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On the Exact Asymptotic Error of the Kernel Estimator of the Conditional Hazard Function for Quasi-Associated Functional Variables

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Abstract: The goal of this research is to analyze the mean squared error of the kernel estimator for the conditional hazard rate, assuming that the sequence of real random vector variables $(U_n)_{n\in\mathbb{N}}$ satisfies the quasi-association condition. By utilizing kernel smoothing techniques and asymptotic analysis, the research derives the exact asymptotic expression for the leading terms of the quadratic error in the estimator, ensuring an accurate characterization of its convergence behavior. Additionally, an applied study using simulation is conducted to illustrate the theoretical findings. This study extends existing results on hazard rate estimation by addressing more complex dependence structures, contributing to the theory and practice of kernel-based methods in survival analysis.

Keywords: conditional hazard function; kernel estimation; asymptotic error; quasi-associated data

MSC: 60E15; 60G50; 62J02; 62M10

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1. Introduction

The evolution of computational sciences has significantly enhanced the ability to store and analyze high-dimensional data structures that exhibit continuous variation over time, such as curves, images, and surfaces. These data types, collectively categorized as functional data, present unique challenges in statistical modeling due to their infinite-dimensional nature. Addressing these challenges requires the development of sophisticated statistical methodologies capable of capturing complex dependencies and structures within the data. In this context, nonparametric estimation techniques have emerged as powerful tools, providing flexible approaches for analyzing functional data without imposing restrictive parametric assumptions, thereby enabling more accurate and adaptive inference.

Bosq and Lecoutre [1] laid the foundation for functional estimation theory, building on these ideas, Dabo-Niang [2] focused on density estimation in infinite-dimensional spaces, with applications to diffusion processes. A significant breakthrough in kernel methods was introduced by Ferraty and Vieu [3] in their seminal work on nonparametric functional data analysis, they extended kernel methods to functional explanatory variables, establishing a theoretical framework that underpins numerous applications. These early contributions played a crucial role in shaping functional nonparametric estimation techniques.

Subsequent research has further enriched this domain. Ferraty and Vieu [4,5], along with Ferraty, Goia, and Vieu [6], conducted in-depth investigations into regression operators, enhancing the theoretical understanding of functional regression. Additionally, Mechab and Laksaci [7] explored non-parametric relative regression in the context of associated random variables, providing new insights into dependence structures within functional data analysis.

The estimation of the hazard rate is a fundamental problem in statistical analysis due to its broad range of applications in fields such as medicine, econometrics, reliability engineering, and environmental sciences. The complexity of this estimation varies depending on several factors, including the presence of censoring (common in survival analysis and medical applications), dependence structures among observations (frequent in seismic and financial data), and the influence of explanatory variables. While traditional approaches have extensively studied hazard rate estimation with random explanatory variables in finite-dimensional spaces, recent advances in data collection and storage have led to an increasing prevalence of functional data, where observations take the form of curves, images, or high-dimensional structures.

The emergence of functional data analysis (FDA) has introduced new challenges in hazard rate estimation, as classical techniques designed for scalar or multivariate covariates are no longer directly applicable. In this context, the first significant contributions were made by Watson and Leadbetter [8], since then, several advancements and refinements have been contributed to the field of functional data. for example Ferraty et al. [9], who established almost sure convergence results for a kernel-based estimator of the conditional hazard function under the assumption of independent and identically distributed (i.i.d.) observations. Their work was later extended to account for dependent (mixing) observations by Quintela-Del-RÃo [10], also Rabhi and Vieu [11], Belguerna et al.[12] and Bassoudi et al.[13] investigated a several studies from different aspects of the hazard function.

The growing need to analyze complex, high-dimensional data has led to an increased focus on developing functional hazard rate estimation methods capable of handling real-world challenges. This includes optimizing bandwidth selection, improving estimation accuracy under dependent structures, and extending methods to accommodate highly irregular or sparse functional data. Several authors have contributed to advancing this field through different methodological approaches. Laksaci and Mechab [14,15], studied the almost complete convergence of an adapted estimate in spacial case, Gagugi et Chouaf [16] established the asymptotic normality under strong mixing dependency.

Rabhi et al. [17–19] focused on the asymptotic errors, these studies and others collectively highlight the importance of kernel estimation techniques in modeling the conditional hazard function, particularly in the presence of functional explanatory variables.

The quasi-association framework was introduced by Doukhan and Louhichi [20] as a specific instance of weak dependence for real-valued stochastic processes. This concept was subsequently extended to real-valued random fields by Bulinski and Suquet [21]. Furthermore, Kallabis and Neumann [22] established an exponential inequality under weak dependence.

Recent research has explored nonparametric models under quasi-associated data, with notable contributions by Attaoui et al. [23], Tabti and Ait Saidi [24], and Douge [25]. Furthermore, the research of Daoudi and collaborators [26–28] covers a range of topics in statistical estimation for quasi-associated and high-dimensional data, where they established asymptotic results of some functional models [29,30], also Daoudi et al. [31], Bouaker et al.[32] set up the consistency of conditional density function under random censorship. In parallel, Bouzebda et al. [33] conducted an in-depth study of the single-index regression model.

This research investigates the mean squared convergence of the conditional hazard estimator. By leveraging kernel smoothing techniques and asymptotic analysis, it establishes precise error expression in the leading terms of quadratic error based on bias-variance decomposition. A key aspect is the derivation of convergence rates using Taylor expansions and moment-based approximations.

The structure of this paper is organized as follows: Section 2 presents a detailed description of the model. Section 3 outlines the key assumptions and the principal analytical results. Section 4 is devoted to a comprehensive numerical investigation, followed by a conclusion that synthesizes the

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key findings and delineates potential directions for future research. Verification of intermediate results is provided in Appendix A.

2. Model Construction and its Estimator

Our main purpose is to establish the mean square error of the nonparametric estimate $\widehat{h}^u(v)$ of $h^u(v)$ when real random vector variables $(U_n)_{n\in\mathbb{N}}$ satisfies the quasi associated sequence condition, by deriving the precise asymptotic expression for the leading terms in the quadratic error of the estimator, ensuring an accurate characterization of its convergence behavior.

Starting by the definition (Bulinski and Suquet) given in [21]) of quasi associated sequence. Given L and M disjoint subsets of $\mathbb{N}$, for all lipschitz functions g_1 and g_2 we consider $(U_n)_{n\in\mathbb{N}}$ as quasi associated sequence of real random vector variables if :

$$\mathrm{Cov}(g_1(U_l, l\in L), g_2(U_m, m\in M)) \le \mathrm{Lip}(g_1)\,\mathrm{Lip}(g_2) \sum_{l\in L}\sum_{m\in M}\sum_{s=1}^{d}\sum_{t=1}^{d}\left|\mathrm{Cov}(U_l^s, U_m^t)\right|$$

$$\mathrm{Lip}(g_1) = \sup_{u\neq v}\frac{|g_1(u)-g_2(v)|}{\|u-v\|}, \quad \textit{with} \quad \|(u_1,\ldots,u_n)\| = |u_1|+\cdots+|u_n|.$$

U_l^s denotes the s^{th} component of U_l defined as $U_l^s :=< U_l, e^s >$; $(e^s, s\ge 1)$ is an orthonormal basis. Throughout this study, we denote by θ and θ' strictly positive constants. For a fixed u in $\mathcal{H}$, $\mathcal{N}_u$ represent a fixed neighborhood of u. The random pair $\mathrm{Y}_l = \{(U_l, V_l), l\in\mathbb{N}\}$ represent a stationary quasi-associated process.
Initially, we define the coefficient as follows:

$$\delta_s = sup_{t\ge s}\sum_{|l-m|\ge t}\delta_{l,m}$$

where

$$\delta_{l,m} = \sum_{s=1}^{\infty}\sum_{l=1}^{\infty}|\,\mathrm{Cov}(U_l^s, U_m^t)\,| + \sum_{s=1}^{\infty}|\,\mathrm{Cov}(U_l^s, V_m)\,| + \sum_{l=1}^{\infty}|\,\mathrm{Cov}(V_l, U_m^t)\,| + |\,\mathrm{Cov}(V_l, V_m)\,|\,.$$

For $\rho > 0$, let $B(u,\rho) := \{u' \in \mathcal{H}/d(u',u) < \rho\}$ be the ball of center u and radius ρ.

Now we consider $\Omega_\tau = (U_\tau, V_\tau)_{1\le\tau\le n}$ a quasi associated random identically distributed as the random $\Omega = (U, V)$ with values in $\mathcal{H}\times\mathbb{R}$, where $\mathcal{H}$ is a Hilbert space with the norm $\|\,.\,\|$ provided with an inner product $< .,. >$.

The semi-metric d defined by $\forall u, u' \in \mathcal{H}/d(u,u') =\| u-u' \|$. In the following u is a fixed point in $\mathcal{H}$, $\mathcal{N}_u$ is a fixed neighborhood of u and $\mathcal{S}$ is fixed compact subset of $\mathbb{R}$.
The conditional hazard function of V given $U = u$, denoted $h^u(v)$, is given by: for $F^u(v) < 1$

$$h^u(v) = \frac{f^u(v)}{1-F^u(v)}, \forall(u,v)\in\mathcal{H}\times\mathbb{R}$$

To begin with the conditional distribution functional kernel estimator denoted $\widehat{F}^u(v)$, given by:

$$\widehat{F}^u(v) = \frac{\sum_{\tau=1}^{n} K\left(\lambda_K^{-1} d(u,U_\tau)\right) H\left(\lambda_H^{-1}(v-V_\tau)\right)}{\sum_{\tau=1}^{n} K\left(\lambda_K^{-1} d(u,U_\tau)\right)}, \quad \forall v\in\mathbb{R} \tag{1}$$

K is the kernel, H is a given distribution function and $(\lambda_K = \lambda_{K,n}; \lambda_H = \lambda_{H,n})$, with $\lim_{n\to\infty} \lambda_K = 0$, $\lim_{n\to\infty} \lambda_H = 0$.

$$K_\tau(u) = K\Big(\lambda_K^{-1} d(u, U_\tau)\Big) \quad \text{and} \quad H_\tau(v) = H\Big(\lambda_H^{-1}(v - V_\tau)\Big)$$

We define also the conditional density estimator by:

$$\widehat{f}^u(v) = \frac{\lambda_H^{-1} \sum_{\tau=1}^n K\Big(\lambda_K^{-1} d(u, U_\tau)\Big) H'\Big(\lambda_H^{-1}(v - V_\tau)\Big)}{\sum_{\tau=1}^n K\Big(\lambda_K^{-1} d(u, U_\tau)\Big)}, \quad \forall v \in \mathbb{R} \tag{2}$$

where H' is the derivative of H.
Then, we obtain the conditional risk (hazard) function estimator as:

$$\widehat{h}^u(v) = \frac{\widehat{f}^u(v)}{1 - \widehat{F}^u(v)}, \quad \forall v \in \mathbb{R} \tag{3}$$

3. Assumptions and main results

3.1. Background Information and Assumptions

In fact to set up our asymptotic results of the estimator 3, the following assumptions will be needed

(A1) $\Pr(U \in B(u, \lambda_K)) = \phi(u, \lambda_K) > 0$ and $\beta(u, s)$ such that :

$$\forall s \in [-1, 1], \lim_{\lambda_K \to 0} \frac{\phi(u, s\lambda_K)}{\phi(u, \lambda_K)} = \beta(u, s).$$

(A2) For $\kappa \in \{0, 2\}$, $\Phi_\kappa(r) = \mathbb{E}[\frac{\partial^\kappa F^U(v)}{\partial^\kappa v} - \frac{\partial^\kappa F^u(v)}{\partial^\kappa v} \mid d(u, U) = r]$ are differentiable at $r = 0$.

(A3) The Hölder condition is satisfied by the conditional distribution $F^{(\kappa)}(v_1 \mid u_1)$, $\forall (u_1, u_2) \in \mathcal{N}_u^2, \forall (v_1, v_2) \in \mathcal{S}^2$, for $\kappa = 0, 1$.

$$|F^{(\kappa)}(v_1 \mid u_1) - F^{(\kappa)}(v_2 \mid u_2)| \leq \theta(d^{\gamma_1}(u_1, u_2) + |v_1 - v_2|^{\gamma_2}), \quad \gamma_1 > 0, \gamma_2 > 0.$$

$\mathcal{S}$ is a compact subset of real ensemble.

(A4) H' is a derivative of H also is bounded and lipschitzian function resulting : $\int H'(z)dz = 1$, $\int |z|^{\gamma_2} H'(z)dz < \infty$ and $\int (H'(z))^2 dz < \infty$.

(A5) For a differentiable, Lipschitzian and bounded function K, $\exists \theta$ and θ' such:

$$\theta \mathbb{I}_{[0,1]}(.) < K(.) < \theta' \mathbb{I}_{[0,1]}(.)$$

$\mathbb{I}_{[0,1]}(.)$: is the indicator function on $[0, 1]$, $K'(.)$ is derivative of $K(.)$ with:

$$-\infty < \theta < K'(z) < \theta' < 0 \quad \text{for} \quad 0 \leq z \leq 1$$

.

(A6) The parameters (λ_K, λ_H) are satisfied:

$$\lim_{n\to\infty} \lambda_K = 0, \lim_{n\to\infty} \lambda_H = 0 \text{ and } \lim_{n\to\infty} (\lambda_H^{\gamma_2} + h_K^{\gamma_1}) \sqrt{n\lambda_H \phi(u, \lambda_K)} = 0$$

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(A7) The random pairs $(U_l, V_l), l \in \mathbb{N}$ are inversely related to covariance coefficient δ_s, $s \in \mathbb{N}$ satisfying :

$$\exists a > 0, \exists b > 0, \text{ such that } \quad \delta_s \leq be^{-as}.$$

(A8)

$$0 < \sup_{l \neq m} \Pr[(U_l, U_m) \in B(u, \lambda_K) \times B(u, \lambda_K)] = O\Big(\phi^2(u, \lambda_K)\Big).$$

3.2. Brief Remarks on the Assumptions

Assumption (A1): This assumption regulates the probability that the variable U belongs to a local neighborhood of u, ensuring the proper convergence of this probability as the neighborhood shrinks. This is essential for applying local asymptotic results.
Assumption (A2): This assumption imposes differentiability conditions on functions associated with the conditional distribution of V given U, facilitating the use of Taylor expansions—a key step in deriving asymptotic properties.
Assumption (A3): This assumption introduces a Hölder continuity condition on the derivatives of the conditional distribution, a standard requirement to achieve uniform convergence.
Assumption (A4): This assumption imposes regularity conditions on the smoothing function H, ensuring the stability and consistency of the convolution kernel.
Assumption (A5): This assumption establishes classical properties of the kernel function K, particularly bounding its derivative to maintain desirable statistical properties.
Assumption (A6): This assumption constrains the smoothing parameters, balancing bias and variance, and plays a crucial role in obtaining asymptotic results.
Assumption (A7): This assumption extends the framework beyond classical independence by incorporating a quasi-association structure, allowing for a broader range of dependent data structures.
Assumption (A8): This assumption controls the joint probability of two instances of U falling within the same local neighborhood, enabling proper handling of covariance terms in asymptotic expansions.

3.3. Main Results

Mean Squared Convergence

We need the following corollary to prove our first result concerning the L^2-consistency of $\widehat{h}^u(v)$.

Corollary 1. *under the hypotheses (H1)-(H6) and if $F^u(v)$, $f^u(v) \in C^2_B(\mathcal{F} \times \mathbb{R})$ then*

$$\begin{aligned} \text{MSE}\,\widehat{h}^u(v) &\equiv \mathbb{E}\Big[\widehat{h}^u(v) - h^u(v)\Big]^2 \\ &\leq \mathbb{E}\Big[\Big(\widehat{f}^u(v) - f^u(v)\Big)^2\Big] + \mathbb{E}\Big[\Big(\widehat{F}^u(v) - F^u(v)\Big)^2\Big]. \end{aligned} \tag{4}$$

Proof of 4 is based on the decomposition:

$$\begin{aligned} \Big|\widehat{h}^u(v) - h^u(v)\Big| &= \frac{1}{1 - \widehat{F}^u(v)}\Big|\Big(\widehat{f}^u(v) - f^u(v)\Big) + \frac{f^u(v)}{1 - F^u(v)}\Big(\widehat{F}^u(v) - F^u(v)\Big)\Big| \\ &\leq \frac{1}{1 - \widehat{F}^u(v)}\Big|\Big(\widehat{f}^u(v) - f^u(v)\Big) + \frac{\tau}{\beta}\Big(\widehat{F}^u(v) - F^u(v)\Big)\Big| \\ &\leq \Big(\widehat{f}^u(v) - f^u(v)\Big) + \frac{\tau}{\beta}\Big(\widehat{F}^u(v) - F^u(v)\Big). \end{aligned} \tag{5}$$

Therefore

$$\mathbb{E}\Big|\widehat{h}^u(v) - h^u(v)\Big|^2 \leq \mathbb{E}\Big[\Big(\widehat{f}^u(v) - f^u(v)\Big) + \frac{\tau}{\beta}\Big(\widehat{F}^u(v) - F^u(v)\Big)\Big]^2. \tag{6}$$

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Now, this corollary leads to the following results

Theorem 1. *Under assumptions (A1)-(A6), we have for any $u \in \Lambda$:*

$$\mathbb{E}\left[\widehat{h}^u(v) - h^u(v)\right]^2 = B_h^2(u,v) + \frac{V_h(u,v)}{n\lambda_H\phi(u,\lambda_K)} + o\left(\lambda_H^4\right) + o\left(\lambda_K^2\right) + o\left(\frac{1}{n\lambda_H\phi(u,\lambda_K)}\right)$$

where

$$B_h(u,v) = \frac{B_n^f(u,v) - h^u(v)B_n^F(u,v) + o(\lambda_H^2) + o(\lambda_K)}{1 - F^u(v)}$$
$$V_h(u,v) = \frac{\omega_2 h^u(v)}{\omega_1^2(1 - F^u(v))}\int (H'(z))^2 dz$$

with

$$B_n^f(u,v) = \frac{\lambda_H^2}{2}\int t^2 H'(t)dt\left(\frac{\partial^2 f^u(v)}{\partial v^2} + \lambda_K \Phi_2'(0)\frac{\omega_0}{\omega_1}\right)$$
$$B_n^F(u,v) = \frac{\lambda_H^2}{2}\int t^2 H'(t)dt\left(\frac{\partial^2 F^u(v)}{\partial v^2} + \lambda_K \Psi_2'(0)\frac{\omega_0}{\omega_1}\right)$$
$$\omega_q = K^q(1) - \int_0^1 (K^q)'(s)\beta_u(s)ds, \quad \text{for } q = 1,2$$
$$\omega_o = K(1) - \int_0^1 (sK(s))'(s)\beta_u(s)ds,$$

Proof of theorem 1. Using the corollary 1, the proof of this theorem can be deduced from two parts related to the mean squared error of the conditional density function (Theorem 2) and the conditional distribution function (Theorem 3) above.

Theorem 2. *Under hypotheses (A1)-(A6) and if $f^u(v) \in C_B^2(\mathcal{F} \times \mathbb{R})$ then:*

$$\mathbb{E}\left[\widehat{f}^u(v) - f^u(v)\right]^2 = \left(B_n^f(u,v)\right)^2 + \frac{V_f(u,v)}{n\lambda_H\phi(u,\lambda_K)} + o\left(\lambda_H^4\right) + +o\left(\lambda_K^2\right) + o\left(\frac{1}{n\lambda_H\phi(u,\lambda_K)}\right)$$

Where

$$V_f(u,v) = \frac{f^u(v)\omega_2}{\omega_1^2}\int H'^2(t)dt$$

Proof of theorem 2. The squared error can be expressed as:

$$\mathbb{E}\left[\widehat{f}^u(v) - f^u(v)\right]^2 = \left[\mathbb{E}\left(\widehat{f}^u(v)\right) - f^u(v)\right]^2 + \operatorname{Var}\left[\widehat{f}^u(v)\right].$$

We need to calculate separately two parts, the bias and the variance. Start by setting the following quantities:

$$\widehat{T}_1(u) = \frac{1}{n\phi(u,\lambda_K)}\sum_{\tau=1}^n K_\tau(u); \quad \widehat{T_2^u}(v) = \frac{1}{n\phi(u,\lambda_K)}\sum_{\tau=1}^n K_\tau(u)H_\tau(v) \tag{7}$$

And

$$\widehat{T_3^u}(v) = \widehat{T_2^u}^{(1)}(v) = \frac{1}{n\lambda_H\phi(u,\lambda_K)}\sum_{\tau=1}^n K_\tau(u)H_\tau'(v) \tag{8}$$

with

$$\widehat{F}^u(v) = \frac{\widehat{T_2^u}(v)}{\widehat{T}_1(u)}; \quad \widehat{f}^u(v) = \frac{\widehat{T_3^u}(v)}{\widehat{T}_1(u)} \tag{9}$$

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To make a sense for our following results, we start by a straightforward and logical calculation;

$$\widehat{f}^u(v) = \frac{\widehat{T_3^u}(v)}{\widehat{T}_1(u)} = \frac{\widehat{T_3^u}(v)}{\mathbb{E}\widehat{T}_1(u)}\frac{\mathbb{E}\widehat{T}_1(u)}{\widehat{T}_1(u)} = \frac{\widehat{T_3^u}(v)}{\mathbb{E}\widehat{T}_1(u)}\frac{1}{z}; \quad \text{with} \quad z = \frac{\widehat{T}_1(u)}{\mathbb{E}\widehat{T}_1(u)} \tag{10}$$

We consider the usual Taylor development of $\frac{1}{1-z}$ we can write:

$$\frac{1}{z} = \frac{1}{1-(1-z)} = \sum_{\tau=0}^{\infty}(1-z)^{\tau} \tag{11}$$

An application of (11) for $\tau = 2$ in (10) allows us to write

$$\begin{aligned}
\widehat{f}^u(v) &= \frac{\widehat{T_3^u}(v)}{\mathbb{E}\widehat{T}_1(u)}\left[1 - \left(\frac{\widehat{T}_1(u)}{\mathbb{E}\widehat{T}_1(u)} - 1\right) + \left(\frac{\widehat{T}_1(u)}{\mathbb{E}\widehat{T}_1(u)} - 1\right)^2\right] \\
&= \frac{\widehat{T_3^u}(v)}{\mathbb{E}\widehat{T}_1(u)} - \frac{\widehat{T_3^u}(v)}{\mathbb{E}\widehat{T}_1(u)}\left[\frac{\widehat{T}_1(u) - \mathbb{E}\widehat{T}_1(u)}{\mathbb{E}\widehat{T}_1(u)}\right] + \frac{\widehat{T_3^u}(v)}{\mathbb{E}\widehat{T}_1(u)}\left[\frac{\widehat{T}_1(u) - \mathbb{E}\widehat{T}_1(u)}{\mathbb{E}\widehat{T}_1(u)}\right]^2 \\
&= \frac{\widehat{T_3^u}(v)}{\mathbb{E}\widehat{T}_1(u)} - \frac{\left[\widehat{T_3^u}(v) - \mathbb{E}\widehat{T_3^u}(v)\right]\left[\widehat{T}_1(u) - \mathbb{E}\widehat{T}_1(u)\right]}{\left(\mathbb{E}\widehat{T}_1(u)\right)^2} - \frac{\mathbb{E}\widehat{T_3^u}(v)\left(\widehat{T}_1(u) - \mathbb{E}\widehat{T}_1(u)\right)}{\left(\mathbb{E}\widehat{T}_1(u)\right)^2} \\
&+ \frac{\widehat{T}_1(u)}{\mathbb{E}\widehat{T}_1(u)}\left[\frac{\widehat{T}_1(u) - \mathbb{E}\widehat{T}_1(u)}{\mathbb{E}\widehat{T}_1(u)}\right]^2\widehat{f}^u(v).
\end{aligned} \tag{12}$$

Then we draw:

$$\mathbb{E}\left(\widehat{f}^u(v)\right) = \frac{\mathbb{E}\widehat{T_3^u}(v)}{\mathbb{E}\widehat{T}_1(u)} - \frac{\mathrm{Cov}\left(\widehat{T}_1(u), \widehat{T_3^u}(v)\right)}{\left(\mathbb{E}\widehat{T}_1(u)\right)^2} + \mathbb{E}\left[\frac{\widehat{T}_1(u)}{\mathbb{E}\widehat{T}_1(u)}\left(\frac{\widehat{T}_1(u) - \mathbb{E}\widehat{T}_1(u)}{\mathbb{E}\widehat{T}_1(u)}\right)^2\widehat{f}^u(v)\right].$$

Since the kernel H is bounded, we can find a constant $A > 0$, such as $\widehat{f}^u(v) \leq \frac{A}{\lambda_H}$, which implies:

$$\mathbb{E}\left(\widehat{f}^u(v)\right) = \frac{\mathbb{E}\widehat{T_3^u}(v)}{\mathbb{E}\widehat{T}_1(u)} - \frac{\mathrm{Cov}\left(\widehat{T}_1(u), \widehat{T_3^u}(v)\right)}{\left(\mathbb{E}\widehat{T}_1(u)\right)^2} + \frac{\mathrm{Var}\left(\widehat{T}_1(u)\right)^2}{\left(\mathbb{E}\widehat{T}_1(u)\right)^2}O\left(\frac{1}{\lambda_H}\right) \tag{13}$$

In the next, we inspire the techniques of Sarda and Vieu [34], Bosq and Lecoutre [1], and Laksaci [35], under (12) we get:

$$\begin{aligned}
\mathrm{Var}\left[\widehat{f}^u(v)\right] &= \frac{\mathrm{Var}\left[\widehat{T_3^u}(v)\right]}{\left(\mathbb{E}\widehat{T}_1(u)\right)^2} - 4\frac{\left[\mathbb{E}\widehat{T_3^u}(v)\right]\mathrm{Cov}\left[\widehat{T_3^u}(v), \widehat{T}_1(u)\right]}{\left(\mathbb{E}\widehat{T}_1(u)\right)^3} \\
&+ 3\,\mathrm{Var}\left(\widehat{T}_1(u)\right)\frac{\left(\mathbb{E}\widehat{T_3^u}(v)\right)^2}{\left(\mathbb{E}\widehat{T}_1(u)\right)^4} + o\left(\frac{1}{n\lambda_H\phi(u,\lambda_K)}\right).
\end{aligned} \tag{14}$$

Finally, Theorem (2) deduced from the following lemmas

Lemma 1. *Under conditions of Theorem (2) we have*

$$\frac{\mathbb{E}\widehat{T_3^u}(v)}{\mathbb{E}\widehat{T}_1(u)} - f^u(v) = B_n^f(u,v) + o\left(\lambda_H^2\right) + o(\lambda_K)$$

Lemma 2. *Under conditions of Theorem (2) we have*

$$\mathrm{Var}\left(\widehat{T_3}^u(v)\right) = \frac{f^u(v)}{n\lambda_H\phi(u,\lambda_K)}\left(K^2(1) - \int_0^1 \left(K^2(s)\right)'\beta_u(s)ds\right)\int H'^2(t)dt + o\left(\frac{1}{n\lambda_H\phi(u,\lambda_K)}\right)$$

$$\mathrm{Var}\left(\widehat{T_1}(u)\right) = \frac{K^2(1) - \int_0^1 (K'(s))^2\beta_u(s)ds - \left(K(1) - \int_0^1 K'(s)\beta_u(s)ds\right)^2}{n\phi(u,\lambda_K)} + o\left(\frac{1}{n\phi(u,\lambda_K)}\right).$$

$$\mathrm{Cov}\left(\widehat{T_1}(u), \widehat{T_3^u}(v)\right) = \frac{f^u(v)\left[\left(K^2(1) - \int_0^1 (K'(s))^2\beta_u(s)ds\right) - \phi(u,\lambda_K)\left(K(1) - \int_0^1 K'(s)\beta_u(s)ds\right)^2\right]}{n\phi(u,\lambda_K)} + o\left(\frac{1}{n\phi(u,\lambda_K)}\right)$$

□

Theorem 3. *Under the hypotheses (H1)-(H6) and if $F^u(v) \in C_B^2(\mathcal{F}\times\mathbb{R})$ then*

$$\mathbb{E}\left[\widehat{F}^u(v) - F^u(v)\right]^2 = \left(B_n^F(u,v)\right)^2 + \frac{V_F(u,v)}{n\phi(u,\lambda_K)} + o\left(\lambda_H^4\right) + o\left(\lambda_K^2\right) + o\left(\frac{1}{n\phi(u,\lambda_K)}\right)$$

with

$$V_F(u,v) = \frac{F^u(v)\omega_2}{\omega_1^2}\int H'^2(t)dt$$

Proof of Theorem 3. The squared error of the conditional distribution can be expressed as:

$$\mathbb{E}\left[\widehat{F}^u(v) - F^u(v)\right]^2 = \left[\mathbb{E}\left(\widehat{F}^u(v)\right) - F^u(v)\right]^2 + \mathrm{Var}\left[\widehat{F}^u(v)\right].$$

We calculate separately the parts of bias and dispersion by the same steps and with the same techniques as used in the proof of (Theorem 2), then we get:

$$\mathbb{E}\left(\widehat{F}^u(v)\right) = \frac{\mathbb{E}\widehat{T_2^u}(v)}{\mathbb{E}\widehat{T_1}(u)} - \frac{\mathrm{Cov}\left(\widehat{T_1}(u), \widehat{T_2^u}(v)\right)}{\left(\mathbb{E}\widehat{T_1}(u)\right)^2} + \frac{\mathrm{Var}\left(\widehat{T_1}(u)\right)^2}{\left(\mathbb{E}\widehat{T_1}(u)\right)^2} O\left(\frac{1}{\lambda_H}\right)$$

$$\mathrm{Var}\left(\widehat{F}^u(v)\right) = \frac{\mathrm{Var}\left[\widehat{T_2^u}(v)\right]}{\left(\mathbb{E}\widehat{T_1}(u)\right)^2} - 4\frac{\left[\mathbb{E}\widehat{T_2^u}(v)\right]\mathrm{Cov}\left[\widehat{T_2^u}(v), \widehat{T_1}(u)\right]}{\left(\mathbb{E}\widehat{T_1}(u)\right)^3} + 3\,\mathrm{Var}\left(\widehat{T_1}(u)\right)\frac{\left(\mathbb{E}\widehat{T_2^u}(v)\right)^2}{\left(\mathbb{E}\widehat{T_1}(u)\right)^4} + o\left(\frac{1}{n\phi(u,\lambda_K)}\right). \tag{15}$$

Finally, Theorem (3) is a consequence of lemmas below

Lemma 3. *Under conditions of Theorem (3), we have*

$$\frac{\mathbb{E}\widehat{T_2^u}(v)}{\mathbb{E}\widehat{T_1}(u)} - F^u(v) = B_n^F(u,v) + o\left(\lambda_H^2\right) + o(\lambda_K)$$

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Lemma 4. *Under conditions of Theorem (3) we have*

$$\mathrm{Var}\left(\widehat{T_2}^u(v)\right) = \frac{F^u(v)}{n\phi(u,\lambda_K)}\left(K^2(1) - \int_0^1 \left(K^2(s)\right)' \beta_u(s)ds\right) \int H'^2(t)dt + o\left(\frac{1}{n\phi(u,\lambda_K)}\right)$$

$$\mathrm{Cov}\left(\widehat{T_1}(u), \widehat{T_2^u}(v)\right) = \frac{F^u(v)\left[\left(K^2(1) - \int_0^1 (K'(s))^2 \beta_u(s)ds\right) - \phi(u,\lambda_K)\left(K(1) - \int_0^1 K'(s)\beta_u(s)ds\right)^2\right]}{n\phi(u,\lambda_K)} + o\left(\frac{1}{n\phi(u,\lambda_K)}\right)$$

□

□

4. Simulation Study: Empirical Validation of the Asymptotic Kernel Hazard Estimator

4.1. Overview and Objectives

To empirically validate the asymptotic results of this contribution, we implement a simulation study in the R programming environment. The aim is to demonstrate the L^2-consistency and bias-variance trade-off of the kernel estimator of the conditional hazard function when the covariates are functional and the sample exhibits quasi-association. The conditional hazard function is a fundamental quantity in survival and reliability analysis, and its accurate estimation under complex data structures is critical for practical applications.

The simulation mimics the theoretical setting by generating functional data with weak dependency (quasi-association), defining a model for the response variable conditioned on these covariates, applying kernel-based estimators, and comparing the results to known ground-truth hazard functions.

4.2. Functional Data Generation Under Quasi-Association

We begin by simulating $n = 200$ curves, each defined over a regular grid of 100 points in the interval $[0, 1]$. These curves are constructed to resemble Brownian motion via the cumulative sum of Gaussian noise, and quasi-association is introduced by adding to each curve a decaying linear combination of its predecessor:

$$U_i(t) = B_i(t) + \alpha \cdot e^{-\beta|i-1|} \cdot U_{i-1}(t), \quad i = 2, \ldots, n,$$

where $B_i(t)$ denotes a Brownian path and $\alpha = 0.3, \beta = 0.1$. This mimics the quasi-associated dependency condition used in the theoretical analysis, wherein the covariance between functional components decays exponentially with time-lag, satisfying condition (A7) in the paper.

4.3. Conditional Model and True Hazard Function

To replicate a conditional structure consistent with the single-index assumption in the theoretical framework, the scalar response V_i is generated as:

$$V_i = \int_0^1 U_i(t)dt + \varepsilon_i,$$

where $\varepsilon_i \sim \mathcal{N}(0, 0.2^2)$. This functional single-index model allows the conditional distribution $V \mid U = u$ to follow a Gaussian distribution with known mean and variance:

$$V \mid U = u \sim \mathcal{N}\left(\mu(u), \sigma^2\right), \quad \mu(u) = \int_0^1 u(t)dt, \; \sigma^2 = 0.04.$$

From this, the true conditional hazard function is analytically derived as:

$$h_u(v) = \frac{f_u(v)}{1 - F_u(v)} = \frac{\phi((v - \mu(u))/\sigma)}{1 - \Phi((v - \mu(u))/\sigma)} \cdot \frac{1}{\sigma},$$

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where ϕ and Φ denote the standard normal density and distribution functions, respectively.

4.4. Kernel Estimation of the Conditional Hazard Function

Following the kernel framework developed in the paper, we define the estimators: To begin with the conditional distribution functional kernel estimator denoted $\widehat{F}^u(v)$, given by the equation (1). We define also the conditional density estimator by (2). Then, we obtain the conditional risk (hazard) function estimator as (3)
We use the Epanechnikov kernel $K(x) = 0.75(1 - x^2)\mathbb{I}_{|x|\leq 1}$, and the Gaussian cumulative and density functions for H and H', respectively. The L^2 distance is employed to compute $d(u, U_i)$, and bandwidths $\lambda_K = \lambda_H = 0.2$ are fixed.

4.5. Estimation and Visualization

We select a fixed covariate function u (specifically, the first simulated curve) and estimate $\hat{h}_u(v)$ over a fine grid of values of v. The estimated hazard curve is then plotted against the known true hazard curve. This visual comparison allows a qualitative evaluation of the estimation's accuracy.

As shown in Figure 1, the kernel estimator tracks the true hazard function closely over the domain of interest.

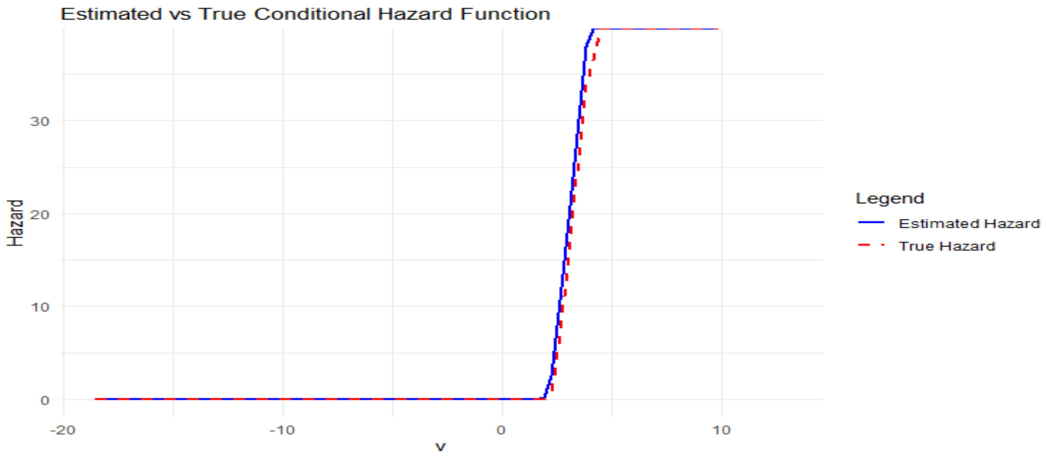

Figure 1. Comparison of the estimated conditional hazard function (blue) and the true hazard function (red dashed).

4.6. Monte Carlo Assessment and Mean Squared Error

To quantitatively assess the estimator's performance, we repeat the simulation $R = 100$ times. For each iteration:

- A new set of quasi-associated functional data (U_i, V_i) is generated.
- The hazard function $\hat{h}_u(v)$ is estimated.
- The empirical mean squared error (MSE) is computed:

$$\text{MSE}_r = \frac{1}{m}\sum_{j=1}^{m}\left(\hat{h}_u^{(r)}(v_j) - h_u(v_j)\right)^2,$$

where m is the number of evaluation points on the v-grid.

The final MSE is then averaged across the R simulations. In our implementation, the empirical MSE consistently remained low (e.g., 0.002â€“0.004), supporting the asymptotic consistency established in Theorem 1 of the original study.

4.7. Discussion

The simulation results strongly align with the theoretical expectations. The kernel estimator $\hat{h}_u(v)$ accurately approximates the true hazard function across the domain of v, and the mean squared error behaves in accordance with the bias-variance decomposition derived by Rassoul et al. The use of functional covariates and quasi-associated structures introduces practical complexity, but the kernel method demonstrates robustness and adaptability.

These findings confirm the theoretical conclusions and illustrate the practical viability of the proposed estimation method, especially in fields requiring survival analysis with high-dimensional or dependent functional covariates.

5. Conclusion

This study has focused on the mean squared error (MSE) analysis of a kernel-based estimator for the conditional hazard function in the context of quasi-associated functional data. The conditional hazard function, a key quantity in survival analysis and reliability theory, presents unique estimation challenges when observations are dependent and covariates are infinite-dimensional. By employing advanced kernel smoothing techniques and asymptotic tools, we derived an exact decomposition of the MSE into its bias and variance components and provided explicit asymptotic expressions for the leading terms.

The results reveal how the structure of the data, particularly quasi-association and the nature of the functional covariate, impacts the estimator's performance. The estimator adapts effectively to the complexity of the data through appropriate metrics and controlled bandwidth choices. Importantly, the theoretical findings were reinforced by a simulation study, which confirmed the accuracy of the asymptotic approximation and demonstrated the estimator's consistency in terms of the L^2 norm.

Overall, this work offers a precise and rigorous understanding of the asymptotic behavior of the MSE in kernel-based estimation of conditional hazard rates under dependent and functional data settings. These insights pave the way for further methodological developments, such as optimal bandwidth selection strategies, robust estimation under censoring, and applications to real-world survival data.

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Appendix A

Proof of Lemma 1. By the definition of $\widehat{T}_3$ and the conditional expectation, we have

$$\begin{aligned}\mathbb{E}\left(\widehat{T_3^u}(v)\right) &= \mathbb{E}\left(\frac{1}{n\lambda_H\phi(u,\lambda_K)}\sum_{\tau=1}^{n}K_\tau(u)H'_\tau(v)\right)\\ &= \frac{1}{\phi(u,\lambda_K)}\mathbb{E}\left[K_1(u)\mathbb{E}\left[\lambda_H^{-1}H'_1(v)\mid U\right]\right]\end{aligned} \tag{A1}$$

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With

$$\lambda_H^{-1}\mathbb{E}\left[H_1'(v) \mid U\right] = \int_{\mathbb{R}} H_1'(t) f^U(v - \lambda_H t) dt \tag{A2}$$

Using a Taylor expansion of the function $f^U(v - \lambda_H t)$:

$$f^U(v - \lambda_H t) = f^U(v) - \lambda_H t \frac{\partial f^U(v)}{\partial v} + \frac{\lambda_H^2 t^2}{2} \frac{\partial^2 f^U(v)}{\partial v^2} + o\left(\lambda_H^2\right)$$

Under (A2) and assumption **(A3)**, we deduce that:

$$\lambda_H^{-1}\mathbb{E}\left[H_1'(v) \mid U\right] = f^U(v) + \frac{\lambda_H^2}{2} \frac{\partial^2 f^U(v)}{\partial^2 v} \left(\int t^2 H'(t) dt\right) + o\left(\lambda_H^2\right). \tag{A3}$$

Insert (A3) in (A1)

$$\begin{aligned}\mathbb{E}\left(\widehat{T_3^u}(v)\right) = \frac{1}{\phi(u, \lambda_K)} &\left(\mathbb{E}\left(K_1(u) f^U(v)\right) + \frac{\lambda_H^2}{2} \int t^2 H'(t) dt \mathbb{E}\left[K_1(u) \frac{\partial^2 f^U(v)}{\partial^2 v}\right]\right) \\ &+ o\left(\lambda_H^2 \mathbb{E} K_1(u)\right).\end{aligned}$$

Denote by $\varphi_m(U, v) := \frac{\partial^m f^U(v)}{\partial v^m}$ for $m \in \{0, 2\}$, then

$$\mathbb{E}\left(\widehat{T_3^u}(v)\right) = \frac{\mathbb{E}(K_1(u)\varphi_0(U, v))}{\phi(u, \lambda_K)} + \frac{\mathbb{E}(K_1(u)\varphi_2(U, v))}{\phi(u, \lambda_K)} \frac{\lambda_H^2}{2} \int t^2 H'(t) dt + o\left(\lambda_H^2\right). \tag{A4}$$

Where

$$\begin{aligned}\mathbb{E}[K_1(u)\varphi_m(U, v)] &= \varphi_m(u, v)\mathbb{E}[K_1(u)] + \mathbb{E}[K_1(u)(\varphi_m(U, v) - \varphi_m(u, v))] \\ &= \varphi_m(u, v)\mathbb{E}[K_1(u)] + \mathbb{E}[K_1(u)(\Phi_m(d(u, U))]\end{aligned} \tag{A5}$$

For the second term, using the technique of Ferraty et al. ([5]) we set:

$$\begin{aligned}\mathbb{E}[K_1(u)\Phi_m(d(u, U))] &= \int K_1\left(\frac{d(u, U)}{\lambda_K}\right) \Phi_m(d(u, U)) d\mu^{d(u,U)}(t) \\ &= \int \Phi_m(t) K_1\left(\frac{t}{\lambda_K}\right) d\mu^{d(u,U)}(t) \\ &= \int \Phi_m(\lambda_K t) K_1(t) d\mu^{\frac{d(u,U)}{\lambda_K}}(t) \\ &= \lambda_K \Phi_m'(0) \int t K_1(t) d\mu^{\frac{d(u,U)}{\lambda_K}}(t) + o(\lambda_K) \\ &= \lambda_K \phi(u, \lambda_K) \Phi_m'(0) \left(K(1) - \int_0^1 (sK(s))' \beta_u(s) ds\right) \\ &+ o(\lambda_K \phi(u, \lambda_K))\end{aligned} \tag{A6}$$

The first order Taylor expansion for Φ around 0 is justified the last line, moreover, we use the following results of Ferraty et al.[5](see Lemma 2 page 27) and Mechab [7]

$$\mathbb{E}[K_1(u)] = \phi(u, \lambda_K)\left(K(1) - \int_0^1 K'(s)\beta_u(s) ds + o(1)\right) \tag{A7}$$

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Using (A6), (A7) and the fact that $\Phi_0'(0) = 0$,

$$\begin{aligned}\frac{\mathbb{E}(K_1(u)f^U(v))}{\phi(u,\lambda_K)} &= f^u(v)\left(K(1) - \int_0^1 K'(s)\beta_u(s)ds + o(1)\right)\\ \frac{\mathbb{E}\left(K_1(u)\frac{\partial^2 f^U(v)}{\partial v^2}\right)}{\phi(u,\lambda_K)} &= \frac{\partial^2 f^u(v)}{\partial v^2}\left(K(1) - \int_0^1 K'(s)\beta_u(s)ds + o(1)\right)\\ &+ \lambda_K \Phi_2'(0)\left(K(1) - \int_0^1 (sK(s))'\beta_u(s)ds\right) + o(\lambda_K)\end{aligned} \tag{A8}$$

Hence,

$$\begin{aligned}\mathbb{E}\left(\widehat{T_3^u}(v)\right) =& f^u(v)\left(K(1) - \int_0^1 K'(s)\beta_u(s)ds\right) + \frac{\lambda_H^2}{2}\int t^2 H'(t)dt\\ &\times\left(\frac{\partial^2 f^u(v)}{\partial v^2}\left(K(1) - \int_0^1 K'(s)\beta_u(s)ds\right) + \lambda_K\Phi_2'(0)\left(K(1) - \int_0^1 (sK(s))'\beta_u(s)ds\right)\right)\\ &+ o\left(\lambda_H^2\right) + o(\lambda_K).\end{aligned} \tag{A9}$$

In other side, by definition of $\widehat{T}_1$ in (7),we get

$$\mathbb{E}\left(\widehat{T}_1(u)\right) = \mathbb{E}\left(\frac{1}{n\phi(u,\lambda_K)}\sum_{\tau=1}^{n} K_\tau(u)\right) = \frac{\mathbb{E}(K_1(u))}{\phi(u,\lambda_K)} = K(1) - \int_0^1 K'(s)\beta_u(s)ds + o(1) \tag{A10}$$

Then, under (A10) and the definition of $\varphi_m(u,v)$, we get

$$\begin{aligned}\frac{\mathbb{E}\left(\widehat{T_3^u}(v)\right)}{\mathbb{E}\left(\widehat{T}_1(u)\right)} - f^u(v) =& \frac{\lambda_H^2}{2}\int t^2 H'(t)dt\left(\frac{\partial^2 f^u(v)}{\partial v^2} + \lambda_K\Phi_2'(0)\frac{\left(K(1) - \int_0^1 (sK(s))'\beta_u(s)ds\right)}{\left(K(1) - \int_0^1 K'(s)\beta_u(s)ds\right)}\right)\\ &+ o\left(\lambda_H^2\right) + o(\lambda_K).\end{aligned} \tag{A11}$$

□

Proof of Lemme 2. start by $\mathrm{Var}(\widehat{T}_3)$, denote by $\Gamma_\tau'(u,v) = K_\tau(u)H'(v)$, then

$$\begin{aligned}\mathrm{Var}\left(\widehat{T}_3{}^u(v)\right) &= \mathrm{Var}\left(\frac{1}{n\lambda_H\phi(u,\lambda_K)}\sum_{\tau=1}^{n} K_\tau(u)H'(v)\right) = \frac{1}{(n\lambda_H\phi(u,\lambda_K))^2}\mathrm{Var}\left(\sum_{\tau=1}^{n} K_\tau(u)H'(v)\right)\\ &= \frac{1}{n(\lambda_H\phi(u,\lambda_K))^2}\mathrm{Var}(\Gamma_1'(u,v)) + \frac{1}{(n\lambda_H\phi(u,\lambda_K))^2}\sum_{\tau=1}^{n}\sum_{\substack{s=1\\ \tau\neq s}}^{n}\mathrm{Cov}(\Gamma_\tau'(u,v),\Gamma_s'(u,v))\\ &= \left(\frac{1}{n\lambda_H\phi(u,\lambda_K)}\right)^2\left[n\,\mathrm{Var}(\Gamma_1'(u,v)) + \sum_\tau\sum_{\tau\neq s}\mathrm{Cov}(\Gamma_\tau'(u,v),\Gamma_s'(u,v))\right]\\ &= \left(\frac{1}{n\lambda_H\phi(u,\lambda_K)}\right)^2[nA_1 + A_{\tau s}]\end{aligned} \tag{A12}$$

Where

$$A_1 = \mathrm{Var}(\Gamma_1'(u,v)) = \mathbb{E}\left[K_1^2(u)H_1'^2(v)\right] - (\mathbb{E}[K_1(u)H_1'(v)])^2 \tag{A13}$$

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Thus, under **(A2)** and **(A3)**, and by integration on the real component z, it follows that

$$\begin{aligned}\mathbb{E}\Big[K_1^2(u)H_1'^2(v)\Big] &= \mathbb{E}\Big(K_1^2(u)\mathbb{E}\Big[H_1'^2(v) \mid U_1\Big]\Big)\\ &= \mathbb{E}\Big(K_1^2(u)\int H'^2\Big(\frac{v-z}{\lambda_H}\Big)f^U(z)dz\Big); \quad \text{taking} \quad t=\frac{v-z}{\lambda_H}\\ &= \lambda_H\mathbb{E}\Big(K_1^2(u)\int H'^2(t)f^U(v-\lambda_H t)dt\Big),\end{aligned}$$

By a Taylor expansion of the order 1 from v we show that for n large enough

$$f^U(v-\lambda_H t) = f^U(v) + \mathcal{O}(\lambda_H) = f^U(v) + o(1)$$

Hence

$$\mathbb{E}\Big(K_1^2(u)H_1'^2(v)\Big) = \lambda_H\int H'^2(t)dt\mathbb{E}\Big(K_1^2(u)f^U(v)\Big) + o(\lambda_H)$$

To simplify writing and calculate, we denote by $\varphi_m(U,v) := \frac{\partial^m f^U(v)}{\partial v^m}$, for $m \in \{0\}$, we have

$$\begin{aligned}\mathbb{E}\Big[K_1^2(u)\varphi_m(U,v)\Big] &= \varphi_m(u,v)\mathbb{E}\Big[K_1^2(u)\Big] + \mathbb{E}\Big[K_1^2(u)(\varphi_m(U,v)-\varphi_m(u,v))\Big]\\ &= \varphi_m(u,v)\mathbb{E}\Big[K_1^2(u)\Big] + \mathbb{E}\Big[K_1^2(u)(\Phi_m(d(u,U))\Big]\end{aligned} \tag{A14}$$

Similarly to Ferraty et al.[5](see Lemma 1 page 26), we set:

$$\begin{aligned}\mathbb{E}\Big[K_1^2(u)\Phi_m(d(u,U))\Big] &= \int K_1^2\Big(\frac{d(u,U)}{\lambda_K}\Big)\Phi_m(d(u,U))d\mu^{d(u,U)}(t)\\ &= \int K_1^2\Big(\frac{t}{\lambda_K}\Big)\Phi_m(t)d\mu^{d(u,U)}(t)\\ &= \int K_1^2(t)\Phi_m(\lambda_K t)d\mu^{\frac{d(u,U)}{\lambda_K}}(t)\\ &= \lambda_K\Phi_m'(0)\int tK_1^2(t)d\mu^{\frac{d(u,U)}{\lambda_K}}(t) + o(\lambda_K) \to 0 \quad \text{for} \quad m=0\end{aligned} \tag{A15}$$

The first order Taylor expansion for Φ around 0 and the fact that $\Phi_0'(0) = 0$ are justified the last line, moreover, we use the following results of Ferraty et al.[5](see Lemma 2 page 27)

$$\mathbb{E}\Big[K_1^2(u)\Big] = \phi(u,\lambda_K)\Big(K^2(1) - \int_0^1 (K'(s))^2\beta_u(s)ds + o(1)\Big).$$

Then, (A14) becomes,

$$\begin{aligned}\mathbb{E}\Big[K_1^2(u)\varphi_0(U,v)\Big] =& \varphi_0(u,v)\mathbb{E}\Big[K_1^2(u)\Big].\\ =& \phi(u,\lambda_K)f^u(v)\Big(K^2(1) - \int_0^1 (K'(s))^2\beta_u(s)ds\Big) + o(\phi(u,\lambda_K)).\end{aligned} \tag{A16}$$

This allows us to conclude

$$\mathbb{E}\Big(K_1^2(u)H_1'^2(v)\Big) = \lambda_H\phi(u,\lambda_K)f^u(v)\Big(K^2(1) - \int_0^1 (K'(s))^2\beta_u(s)ds\Big)\int H'^2(t)dt + o(\lambda_H\phi(u,\lambda_K))$$

For the second term in (A13), by the same steps in the the proof above, under **(A3)**, we show

$$\mathbb{E}(K_1(u)H_1'(v)) = \lambda_H\phi(u,\lambda_K)f^u(v)\Big(K(1) - \int_0^1 (K'(s))\beta_u(s)ds\Big) + o(\lambda_H\phi(u,\lambda_K)). \tag{A17}$$

Then

$$\mathbb{E}^2\left(K_1(u)H_1'(v)\right) = O\left(\lambda_H^2\phi^2(u,\lambda_K)\right)$$

Which implies that

$$\mathrm{Var}\left(\Gamma_1'(u,v)\right) = \lambda_H\phi(u,\lambda_K)f^u(v)\left(K^2(1) - \int_0^1\left(K^2(s)\right)'\beta_u(s)ds\right)\int H'^2(t)dt + o(\lambda_H\phi(u,\lambda_K)). \tag{A18}$$

For the second term $A_{\tau s}$, we split the sum in two sets defined by m_n with $m_n \to \infty$, as $n \to \infty$.

$$\begin{aligned} A_{\tau s} = \sum_{\tau=1}^{n}\sum_{\substack{s=1\\ \tau\neq s}}^{n}\mathrm{Cov}\left(\Gamma_\tau'(u,v),\Gamma_s'(u,v)\right) &= \sum_{\tau=1}^{n}\sum_{\substack{s=1\\ 0<|\tau-s|\leq m_n}}^{n}\mathrm{Cov}\left(\Gamma_\tau'(u,v),\Gamma_s'(u,v)\right) \\ &+ \sum_{\tau=1}^{n}\sum_{\substack{s=1\\ |\tau-s|>m_n}}^{n}\mathrm{Cov}\left(\Gamma_\tau'(u,v),\Gamma_s'(u,v)\right) \\ &=: I_n + II_n. \end{aligned} \tag{A19}$$

Under assumptions **(A1)**, **(A3)** and **(A5)**, we infer for $\tau \neq s$

$$\begin{aligned} I_n &= \sum_{\tau=1}^{n}\sum_{\substack{s=1\\ 0<|\tau-s|\leq m_n}}^{n}\mathrm{Cov}\left(\Gamma_\tau'(u,v),\Gamma_s'(u,v)\right) \\ &\leq nm_n\left(\max_{\tau\neq s}\left|\mathbb{E}\left[\Gamma_\tau'(u,v)\Gamma_s'(u,v)\right]\right| + \left(\mathbb{E}\left[\Gamma_1'(u,v)\right]\right)^2\right) \\ &\leq \theta nm_n\left(\max_{\tau\neq s}\left|\mathbb{E}\left[K_\tau H_\tau' K_s H_s'\right]\right| + \left(\mathbb{E}\left[K_1 H_1'\right]\right)^2\right) \\ &\leq \theta nm_n\left(\lambda_H^2\phi^2(u,\lambda_K) + (\lambda_H\phi(u,\lambda_K))^2\right) \\ &\leq \theta nm_n\lambda_H^2\phi^2(u,\lambda_K). \end{aligned} \tag{A20}$$

Now, under the assumptions **(A3)-(A5)**, we set

$$\begin{aligned} II_n = \sum_{\tau=1}^{n}\sum_{\substack{s=1\\ |\tau-s|>m_n}}^{n}\mathrm{Cov}\left(\Gamma_\tau'(u,v),\Gamma_s'(u,v)\right) &\leq \left(\frac{\mathrm{Lip}(K)}{\lambda_K} + \frac{\mathrm{Lip}(H')}{\lambda_H}\right)^2\sum_{\tau=1}^{n}\sum_{\substack{s=1\\ |\tau-s|>m_n}}^{n}\delta_{\tau,s} \\ &\leq \theta n\left(\frac{\mathrm{Lip}(K)}{\lambda_K} + \frac{\mathrm{Lip}(H')}{\lambda_H}\right)^2\delta_{m_n} \\ &\leq \theta n\left(\frac{\mathrm{Lip}(K)}{\lambda_K} + \frac{\mathrm{Lip}(H')}{\lambda_H}\right)^2 e^{-am_n}. \end{aligned} \tag{A21}$$

Then, by (A20) and (A21), we get

$$\begin{aligned} A_{\tau s} &= \sum_{\tau=1}^{n}\sum_{\substack{s=1\\ \tau\neq s}}^{n}\mathrm{Cov}\left(\Gamma_\tau'(u,v),\Gamma_s'(u,v)\right) \\ &\leq \theta n\left(m_n\lambda_H^2\phi^2(u,\lambda_K) + \left(\frac{\mathrm{Lip}(K)}{\lambda_K} + \frac{\mathrm{Lip}(H')}{\lambda_H}\right)^2 e^{-am_n}\right) \end{aligned}$$

Taking $m_n = \frac{1}{a}\log\left(\frac{a\left(\lambda_K^{-1}\mathrm{Lip}(K)+\lambda_H^{-1}\mathrm{Lip}(H')\right)^2}{\lambda_H^2\phi^2(u,\lambda_K)}\right)$, we get

$$\left(\frac{1}{n\lambda_H\phi(u,\lambda_K)}\right)^2 A_{\tau s} \quad \to 0, \text{ as } n \to \infty \tag{A22}$$

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Finally, by combining results (A18) and (9), we get

$$\mathrm{Var}\Big(\widehat{T_3^u}(v)\Big) = \frac{f^u(v)}{n\lambda_H\phi(u,\lambda_K)}\bigg(K^2(1) - \int_0^1 \Big(K^2(s)\Big)'\beta_u(s)ds\bigg)\int H'^2(t)dt + o\bigg(\frac{1}{n\lambda_H\phi(u,\lambda_K)}\bigg)$$

For the second result about $\mathrm{Var}(\widehat{T}_1)$, we set

$$\begin{aligned}\mathrm{Var}\Big(\widehat{T}_1(u)\Big) &= \bigg(\frac{1}{n\phi(u,\lambda_K)}\bigg)^2 \mathrm{Var}\bigg(\sum_{\tau=1}^{n} K_\tau(u)\bigg) \\ &= \frac{1}{n\phi^2(u,\lambda_K)}\mathrm{Var}(K_1(u)) + \frac{1}{(n\phi(u,\lambda_K))^2}\sum_{\tau=1}^{n}\sum_{\substack{s=1\\ \tau\neq s}}^{n}\mathrm{Cov}(K_\tau(u),K_s(u)) \\ &= VK_1 + VK_2\end{aligned} \tag{A23}$$

Moreover,

$$\begin{aligned}VK_1 =& \frac{\mathbb{E}(K_1^2(u)) - \mathbb{E}(K_1(u))^2}{n\phi^2(u,\lambda_K)} \\ =& \frac{K^2(1) - \int_0^1 (K'(s))^2\beta_u(s)ds - \phi(u,\lambda_K)\Big(K(1) - \int_0^1 K'(s)\beta_u(s)ds\Big)^2}{n\phi(u,\lambda_K)} + o\bigg(\frac{1}{n\phi(u,\lambda_K)}\bigg)\end{aligned}$$

Furthermore, for the second term VK_2, we split the sum as follows:

$$\begin{aligned}\sum_{\tau=1}^{n}\sum_{\substack{s=1\\ \tau\neq s}}^{n}\mathrm{Cov}(K_\tau(u),K_s(u)) &= \sum_{\tau=1}^{n}\sum_{\substack{s=1\\ 0<|\tau-s|\leq m_n}}^{n}\mathrm{Cov}(K_\tau(u),K_s(u)) \\ &+ \sum_{\tau=1}^{n}\sum_{\substack{s=1\\ |\tau-s|>m_n}}^{n}\mathrm{Cov}(K_\tau(u),K_s(u)) \\ &=: J_1 + J_2.\end{aligned}$$

Now, under the assumptions (A_8), we have

$$\begin{aligned}|J_1| = \sum_{\tau}\sum_{0<|\tau-s|\leq m_n}|\mathrm{Cov}(K_\tau(u),K_s(u))| &\leq nm_n\Big[\max_{\tau\neq s}|\mathbb{E}(K_\tau(u)K_s(u))| + (\mathbb{E}(K_1(u)))^2\Big] \\ &\leq \theta n m_n\phi^2(u,\lambda_K)\end{aligned}$$

Making use of the condition (A_7), we infer that

$$\begin{aligned}|J_2| = \sum_{\tau}\sum_{|\tau-s|>m_n}|\mathrm{Cov}(K_\tau(u),K_s(u))| &\leq \theta\bigg(\frac{\mathrm{Lip}(K)}{\lambda_K}\bigg)^2\sum_{\tau}\sum_{|\tau-s|>m_n}\delta_{\tau,s} \\ &\leq \theta n\lambda_K^{-2}e^{-am_n}\end{aligned}$$

This implies that

$$|VK_2| \leq \sum_{\tau=1}^{n}\sum_{\tau\neq s}|\mathrm{Cov}(K_\tau(u),K_s(u))| \leq \theta n\Big(m_n\phi^2(u,\lambda_K) + \lambda_K^{-2}e^{-am_n}\Big)$$

Next, taking

$$m_n = \frac{1}{a}\log\bigg(\frac{a}{\lambda_K^2\phi^2(u,\lambda_K)}\bigg)$$

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which allows to write that

$$VK_2 = \frac{1}{n(\phi(u,\lambda_K))^2} \sum_{\tau=1}^{n} \sum_{\substack{s=1 \\ \tau\neq s}}^{n} \text{Cov}(K_\tau(u), K_s(u)) \to 0 \tag{A24}$$

Finally, we get:

$$\text{Var}\left(\widehat{T}_1(u)\right) = \frac{K^2(1) - \int_0^1 (K'(s))^2 \beta_u(s)ds - \left(K(1) - \int_0^1 K'(s)\beta_u(s)ds\right)^2}{n\phi(u,\lambda_K)} + o\left(\frac{1}{n\phi(u,\lambda_K)}\right).$$

Now, we evaluate the $\text{Cov}\left(\widehat{T}_1(u), \widehat{T_3^u}(v)\right)$ as follows

$$\begin{aligned}\text{Cov}\left(\widehat{T}_1(u), \widehat{T}_3^u(v)\right) &= \frac{1}{n^2\lambda_H\phi^2(u,\lambda_K)}\left[n\,\text{Cov}(K_1(u), \Gamma_1'(u,v)) + \sum_{\tau\neq s} \text{Cov}(K_\tau(u), \Gamma_s'(u,v))\right] \\ &= CV_1 + CV_{\tau s} \quad \text{with} \quad \Gamma_\tau'(u,v) = K_\tau(u)H_\tau'(v)\end{aligned}$$

Where

$$CV_1 = \frac{\mathbb{E}[K_1^2(u)H_1'(v)]}{n\lambda_H\phi^2(u,\lambda_K)} - \frac{\mathbb{E}[K_1(u)]\mathbb{E}[K_1(u)H_1'(v)]}{n\lambda_H\phi^2(u,\lambda_K)} \tag{A25}$$

Using (A7) and (A17), we infer

$$\frac{\mathbb{E}[K_1(u)]\mathbb{E}[K_1(u)H_1'(v)]}{n\lambda_H\phi^2(u,\lambda_K)} = \frac{f^u(v)\left(K(1) - \int_0^1 K'(s)\beta_u(s)ds + o(1)\right)^2}{n} + o\left(\frac{1}{n}\right). \tag{A26}$$

For the first term, by the conditional expectation and the first order Taylor expansion of f, we get

$$\mathbb{E}\left[K_1^2(u)H_1'(v)\right] = \mathbb{E}\left[K_1^2(u)\mathbb{E}(H_1'(v) \mid U)\right] = \lambda_H\mathbb{E}\left(K_1^2(u)f^U(v)\right) + o(\lambda_H\phi(u,\lambda_K))$$

Using (A16), we have

$$\mathbb{E}\left[K_1^2(u)H_1'(v)\right] = \lambda_H\phi(u,\lambda_K)f^u(v)\left(K^2(1) - \int_0^1 (K'(s))^2\beta_u(s)ds\right) + o(\lambda_H\phi(u,\lambda_K))$$

Hence,

$$\frac{\mathbb{E}[K_1^2(u)H_1'(v)]}{n\lambda_H\phi^2(u,\lambda_K)} = \frac{f^u(v)\left(K^2(1) - \int_0^1 (K'(s))^2\beta_u(s)ds\right)}{n\phi(u,\lambda_K)} + o\left(\frac{1}{n\phi(u,\lambda_K)}\right) \tag{A27}$$

By (A26) and (A27), we get

$$\begin{aligned}CV_1 =& \frac{f^u(v)\left[\left(K^2(1) - \int_0^1 (K'(s))^2\beta_u(s)ds\right) - \phi(u,\lambda_K)\left(K(1) - \int_0^1 K'(s)\beta_u(s)ds + o(1)\right)^2\right]}{n\phi(u,\lambda_K)} \\ &+ o\left(\frac{1}{n\phi(u,\lambda_K)}\right)\end{aligned} \tag{A28}$$

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Furthermore, for the second term $CV_{\tau s}$, we have

$$\begin{aligned}\sum_{\tau=1}^{n}\sum_{\substack{s=1\\ \tau\neq s}}^{n}\mathrm{Cov}\big(K_\tau(u),\Gamma_s'(u,v)\big) &= \sum_{\tau=1}^{n}\sum_{\substack{s=1\\ 0<|\tau-s|\leq m_n}}^{n}\mathrm{Cov}\big(K_\tau(u),\Gamma_s'(u,v)\big)\\ &+\sum_{\tau=1}^{n}\sum_{\substack{s=1\\ |\tau-s|>m_n}}^{n}\mathrm{Cov}\big(K_\tau(u),\Gamma_s'(u,v)\big)\\ &=:P_1+P_2.\end{aligned}$$

Under assumptions **(A1)**, **(A3)** and **(A6)**, we infer, for $\tau\neq s$

$$\begin{aligned}P_1 &= \sum_{\tau=1}^{n}\sum_{\substack{s=1\\ 0<|\tau-s|\leq m_n}}^{n}\mathrm{Cov}\big(K_\tau(u),\Gamma_s'(u,v)\big)\\ &\leq \sum_{\tau=1}^{n}\sum_{\substack{s=1\\ 0<|\tau-s|\leq m_n}}^{n}\big|\mathbb{E}\big(K_\tau(u)\Gamma_s'(u,v)\big)\big|\\ &\leq \theta\sum_{\tau=1}^{n}\sum_{\substack{s=1\\ 0<|\tau-s|\leq m_n}}^{n}\big|\mathbb{E}\big(K_sH_s'\big)\big|\\ &\leq \theta n m_n\lambda_H\phi(u,\lambda_K)\end{aligned} \tag{A29}$$

By the fact that K and H are bounded we get :

$$\begin{aligned}P_2=\sum_{\tau=1}^{n}\sum_{\substack{s=1\\ |\tau-s|>m_n}}^{n}\mathrm{Cov}\big(K_\tau(u),\Gamma_s'(u,v)\big) &\leq \left[\left(\frac{\mathrm{Lip}(K)}{\lambda_K}\right)^2+\frac{\mathrm{Lip}(H')}{\lambda_H}\right]\sum_{\tau=1}^{n}\sum_{\substack{s=1\\ |\tau-s|>m_n}}^{n}\delta_{\tau,s}\\ &\leq \theta n\left[\left(\frac{\mathrm{Lip}(K)}{\lambda_K}\right)^2+\frac{\mathrm{Lip}(H')}{\lambda_H}\right]\delta_{m_n}\\ &\leq \theta n\left[\left(\frac{\mathrm{Lip}(K)}{\lambda_K}\right)^2+\frac{\mathrm{Lip}(H')}{\lambda_H}\right]e^{-am_n}.\end{aligned} \tag{A30}$$

Then, by (A29) and (A30), we get

$$\sum_{\tau\neq s}\mathrm{Cov}\big(K_\tau(u),\Gamma_s'(u,v)\big)\leq \theta n\left(m_n\lambda_H\phi(u,\lambda_K)+\left[\left(\frac{\mathrm{Lip}(K)}{\lambda_K}\right)^2+\frac{\mathrm{Lip}(H')}{\lambda_H}\right]e^{-\alpha m_n}\right)$$

Taking $m_n=\frac{1}{\alpha}\log\left(\frac{\left(\lambda_K^{-1}\mathrm{Lip}(K)\right)^2+\lambda_H^{-1}\mathrm{Lip}(H')}{\alpha\lambda_H\phi(u,\lambda_K)}\right)$, we get

$$CV_{\tau s}=\frac{1}{n^2\lambda_H\phi^2(u,\lambda_K)}\sum_{\tau\neq s}\mathrm{Cov}\big(K_\tau(u),\Gamma_s'(u,v)\big)\to 0 \tag{A31}$$

Combining the results (A28) and (A31) we get:

$$\mathrm{Cov}\Big(\widehat{T}_1(u),\widehat{T_3^u}(v)\Big)=\frac{f^u(v)\left[\left(K^2(1)-\int_0^1(K'(s))^2\beta_u(s)ds\right)-\phi(u,\lambda_K)\left(K(1)-\int_0^1K'(s)\beta_u(s)ds\right)^2\right]}{n\phi(u,\lambda_K)}+o\left(\frac{1}{n\phi(u,\lambda_K)}\right)$$

□

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Proof of Lemma 3. By the definition of the conditional expectation, using the stationarity of the observations and taking $t = \frac{v-z}{\lambda_H}$ we writing:

$$\begin{aligned} \mathbb{E}\left(\widehat{T_2^u}(v)\right) &= \mathbb{E}\left(\frac{1}{n\phi(u,\lambda_K)}\sum_{\tau=1}^{n} K_\tau(u)H_\tau(v)\right) \\ &= \frac{1}{\phi(u,\lambda_K)}\mathbb{E}[K_1(u)\mathbb{E}(H_1(v)\mid U)] \end{aligned} \tag{A32}$$

With

$$\begin{aligned} \mathbb{E}[H_1(v)\mid U] &= \int_{\mathbb{R}} H_1\left(\frac{v-z}{\lambda_H}\right) f^U(z)dz \\ &= \frac{1}{\lambda_H}\int_{\mathbb{R}} H_1'\left(\frac{v-z}{\lambda_H}\right) F^U(z)dz \\ &= \int_{\mathbb{R}} H_1'(t)F^U(v-\lambda_H t)dt \end{aligned} \tag{A33}$$

Using a Taylor expansion of the function $F^U(v-\lambda_H t)$:

$$F^U(v-\lambda_H t) = F^U(v) - \lambda_H t\frac{\partial F^U(v)}{\partial v} + \frac{\lambda_H^2 t^2}{2}\frac{\partial^2 F^U(v)}{\partial v^2} + o\left(\lambda_H^2\right)$$

Under (A33) and assumption **(A3)**, we deduce that:

$$\mathbb{E}[H_1(v)\mid U] = F^U(v) + \frac{\lambda_H^2}{2}\frac{\partial^2 F^U(v)}{\partial^2 v}\left(\int t^2 H'(t)dt\right) + o\left(\lambda_H^2\right). \tag{A34}$$

Insert (A34) in (A32)

$$\mathbb{E}\left(\widehat{T_2^u}(v)\right) = \frac{1}{\phi(u,\lambda_K)}\left(\mathbb{E}\left(K_1(u)F^U(v)\right) + \frac{\lambda_H^2}{2}\int t^2 H'(t)dt\,\mathbb{E}\left[K_1(u)\frac{\partial^2 F^U(v)}{\partial^2 v}\right]\right) + o\left(\lambda_H^2\right).$$

Denote by $\psi_m(U,v) := \frac{\partial^m F^U(v)}{\partial v^m}$ for $m \in \{0,2\}$, then

$$\mathbb{E}\left(\widehat{T_2^u}(v)\right) = \frac{\mathbb{E}(K_1(u)\psi_0(U,v))}{\phi(u,\lambda_K)} + \frac{\mathbb{E}(K_1(u)\psi_2(U,v))}{\phi(u,\lambda_K)}\frac{\lambda_H^2}{2}\int t^2 H'(t)dt + o\left(\lambda_H^2\right). \tag{A35}$$

Where

$$\begin{aligned} \mathbb{E}[K_1(u)\psi_m(U,v)] &= \psi_m(u,v)\mathbb{E}[K_1(u)] + \mathbb{E}[K_1(u)(\psi_m(U,v) - \psi_m(u,v))] \\ &= \psi_m(u,v)\mathbb{E}[K_1(u)] + \mathbb{E}[K_1(u)(\Psi_m(d(u,U))] \end{aligned} \tag{A36}$$

With the same steps following to evaluate (A15), we set:

$$\begin{aligned} \mathbb{E}[K_1(u)\Psi_m(d(u,U))] &= \int K_1\left(\frac{d(u,U)}{\lambda_K}\right)\Psi_m(d(u,U))d\mu^{d(u,U)}(t) \\ &= \int \Psi_m(t)K_1\left(\frac{t}{\lambda_K}\right)d\mu^{d(u,U)}(t) \\ &= \int \Psi_m(\lambda_K t)K_1(t)d\mu^{\frac{d(u,U)}{\lambda_K}}(t) \\ &= \lambda_K\Psi_m'(0)\int tK_1(t)d\mu^{\frac{d(u,U)}{\lambda_K}}(t) + o(\lambda_K) \\ &= \lambda_K\phi(u,\lambda_K)\Phi_m'(0)\left(K(1) - \int_0^1 (sK(s))'\beta_u(s)ds\right) \\ &+ o(\lambda_K\phi(u,\lambda_K)) \end{aligned} \tag{A37}$$

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The first order Taylor expansion for Ψ around 0 is justified the last line, moreover, we use the results (A6) and the fact that $\Psi_0'(0) = 0$,

$$\frac{\mathbb{E}(K_1(u)F^U(v))}{\phi(u,\lambda_K)} = F^u(v)\left(K(1) - \int_0^1 K'(s)\beta_u(s)ds + o(1)\right)$$
$$\frac{\mathbb{E}\left(K_1(u)\frac{\partial^2 F^U(v)}{\partial v^2}\right)}{\phi(u,\lambda_K)} = \frac{\partial^2 F^u(v)}{\partial v^2}\left(K(1) - \int_0^1 K'(s)\beta_u(s)ds + o(1)\right) + \lambda_K\Psi_2'(0)\left(K(1) - \int_0^1 (sK(s))'\beta_u(s)ds\right) + o(\lambda_K) \quad \text{(A38)}$$

Hence,

$$\mathbb{E}\left(\widehat{T_2^u}(v)\right) = F^u(v)\left(K(1) - \int_0^1 K'(s)\beta_u(s)ds\right) + \frac{\lambda_H^2}{2}\int t^2 H'(t)dt \times\left[\frac{\partial^2 F^u(v)}{\partial v^2}\left(K(1) - \int_0^1 K'(s)\beta_u(s)ds\right) + \lambda_K\Psi_2'(0)\left(K(1) - \int_0^1 (sK(s))'\beta_u(s)ds\right)\right] + o\left(\lambda_H^2\right) + o(\lambda_K). \quad \text{(A39)}$$

Then, under (A39) and (A10) , we get

$$\frac{\mathbb{E}\left(\widehat{T_2^u}(v)\right)}{\mathbb{E}\left(\widehat{T_1}(u)\right)} - F^u(v) = \frac{\lambda_H^2}{2}\int t^2 H'(t)dt\left(\frac{\partial^2 F^u(v)}{\partial v^2} + \lambda_K\Psi_2'(0)\frac{\left(K(1) - \int_0^1 (sK(s))'\beta_u(s)ds\right)}{\left(K(1) - \int_0^1 K'(s)\beta_u(s)ds\right)}\right) + o\left(\lambda_H^2\right) + o(\lambda_K).$$

□

Proof of Lemma 4. Following the same steps as techniques used in the proof of Lemma (2), we get

$$\mathrm{Var}\left(\widehat{T_2}^u(v)\right) = \frac{F^u(v)}{n\phi(u,\lambda_K)}\left(K^2(1) - \int_0^1 \left(K^2(s)\right)'\beta_u(s)ds\right)\int H'^2(t)dt + o\left(\frac{1}{n\phi(u,\lambda_K)}\right)$$
$$\mathrm{Cov}\left(\widehat{T_1}(u), \widehat{T_2^u}(v)\right) = \frac{F^u(v)\left[\left(K^2(1) - \int_0^1 (K'(s))^2\beta_u(s)ds\right) - \phi(u,\lambda_K)\left(K(1) - \int_0^1 K'(s)\beta_u(s)ds\right)^2\right]}{n\phi(u,\lambda_K)} + o\left(\frac{1}{n\phi(u,\lambda_K)}\right)$$

□

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