

SOME WEIGHTED INTEGRAL INEQUALITIES FOR SUB/SUPERADDITIVE FUNCTIONS ON LINEAR SPACES

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ABSTRACT. Assume that $f : C \rightarrow \mathbb{R}$ is subadditive (superadditive) and hemi-Lebesgue integrable on C , a cone in the linear space X with $0 \in C$. Then for all $x, y \in C$ and a symmetric Lebesgue integrable and nonnegative function $p : [0, 1] \rightarrow [0, \infty)$,

$$\begin{aligned} \frac{1}{2}f(x+y) \int_0^1 p(t) dt &\leq (\geq) \int_0^1 p(t) f((1-t)x + ty) dt \\ &\leq (\geq) \int_0^1 p(t) f(tx) dt + \int_0^1 p(t) f(ty) dt. \end{aligned}$$

In particular, for $p \equiv 1$, we have

$$\frac{1}{2}f(x+y) \leq (\geq) \int_0^1 f((1-t)x + ty) dt \leq (\geq) \int_0^1 f(tx) dt + \int_0^1 f(ty) dt.$$

Some particular inequalities related to Jensen's discrete inequality for convex functions are also given.

1. INTRODUCTION

Let X be a real linear space, $x, y \in X$, $x \neq y$ and let $[x, y] := \{(1-\lambda)x + \lambda y, \lambda \in [0, 1]\}$ be the *segment* generated by x and y . We consider the function $f : [x, y] \rightarrow \mathbb{R}$ and the attached function $\varphi_{(x,y)} : [0, 1] \rightarrow \mathbb{R}$, $\varphi_{(x,y)}(t) := f[(1-t)x + ty]$, $t \in [0, 1]$.

The following inequality is the well-known Hermite-Hadamard integral inequality for convex functions defined on a segment $[x, y] \subset X$:

$$(HH) \quad f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty] dt \leq \frac{f(x) + f(y)}{2},$$

which easily follows by the classical Hermite-Hadamard inequality for the convex function $\varphi_{(x,y)} : [0, 1] \rightarrow \mathbb{R}$

$$\varphi_{(x,y)}\left(\frac{1}{2}\right) \leq \int_0^1 \varphi_{(x,y)}(t) dt \leq \frac{\varphi_{(x,y)}(0) + \varphi_{(x,y)}(1)}{2}.$$

For other related results see the monograph on line [6]. For some recent results in linear spaces, see [1], [2] and [14]-[17].

By the convexity of f we have for all $t \in [0, 1]$ that

$$f\left(\frac{x+y}{2}\right) \leq \frac{f[(1-t)x + ty] + f[(1-t)y + tx]}{2} \leq \frac{f(x) + f(y)}{2}.$$

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If we multiply this inequality by $p : [0, 1] \rightarrow [0, \infty)$, a Lebesgue integrable function on $[0, 1]$, and integrate on $[0, 1]$ over $t \in [0, 1]$, then we get

$$(1.1) \quad \begin{aligned} f\left(\frac{x+y}{2}\right) \int_0^1 p(t) dt \\ \leq \frac{\int_0^1 f[(1-t)x+ty]p(t) dt + \int_0^1 f[(1-t)y+tx]p(t) dt}{2} \\ \leq \frac{f(x) + f(y)}{2} \int_0^1 p(t) dt. \end{aligned}$$

If p is symmetric on $[0, 1]$, namely $p(t) = p(1-t)$ for $t \in [0, 1]$, then by changing the variable $s = 1-t$, we get

$$\begin{aligned} \int_0^1 f[(1-t)y+tx]p(t) dt &= \int_0^1 f[sy+(1-s)x]p(1-s) dt \\ &= \int_0^1 f[(1-t)x+ty]p(t) dt \end{aligned}$$

and by (1.1) we obtain the *Féjer's inequality*

$$(1.2) \quad \begin{aligned} f\left(\frac{x+y}{2}\right) \int_0^1 p(t) dt \leq \int_0^1 f[(1-t)x+ty]p(t) dt \\ \leq \frac{f(x) + f(y)}{2} \int_0^1 p(t) dt. \end{aligned}$$

If $(X; \|\cdot\|)$ is a normed linear space, then $f(x) = \|x\|^r$, $r \geq 1$ is convex and by (1.2) we get

$$(1.3) \quad \begin{aligned} \left\|\frac{x+y}{2}\right\|^r \int_0^1 p(t) dt \leq \int_0^1 \|(1-t)x+ty\|^r p(t) dt \\ \leq \frac{\|x\|^r + \|y\|^r}{2} \int_0^1 p(t) dt, \end{aligned}$$

for all $x, y \in X$.

For $r = 1$ we get

$$(1.4) \quad \begin{aligned} \left\|\frac{x+y}{2}\right\| \int_0^1 p(t) dt \leq \int_0^1 \|(1-t)x+ty\| p(t) dt \\ \leq \frac{\|x\| + \|y\|}{2} \int_0^1 p(t) dt, \end{aligned}$$

for all $x, y \in X$.

Let X be a linear space. A subset $C \subseteq X$ is called a *convex cone* in X provided the following conditions hold:

- (i) $x, y \in C$ imply $x+y \in C$;
- (ii) $x \in C, \alpha \geq 0$ imply $\alpha x \in C$.

A functional $h : C \rightarrow \mathbb{R}$ is called *superadditive (subadditive)* on C if

- (iii) $h(x+y) \geq (\leq) h(x) + h(y)$ for any $x, y \in C$

and *nonnegative (strictly positive)* on C if, obviously, it satisfies

- (iv) $h(x) \geq (>) 0$ for each $x \in C$.

The functional h is *s-positive homogeneous* on C , for a given $s > 0$, if

(v) $h(\alpha x) = \alpha^s h(x)$ for any $\alpha \geq 0$ and $x \in C$.

In [9] we obtained further results concerning the quasilinearity of some composite functionals:

Theorem 1. *Let C be a convex cone in the linear space X and $v : C \rightarrow (0, \infty)$ an additive functional on C . If $h : C \rightarrow [0, \infty)$ is a superadditive (subadditive) functional on C and $p, q \geq 1$ ($0 < p, q < 1$) then the functional*

$$(1.5) \quad \Psi_{p,q} : C \rightarrow [0, \infty), \quad \Psi_{p,q}(x) = h^q(x) v^{q(1-\frac{1}{p})}(x)$$

is superadditive (subadditive) on C .

Theorem 2. *Let C be a convex cone in the linear space X and $v : C \rightarrow (0, \infty)$ an additive functional on C . If $h : C \rightarrow [0, \infty)$ is a superadditive functional on C and $0 < p, q < 1$ then the functional*

$$(1.6) \quad \Phi_{p,q} : C \rightarrow [0, \infty), \quad \Phi_{p,q}(x) = \frac{v^{q(1-\frac{1}{p})}(x)}{h^q(x)}$$

is subadditive on C .

The following result holds [11].

Theorem 3. *Let C be a convex cone in the linear space X and $v : C \rightarrow (0, \infty)$ an additive functional on C .*

(i) *If $p \geq q \geq 0$, $p \geq 1$ and $h : C \rightarrow [0, \infty)$ is superadditive on C , then the new mapping*

$$(1.7) \quad \Lambda_{p,q} : C \rightarrow [0, \infty), \quad \Lambda_{p,q}(x) := v^{\frac{p-q}{p}}(x) h^q(x)$$

is superadditive on C ;

(ii) *If $p \leq q$, $p \in (0, 1)$ and $h : C \rightarrow [0, \infty)$ is subadditive on C , then the mapping $\Lambda_{p,q}$ is subadditive on C .*

Now, if we assume that $p \geq q \geq 0$, $p \geq 1$, then by denoting $r := \frac{q}{p} \in [0, 1]$, we deduce that the functional

$$\Theta_{p,r}(x) := v^{1-r}(x) h^{pr}(x)$$

is superadditive, provided v is additive and h is superadditive on C . In particular, the functional

$$\Upsilon_t(x) := v^{\frac{1}{2}}(x) h^t(x)$$

is superadditive for $t \geq \frac{1}{2}$.

If $p \leq q$, $p \in (0, 1)$ and if we denote $s := \frac{q}{p} \in [1, \infty)$, then the functional

$$F_{p,s}(x) := \frac{h^{sp}(x)}{v^{s-1}(x)}$$

is subadditive provided v is additive and h is subadditive on C . In particular, the functional

$$\Xi_z(x) := \frac{h^z(x)}{v(x)}$$

is subadditive for $z \in (0, 2)$.

Motivated by the above results, in this paper we establish some weighted integral inequalities for subadditive (superadditive) functions defined on cones from linear spaces that are hemi-Lebesgue integrable.

2. MAIN RESULTS

A superadditive or a subadditive function f defined on a cone C in the linear space X will be called *hemi-Lebesgue integrable* on C if for any $x \in C$ the function $[0, 1] \ni t \mapsto f(tx) \in \mathbb{R}$ is Lebesgue integrable on $[0, 1]$.

Theorem 4. *Assume that $f : C \rightarrow \mathbb{R}$ is subadditive (superadditive) and hemi-Lebesgue integrable on C , a cone in the linear space X with $0 \in C$. Then for all $x, y \in C$ and a symmetric Lebesgue integrable and nonnegative function $p : [0, 1] \rightarrow [0, \infty)$,*

$$(2.1) \quad \frac{1}{2}f(x+y) \int_0^1 p(t) dt \leq (\geq) \int_0^1 p(t) f((1-t)x + ty) dt \\ \leq (\geq) \int_0^1 p(t) f(tx) dt + \int_0^1 p(t) f(ty) dt.$$

In particular, for $p \equiv 1$, we have

$$(2.2) \quad \frac{1}{2}f(x+y) \leq (\geq) \int_0^1 f((1-t)x + ty) dt \leq (\geq) \int_0^1 f(tx) dt + \int_0^1 f(ty) dt.$$

Proof. From the subadditivity of f we have for $x, y \in C$ and $t \in [0, 1]$ that

$$f(x+y) = f((1-t)x + ty + tx + (1-t)y) \\ \leq f((1-t)x + ty) + f(tx + (1-t)y) \\ \leq f((1-t)x) + f(ty) + f(tx) + f((1-t)y).$$

If we multiply this inequality by $p(t) \geq 0$ and integrate over $t \in [0, 1]$ we get

$$(2.3) \quad f(x+y) \int_0^1 p(t) dt \\ \leq \int_0^1 p(t) f((1-t)x + ty) dt + \int_0^1 p(t) f(tx + (1-t)y) dt \\ \leq \int_0^1 p(t) f((1-t)x) dt + \int_0^1 p(t) f(ty) dt \\ + \int_0^1 p(t) f(tx) dt + \int_0^1 p(t) f((1-t)y) dt.$$

By the symmetry of p and changing the variable, we have

$$\int_0^1 p(t) f(tx + (1-t)y) dt = \int_0^1 p(1-s) f((1-s)x + sy) ds \\ = \int_0^1 p(t) f((1-t)x + ty) dt,$$

$$\int_0^1 p(t) f((1-t)x) dt = \int_0^1 p(1-s) f(sx) ds = \int_0^1 p(t) f(tx) dt$$

and

$$\int_0^1 p(t) f((1-t)y) dt = \int_0^1 p(t) f(ty) dt.$$

Then by (2.3) we obtain

$$\begin{aligned} f(x+y) \int_0^1 p(t) dt &\leq 2 \int_0^1 p(t) f((1-t)x + ty) dt \\ &\leq 2 \int_0^1 p(t) f(tx) dt + 2 \int_0^1 p(t) f(ty) dt, \end{aligned}$$

which is equivalent to (2.1). \square

Remark 1. We observe, for the simple symmetrical weight $p(t) = |t - \frac{1}{2}|$, $t \in [0, 1]$, we get from (2.1) that

$$\begin{aligned} (2.4) \quad \frac{1}{8} f(x+y) &\leq (\geq) \int_0^1 \left| t - \frac{1}{2} \right| f((1-t)x + ty) dt \\ &\leq (\geq) \int_0^1 \left| t - \frac{1}{2} \right| f(tx) dt + \int_0^1 \left| t - \frac{1}{2} \right| f(ty) dt, \end{aligned}$$

while for $p(t) = t(1-t)$, $t \in [0, 1]$, we get

$$\begin{aligned} (2.5) \quad \frac{1}{12} f(x+y) &\leq (\geq) \int_0^1 t(1-t) f((1-t)x + ty) dt \\ &\leq (\geq) \int_0^1 t(1-t) f(tx) dt + \int_0^1 t(1-t) f(ty) dt \end{aligned}$$

for $x, y \in C$, where $f : C \rightarrow \mathbb{R}$ is subadditive (superadditive) and hemi-Lebesgue integrable on C , a cone C in the linear space X with $0 \in C$.

Definition 1. Let C be a cone in the linear space X with $0 \in C$. The function f defined on C is called convex-starshaped if $f(tx) \leq tf(x)$ for all $t \in [0, 1]$ and $x \in C$. It is called concave-starshaped if $f(tx) \geq tf(x)$ for all $t \in [0, 1]$ and $x \in C$.

Corollary 1. With the assumptions of Theorem 4 and, in addition, f is convex-starshaped (concave-starshaped), then for all $x, y \in C$

$$\begin{aligned} (2.6) \quad f\left(\frac{x+y}{2}\right) \int_0^1 p(t) dt &\leq (\geq) \frac{1}{2} f(x+y) \int_0^1 p(t) dt \\ &\leq (\geq) \int_0^1 p(t) f((1-t)x + ty) dt \\ &\leq (\geq) \int_0^1 p(t) f(tx) dt + \int_0^1 p(t) f(ty) dt \\ &\leq (\geq) [f(x) + f(y)] \int_0^1 tp(t) dt. \end{aligned}$$

In particular, for $p \equiv 1$, we have

$$\begin{aligned} (2.7) \quad f\left(\frac{x+y}{2}\right) &\leq (\geq) \frac{1}{2} f(x+y) \leq (\geq) \int_0^1 f((1-t)x + ty) dt \\ &\leq (\geq) \int_0^1 f(tx) dt + \int_0^1 f(ty) dt \leq (\geq) \frac{f(x) + f(y)}{2} \end{aligned}$$

for all $x, y \in C$.

If $(X; \|\cdot\|)$ is a normed space, then the function f is subadditive and convex-starshaped, and for all $x, y \in X$ we have

$$(2.8) \quad \left\| \frac{x+y}{2} \right\| \int_0^1 p(t) dt \leq \int_0^1 p(t) \|(1-t)x + ty\| dt \\ \leq [\|x\| + \|y\|] \int_0^1 tp(t) dt.$$

Since p is symmetric, then

$$\int_0^1 tp(t) dt = \int_0^1 (1-t)p(1-t) dt = \int_0^1 (1-t)p(t) dt \\ = \int_0^1 p(t) dt - \int_0^1 tp(t) dt,$$

which shows that

$$\int_0^1 tp(t) dt = \frac{1}{2} \int_0^1 p(t) dt.$$

Therefore the inequality (2.8) is the same with the inequality (1.4).

Remark 2. If f is a function of real variable defined on $[0, \infty)$ that is subadditive (superadditive) and continuous, and since for $0 < a < b$

$$\int_0^1 p(t) f((1-t)a + ty) dt = \frac{1}{b-a} \int_a^b p\left(\frac{u-a}{b-a}\right) f(u) du,$$

$$\int_0^1 p(t) f(ta) dt = \frac{1}{a} \int_0^a p\left(\frac{v}{a}\right) f(v) dv$$

and

$$\int_0^1 p(t) f(tb) dt = \frac{1}{b} \int_0^b p\left(\frac{v}{b}\right) f(v) dv,$$

then by (2.1) we get

$$(2.9) \quad \frac{1}{2} f(a+b) \int_0^1 p(t) dt \leq (\geq) \frac{1}{b-a} \int_a^b p\left(\frac{u-a}{b-a}\right) f(u) du \\ \leq (\geq) \frac{1}{a} \int_0^a p\left(\frac{v}{a}\right) f(v) dv + \frac{1}{b} \int_0^b p\left(\frac{v}{b}\right) f(v) dv$$

for any symmetric Lebesgue integrable and nonnegative function $p : [0, 1] \rightarrow [0, \infty)$.

In particular, if $p \equiv 1$, then we have the inequality

$$(2.10) \quad \frac{1}{2} f(a+b) \leq (\geq) \frac{1}{b-a} \int_a^b f(u) du \leq (\geq) \frac{1}{a} \int_0^a f(v) dv + \frac{1}{b} \int_0^b f(v) dv,$$

which was obtained in [18].

Corollary 2. Let C be a convex cone in the linear space X with $0 \in C$ and $v : C \rightarrow (0, \infty)$ an additive functional on C . Assume that $h : C \rightarrow [0, \infty)$ is a

superadditive (subadditive) functional on C and $p, q \geq 1$ ($0 < p, q < 1$). If h and v are hemi-Lebesgue integrable on C , then

$$(2.11) \quad \begin{aligned} & \frac{1}{2} h^q(x+y) v^{q(1-\frac{1}{p})}(x+y) \int_0^1 w(t) dt \\ & \leq (\geq) \int_0^1 w(t) h^q((1-t)x+ty) v^{q(1-\frac{1}{p})}((1-t)x+ty) dt \\ & \leq (\geq) \int_0^1 w(t) h^q(tx) v^{q(1-\frac{1}{p})}(tx) dt + \int_0^1 w(t) h^q(ty) v^{q(1-\frac{1}{p})}(ty) dt, \end{aligned}$$

where $x, y \in C$ and a symmetric Lebesgue integrable and nonnegative function $w : [0, 1] \rightarrow [0, \infty)$.

In particular, we have

$$(2.12) \quad \begin{aligned} & \frac{1}{2} h^q(x+y) v^{q(1-\frac{1}{p})}(x+y) \\ & \leq (\geq) \int_0^1 h^q((1-t)x+ty) v^{q(1-\frac{1}{p})}((1-t)x+ty) dt \\ & \leq (\geq) \int_0^1 h^q(tx) v^{q(1-\frac{1}{p})}(tx) dt + \int_0^1 h^q(ty) v^{q(1-\frac{1}{p})}(ty) dt, \end{aligned}$$

where $x, y \in C$.

Proof. Observe, by Theorem 1, that the functional

$$\Psi_{p,q} : C \rightarrow [0, \infty), \quad \Psi_{p,q}(x) = h^q(x) v^{q(1-\frac{1}{p})}(x)$$

is superadditive (subadditive) on C .

If we write Theorem 4 for the function $f = \Psi_{p,q}$ and $p = w$, we get (2.11). \square

Corollary 3. Let C be a convex cone in the linear space X with $0 \in C$ and $v : C \rightarrow (0, \infty)$ an additive functional on C . Assume that $h : C \rightarrow [0, \infty)$ is a superadditive functional on C and $0 < p, q < 1$. If h and v are hemi-Lebesgue integrable on C , then

$$(2.13) \quad \begin{aligned} & \frac{1}{2} \frac{v^{q(1-\frac{1}{p})}(x+y)}{h^q(x+y)} \int_0^1 w(t) dt \\ & \leq \int_0^1 w(t) \frac{v^{q(1-\frac{1}{p})}((1-t)x+ty)}{h^q((1-t)x+ty)} dt \\ & \leq \int_0^1 w(t) \frac{v^{q(1-\frac{1}{p})}(tx)}{h^q(tx)} dt + \int_0^1 w(t) \frac{v^{q(1-\frac{1}{p})}(ty)}{h^q(ty)} dt, \end{aligned}$$

where $x, y \in C$, for a symmetric Lebesgue integrable and nonnegative function $w : [0, 1] \rightarrow [0, \infty)$.

In particular, for $w \equiv 1$, we have

$$(2.14) \quad \begin{aligned} & \frac{1}{2} \frac{v^{q(1-\frac{1}{p})}(x+y)}{h^q(x+y)} \leq \int_0^1 \frac{v^{q(1-\frac{1}{p})}((1-t)x+ty)}{h^q((1-t)x+ty)} dt \\ & \leq \int_0^1 \frac{v^{q(1-\frac{1}{p})}(tx)}{h^q(tx)} dt + \int_0^1 \frac{v^{q(1-\frac{1}{p})}(ty)}{h^q(ty)} dt, \end{aligned}$$

where $x, y \in C$.

Similar results may be obtained by the use of Theorem 3 and its consequences, however we do not provide them here.

We also have the double integral inequalities:

Theorem 5. *Assume that $f : C \rightarrow \mathbb{R}$ is subadditive (superadditive) and hemi-Lebesgue integrable on C , a cone in the linear space X with $0 \in C$. Then for all $x, y \in C$ and symmetric Lebesgue integrable and nonnegative functions $p, q : [0, 1] \rightarrow [0, \infty)$, we have*

$$\begin{aligned}
 (2.15) \quad & \frac{1}{2}f(x+y) \int_0^1 p(t) dt \int_0^1 q(t) dt \\
 & \leq \int_0^1 \int_0^1 p(t)q(s) f((1-t-s+2ts)x + (s+t-2st)y) dt ds \\
 & \leq \int_0^1 \int_0^1 p(t)q(s) f(t(1-s)x + tsy) dt ds \\
 & \quad + \int_0^1 \int_0^1 p(t)q(s) ftsx + t(1-s)y) dt ds \\
 & \leq 2 \int_0^1 \int_0^1 p(t)q(s) ftsx) dt ds + 2 \int_0^1 \int_0^1 p(t)q(s) ftsy) dt ds.
 \end{aligned}$$

In particular, for $p, q \equiv 1$, we have

$$\begin{aligned}
 (2.16) \quad & \frac{1}{2}f(x+y) \\
 & \leq \int_0^1 \int_0^1 f((1-t-s+2ts)x + (s+t-2st)y) dt ds \\
 & \leq \int_0^1 \int_0^1 f(t(1-s)x + tsy) dt ds + \int_0^1 \int_0^1 ftsx + t(1-s)y) dt ds \\
 & \leq 2 \int_0^1 \int_0^1 ftsx) dt ds + 2 \int_0^1 \int_0^1 ftsy) dt ds.
 \end{aligned}$$

Proof. If we replace x with $(1-s)x + sy$ and y with $sx + (1-s)y$, $s \in [0, 1]$ in the inequality (2.1), then we get

$$\begin{aligned}
 & \frac{1}{2}f(x+y) \int_0^1 p(t) dt \\
 & \leq (\geq) \int_0^1 p(t) f((1-t)((1-s)x + sy) + t(sx + (1-s)y)) dt \\
 & \leq (\geq) \int_0^1 p(t) f(t((1-s)x + sy)) dt + \int_0^1 p(t) f(t(sx + (1-s)y)) dt.
 \end{aligned}$$

If we multiply this inequality by $q(t) \geq 0$, $s \in [0, 1]$, integrate and use Fubini's theorem, then we get

$$\begin{aligned}
 (2.17) \quad & \frac{1}{2} f(x+y) \int_0^1 p(t) dt \int_0^1 q(t) dt \\
 & \leq (\geq) \int_0^1 \int_0^1 p(t) q(s) f((1-t)((1-s)x+sy) + t(sx+(1-s)y)) dt ds \\
 & \leq (\geq) \int_0^1 \int_0^1 p(t) q(s) f(t((1-s)x+sy)) dt ds \\
 & \quad + \int_0^1 \int_0^1 p(t) q(s) f(t(sx+(1-s)y)) dt ds.
 \end{aligned}$$

Observe that

$$\begin{aligned}
 & \int_0^1 \int_0^1 p(t) q(s) f((1-t)((1-s)x+sy) + t(sx+(1-s)y)) dt ds \\
 & = \int_0^1 \int_0^1 p(t) q(s) f((1-t)(1-s)x + (1-t)sy + tsx + t(1-s)y) dt ds \\
 & = \int_0^1 \int_0^1 p(t) q(s) f((1-t-s+2ts)x + (s+t-2st)y) dt ds,
 \end{aligned}$$

$$\begin{aligned}
 & \int_0^1 \int_0^1 p(t) q(s) f(t((1-s)x+sy)) dt ds \\
 & = \int_0^1 \int_0^1 p(t) q(s) f(t(1-s)x + tsy) dt ds \\
 & \leq \int_0^1 \int_0^1 p(t) q(s) f(t(1-s)x) dt ds + \int_0^1 \int_0^1 p(t) q(s) f(tsy) dt ds \\
 & = \int_0^1 \int_0^1 p(t) q(1-u) f(tux) dt du + \int_0^1 \int_0^1 p(t) q(s) f(tsy) dt ds \\
 & = \int_0^1 \int_0^1 p(t) q(u) f(tux) dt du + \int_0^1 \int_0^1 p(t) q(s) f(tsy) dt ds
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_0^1 \int_0^1 p(t) q(s) f(t(sx+(1-s)y)) dt ds \\
 & = \int_0^1 \int_0^1 p(t) q(s) f(tsx + t(1-s)y) dt ds \\
 & \leq \int_0^1 \int_0^1 p(t) q(s) f(tsx) dt ds + \int_0^1 \int_0^1 p(t) q(s) f(t(1-s)y) dt ds \\
 & = \int_0^1 \int_0^1 p(t) q(s) f(tsx) dt ds + \int_0^1 \int_0^1 p(t) q(1-u) f(tuy) dt ds \\
 & = \int_0^1 \int_0^1 p(t) q(s) f(tsx) dt ds + \int_0^1 \int_0^1 p(t) q(u) f(tuy) dt du.
 \end{aligned}$$

By utilising (2.17) we get the desired result (2.15). \square

3. SOME RESULTS RELATED TO JENSEN'S INEQUALITY

Let C be a convex subset of the real linear space X and let $\varphi : C \rightarrow \mathbb{R}$ be a convex mapping. Here we consider the following well-known form of *Jensen's discrete inequality*:

$$(3.1) \quad \varphi \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \leq \frac{1}{P_I} \sum_{i \in I} p_i \varphi(x_i),$$

where I denotes a finite subset of the set \mathbb{N} of natural numbers, $x_i \in C$, $p_i \geq 0$ for $i \in I$ and $P_I := \sum_{i \in I} p_i > 0$.

Let us fix $I \in \mathcal{P}_f(\mathbb{N})$ (the class of finite parts of \mathbb{N}) and $x_i \in C$ ($i \in I$). Now consider the functional $f_I : S_+(I) \rightarrow \mathbb{R}$ given by

$$(3.2) \quad f_I(\mathbf{p}) := \sum_{i \in I} p_i \varphi(x_i) - P_I \varphi \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \geq 0,$$

where $S_+(I) := \{\mathbf{p} = (p_i)_{i \in I} \mid p_i \geq 0, i \in I \text{ and } P_I > 0\}$ and f is convex on C .

We observe that $S_+(I)$ is a convex cone and the functional J_I is nonnegative and positive homogeneous on $S_+(I)$.

Lemma 1 ([13]). *The functional $f_I(\cdot)$ is a superadditive functional on $S_+(I)$.*

We have for $\mathbf{p}, \mathbf{q} \in S_+(I)$ that

$$f_I(\mathbf{p} + \mathbf{q}) := \sum_{i \in I} (p_i + q_i) \varphi(x_i) - (P_I + Q_I) \varphi \left(\frac{1}{P_I + Q_I} \sum_{i \in I} (p_i + q_i) x_i \right),$$

and for a symmetric nonnegative Lebesgue integrable function $w : [0, 1] \rightarrow [0, \infty)$ we have

$$\begin{aligned} & \int_0^1 w(t) f_I((1-t)\mathbf{p} + t\mathbf{q}) dt \\ &= \sum_{i \in I} \int_0^1 w(t) ((1-t)p_i + tq_i) dt \varphi(x_i) \\ & - \int_0^1 w(t) ((1-t)P_I + tQ_I) \varphi \left(\frac{1}{(1-t)P_I + tQ_I} \sum_{i \in I} ((1-t)p_i + tq_i) x_i \right) dt \\ &= \sum_{i \in I} p_i \varphi(x_i) \int_0^1 w(t) (1-t) dt + \sum_{i \in I} q_i \varphi(x_i) \int_0^1 w(t) t dt \\ & - \int_0^1 w(t) ((1-t)P_I + tQ_I) \varphi \left(\frac{1}{(1-t)P_I + tQ_I} \sum_{i \in I} ((1-t)p_i + tq_i) x_i \right) dt \\ &= \left[\sum_{i \in I} p_i \varphi(x_i) + \sum_{i \in I} q_i \varphi(x_i) \right] \int_0^1 w(t) t dt \\ & - \int_0^1 w(t) ((1-t)P_I + tQ_I) \varphi \left(\frac{1}{(1-t)P_I + tQ_I} \sum_{i \in I} ((1-t)p_i + tq_i) x_i \right) dt, \end{aligned}$$

$$\begin{aligned}
& \int_0^1 w(t) f_I(t\mathbf{p}) dt \\
&= \sum_{i \in I} p_i \varphi(x_i) \int_0^1 tw(t) dt - P_I \varphi \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \int_0^1 tw(t) dt \\
&= \int_0^1 tw(t) dt \left[\sum_{i \in I} p_i \varphi(x_i) - P_I \varphi \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right]
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^1 w(t) f_I(t\mathbf{q}) dt \\
&= \sum_{i \in I} q_i \varphi(x_i) \int_0^1 tw(t) dt - Q_I \varphi \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \int_0^1 tw(t) dt \\
&= \left[\sum_{i \in I} q_i \varphi(x_i) - Q_I \varphi \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right] \int_0^1 tw(t) dt.
\end{aligned}$$

From the inequality (2.1) we have

$$\begin{aligned}
(3.3) \quad \frac{1}{2} f_I(\mathbf{p} + \mathbf{q}) \int_0^1 w(t) dt &\geq \int_0^1 w(t) f_I((1-t)\mathbf{p} + t\mathbf{q}) dt \\
&\geq \int_0^1 w(t) f_I(t\mathbf{p}) dt + \int_0^1 w(t) f_I(t\mathbf{q}) dt,
\end{aligned}$$

for all $\mathbf{p}, \mathbf{q} \in S_+(I)$.

Therefore

$$\begin{aligned}
(3.4) \quad & \frac{1}{2} \left[\sum_{i \in I} (p_i + q_i) \varphi(x_i) - (P_I + Q_I) \varphi \left(\frac{1}{P_I + Q_I} \sum_{i \in I} (p_i + q_i) x_i \right) \right] \\
&\geq \left[\sum_{i \in I} p_i \varphi(x_i) + \sum_{i \in I} q_i \varphi(x_i) \right] \frac{\int_0^1 w(t) t dt}{\int_0^1 w(t) dt} \\
&\quad - \frac{1}{\int_0^1 w(t) dt} \int_0^1 w(t) ((1-t)P_I + tQ_I) \\
&\quad \times \varphi \left(\frac{1}{(1-t)P_I + tQ_I} \sum_{i \in I} ((1-t)p_i + tq_i) x_i \right) dt \\
&\geq \left[\sum_{i \in I} p_i \varphi(x_i) - P_I \varphi \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right] \frac{\int_0^1 w(t) t dt}{\int_0^1 w(t) dt} \\
&\quad + \left[\sum_{i \in I} q_i \varphi(x_i) - Q_I \varphi \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right] \frac{\int_0^1 w(t) t dt}{\int_0^1 w(t) dt}
\end{aligned}$$

for all $\mathbf{p}, \mathbf{q} \in S_+(I)$.

If $w \equiv 1$ in (3.4), then we get, after some calculations, that

$$(3.5) \quad \begin{aligned} & \frac{P_I + Q_I}{2} \varphi \left(\frac{1}{P_I + Q_I} \sum_{i \in I} (p_i + q_i) x_i \right) \\ & \leq \int_0^1 ((1-t)P_I + tQ_I) \varphi \left(\frac{1}{(1-t)P_I + tQ_I} \sum_{i \in I} ((1-t)p_i + tq_i) x_i \right) dt \\ & \leq \frac{1}{2} \left[P_I \varphi \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) + Q_I \varphi \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right] \end{aligned}$$

for all $\mathbf{p}, \mathbf{q} \in S_+(I)$.

If $(X, \|\cdot\|)$ is a normed space and $\varphi(x) = \|x\|^r$, $r \geq 1$, then φ is convex and by (3.5) we get

$$(3.6) \quad \begin{aligned} & \frac{(P_I + Q_I)^{r-1}}{2} \left\| \sum_{i \in I} (p_i + q_i) x_i \right\|^r \\ & \leq \int_0^1 ((1-t)P_I + tQ_I)^{r-1} \|((1-t)p_i + tq_i) x_i\|^r dt \\ & \leq \frac{1}{2} \left[P_I^{r-1} \left\| \sum_{i \in I} p_i x_i \right\|^r + Q_I^{r-1} \left\| \sum_{i \in I} q_i x_i \right\|^r \right] \end{aligned}$$

for all $\mathbf{p}, \mathbf{q} \in S_+(I)$ and $x_i \in C$ ($i \in I$).

If $x_i \in \mathbb{R}$, ($i \in I$) and $\mathbf{p}, \mathbf{q} \in S_+(I)$, then by taking $\varphi(x) = \exp x$, we get

$$(3.7) \quad \begin{aligned} & \frac{P_I + Q_I}{2} \exp \left(\frac{1}{P_I + Q_I} \sum_{i \in I} (p_i + q_i) x_i \right) \\ & \leq \int_0^1 ((1-t)P_I + tQ_I) \exp \left(\frac{1}{(1-t)P_I + tQ_I} \sum_{i \in I} ((1-t)p_i + tq_i) x_i \right) dt \\ & \leq \frac{1}{2} \left[P_I \exp \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) + Q_I \exp \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right]. \end{aligned}$$

If $x_i > 0$, ($i \in I$) and $\mathbf{p}, \mathbf{q} \in S_+(I)$, then by taking $\varphi(x) = -\ln x$ in (3.5) we get

$$(3.8) \quad \begin{aligned} & \frac{P_I + Q_I}{2} \ln \left(\frac{1}{P_I + Q_I} \sum_{i \in I} (p_i + q_i) x_i \right) \\ & \geq \int_0^1 ((1-t)P_I + tQ_I) \ln \left(\frac{1}{(1-t)P_I + tQ_I} \sum_{i \in I} ((1-t)p_i + tq_i) x_i \right) dt \\ & \geq \frac{1}{2} \left[P_I \ln \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) + Q_I \ln \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right) \right] \end{aligned}$$

or, equivalently,

$$\begin{aligned}
 (3.9) \quad & \left(\frac{1}{P_I + Q_I} \sum_{i \in I} (p_i + q_i) x_i \right)^{\frac{P_I + Q_I}{2}} \\
 & \geq \exp \left[\int_0^1 ((1-t) P_I + t Q_I) \right. \\
 & \quad \times \ln \left(\frac{1}{(1-t) P_I + t Q_I} \sum_{i \in I} ((1-t) p_i + t q_i) x_i \right) dt \left. \right] \\
 & \geq \sqrt{\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right)^{P_I} \left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i \right)^{Q_I}}
 \end{aligned}$$

for $\mathbf{p}, \mathbf{q} \in S_+(I)$ and $x_i > 0$, ($i \in I$).

Define the following functional

$$(3.10) \quad L_{p,q,I}(\mathbf{p}) := P_I^{\frac{p-q}{p}} \left[\sum_{i \in I} p_i f(x_i) - P_I f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right]^q$$

for $p \geq 1$ and $p \geq q \geq 0$.

The following proposition can be stated via Theorem 3:

Proposition 1. *The functional $L_{p,q,I}(\cdot)$ is superadditive on $S_+(I)$ for any $p \geq 1$ and $p \geq q \geq 0$.*

Remark 3. *We observe that, in particular, the following functionals*

$$L_{p,\alpha,I}(\mathbf{p}) := P_I^{1-\alpha} \left[\sum_{i \in I} p_i f(x_i) - P_I f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right]^{\alpha p}$$

and

$$\tilde{L}_{p,I}(\mathbf{p}) := P_I^{\frac{1}{p}} \left[\sum_{i \in I} p_i f(x_i) - P_I f \left(\frac{1}{P_I} \sum_{i \in I} p_i x_i \right) \right]^{\frac{p}{2}}$$

are superadditive on $S_+(I)$ for any $p \geq 1$ and $\alpha \in (0, 1)$.

One can state similar results by utilising the functionals $L_{p,q,I}$, $L_{p,\alpha,I}$ and $\tilde{L}_{p,I}$, however we do not provide the details here.

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