, $S(\cdot,\cdot,\cdot) : \widetilde{E} \times \widetilde{E} \times \widetilde{E} \to \widetilde{E}$ *be an operator,* $\widetilde{A}, \widetilde{B}, \widetilde{C} : \widetilde{E} \to \widetilde{C}(\widetilde{E})$ *be the multi-valued operators,* $\widetilde{m} : \widetilde{E} \to \widetilde{E}$ *be an operator. Let* $Q_F : \widetilde{E} \to F$ *be a nonexpansive sunny retraction and* $\widetilde{K} : \widetilde{E} \to 2^{\widetilde{E}}$ *be a multi-valued operator such that* $\widetilde{K}(\dot{a}) = \widetilde{m}(\dot{a}) + F$, *for all* $\dot{a} \in \widetilde{E}$. *Let* $\widehat{M}, \widehat{N} : \widetilde{E} \to 2^{\widetilde{E}}$ *be the multi-valued operators,* $\widehat{h}_1, \widehat{h}_2 : \widetilde{E} \to \widetilde{E}$ *be operators. Let* $Y^{\widehat{M}}_{\widehat{h}_1,\lambda}$ *be the generalized Yosida approximation operator associated with the generalized resolvent operator* $R^{\widehat{M}}_{\widehat{h}_1,\lambda}$ *and* $Y^{\widehat{N}}_{\widehat{h}_2,\lambda}$ *be* *the generalized Yosida approximation operators associated with the generalized resolvent operator* $R^{\widehat{N}}_{\widehat{h}_2,\lambda}$. *Suppose that the following assertions are satisfied:*

(i) $S(\cdot,\cdot,\cdot)$ *is* λ_{S_1}*-strongly accretive with respect to* $\widetilde{A}$ *in the first slot,* λ_{S_2}*-strongly accretive with respect to* $\widetilde{B}$ *in the second slot,* λ_{S_3}*-strongly accretive with respect to* $\widetilde{C}$ *in the third slot and* δ_{S_1}*-Lipschitz continuous in first slot,* δ_{S_2}*-Lipschitz continuous in second slot,* δ_{S_3}*-Lipschitz continuous in third slot.*

(ii) $\widetilde{A}$ *is* $\alpha_{\widetilde{A}}$*-D-Lipschitz continuous,* $\widetilde{B}$ *is* $\alpha_{\widetilde{B}}$*-D-Lipschitz continuous and* $\widetilde{C}$ *is* $\alpha_{\widetilde{C}}$*-D-Lipschitz continuous.*

(iii) $\widetilde{m}$ *is* λ_m*-Lipschitz continuous.*

(iv) $\widehat{h}_1$ *is* r_1*-strongly accretive,* $\beta_{\widehat{h}_1}$*-expansive and* $\lambda_{\widehat{h}_1}$*-Lipschitz continuous and* $\widehat{h}_2$ *is* r_2*-strongly accretive,* $\beta_{\widehat{h}_2}$*-expansive and* $\lambda_{\widehat{h}_2}$*-Lipschitz continuous.*

(v) $R^{\widehat{M}}_{\widehat{h}_1,\lambda}$ *is* $\frac{1}{r_1}$*-Lipschitz continuous and* $R^{\widehat{N}}_{\widehat{h}_2,\lambda}$ *is* $\frac{1}{r_2}$*-Lipschitz continuous.*

(vi) $Y^{\widehat{M}}_{\widehat{h}_1,\lambda}$ *is* $\delta_{Y_{\widehat{h}_1}}$*-strongly accretive,* $\lambda_{Y_{\widehat{h}_1}}$*-Lipschitz continuous and* $Y^{\widehat{N}}_{\widehat{h}_2,\lambda}$ *is* $\delta_{Y_{\widehat{h}_2}}$*-strongly accretive,* $\lambda_{Y_{\widehat{h}_2}}$*-Lipschitz continuous.*

(vii) *Suppose that*

$$0 < \Bigg[\sqrt{\Big[\big(1-2(\lambda_{S_1}+\lambda_{S_2}+\lambda_{S_3})\big)+64C\big(\delta^2_{S_1}\alpha^2_{\widetilde{A}}+\delta^2_{S_2}\alpha^2_{\widetilde{B}}+\delta^2_{S_3}\alpha^2_{\widetilde{C}}\big)\Big]}$$
$$+2\lambda_m+(\delta_{S_1}\alpha_{\widetilde{A}}+\delta_{S_2}\alpha_{\widetilde{B}}+\delta_{S_3}\alpha_{\widetilde{C}})+\sqrt{1-2\lambda\delta_{Y_{\widehat{h}_1}}+64C\lambda^4\lambda^2_{Y_{\widehat{h}_1}}}$$
$$+\sqrt{1-2\lambda\delta_{Y_{\widehat{h}_2}}+64C\lambda^4\lambda^2_{Y_{\widehat{h}_2}}}\Bigg] < 1,$$

where

$$\delta_{Y_{\widehat{h}_1}} = \frac{\beta^2_{\widehat{h}_1}r_1-\lambda_{\widehat{h}_1}}{\lambda r_1}, \qquad \delta_{Y_{\widehat{h}_2}} = \frac{\beta^2_{\widehat{h}_2}r_2-\lambda_{\widehat{h}_2}}{\lambda r_2},$$
$$\lambda_{Y_{\widehat{h}_1}} = \frac{\lambda_{\widehat{h}_1}r_1+1}{\lambda r_1}, \qquad \lambda_{Y_{\widehat{h}_2}} = \frac{\lambda_{\widehat{h}_2}r_2+1}{\lambda r_2},$$
$$\beta^2_{\widehat{h}_1}r_1 > \lambda_{\widehat{h}_1} \quad \text{and} \quad \beta^2_{\widehat{h}_2}r_2 > \lambda_{\widehat{h}_2}.$$

Then, there exist $\dot{a} \in \widetilde{E}, \widehat{u} \in \widetilde{A}(\dot{a}), \widehat{v} \in \widetilde{B}(\dot{a})$ *and* $\widehat{w} \in \widetilde{C}(\dot{a})$*, the solution of problem* (1)*. Also sequences* $\{\dot{a}_n\}, \{\widehat{u}_n\}, \{\widehat{v}_n\}$ *and* $\{\widehat{w}_n\}$ *converge strongly to* $\dot{a}, \widehat{u}, \widehat{v}$ *and* $\widehat{w}$*, respectively.*

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Proof. Using (3) of iterative method 3 and nonexpansive retraction property of Q_F, we estimate

$$\begin{aligned}
\|\dot{a}_{n+1} - \dot{a}_n\| &= \Big\| \Big[\dot{a}_n + \widetilde{m}(\dot{a}_n) - S(\widehat{u}_n, \widehat{v}_n, \widehat{w}_n) + Q_F\big[S(\widehat{u}_n, \widehat{v}_n, \widehat{w}_n) \\
&\qquad -\lambda\left(Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_n) - Y^{\widehat{N}}_{\widehat{h}_2,\lambda}(\dot{a}_n)\right) - \widetilde{m}(\dot{a}_n)\big]\Big] \\
&\quad - \Big[\dot{a}_{n-1} + \widetilde{m}(\dot{a}_{n-1}) - S(\widehat{u}_{n-1}, \widehat{v}_{n-1}, \widehat{w}_{n-1}) + Q_F\big[S(\widehat{u}_{n-1}, \widehat{v}_{n-1}, \widehat{w}_{n-1}) \\
&\qquad -\lambda\left(Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_{n-1}) - Y^{\widehat{N}}_{\widehat{h}_2,\lambda}(\dot{a}_{n-1})\right) - \widetilde{m}(\dot{a}_{n-1})\big]\Big]\Big\| \\
&\leq \|\dot{a}_n - \dot{a}_{n-1} - \big(S(\widehat{u}_n, \widehat{v}_n, \widehat{w}_n) - S(\widehat{u}_{n-1}, \widehat{v}_{n-1}, \widehat{w}_{n-1})\big)\| + \|\widetilde{m}(\dot{a}_n) - \widetilde{m}(\dot{a}_{n-1})\| \\
&\quad + \Big\|Q_F\big[S(\widehat{u}_n, \widehat{v}_n, \widehat{w}_n) - \lambda\left(Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_n) - Y^{\widehat{N}}_{\widehat{h}_2,\lambda}(\dot{a}_n)\right) - \widetilde{m}(\dot{a}_n)\big] \\
&\quad - Q_F\big[S(\widehat{u}_{n-1}, \widehat{v}_{n-1}, \widehat{w}_{n-1}) - \lambda\left(Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_{n-1}) - Y^{\widehat{N}}_{\widehat{h}_2,\lambda}(\dot{a}_{n-1})\right) - \widetilde{m}(\dot{a}_{n-1})\big]\Big\| \\
&\leq \|\dot{a}_n - \dot{a}_{n-1} - \big(S(\widehat{u}_n, \widehat{v}_n, \widehat{w}_n) - S(\widehat{u}_{n-1}, \widehat{v}_{n-1}, \widehat{w}_{n-1})\big)\| \\
&\quad + 2\|\widetilde{m}(\dot{a}_n) - \widetilde{m}(\dot{a}_{n-1})\| + \|S(\widehat{u}_n, \widehat{v}_n, \widehat{w}_n) - S(\widehat{u}_{n-1}, \widehat{v}_{n-1}, \widehat{w}_{n-1})\| \\
&\quad + \|\dot{a}_n - \dot{a}_{n-1} - \lambda\left(Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_n) - Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_{n-1})\right)\| \\
&\quad + \|\dot{a}_n - \dot{a}_{n-1} - \lambda\left(Y^{\widehat{N}}_{\widehat{h}_2,\lambda}(\dot{a}_n) - Y^{\widehat{N}}_{\widehat{h}_2,\lambda}(\dot{a}_{n-1})\right)\|.
\end{aligned} \tag{7}$$

Applying Proposition 1, we evaluate

$$\begin{aligned}
&\|\dot{a}_n - \dot{a}_{n-1} - \big(S(\widehat{u}_n, \widehat{v}_n, \widehat{w}_n) - S(\widehat{u}_{n-1}, \widehat{v}_{n-1}, \widehat{w}_{n-1})\big)\|^2 \\
&\leq \|\dot{a}_n - \dot{a}_{n-1}\|^2 \\
&\quad -2\langle S(\widehat{u}_n, \widehat{v}_n, \widehat{w}_n) - S(\widehat{u}_{n-1}, \widehat{v}_{n-1}, \widehat{w}_{n-1}), J\big(\dot{a}_n - \dot{a}_{n-1} - \big(S(\widehat{u}_n, \widehat{v}_n, \widehat{w}_n) - S(\widehat{u}_{n-1}, \widehat{v}_{n-1}, \widehat{w}_{n-1})\big)\big)\rangle \\
&= \|\dot{a}_n - \dot{a}_{n-1}\|^2 - 2\langle S(\widehat{u}_n, \widehat{v}_n, \widehat{w}_n) - S(\widehat{u}_{n-1}, \widehat{v}_{n-1}, \widehat{w}_{n-1}), J(\dot{a}_n - \dot{a}_{n-1})\rangle \\
&\quad -2\langle S(\widehat{u}_n, \widehat{v}_n, \widehat{w}_n) - S(\widehat{u}_{n-1}, \widehat{v}_{n-1}, \widehat{w}_{n-1}), \\
&\qquad\qquad J\big(\dot{a}_n - \dot{a}_{n-1} - \big(S(\widehat{u}_n, \widehat{v}_n, \widehat{w}_n) - S(\widehat{u}_{n-1}, \widehat{v}_{n-1}, \widehat{w}_{n-1})\big)\big) - J(\dot{a}_n - \dot{a}_{n-1})\rangle \\
&= \|\dot{a}_n - \dot{a}_{n-1}\|^2 - 2\langle S(\widehat{u}_n, \widehat{v}_n, \widehat{w}_n) - S(\widehat{u}_{n-1}, \widehat{v}_n, \widehat{w}_n) \\
&\quad + S(\widehat{u}_{n-1}, \widehat{v}_n, \widehat{w}_n) - S(\widehat{u}_{n-1}, \widehat{v}_{n-1}, \widehat{w}_n) + S(\widehat{u}_{n-1}, \widehat{v}_{n-1}, \widehat{w}_n) - S(\widehat{u}_{n-1}, \widehat{v}_{n-1}, \widehat{w}_{n-1}), J(\dot{a}_n - \dot{a}_{n-1})\rangle \\
&\quad -2\langle S(\widehat{u}_n, \widehat{v}_n, \widehat{w}_n) - S(\widehat{u}_{n-1}, \widehat{v}_{n-1}, \widehat{w}_{n-1}), J\big(\dot{a}_n - \dot{a}_{n-1} \\
&\qquad\qquad - \big(S(\widehat{u}_n, \widehat{v}_n, \widehat{w}_n) - S(\widehat{u}_{n-1}, \widehat{v}_{n-1}, \widehat{w}_{n-1})\big)\big) - J(\dot{a}_n - \dot{a}_{n-1})\rangle \\
&= \|\dot{a}_n - \dot{a}_{n-1}\|^2 - 2\langle S(\widehat{u}_n, \widehat{v}_n, \widehat{w}_n) - S(\widehat{u}_{n-1}, \widehat{v}_n, \widehat{w}_n), J(\dot{a}_n - \dot{a}_{n-1})\rangle \\
&\quad -2\langle S(\widehat{u}_{n-1}, \widehat{v}_n, \widehat{w}_n) - S(\widehat{u}_{n-1}, \widehat{v}_{n-1}, \widehat{w}_n), J(\dot{a}_n - \dot{a}_{n-1})\rangle \\
&\quad -2\langle S(\widehat{u}_{n-1}, \widehat{v}_{n-1}, \widehat{w}_n) - S(\widehat{u}_{n-1}, \widehat{v}_{n-1}, \widehat{w}_{n-1}), J(\dot{a}_n - \dot{a}_{n-1})\rangle \\
&\quad -2\langle S(\widehat{u}_n, \widehat{v}_n, \widehat{w}_n) - S(\widehat{u}_{n-1}, \widehat{v}_{n-1}, \widehat{w}_{n-1}), \\
&\qquad\qquad J\big(\dot{a}_n - \dot{a}_{n-1} - \big(S(\widehat{u}_n, \widehat{v}_n, \widehat{w}_n) - S(\widehat{u}_{n-1}, \widehat{v}_{n-1}, \widehat{w}_{n-1})\big) - J(\dot{a}_n - \dot{a}_{n-1})\rangle.
\end{aligned} \tag{8}$$

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Since $S(\cdot,\cdot,\cdot)$ is λ_{S_1}-strongly accretive with respect to $\widetilde{A}$ in the first slot, λ_{S_2}-strongly accretive with respect to $\widetilde{B}$ in the second slot, λ_{S_3}-strongly accretive with respect to $\widetilde{C}$ in the third slot and applying (ii) of Proposition 1, (8) becomes

$$\begin{aligned}
&\|(\dot{a}_n-\dot{a}_{n-1})-\big(S(\widehat{u}_n,\widehat{v}_n,\widehat{w}_n)-S(\widehat{u}_{n-1},\widehat{v}_{n-1},\widehat{w}_{n-1})\big)\|^2\\
&\quad\leq \|\dot{a}_n-\dot{a}_{n-1}\|^2-2(\lambda_{S_1}+\lambda_{S_2}+\lambda_{S_3})\|\dot{a}_n-\dot{a}_{n-1}\|^2\\
&\quad -2\langle S(\widehat{u}_n,\widehat{v}_n,\widehat{w}_n)-S(\widehat{u}_{n-1},\widehat{v}_{n-1},\widehat{w}_{n-1}),\\
&\qquad\qquad J\big(\dot{a}_n-\dot{a}_{n-1}-\big(S(\widehat{u}_n,\widehat{v}_n,\widehat{w}_n)-S(\widehat{u}_{n-1},\widehat{v}_{n-1},\widehat{w}_{n-1})\big)\big)-J(\dot{a}_n-\dot{a}_{n-1})\rangle\\
&\quad\leq (1-2(\lambda_{S_1}+\lambda_{S_2}+\lambda_{S_3}))\|\dot{a}_n-\dot{a}_{n-1}\|^2\\
&\qquad\qquad +4d^2\rho_{\widetilde{E}}\left(\frac{4\|S(\widehat{u}_n,\widehat{v}_n,\widehat{w}_n)-S(\widehat{u}_{n-1},\widehat{v}_{n-1},\widehat{w}_{n-1})\|}{d}\right).
\end{aligned}\tag{9}$$

As $S(\cdot,\cdot,\cdot)$ is δ_{S_1}-Lipschitz continuous in first slot, δ_{S_2}-Lipschitz continuous in second slot, δ_{S_3}-Lipschitz continuous in third slot and $\widetilde{A}$ is $\alpha_{\widetilde{A}}$-D-Lipschitz continuous, $\widetilde{B}$ is $\alpha_{\widetilde{B}}$-D-Lipschitz continuous, $\widetilde{C}$ is $\alpha_{\widetilde{C}}$-D-Lipschitz continuous, we have

$$\begin{aligned}
&\|S(\widehat{u}_n,\widehat{v}_n,\widehat{w}_n)-S(\widehat{u}_{n-1},\widehat{v}_{n-1},\widehat{w}_{n-1})\|\\
&\quad=\|S(\widehat{u}_n,\widehat{v}_n,\widehat{w}_n)-S(\widehat{u}_{n-1},\widehat{v}_n,\widehat{w}_n)+S(\widehat{u}_{n-1},\widehat{v}_n,\widehat{w}_n)-S(\widehat{u}_{n-1},\widehat{v}_{n-1},\widehat{w}_n)\\
&\qquad\qquad +S(\widehat{u}_{n-1},\widehat{v}_{n-1},\widehat{w}_n)-S(\widehat{u}_{n-1},\widehat{v}_{n-1},\widehat{w}_{n-1})\|\\
&\quad\leq \delta_{S_1}\|\widehat{u}_n-\widehat{u}_{n-1}\|+\delta_{S_2}\|\widehat{v}_n-\widehat{v}_{n-1}\|+\delta_{S_3}\|\widehat{w}_n-\widehat{w}_{n-1}\|\\
&\quad\leq \delta_{S_1}D(\widetilde{A}(\dot{a}_n),\widetilde{A}(\dot{a}_{n-1}))+\delta_{S_2}D(\widetilde{B}(\dot{a}_n),\widetilde{B}(\dot{a}_{n-1}))+\delta_{S_3}D(\widetilde{C}(\dot{a}_n),\widetilde{C}(\dot{a}_{n-1}))\\
&\quad\leq \delta_{S_1}\alpha_{\widetilde{A}}\|\dot{a}_n-\dot{a}_{n-1}\|+\delta_{S_2}\alpha_{\widetilde{B}}\|\dot{a}_n-\dot{a}_{n-1}\|+\delta_{S_3}\alpha_{\widetilde{C}}\|\dot{a}_n-\dot{a}_{n-1}\|\\
&\quad\leq (\delta_{S_1}\alpha_{\widetilde{A}}+\delta_{S_2}\alpha_{\widetilde{B}}+\delta_{S_3}\alpha_{\widetilde{C}})\|\dot{a}_n-\dot{a}_{n-1}\|.
\end{aligned}\tag{10}$$

Using equation (10) and (ii) of Proposition 1, we evaluate

$$\begin{aligned}
&4d^2\rho_{\widetilde{E}}\left(\frac{4\|S(\widehat{u}_n,\widehat{v}_n,\widehat{w}_n)-S(\widehat{u}_{n-1},\widehat{v}_{n-1},\widehat{w}_{n-1})\|}{d}\right)\\
&\quad=4d^2\rho_{\widetilde{E}}\bigg(\frac{4}{d}(\|S(\widehat{u}_n,\widehat{v}_n,\widehat{w}_n)-S(\widehat{u}_{n-1},\widehat{v}_n,\widehat{w}_n)+S(\widehat{u}_{n-1},\widehat{v}_n,\widehat{w}_n)\\
&\qquad -S(\widehat{u}_{n-1},\widehat{v}_{n-1},\widehat{w}_n)+S(\widehat{u}_{n-1},\widehat{v}_{n-1},\widehat{w}_n)-S(\widehat{u}_{n-1},\widehat{v}_{n-1},\widehat{w}_{n-1})\|)\bigg)\\
&\quad\leq 64C\Big(\|S(\widehat{u}_n,\widehat{v}_n,\widehat{w}_n)-S(\widehat{u}_{n-1},\widehat{v}_n,\widehat{w}_n)\|^2+\|S(\widehat{u}_{n-1},\widehat{v}_n,\widehat{w}_n)\\
&\qquad -S(\widehat{u}_{n-1},\widehat{v}_{n-1},\widehat{w}_n)\|^2+\|S(n_{n-1},\widehat{v}_{n-1},\widehat{w}_n)-S(\widehat{u}_{n-1},\widehat{v}_{n-1},\widehat{w}_{n-1})\|^2\Big)\\
&\quad\leq 64C(\delta_{S_1}^2\|\widehat{u}_n-\widehat{u}_{n-1}\|^2+\delta_{S_2}^2\|\widehat{v}_n-\widehat{v}_{n-1}\|^2+\delta_{S_3}^2\|\widehat{w}_n-\widehat{w}_{n-1}\|^2)\\
&\quad\leq 64C(\delta_{S_1}^2D^2(\widetilde{A}(\dot{a}_n),\widetilde{A}(\dot{a}_{n-1}))+\delta_{S_2}^2D^2(\widetilde{B}(\dot{a}_n),\widetilde{B}(\dot{a}_{n-1}))+\delta_{S_3}^2D^2(\widetilde{C}(\dot{a}_n),\widetilde{C}(\dot{a}_{n-1})))\\
&\quad\leq 64C(\delta_{S_1}^2\alpha_{\widetilde{A}}^2\|\dot{a}_n-\dot{a}_{n-1}\|^2+\delta_{S_2}^2\alpha_{\widetilde{B}}^2\|\dot{a}_n-\dot{a}_{n-1}\|^2+\delta_{S_3}^2\alpha_{\widetilde{C}}^2\|\dot{a}_n-\dot{a}_{n-1}\|^2)\\
&\quad=64C(\delta_{S_1}^2\alpha_{\widetilde{A}}^2+\delta_{S_2}^2\alpha_{\widetilde{B}}^2+\delta_{S_3}^2\alpha_{\widetilde{C}}^2)\|\dot{a}_n-\dot{a}_{n-1}\|^2.
\end{aligned}\tag{11}$$

Combining (9) and (11), we have

$$\begin{aligned}
\|(\dot{a}_n-\dot{a}_{n-1})-\big(S(\widehat{u}_n,\widehat{v}_n,\widehat{w}_n)-S(\widehat{u}_{n-1},\widehat{v}_{n-1},\widehat{w}_{n-1})\big)\|^2\leq\Big[&(1-2(\lambda_{S_1}+\lambda_{S_2}+\lambda_{S_3}))\\
&+64C(\delta_{S_1}^2\alpha_{\widetilde{A}}^2+\delta_{S_2}^2\alpha_{\widetilde{B}}^2+\delta_{S_3}^2\alpha_{\widetilde{C}}^2)\Big]\|\dot{a}_n-\dot{a}_{n-1}\|^2,
\end{aligned}$$

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which implies that

$$\begin{aligned}\|(\dot{a}_n - \dot{a}_{n-1}) - (S(\widehat{u}_n, \widehat{v}_n, \widehat{w}_n) - S(\widehat{u}_{n-1}, \widehat{v}_{n-1}, \widehat{w}_{n-1}))\| \\ \le \sqrt{\left[(1 - 2(\lambda_{S_1} + \lambda_{S_2} + \lambda_{S_3})) + 64C\big(\delta_{S_1}^2 \alpha_{\widetilde{A}}^2 + \delta_{S_2}^2 \alpha_{\widetilde{B}}^2 + \delta_{S_3}^2 \alpha_{\widetilde{C}}^2\big)\right]} \|\dot{a}_n - \dot{a}_{n-1}\|. \quad (12)\end{aligned}$$

Since $\widetilde{m}$ is λ_m-Lipschitz continuous, we have

$$\|\widetilde{m}(\dot{a}_n) - \widetilde{m}(\dot{a}_{n-1})\| \le \lambda_m \|\dot{a}_n - \dot{a}_{n-1}\|. \quad (13)$$

As Yosida approximation operator $Y^{\widehat{M}}_{\widehat{h}_1,\lambda}$ is $\delta_{Y_{\widehat{h}_1}}$-strongly accretive, $\lambda_{Y_{\widehat{h}_1}}$-Lipschitz continuous and applying Proposition 1, we evaluate

$$\begin{aligned}&\left\|(\dot{a}_n - \dot{a}_{n-1}) - \lambda\left(Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_n) - Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_{n-1})\right)\right\|^2 \\ &\le \|\dot{a}_n - \dot{a}_{n-1}\|^2 - 2\lambda\left\langle Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_n) - Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_{n-1}), J\left(\dot{a}_n - \dot{a}_{n-1} - \lambda\left(Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_n) - Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_{n-1})\right)\right)\right\rangle \\ &= \|\dot{a}_n - \dot{a}_{n-1}\|^2 - 2\lambda\langle Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_n) - Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_{n-1}), J(\dot{a}_n - \dot{a}_{n-1})\rangle \\ &-2\lambda\langle Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_n) - Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_{n-1}), J\left(\dot{a}_n - \dot{a}_{n-1} - \left(Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_n) - Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_{n-1})\right)\right) - J(\dot{a}_n - \dot{a}_{n-1})\rangle \\ &\le \|\dot{a}_n - \dot{a}_{n-1}\|^2 - 2\lambda\delta_{Y_{\widehat{h}_1}}\|\dot{a}_n - \dot{a}_{n-1}\|^2 + 4d^2\rho_{\widetilde{E}}\left(\frac{4\lambda^2\left\|Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_n) - Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_{n-1})\right\|}{d}\right) \\ &= \|\dot{a}_n - \dot{a}_{n-1}\|^2 - 2\lambda\delta_{Y_{\widehat{h}_1}}\|\dot{a}_n - \dot{a}_{n-1}\|^2 + 64C\lambda^4\left\|Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_n) - Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_{n-1})\right\|^2 \\ &= \|\dot{a}_n - \dot{a}_{n-1}\|^2 - 2\lambda\delta_{Y_{\widehat{h}_1}}\|\dot{a}_n - \dot{a}_{n-1}\|^2 + 64C\lambda^4\lambda^2_{Y_{\widehat{h}_1}}\|\dot{a}_n - \dot{a}_{n-1}\|^2 \\ &= (1 - 2\lambda\delta_{Y_{\widehat{h}_1}} + 64C\lambda^4\lambda^2_{Y_{\widehat{h}_1}})\|\dot{a}_n - \dot{a}_{n-1}\|^2,\end{aligned}$$

that is

$$\|(\dot{a}_n - \dot{a}_{n-1}) - \lambda\left(Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_n) - Y^{\widehat{M}}_{\widehat{h}_1,\lambda}(\dot{a}_{n-1})\right)\| \le \sqrt{1 - 2\lambda\delta_{Y_{\widehat{h}_1}} + 64C\lambda^4\lambda^2_{Y_{\widehat{h}_1}}}\,\|\dot{a}_n - \dot{a}_{n-1}\|. \quad (14)$$

Using the same arguments as for (14), we have

$$\|(\dot{a}_n - \dot{a}_{n-1}) - \lambda\left(Y^{\widehat{N}}_{\widehat{h}_2,\lambda}(\dot{a}_n) - Y^{\widehat{N}}_{\widehat{h}_2,\lambda}(\dot{a}_{n-1})\right)\| \le \sqrt{1 - 2\lambda\delta_{Y_{\widehat{h}_2}} + 64C\lambda^4\lambda^2_{Y_{\widehat{h}_2}}}\,\|\dot{a}_n - \dot{a}_{n-1}\|. \quad (15)$$

Using (10), (12), (13), (14), (15), (7) becomes

$$\begin{aligned}\|\dot{a}_{n+1} - \dot{a}_n\| \le\ & \sqrt{\left[(1 - 2(\lambda_{S_1} + \lambda_{S_2} + \lambda_{S_3})) + 64C\big(\delta_{S_1}^2 \alpha_{\widetilde{A}}^2 + \delta_{S_2}^2 \alpha_{\widetilde{B}}^2 + \delta_{S_3}^2 \alpha_{\widetilde{C}}^2\big)\right]}\|\dot{a}_n - \dot{a}_{n-1}\| \\ & +2\lambda_m\|\dot{a}_n - \dot{a}_{n-1}\| + (\delta_{S_1}\alpha_{\widetilde{A}} + \delta_{S_2}\alpha_{\widetilde{B}} + \delta_{S_3}\alpha_{\widetilde{C}})\|\dot{a}_n - \dot{a}_{n-1}\| \\ & +\sqrt{1 - 2\lambda\delta_{Y_{\widehat{h}_1}} + 64C\lambda^4\lambda^2_{Y_{\widehat{h}_1}}}\,\|\dot{a}_n - \dot{a}_{n-1}\| \\ & +\sqrt{1 - 2\lambda\delta_{Y_{\widehat{h}_2}} + 64C\lambda^4\lambda^2_{Y_{\widehat{h}_2}}}\,\|\dot{a}_n - \dot{a}_{n-1}\| \\ =\ & \theta\|\dot{a}_n - \dot{a}_{n-1}\|, \quad (16)\end{aligned}$$

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$$\text{where} \quad \theta = \Bigg[\sqrt{\Big[\big(1-2(\lambda_{S_1}+\lambda_{S_2}+\lambda_{S_3})\big)+64C\big(\delta_{S_1}^2\alpha_{\widetilde{A}}^2+\delta_{S_2}^2\alpha_{\widetilde{B}}^2+\delta_{S_3}^2\alpha_{\widetilde{C}}^2\big)\Big]}$$
$$+2\lambda_m+(\delta_{S_1}\alpha_{\widetilde{A}}+\delta_{S_2}\alpha_{\widetilde{B}}+\delta_{S_3}\alpha_{\widetilde{C}})+\sqrt{1-2\lambda\delta_{Y_{\widehat{h}_1}}+64C\lambda^4\lambda_{Y_{\widehat{h}_1}}^2}$$
$$+\sqrt{1-2\lambda\delta_{Y_{\widehat{h}_2}}+64C\lambda^4\lambda_{Y_{\widehat{h}_2}}^2}\Bigg]. \tag{17}$$

In view of the assumption (vii), $0<\theta<1$ and clearly $\{\dot{a}_n\}$ is a Cauchy sequence such that we have $\dot{a}\in\widetilde{E}$ and $\dot{a}_n\to\dot{a}$. Using (4), (5), (6) of iterative method 3, D-Lipschitz continuity of $\widetilde{A},\widetilde{B}$, $\widetilde{C}$ and the techniques of Ahmad and Irfan [12], it can be shown easily that $\{\widehat{u}_n\},\{\widehat{v}_n\}$ and $\{\widehat{w}_n\}$ are all Cauchy sequences in $\widetilde{E}$. Thus, $\widehat{u}_n\to\widehat{u}\in\widetilde{E},\widehat{v}_n\to\widehat{v}\in\widetilde{E}$ and $\widehat{w}_n\to\widehat{w}\in\widetilde{E}$. Since $Q_F, S(\cdot,\cdot,\cdot),\widetilde{A},\widetilde{B},\widetilde{C},\widehat{h}_1,\widehat{h}_2,\widehat{M},N,Y_{\widehat{h}_1,\lambda}^{\widehat{M}}$ and $Y_{\widehat{h}_2,\lambda}^{\widehat{N}}$ are all continuous operators in $\widetilde{E}$, we have

$$\dot{a}=\dot{a}+\widetilde{m}(\dot{a})-S(\widehat{u},\widehat{v},\widehat{w})+Q_F[S(\widehat{u},\widehat{v},\widehat{w})-\lambda\left(Y_{\widehat{h}_1,\lambda}^{\widehat{M}}(\dot{a})-Y_{\widehat{h}_2,\lambda}^{\widehat{N}}(\dot{a})\right)].$$

It is remaining to show that $\widehat{u}\in\widetilde{A}(\dot{a}),\widehat{v}\in\widetilde{B}(\dot{a})$ and $\widehat{w}_n\to\widehat{w}\in\widetilde{C}(\dot{a})$. In fact

$$\begin{aligned} d(\widehat{u},\widetilde{A}(\dot{a})) &= \inf\{\|\widehat{u}-h\|:h\in\widetilde{A}(\dot{a})\} \\ &\le \|\widehat{u}-\widehat{u}_n\|+d(\widehat{u}_n,\widetilde{A}(\dot{a})) \\ &\le \|\widehat{u}-\widehat{u}_n\|+D(\widetilde{A}(\dot{a}_n),\widetilde{A}(\dot{a})) \\ &\le \|\widehat{u}-\widehat{u}_n\|+\alpha_{\widetilde{A}}\|\dot{a}_n-\dot{a}\|\to 0. \end{aligned}$$

Hence, $d(\widehat{u},\widetilde{A}(\dot{a}))=0$ and thus $\widehat{u}\in\widetilde{A}(\dot{a})$. Similarly, it can be shown that $\widehat{v}\in\widetilde{B}(\dot{a})$ and $\widehat{w}\in\widetilde{C}(\dot{a})$. From Theorem 2, the result follows. □

5. Conclusion

In this work, we consider a different version of co-variational inequalities existing in available literature. We call it co-variational inequality problem which involves two generalized Yosida approximation operators depending on different generalized resolvent operators. Some properties of generalized Yosida approximation operators are proved. Using the concept of nonexpansive sunny retraction, we prove an existence and convergence result for problem (1).

Our results may be used for further generalization and experimental purposes.

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