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Article

Study of the ρ^0 Decay into $\pi^0\pi^0\gamma$ Within Framework of Chiral Perturbation Theory with Resonances

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Abstract

In this work we calculate the branching ratio (BR) of the process $\rho^0 \rightarrow \pi^0\pi^0\gamma$ at tree level within the theoretical framework of chiral perturbation theory with resonances, including the odd intrinsic parity sector. Owing to the nature of an effective field theory, the BR depends on three unknown couplings of the model: d_4 and the combination $c_{57} = c_5 + c_7$. To constrain this unknown couplings, we use the experimental BR of this decay and explore the possibility of reducing the number of unknown coupling through the on-shell condition at one vertex. Our result suggests that off-shell conditions must be applied to both vertices, leading to a space parameter for the pair (c_{57}, d_4) .

Keywords: QCD; chiral perturbation theory (ChPT); effective field theories of QCD; resonance chiral theory ($R\chi T$)

1. Introduction

The Standard Model of Particle Physics (SM) [1,2] is one of the most successful theories, it predicts almost all phenomena in particle physics between matter and its interactions: electromagnetic, weak, and strong. However, it is incomplete because it does not incorporate dark matter, gravity, neutrino masses and it does not explain the asymmetry matter -antimatter in the universe. Nowadays SM is considered an effective field theory with 19 free parameters that cannot be derived from the theory itself. Instead, these parameters must be determined phenomenologically, which is accomplished through the advancement of particle colliders experiments. The current status of these parameters can be found in Ref. [3].

At low energies, below 1 GeV, the particle spectrum consists predominantly of hadrons, which are composed of quarks embedded in a gluon sea. Gluons are the massless gauge bosons of spin 1 that mediate the strong interactions between quarks. In this regime, the strong coupling constant α_s becomes sufficiently large to prevent the use of perturbative methods. Consequently the dynamics of hadronic structure cannot be described within the framework of SM at such energy scales.

As an alternative approach to the study of strong interactions, effective field theories (EFT) provide an ideal framework. They are constructed from the underlying fundamental theory in accordance with the Weinberg's theorem [4], which states that the effective Lagrangian must respect the symmetries relevant to the energy range of interest. However, the predictive power of an EFT relies on the knowledge of its low energy coupling constants (LECs), which cannot be determined by symmetry requirements and must therefore be obtained either from the underlying theory or through phenomenological methods.

Chiral perturbation theory (ChPT) is an EFT of quantum chromodynamics (QCD) in the chiral limit¹, formulated as a perturbative expansion in powers of momenta. The relevant degrees of freedom

¹ The chiral limit denotes the theoretical scenario in which the light quarks u , d , and s are massless.

correspond to the Goldstone bosons arising from the spontaneous breaking of chiral symmetry, which can be identified with the lightest pseudoscalar mesons [5]. At leading order, corresponding to $\mathcal{O}(p^2)$, the effective Lagrangian depends on two LECs, whose values are determined phenomenologically and are well established [5,6]. However, at higher orders, such as $\mathcal{O}(p^4)$, the number of LECs increases significantly, making their determination a challenge.

As the energy increases up to 1 GeV, the role of resonance particles becomes significant, and they must be included as additional degrees of freedom. This leads to the framework of Resonance Chiral Theory ($R\chi T$) [6,7], which introduces new LECs. In the even parity sector [7], two LECs arise, and their corresponding values are well known. On the other hand, the odd intrinsic parity sector, formulated in Ref. [8], includes interactions among two light resonances (vectors) and one light pseudoscalar field, namely the VVP operators. Moreover, the interaction involving a vector, a pseudoscalar, and an external source (VJP) allows one to describe electromagnetic and weak interactions. This formulation incorporates eleven LECs, which are a priori unknown. Fortunately, by constructing the VVP and VJP Green's three-point functions at leading order in $1/N_c$, some combinations of these LECs can be predicted by requiring that the short-distance (high-energy) behavior of these Green's functions matches the corresponding Operator Product Expansion (OPE) [8]. Nonetheless, some couplings remain unknown, in particular c_5, c_7 and d_4 , thus requiring a phenomenological analysis to set reasonable bounds on LECs.

In computing the isospin symmetry-breaking (IB) effects in the process $\tau^+ \rightarrow \pi^+ \pi^0 \bar{\nu}_\tau$, an upper bound of $|d_4| < 0.15$ is assumed in Ref. [9]. This IB corrections are important in the tau-based determination of the hadronic vacuum polarization contribution to the anomalous magnetic moment of the muon [10]. Following the same line, Ref. [11] found that the $\omega^0 \rightarrow \pi^0 \pi^0 \gamma$ decay can provide information on d_4 , assuming the on-shell condition for the $\rho^0 \omega^0 \pi^0$ vertex, which eliminates the c_5, c_7 , dependence. This coupling is an input in their analysis of the triple-product asymmetries in the τ decay and in the radiative two-pion decay, namely $\tau^+ \rightarrow \pi^+ \pi^0 \nu_\tau \gamma$. They report two possible values, $d_4 = 0.82 \pm 0.05$ and $d_4 = -0.12 \pm 0.05$ and choose² the positive one in their analysis to maintain consistency with the result in [9].

With this proviso, it is clear that a more general analysis of this pair of couplings can be performed. The aim of this work is to determine the values of these couplings through the analysis of the branching ratio $\mathcal{BR}(\rho^0 \rightarrow \pi^0 \pi^0 \gamma)$, described at tree level within the framework of the odd intrinsic parity Lagrangian, considering both vertices to be off-shell.

This paper is organized as follows. In Section 2, we review the foundations of Chiral Perturbation Theory (ChPT) and the inclusion of light resonances that give rise to $R\chi T$, covering both the even- and odd-intrinsic-parity sectors, as described in Ref. [8]. In Section 3, we compute the invariant amplitude of the process $\rho^0 \rightarrow \pi^0 \pi^0 \gamma$, showing that only depends on $c_{57} \equiv c_5 + c_7$ and d_4 . In addition, the on-shell and off-shell vertices are defined, and their impact on the determination of the BR is discussed. In Section 4, we present our results and conclusions and in the last section, Appendix A, the full analytical expressions are reported.

2. Chiral Perturbation Theory with Resonances

ChPT is an EFT of QCD that incorporates all the essential features of QCD at low energies, well below 1 GeV. It is formulated as an expansion in powers of momentum, where the relevant degree of freedom are the eight Goldstone bosons corresponding to the lightest states in the hadronic spectrum. According to the power-counting scheme, the ChPT Lagrangian at lowest order $\mathcal{O}(p^2)$ is given by³

$$\mathcal{L}_\chi^{(2)} = \frac{F^2}{4} \text{Tr} \left(D_\mu U^\dagger D^\mu U + U^\dagger \chi + \chi^\dagger U \right), \quad (1)$$

² see discussion in appendix B of Ref. [11])

³ Tr denotes the trace.

where

$$U(\phi) = \exp\left(i\frac{\sqrt{2}\phi}{F}\right), \quad (2)$$

is the parameterization of the Goldstone fields ϕ_a , which transform under the group⁴ G as

$$U \xrightarrow{G} g_R U g_L^\dagger, \quad g \equiv (g_L, g_R) \in G. \quad (3)$$

The field ϕ is expressed as follows

$$\phi = \sum_{a=1}^8 \phi_a \frac{\lambda_a}{\sqrt{2}}, \quad (4)$$

where λ_a denote the Gell-Mann matrices and the covariant derivative

$$D_\mu U = \partial_\mu U - i r_\mu U + i U \ell_\mu, \quad (5)$$

is defined to transform as

$$D_\mu U(x) \longrightarrow g_R (D_\mu U) g_L^\dagger. \quad (6)$$

The symmetry breaking is introduced by the last two terms in Equation (1) with

$$\chi = 2B_0(s + ip), \quad (7)$$

where s and p are the scalar and pseudoscalar external fields, respectively.

The constants F and B_0 are LECs that can be determined phenomenologically [5,8]. For instance $F \approx F_\pi = 92.4$ MeV is associated with the pion decay constant, which can be extracted from the decay $\pi^+ \rightarrow \mu^+ \nu_\mu$. On the other hand, B_0 is related to the scalar quark condensate $\langle \bar{q}q \rangle$ and plays a central role in the spontaneous breaking of chiral symmetry.

Interactions with external fields are incorporated through the fields l_μ and r_μ . In the case of the electromagnetic interaction, these fields are given by

$$r_\mu = l_\mu = -e \mathcal{A}_\mu(x) Q, \quad (8)$$

where $\mathcal{A}_\mu(x)$ is the electromagnetic field and

$$Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}, \quad (9)$$

is the quark charge matrix.

As the energy approaches 1 GeV, $R\chi T$ incorporates the light resonances as explicit degrees of freedom. In Ref. [12], it is shown that the vector and the antisymmetric tensor representation are equivalent. In this work, we adopt the antisymmetric tensor formalism as given in Ref. [7]. The Lagrangian of $R\chi T$ in the even intrinsic parity sector reads

$$\mathcal{L}_{\chi V} = \mathcal{L}_\chi^{(2)} + \mathcal{L}_V^{(2)} + \mathcal{L}_{Kin}(V). \quad (10)$$

To facilitate the construction of invariant terms involving resonances, the following parameterization is used

$$U(\phi) = u^2(\phi), \quad (11)$$

⁴ The symmetry group $G := SU(3)_L \times SU(3)_R$ is referred to as the Chiral Symmetry

which transforms under the chiral group G as follows

$$u(\phi) \rightarrow g_R u(\phi) h^\dagger(g, \phi) = h(g, \phi) u(\phi) g_L^\dagger, \quad (12)$$

with $h(g, \phi) \in SU(3)_V$. In terms of this parameterization, the lowest-order chiral Lagrangian in Equation (1) becomes

$$\mathcal{L}_\chi^{(2)} = \frac{F^2}{4} \text{Tr}(u_\mu u^\mu + \chi_+), \quad (13)$$

where

$$u_\mu = i \left[u^\dagger (\partial_\mu - i r_\mu) u - u (\partial_\mu - i \ell_\mu) u^\dagger \right], \quad (14)$$

$$\chi_\pm = u^\dagger \chi u^\dagger \pm u \chi^\dagger u. \quad (15)$$

The interaction of the vector resonances with the light mesons and external fields is described by

$$\mathcal{L}_V^{(2)} = \frac{F_V}{2\sqrt{2}} \text{Tr}(V_{\mu\nu} f_+^{\mu\nu}) + i \frac{G_V}{\sqrt{2}} \text{Tr}(V_{\mu\nu} u^\mu u^\nu), \quad (16)$$

where

$$f_\pm^{\mu\nu} = u F_L^{\mu\nu} u^\dagger \pm u^\dagger F_R^{\mu\nu} u, \quad (17)$$

and

$$F_L^{\mu\nu} = \partial^\mu \ell^\nu - \partial^\nu \ell^\mu - i[\ell^\mu, \ell^\nu], \quad (18)$$

$$F_R^{\mu\nu} = \partial^\mu r^\nu - \partial^\nu r^\mu - i[r^\mu, r^\nu]. \quad (19)$$

Finally, the kinetic part for the vector resonances is introduced as

$$\mathcal{L}_{Kin}(V) = -\frac{1}{2} \text{Tr} \left(\nabla^\lambda V_{\lambda\mu} \nabla_\nu V^{\nu\mu} - \frac{M_V^2}{2} V_{\mu\nu} V^{\mu\nu} \right), \quad (20)$$

where M_V is the mass of the vector octet and

$$\nabla_\mu V = \partial_\mu V + [\Gamma_\mu, V], \quad (21)$$

is the covariant derivative, defined to ensure invariance under the transformation of the vector field,

$$V \xrightarrow{G} h(\phi, g) V h(\phi, g)^\dagger. \quad (22)$$

The natural connection Γ_μ in the coset space [6] is given by

$$\Gamma_\mu = \frac{1}{2} \left\{ u^\dagger (\partial_\mu - i r_\mu) u + u (\partial_\mu - i \ell_\mu) u^\dagger \right\}. \quad (23)$$

The representation of the octets of Goldstone bosons and vector resonances are written as follows

$$\Phi(x) = \begin{pmatrix} \frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}} \eta_8 \end{pmatrix}, \quad (24)$$

$$V_{\mu\nu} = \begin{pmatrix} \frac{1}{\sqrt{2}} \rho^0 + \frac{1}{\sqrt{6}} \omega_8 & \rho^+ & K^{*+} \\ \rho^- & -\frac{1}{\sqrt{2}} \rho^0 + \frac{1}{\sqrt{6}} \omega_8 & K^{*0} \\ K^{*-} & \bar{K}^{*0} & -\frac{2}{\sqrt{6}} \omega_8 \end{pmatrix}_{\mu\nu}. \quad (25)$$

In Ref. [8], the Lagrangian of the odd intrinsic parity sector is introduced, namely

$$\mathcal{L}_V^{odd} = \sum_{a=1}^7 \frac{c_a}{M_V} \mathcal{O}_{VJP}^a + \sum_{a=1}^4 d_a \mathcal{O}_{VVP}^a, \quad (26)$$

where the operators $\mathcal{O}_{VVP}^a$ are given by

$$\mathcal{O}_{VVP}^1 = \epsilon_{\mu\nu\rho\sigma} \text{Tr}(\{V^{\mu\nu}, V^{\rho\alpha}\} \nabla_\alpha u^\sigma), \quad (27)$$

$$\mathcal{O}_{VVP}^2 = i\epsilon_{\mu\nu\rho\sigma} \text{Tr}(\{V^{\mu\nu}, V^{\rho\sigma}\} \chi_-), \quad (28)$$

$$\mathcal{O}_{VVP}^3 = \epsilon_{\mu\nu\rho\sigma} \text{Tr}(\{\nabla_\alpha V^{\mu\nu}, V^{\rho\alpha}\} u^\sigma), \quad (29)$$

$$\mathcal{O}_{VVP}^4 = \epsilon_{\mu\nu\rho\sigma} \text{Tr}(\{\nabla^\sigma V^{\mu\nu}, V^{\rho\alpha}\} u_\alpha). \quad (30)$$

These operators describe the interaction between two vector resonances and one pseudoscalar (VVP). On the other hand, the operators $\mathcal{O}_{VJP}^a$ given by

$$\mathcal{O}_{VJP}^1 = \epsilon_{\mu\nu\rho\sigma} \text{Tr}(\{V^{\mu\nu}, f_+^{\rho\sigma}\} \nabla_\alpha u^\sigma), \quad (31)$$

$$\mathcal{O}_{VJP}^2 = \epsilon_{\mu\nu\rho\sigma} \text{Tr}(\{V^{\mu\alpha}, f_+^{\rho\sigma}\} \nabla_\alpha u^\nu), \quad (32)$$

$$\mathcal{O}_{VJP}^3 = i\epsilon_{\mu\nu\rho\sigma} \text{Tr}(\{V^{\mu\nu}, f_+^{\rho\sigma}\} \chi_-), \quad (33)$$

$$\mathcal{O}_{VJP}^4 = i\epsilon_{\mu\nu\rho\sigma} \text{Tr}(V^{\mu\nu} [f_-^{\rho\sigma}, \chi_+]), \quad (34)$$

$$\mathcal{O}_{VJP}^5 = \epsilon_{\mu\nu\rho\sigma} \text{Tr}(\{\nabla_\alpha V^{\mu\nu}, f_+^{\rho\alpha}\} u^\sigma), \quad (35)$$

$$\mathcal{O}_{VJP}^6 = \epsilon_{\mu\nu\rho\sigma} \text{Tr}(\{\nabla_\alpha V^{\mu\alpha}, f_+^{\rho\sigma}\} u^\nu), \quad (36)$$

$$\mathcal{O}_{VJP}^7 = \epsilon_{\mu\nu\rho\sigma} \text{Tr}(\{\nabla^\sigma V^{\mu\nu}, f_+^{\rho\alpha}\} u_\alpha), \quad (37)$$

describe the interaction between a vector resonance, an external vector source and a pseudoscalar (VJP). Some of the couplings c_a, d_a can be constrained by using the large- N_c limit and by requiring compatibility with the short-distance behavior of QCD [8] and the result reads,

$$c_1 = -4c_3, \quad (38)$$

$$c_2 = c_5 - 4c_3, \quad (39)$$

$$c_6 = c_5 - \frac{N_C M_V}{64\pi^2 \sqrt{2} F_V}, \quad (40)$$

$$d_1 + 8d_2 = -\frac{N_C M_V^2}{64\pi^2 F_V^2} + \frac{F^2}{4F_V^2}, \quad (41)$$

$$d_3 = -\frac{N_C M_V^2}{64\pi^2 F_V^2} + \frac{F^2}{8F_V^2}, \quad (42)$$

$$F_V^2 = 2F^2. \quad (43)$$

It is worth mentioning that in [13], an updated set of high-energy constraints on the LECs is provided,

$$d_1 + 8d_2 = -\frac{N_C M_V^2}{64\pi^2 F_V^2} + \frac{F^2}{8F_V^2}, \quad (44)$$

$$d_3 = -\frac{N_C M_V^2}{64\pi^2 F_V^2}, \quad (45)$$

$$F_V^2 = 3F^2. \quad (46)$$

Therefore, the total Lagrangian

$$\mathcal{L} = \mathcal{L}_{\chi V} + \mathcal{L}_V^{odd}, \quad (47)$$

is sufficient to describe the decay process $\rho^0 \rightarrow \pi^0 \pi^0 \gamma$ at tree level. Interactions that are not relevant to this study, such as those involving a vector resonance, an axial vector resonances and a pseudoscalar, among other are discussed in Refs. [7,14–18].

3. Study of the ρ Decay into $\pi^0 \pi^0 \gamma$ Within $R\chi T$

The decay process $\rho \rightarrow \pi^0 \pi^0 \gamma$ has been extensively studied. The first measurement of $\mathcal{BR}(\rho^0 \rightarrow \pi^0 \pi^0 \gamma)$ was obtained from the radiative process $e^+ e^- \rightarrow \pi^0 \pi^0 \gamma$ below 1 GeV [19]. This process is described within the vector meson dominance framework, including kaon loops and an intermediate scalar state, the σ meson [20–24]. However, a discrepancy with experimental data persist. Further efforts have been made [25] by incorporating additional intermediate states, such as the $f_0(600)$ (now referred to as $f_0(500)$), but the disagreement remains unsolved.

To further investigate possible contributions that could account for this discrepancy, we analyze the decay process $\rho^0 \rightarrow \omega \pi^0 \rightarrow \pi^0 \pi^0 \gamma$ within the framework of Resonance Chiral Theory ($R\chi T$), focusing on the role of the unknown coupling constants.

3.1. Interaction Vertex Among the ρ^0 , ω^0 , and π^0 Particles

The $V_{\rho\omega\pi}$ interaction vertices in terms of the fields are obtained from the VVP operators in Equation (26), considering only the relevant degrees of freedom (ρ , ω , π) and neglecting higher-order terms. Given this convention, the required expressions are written as,

$$\phi(x) = \frac{\pi}{\sqrt{2}} \text{diag}(1, -1, 0), \quad (48)$$

$$u(\phi) = \text{diag}\left(1 + i\frac{\pi}{2F}, 1 - i\frac{\pi}{2F}, 1\right), \quad (49)$$

$$V^{\mu\nu} = \frac{1}{\sqrt{2}} \text{diag}(\omega^{\mu\nu} + \rho^{\mu\nu}, \omega^{\mu\nu} - \rho^{\mu\nu}, 0), \quad (50)$$

$$\nabla_\alpha V^{\mu\nu} = \frac{1}{\sqrt{2}} \text{diag}(\partial_\alpha \omega^{\mu\nu} + \partial_\alpha \rho^{\mu\nu}, \partial_\alpha \omega^{\mu\nu} - \partial_\alpha \rho^{\mu\nu}, 0), \quad (51)$$

$$u_\mu = \frac{\partial_\mu \pi}{F} \text{diag}(-1, 1, 0), \quad (52)$$

$$\nabla_\alpha u_\mu = \frac{\partial_\alpha \partial_\mu \pi}{F} \text{diag}(-1, 1, 0), \quad (53)$$

$$\chi_- = -\frac{2i m_\pi^2}{F} \pi \text{diag}(1, -1, 0). \quad (54)$$

where diag denotes a diagonal matrix, for instance

$$\text{diag}(a_1, a_2, a_3) \equiv \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}. \quad (55)$$

Consequently, the VVP (27)–(30) operators take the following form

$$\tilde{\mathcal{O}}_{\rho\omega\pi}^1 = -\frac{2}{F} \epsilon_{\mu\nu\delta\sigma} (\rho^{\mu\nu} \omega^{\delta\alpha} + \rho^{\delta\alpha} \omega^{\mu\nu}) \partial_\alpha \partial^\sigma \pi, \quad (56)$$

$$\tilde{\mathcal{O}}_{\rho\omega\pi}^2 = \frac{8m_\pi^2}{F} \epsilon_{\mu\nu\delta\alpha} \rho^{\mu\nu} \omega^{\delta\alpha} \pi, \quad (57)$$

$$\tilde{\mathcal{O}}_{\rho\omega\pi}^3 = -\frac{2}{F} \epsilon_{\mu\nu\delta\sigma} (\partial_\alpha \rho^{\mu\nu} \omega^{\delta\alpha} + \partial_\alpha \omega^{\mu\nu} \rho^{\delta\alpha}) \partial^\sigma \pi, \quad (58)$$

$$\tilde{\mathcal{O}}_{\rho\omega\pi}^4 = -\frac{2}{F} \epsilon_{\mu\nu\delta\sigma} (\partial^\sigma \rho^{\mu\nu} \omega^{\delta\alpha} + \partial^\sigma \omega^{\mu\nu} \rho^{\delta\alpha}) \partial_\alpha \pi. \quad (59)$$

From now on, the subscripts $\rho, \omega, \gamma, \pi$ on $\tilde{\mathcal{O}}$ will be used exclusively as identifying labels and shall not be interpreted as tensor indices. The convention for momenta is shown in Figure 1. Consequently, the Feynman rule for the $\rho\omega\pi$ vertex is given by

$$V_{\rho\omega\pi}^{\mu\nu\delta\alpha} = \sum_{j=1}^4 d_j \mathcal{V}_{j\rho\omega\pi}^{\mu\nu\delta\alpha}, \quad (60)$$

where

$$\mathcal{V}_{1\rho\omega\pi}^{\mu\nu\delta\alpha} = \frac{2}{F} (\epsilon^{\mu\nu\delta\lambda} P^\alpha + \epsilon^{\mu\lambda\delta\alpha} P^\nu) P_\lambda, \quad (61)$$

$$\mathcal{V}_{2\rho\omega\pi}^{\mu\nu\delta\alpha} = \frac{8m_\pi^2}{F} \epsilon^{\mu\nu\delta\alpha}, \quad (62)$$

$$\mathcal{V}_{3\rho\omega\pi}^{\mu\nu\delta\alpha} = \frac{2}{F} (\epsilon^{\mu\nu\delta\lambda} Q^\alpha + \epsilon^{\mu\lambda\delta\alpha} q^\nu) P_\lambda, \quad (63)$$

$$\mathcal{V}_{4\rho\omega\pi}^{\mu\nu\delta\alpha} = \frac{2}{F} (\epsilon^{\mu\nu\delta\lambda} Q_\lambda P^\alpha + \epsilon^{\mu\lambda\delta\alpha} q_\lambda P^\nu). \quad (64)$$

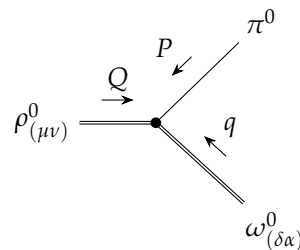


Figure 1. Convention of momentum and Lorentz index in the vertex $\rho^0\omega^0\pi^0$ where arrows indicate the direction of annihilation.

3.1.1. $\rho^0\omega^0\pi^0$ On-Shell Vertex

The decay $\rho^0 \rightarrow \pi^0\pi^0\gamma$ can be described within the $R\chi T$ framework, but its evaluation is hindered by its dependence on a set of LECs that remain unknown. Nonetheless, the number of independent couplings can be significantly reduced by employing on-shell vertices [11]. An on-shell vertex scenario corresponds to applying the Feynman rule for a vertex under the assumption that all external particles are on-shell, even when, in the Feynman amplitude, one of the legs corresponds to a virtual (off-shell) particle.

To define the on-shell vertex, we consider the process $\rho^0(Q) \rightarrow \omega^0(q)\pi^0(P)$, where all the particles are on-shell and their corresponding momentum satisfy

$$Q^2 = m_\rho^2, \quad (65)$$

$$q^2 = m_\omega^2, \quad (66)$$

$$P^2 = m_\pi^2. \quad (67)$$

We use the FeynCalc package [26,27] for tensor manipulations, which enables the reduction of the Feynman amplitude to

$$\mathcal{M}_{\rho \rightarrow \omega\pi} = g_{\rho\omega\pi} \epsilon_{\mu\nu\alpha\beta} \frac{2Q^\nu \epsilon_\rho^\beta}{m_\rho} \frac{2q^\mu \epsilon_\omega^\alpha}{m_\omega}, \quad (68)$$

where ϵ_ρ and ϵ_ω are the polarization vector of the ρ and ω mesons, respectively. From the last expression, the on-shell Feynman rule for the ρ - ω - π vertex is given by

$$V_{\rho\omega\pi_{on-shell}}^{\mu\nu\delta\alpha} = g_{\rho\omega\pi} \epsilon^{\delta\mu\alpha\nu}, \quad (69)$$

where

$$g_{\rho\omega\pi} = \frac{m_\pi^2(d_1 + 8d_2) + d_3(m_\rho^2 + m_\omega^2 - m_\pi^2)}{F}. \quad (70)$$

The tensor indices are assigned according to Figure 1. As shown, the on-shell vertex derived depends on a specific combination of LECs, whose values, according to Section 2, are well known.

3.2. Interaction Vertex Among the ω^0 , γ^0 , and π^0 Particles

In this section, we closely follow the treatment presented in Section 3.1. The interaction vertex involving the ω, π, γ particles is derived from the Lagrangian given in Equation (26), taking into account only the relevant degrees of freedom and neglecting higher-order terms. The expressions thus derived are

$$V^{\mu\nu} = \frac{\omega^{\mu\nu}}{\sqrt{2}} \text{diag}(1, 1, 0), \quad (71)$$

$$\bar{F}^{\delta\alpha} \equiv \partial^\delta A^\alpha - \partial^\alpha A^\delta, \quad (72)$$

$$\mathcal{F}_+^{\delta\alpha} = \frac{2}{3} e \bar{F}^{\delta\alpha} \text{diag}(2, -1, -1), \quad (73)$$

$$\mathcal{F}_-^{\delta\alpha} = 0, \quad (74)$$

$$\nabla_\alpha V^{\mu\nu} = \frac{\partial_\alpha \omega^{\mu\nu}}{\sqrt{2}} \text{diag}(1, 1, 0), \quad (75)$$

$$\chi_+ = 2m_\pi^2 \text{diag}(1, 1, -1). \quad (76)$$

Consequently, the VJP operators in Equations (31)–(37) in terms of fields take the form

$$\tilde{\mathcal{O}}_{\omega\gamma\pi}^1 = -\frac{2\sqrt{2}e}{F} \epsilon_{\mu\nu\delta\sigma} \bar{F}^{\delta\alpha} \omega^{\mu\nu} \partial_\alpha \partial^\sigma \pi, \quad (77)$$

$$\tilde{\mathcal{O}}_{\omega\gamma\pi}^2 = -\frac{2\sqrt{2}e}{F} \epsilon_{\mu\nu\delta\sigma} \bar{F}^{\delta\sigma} \omega^{\mu\alpha} \partial_\alpha \partial^\nu \pi, \quad (78)$$

$$\tilde{\mathcal{O}}_{\omega\gamma\pi}^3 = \frac{4\sqrt{2}e m_\pi^2}{F} \epsilon_{\mu\nu\delta\sigma} \bar{F}^{\delta\sigma} \omega^{\mu\nu} \pi, \quad (79)$$

$$\tilde{\mathcal{O}}_{\omega\gamma\pi}^4 = 0, \quad (80)$$

$$\tilde{\mathcal{O}}_{\omega\gamma\pi}^5 = -\frac{2\sqrt{2}e}{F} \epsilon_{\mu\nu\delta\sigma} \bar{F}^{\delta\alpha} \partial_\alpha \omega^{\mu\nu} \partial^\sigma \pi, \quad (81)$$

$$\tilde{\mathcal{O}}_{\omega\gamma\pi}^6 = -\frac{2\sqrt{2}e}{F} \epsilon_{\mu\nu\delta\sigma} \bar{F}^{\delta\sigma} \partial_\alpha \omega^{\mu\alpha} \partial^\nu \pi, \quad (82)$$

$$\tilde{\mathcal{O}}_{\omega\gamma\pi}^7 = -\frac{2\sqrt{2}e}{F} \epsilon_{\mu\nu\delta\sigma} \bar{F}^{\delta\alpha} \partial^\sigma \omega^{\mu\nu} \partial_\alpha \pi. \quad (83)$$

Therefore, the Feynman rule for the vertex $\omega^0 \pi^0 \gamma$, shown in Figure 2, is written as

$$V_{\omega\gamma\pi}^{\mu\nu l} = \sum_{a=1}^7 \frac{c_j}{M_V} \mathcal{V}_{j\omega\pi\gamma'}^{\mu\nu l} \quad (84)$$

where

$$\mathcal{V}_{1\omega\gamma\pi}^{\mu\nu l} = \frac{2\sqrt{2}e}{F} i (\epsilon^{\mu\nu l\lambda} k_\alpha P^\alpha - \epsilon^{\mu\nu\alpha\lambda} k_\alpha P^l) P_\lambda, \quad (85)$$

$$\mathcal{V}_{2\omega\gamma\pi}^{\mu\nu l} = \frac{4\sqrt{2}e}{F} i \epsilon^{\mu\delta\lambda l} k_\delta P_\lambda P^\nu, \quad (86)$$

$$\mathcal{V}_{3\omega\gamma\pi}^{\mu\nu l} = \frac{8\sqrt{2}e}{F} i m_\pi^2 \epsilon^{\mu\nu l\lambda} k_\lambda, \quad (87)$$

$$\mathcal{V}_{4\omega\gamma\pi}^{\mu\nu l} = 0, \quad (88)$$

$$\mathcal{V}_{5\omega\gamma\pi}^{\mu\nu l} = \frac{2\sqrt{2}e}{F} i (\epsilon^{\mu\nu l\lambda} k_\alpha q^\alpha - \epsilon^{\mu\nu\alpha\lambda} k_\alpha q^l) P_\lambda, \quad (89)$$

$$\mathcal{V}_{6\omega\gamma\pi}^{\mu\nu l} = \frac{4\sqrt{2}e}{F} i \epsilon^{\mu\lambda l\alpha} k_\alpha q^\nu P_\lambda, \quad (90)$$

$$\mathcal{V}_{7\omega\gamma\pi}^{\mu\nu l} = \frac{2\sqrt{2}e}{F} i (\epsilon^{\mu\nu l\lambda} k_\alpha P^\alpha - \epsilon^{\mu\nu\alpha\lambda} k_\alpha P^l) q_\lambda. \quad (91)$$

The Lorentz indices μ, ν and l are assigned according to Figure 2.

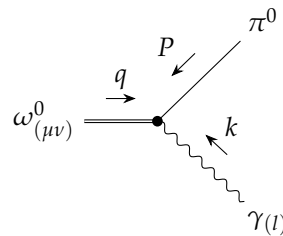


Figure 2. Convention of momentum and Lorentz index for the ω^0 , γ and π^0 vertex, where arrows indicate the direction of annihilation.

3.2.1. $\omega^0\gamma\pi^0$ On-Shell Vertex

Considering the $\omega^0 \rightarrow \pi^0\gamma$ process, the Feynman amplitude, provided in Ref. [8], is given by

$$\mathcal{M}_{\omega^0 \rightarrow \pi^0\gamma} = e g_{\omega\pi} \epsilon_{\alpha\beta\rho\sigma} k^\sigma \frac{2\epsilon_\omega^\alpha q^\rho}{m_\omega} \epsilon_\gamma^\beta, \quad (92)$$

where $g_{\omega\pi}$ can be reduced, after using Equations (38)–(40), as follows,

$$g_{\omega\pi} = \frac{N_c}{32\pi^2 F_V F} m_\omega^2. \quad (93)$$

Therefore, the on-shell vertex is given by

$$V_{\omega\gamma\pi \text{ on shell}}^{\mu\nu l} = e g_{\omega\pi} \epsilon_{\mu l\nu\sigma} k^\sigma, \quad (94)$$

which corresponds to the expression reported in [11].

3.3. ρ^0 Decay into $\pi^0\pi^0\gamma$

In this section, we analyze the process $\rho^0 \rightarrow \pi^0\pi^0\gamma$ using the interaction Lagrangian given in Equation (47). At lowest order, the Feynman amplitude requires a VJP and a VVP vertex, with the ω treated as a virtual particle, as shown in Figure 3. It is found that the $\mathcal{BR}(\rho^0 \rightarrow \pi^0\pi^0\gamma)$ depends on the specific combination of LECs $c_{57} \equiv c_5 + c_7$ and d_4 . In order to reduce the number of unknown couplings, one can use the $\rho\omega\pi$ on-shell vertex, the $\omega\pi\gamma$ on-shell vertex or considering both vertices on shell.

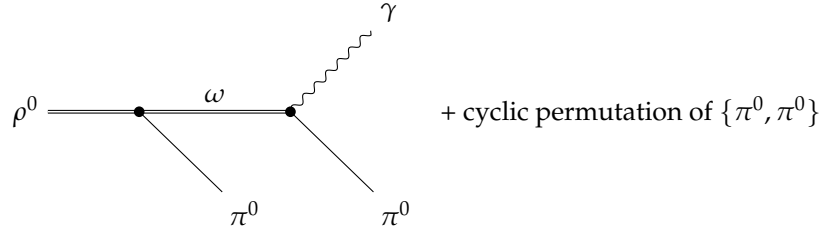


Figure 3. Lowest-order Feynman diagram for the process $\rho^0(Q) \rightarrow \pi^0(P_1)\pi^0(P_2)\gamma(k)$, proceeding through a virtual ω resonance.

Scenario 1: The Decay $\rho^0 \rightarrow \pi^0\pi^0\gamma$ via Purely on-Shell Vertices

First we calculate the branching ratio using both vertices on-shell, which are given in Equations (69) and (94). The corresponding Feynman amplitude is given by

$$\mathcal{M} = \mathcal{M}|_{q=P_1+k} + \mathcal{M}|_{q=P_2+k} \quad (95)$$

where

$$\mathcal{M}|_q = \frac{8e g_\rho g_{\omega\pi\pi}}{m_\rho(M_V^2 - q^2) M_V^2} [(\epsilon_\gamma \cdot \epsilon_\rho) f_1 + (\epsilon_\rho \cdot k)(\epsilon_\gamma \cdot f_2) + (\epsilon_\rho \cdot q)(\epsilon_\gamma \cdot f_3)], \quad (96)$$

and

$$f_1 = (k \cdot q)(q \cdot Q) - M_V^2(k \cdot Q), \quad (97)$$

$$f_2^\alpha = M_V^2 Q^\alpha - q^\alpha(q \cdot Q), \quad (98)$$

$$f_3^\alpha = q^\alpha(k \cdot Q) - Q^\alpha(k \cdot q). \quad (99)$$

The scalar products, expressed in terms of the Mandelstam variables defined as $m_{31} \equiv (k + P_1)^2$, $m_{23} \equiv (k + P_2)^2$, and $m_{12} \equiv (P_1 + P_2)^2$, are written as follows

$$Q \cdot P_1 = \frac{m_\rho^2 + m_\pi^2 - m_{23}}{2}, \quad (100)$$

$$Q \cdot k = \frac{m_\rho^2 - m_{12}}{2}, \quad (101)$$

$$Q \cdot P_2 = \frac{m_\rho^2 + m_\pi^2 - m_{31}}{2}, \quad (102)$$

$$k \cdot P_2 = \frac{m_{23} - m_\pi^2}{2}, \quad (103)$$

$$P_1 \cdot P_2 = \frac{m_{12} - 2m_\pi^2}{2}. \quad (104)$$

The kinematics of a three-body decay can be described by only two independent variables. In this work, we chose m_{12} and m_{23} ; therefore, the branching ratio is calculated as

$$\mathcal{BR}(\rho^0 \rightarrow \pi^0\pi^0\gamma) = \frac{1}{\Gamma_\rho} \frac{1}{(2\pi)^3} \frac{1}{32m_\rho^3} \int_{m_\pi^2}^{(m_\rho - m_\pi)^2} \int_{m_{23-}}^{m_{23+}} X dm_{12} dm_{23}, \quad (105)$$

where

$$X = \frac{1}{3} \sum_{r=1}^3 \sum_{s=1}^2 |\mathcal{M}|^2, \quad (106)$$

and Γ_ρ denotes the total decay width. The minimum and maximum values of m_{12} and m_{23} define the boundaries of the region known as the Dalitz plot (Figure 4), which are determined by the following relations

$$m_{23\pm} = (E_2^* + E_3^*)^2 - \left(E_2^* \mp \sqrt{E_3^{*2} - m_\pi^2} \right)^2, \quad (107)$$

$$E_2^* = \frac{m_{12} - m_\pi^2}{2\sqrt{m_{12}}}, \quad (108)$$

$$E_3^* = \frac{m_\rho^2 - m_{12} - m_\pi^2}{2\sqrt{m_{12}}}. \quad (109)$$

The values of the meson masses and decay widths, according to the Particle Data Group [3], are $m_\omega = 782.66(13)$ MeV, $m_\rho = 775.26(23)$ MeV, $m_\pi = 134.9768(5)$ MeV, $\Gamma_\rho = 147.4(8)$ MeV and $\Gamma_\omega = 8.68(13)$ MeV, where the numbers in parentheses indicate the uncertainties. Additionally, we take from Ref. [8] the values $M_V = m_\rho$, $F_V = 92.4$ MeV, and $N_c = 3$.

The prediction with both vertices on-shell, using the relations (41)–(43), gives the value

$$\mathcal{BR}(\rho^0 \rightarrow \pi^0 \pi^0 \gamma)_{on\ shell} = 2.111(10) \times 10^{-4}, \quad (110)$$

while using Equations (44)–(46) yields

$$\mathcal{BR}(\rho^0 \rightarrow \pi^0 \pi^0 \gamma)_{on\ shell} = 1.606(6) \times 10^{-4}. \quad (111)$$

The propagated error reported in Equations (111) and (110) has been estimated as follows. The inputs parameters reported by the PDG have an uncertainty σ ; thus the evaluation is performed using the central value of all inputs, as well as the central value plus 1σ and minus 1σ . Finally, the uncertainty is collected as the sum of the deviations from the central value.

Comparing our prediction with the experimental value $\mathcal{BR}(\rho^0 \rightarrow \pi^0 \pi^0 \gamma) = 4.5(8) \times 10^{-5}$ clearly shows that taking both vertices on-shell is not consistent.

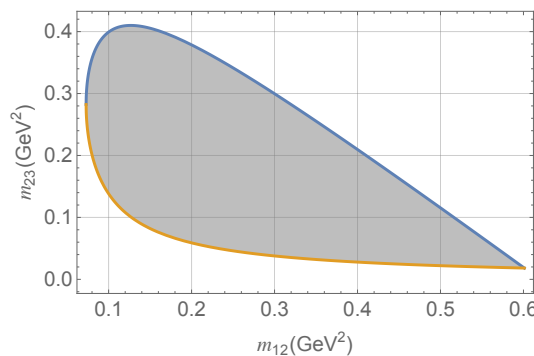


Figure 4. Dalitz plot for the decay process $\rho^0 \rightarrow \pi^0 \pi^0 \gamma$.

Scenario 2: The decay $\rho^0 \rightarrow \pi^0 \pi^0 \gamma$ via the on-shell $V_{\rho\omega\pi}$ vertex

In this section we treat the vertex $V_{\omega\gamma\pi}$ (85)–(91) as off-shell, whereas the $V_{\rho\omega\pi}$ vertex remains on-shell (69). The corresponding Feynman amplitude for the $\rho^0 \rightarrow \pi^0 \pi^0 \gamma$ decay is then given by

$$\mathcal{M} = \frac{16 i \sqrt{2} e g_{\rho\omega\pi}}{F m_\rho M_V^2} \left[-c_{57} H_1 + \frac{c_{56} M_V^2 H_2 - c_{13} m_{\pi^2} H_3 + c_{125} H_4}{M_V^2 - (k + P_2)^2} \right] + \{P_1 \longleftrightarrow P_2\}, \quad (112)$$

where the functions H_1, H_2, H_3 , and H_4 are given in Appendix A. For simplicity, we defined the following notation

$$c_1 + 4c_3 \equiv c_{13}, \quad (113)$$

$$c_1 - c_2 + c_5 \equiv c_{125}, \quad (114)$$

$$c_5 - c_6 \equiv c_{56}, \quad (115)$$

$$c_5 + c_7 \equiv c_{57}. \quad (116)$$

When the relations (38)–(40) are applied, the Equation (112) depend exclusively on c_{57} . Assuming that the theoretical prediction matches the experimental branching ratio, one obtains a quadratic expression in c_{57} . Using the constraints (43), we find the roots $c_{57} = \{0.032(3), -0.025(3)\}$. Similarly, applying the constraints (46) yields the solutions $c_{57} = \{0.030(3), -0.025(3)\}$.

Scenario 3: The Decay $\rho^0 \rightarrow \pi^0 \pi^0 \gamma$ via the on-Shell $V_{\omega\gamma\pi}$ vertex

Here we take the $V_{\omega\gamma\pi}$ vertex (94) to be on-shell, whereas the $V_{\rho\omega\pi}$ vertex (61)–(64) is kept off-shell. Consequently, the Feynman amplitude simplifies to

$$\mathcal{M} = \frac{8 e g_{\omega\pi}}{F m_\rho M_V^2} \left[d_4 h_1 + \frac{-d_{12} m_\pi^2 h_2 + d_3 h_3}{M_V^2 - (k + P_2)^2} \right] + \{P_1 \longleftrightarrow P_2\}, \quad (117)$$

where, for simplicity it was defined the relation

$$d_1 + 8d_2 \equiv d_{12}, \quad (118)$$

and the functions h_1, h_2 and h_3 are defined in the Appendix A.

When the relations (41)–(42) or, alternatively, (44)–(45) are applied, Equation (117) simplifies and the BR becomes a quadratic function of d_4 . Analogous to the analysis performed in the previous scenario, the solutions are $d_4 = \{0.901(54), -0.221(54)\}$ using the constraints (41)–(43) and $d_4 = \{1.059(65), -0.327(65)\}$ using the constraints (44)–(46). A similar analysis was carried out in Ref. [11] for the decay $\omega \rightarrow \pi^0 \pi^0 \gamma$, and our central values are in reasonable agreement with their results, $d_4 = \{0.82(5), -0.12(5)\}$.

Scenario 4: The Decay $\rho^0 \rightarrow \pi^0 \pi^0 \gamma$ with Off-Shell Vertices

Finally, we consider the scenario in which both vertices are off-shell. In this case, the corresponding Feynman amplitude is given by

$$\mathcal{M} = \frac{16 i e \sqrt{2}}{F^2 m_\rho M_V^3} \left[\frac{c_{56} M_V^2 l_1 + c_{13} m_\pi^2 l_2 - c_{57} l_3 - c_{125} l_4}{M_V^2 - (k + P_2)^2} \right] + P_1 \longleftrightarrow P_2. \quad (119)$$

By applying the large- N_c constraints discussed in Section 2, Equation (119) is found to depend only on c_{57} and d_4 . The dependence on d_4 is contained within the functions l_1, l_2, l_3 , and l_4 , which are defined in the Appendix A.

Prior to deriving the solution for c_{57} and d_4 in this scenario, we analyze the impact of using the value of c_{57} obtained in scenarios 2 together with the value of d_4 from scenario 3. The results of this combination are presented in Table 1.

Table 1. The prediction for $\mathcal{BR} \equiv \mathcal{BR}(\rho^0 \rightarrow \pi^0 \pi^0 \gamma)$ computed using a combination of the values of c_{57} and d_4 from scenarios 2 and 3.

Constraints (38)–(43)			Constraints (44)–(46)		
c_{57}	d_4	$\mathcal{BR} \times 10^{-5}$	c_{57}	d_4	$\mathcal{BR} \times 10^{-5}$
0.032(3)	0.901(54)	3.08(63)	0.030(3)	1.059(65)	3.65(88)
-0.025(3)	0.901(54)	1.17(35)	-0.025(3)	1.059(65)	1.74(53)
0.032(3)	-0.221(54)	0.63(19)	0.030(3)	-0.327(65)	0.93(30)
-0.025(3)	-0.221(54)	0.78(20)	-0.025(3)	-0.327(65)	1.09(30)

The results presented in Table 1 indicate that the combination of on-shell and off-shell vertices provides a more accurate estimation of the branching ratio compared to the scenario that employs solely on-shell vertices. For instance, the parameter pair $(c_{57}, d_4) = (0.030(3), 1.059(65))$ yields a $\mathcal{BR}(\rho^0 \rightarrow \pi^0 \pi^0 \gamma) = 3.65(88) \times 10^{-5}$, representing a notable improvement with respect to the result obtained in scenario 1. However, this value remains below the experimental measurement, $\mathcal{BR}(\rho^0 \rightarrow \pi^0 \pi^0 \gamma) = 4.5(8) \times 10^{-5}$, corresponding to a relative deviation of approximately 19% with respect to the central value. However, the procedure used to construct the data in Table 1 is not fully consistent. Consequently, the most effective approach appears to be the use of purely off-shell vertices. Within this framework, a numerical contour solution in the (c_{57}, d_4) parameter space can be derived, as shown in Figure 5, which identifies the region consistent with the experimental branching ratio.

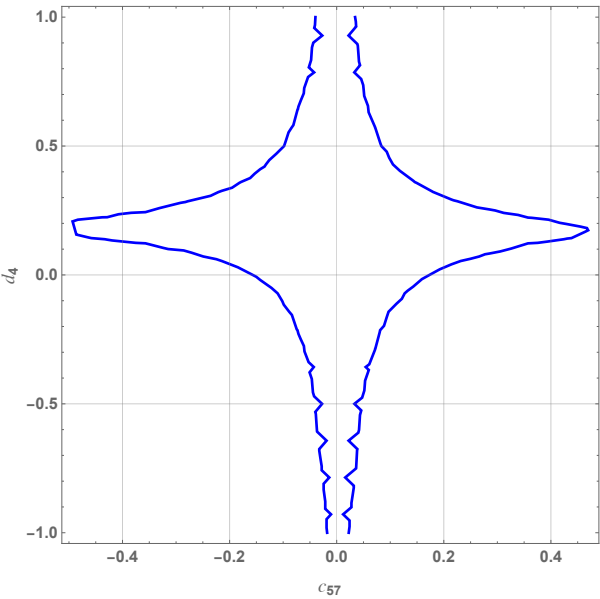


Figure 5. Geometric locus in the (c_{57}, d_4) parameter space that matches the experimental $\mathcal{BR}(\rho \rightarrow \pi^0 \pi^0 \gamma) = 4.5 \times 10^{-5}$, under the constraints (38)–(43).

Selected values from this solution are listed in Table 2.

Table 2. In the first column, the value of d_4 is selected, while the two solutions shown in the second and third columns are obtained by applying the constraints (38)–(43), assuming that the theoretical prediction matches the experimental BR.

d_4	$c_{57}(sol1)$	$c_{57}(sol2)$
-0.98	0.03	-0.03
0.90	0.03	-0.03
-0.80	0.03	-0.03
-0.70	0.04	-0.03
-0.60	0.04	-0.04
-0.50	0.05	-0.04
-0.40	0.06	-0.05
-0.30	0.07	-0.06
-0.20	0.09	-0.08
-0.10	0.12	-0.10
0.02	0.20	-0.18
0.10	0.33	-0.31
0.20	0.44	-0.53
0.30	0.21	-0.25
0.40	0.12	-0.15
0.50	0.09	-0.10
0.60	0.07	-0.08
0.70	0.05	-0.07
0.80	0.05	-0.05
0.90	0.04	-0.05
0.98	0.04	-0.04

It is worth noting that the pairs (c_{57}, d_4) in Figure 5 exhibit three predominant features in the differential branching ratio $dB/d\sqrt{m_{12}} \equiv d\mathcal{BR}(\rho \rightarrow \pi^0 \pi^0 \gamma)/d\sqrt{m_{12}}$, as illustrated in Figure 6. The blue curve appears to qualitatively agree with the results reported in Ref. [20], where the interaction is mediated by pion loops within a framework that combines the Vector Meson Dominance model and the linear sigma model.

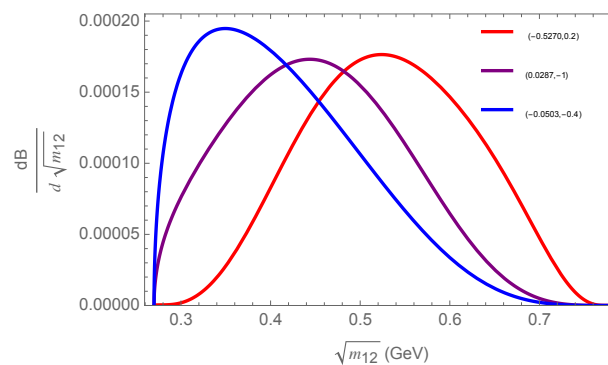


Figure 6. In this plot we show the $dB/d\sqrt{m_{12}}$ using three specific pairs (c_{57}, d_4) from Figure 5.

4. Conclusions

In this work, we have studied the decay process $\rho^0 \rightarrow \pi^0 \pi^0 \gamma$ within the framework of Chiral Perturbation Theory with resonances. Our analysis shows that the branching ratio (BR) depends exclusively on the low-energy couplings c_{57} and d_4 . To address this, we analyzed all possible scenarios arising from different combinations of on-shell and off-shell interaction vertices. Our results indicate that the purely on-shell scenario yields a value significantly below the experimental branching ratio, suggesting that this scenario does not provide a reliable description of the process. On the other hand, scenarios in which one of the vertices is taken on-shell reduce the number of unknown coupling, thereby enabling the determination of the remaining coupling from the roots of a quadratic equation.

It is worth noting that this approach does not allow us to discriminate between the two roots. Pursuing this line of reasoning, we use both couplings simultaneously and find that, even in the best scenario, the theoretical prediction differs by 19 % from the experimental value of 4.5×10^{-5} (Table 1). This discrepancy suggests that the vertices must be off-shell. Our results are presented as a region of possible values for c_{57} and d_4 , consistent with the experimental BR, as shown in Figure 5. Considering the result $d_4 = \{0.82(5), -0.12(5)\}$ reported in Ref. [11], obtained through the decay $\omega \rightarrow \pi^0 \pi^0 \gamma$, we get the values $c_{57} = \{0.04, -0.05\}$ for $d_4 = 0.82$ and $c_{57} = \{0.11, -0.10\}$ for $d_4 = -0.12$. However, these values do not fall within the estimate $|c_i| \leq 0.015$ of Ref. [9]. It must be emphasized that the value of d_4 obtained in Ref. [11] assumes the $\rho\omega\pi$ vertex to be on-shell, whereas the estimate for c_i in Ref. [9] is based on an informed estimate inferred from the behaviour of the remaining couplings. Consequently, these estimates must therefore be interpreted with caution. In addition, we provide the differential branching ratio (dBR), which depends on the precise values of the LECs. In particular, Figure 6 shows the dBR for three representative sets of (c_{57}, d_4) , where the shape of the distribution can be clearly observed. This information can be used to further constrain the parameter space (work in progress) by fitting to the experimental dBR data.

Supplementary Materials: The following supporting information can be downloaded at the website of this paper posted on [Preprints.org](https://www.preprints.org).

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Abbreviations

The following abbreviations are used in this manuscript:

BR	Branching Ratio.
dBR	Differential Branching Ratio.
LEC	Low-energy Coupling Constant.
SM	Standard Model.
EFT	Effective Field Theory.
ChPT	Chiral Perturbation Theory.
QCD	Quantum Chromodynamics.
$R_\chi T$	Resonance Chiral Theory.
VVP	Two vector sources and a pseudoscalar source.
VJP	A vector source, an external source and pseudoscalar source.

Appendix A. Detailed and Supplementary Expressions

This appendix contains the lengthy analytical expression obtained from computing the Feynman amplitudes for the different scenarios involving on-shell and off-shell vertices.

The explicit expressions for the functions H_1, H_2, H_3 and H_4 are:

$$\begin{aligned}
H_1 = & -(k \cdot P_2)(\varepsilon_\gamma \cdot \varepsilon_\rho)(Q \cdot (k + P_2)) \\
& + (k \cdot \varepsilon_\rho)[(k \cdot P_2)(\varepsilon_\gamma \cdot Q) + (Q \cdot P_2)(\varepsilon_\gamma \cdot P_2)] \\
& + (\varepsilon_\rho \cdot P_2)[(k \cdot P_2)(\varepsilon_\gamma \cdot Q) - (k \cdot Q)(\varepsilon_\gamma \cdot P_2)],
\end{aligned} \tag{A1}$$

$$\begin{aligned}
H_2 = & -(\varepsilon_\gamma \cdot \varepsilon_\rho) \left[(k \cdot Q) \left((k \cdot P_2) + m_\pi^2 \right) - (k \cdot P_2)(Q \cdot P_2) \right] \\
& + (k \cdot \varepsilon_\rho) \left[\left((k \cdot P_2) + m_\pi^2 \right) (\varepsilon_\gamma \cdot Q) - (Q \cdot P_2)(\varepsilon_\gamma \cdot P_2) \right] \\
& + (\varepsilon_\rho \cdot P_2) [(k \cdot Q)(\varepsilon_\gamma \cdot P_2) - (k \cdot P_2)(\varepsilon_\gamma \cdot Q)],
\end{aligned} \tag{A2}$$

$$\begin{aligned}
H_3 = & (\varepsilon_\gamma \cdot \varepsilon_\rho) \left[(k \cdot Q) \left(M_V^2 - (k \cdot P_2) \right) - (k \cdot P_2)(Q \cdot P_2) \right] \\
& + (k \cdot \varepsilon_\rho) \left[\left((k \cdot P_2) - M_V^2 \right) (\varepsilon_\gamma \cdot Q) + (Q \cdot P_2)(\varepsilon_\gamma \cdot P_2) \right] \\
& + (\varepsilon_\rho \cdot P_2) [(k \cdot P_2)(\varepsilon_\gamma \cdot Q) - (k \cdot Q)(\varepsilon_\gamma \cdot P_2)],
\end{aligned} \tag{A3}$$

$$\begin{aligned}
H_4 = & (\varepsilon_\gamma \cdot \varepsilon_\rho) \left[(k \cdot Q) \left((k \cdot P_2)^2 + m_\pi^2 M_V^2 \right) + (k \cdot P_2)(Q \cdot P_2) \left((k \cdot P_2) - M_V^2 \right) \right] \\
& + (k \cdot \varepsilon_\rho) \left[(Q \cdot P_2) \left(M_V^2 - (k \cdot P_2) \right) (\varepsilon_\gamma \cdot P_2) - (\varepsilon_\gamma \cdot Q) \left((k \cdot P_2)^2 + m_\pi^2 M_V^2 \right) \right] \\
& + (M_V^2 - k \cdot P_2)(\varepsilon_\rho \cdot P_2) [(k \cdot P_2)(\varepsilon_\gamma \cdot Q) - (k \cdot Q)(\varepsilon_\gamma \cdot P_2)].
\end{aligned} \tag{A4}$$

Similarly, the explicit expressions for the functions h_1 , h_2 , and h_3 are:

$$\begin{aligned}
h_1 = & (\varepsilon_\gamma \cdot P_2) [(k \cdot Q)(\varepsilon_\rho \cdot P_1) - (Q \cdot P_1)(k \cdot \varepsilon_\rho)] \\
& + (k \cdot P_2) [(Q \cdot P_1)(\varepsilon_\gamma \cdot \varepsilon_\rho) - (Q \cdot \varepsilon_\gamma)(\varepsilon_\rho \cdot P_1)],
\end{aligned} \tag{A5}$$

$$\begin{aligned}
h_2 = & (\varepsilon_\gamma \cdot \varepsilon_\rho) \left[(k \cdot Q) \left((k \cdot P_2) - M_V^2 \right) + (k \cdot P_2)(Q \cdot P_2) \right] \\
& + (k \cdot \varepsilon_\rho) \left[\left(M_V^2 - (k \cdot P_2) \right) (\varepsilon_\gamma \cdot Q) - (Q \cdot P_2)(\varepsilon_\gamma \cdot P_2) \right] \\
& + (\varepsilon_\rho \cdot P_2) [(k \cdot Q)(\varepsilon_\gamma \cdot P_2) - (k \cdot P_2)(\varepsilon_\gamma \cdot Q)],
\end{aligned} \tag{A6}$$

$$\begin{aligned}
h_3 = & -(P_1 \cdot P_2) \left[(k \cdot P_2) \left((\epsilon_\gamma \cdot \epsilon_\rho) (Q \cdot (k + P_2) + m_\rho^2) - (Q \cdot \epsilon_\gamma) (\epsilon_\rho \cdot (P_2 + Q)) \right) \right. \\
& \left. + (k \cdot Q) (\epsilon_\gamma \cdot P_2) (\epsilon_\rho \cdot (P_2 + Q)) \right] \\
& + (k \cdot P_1) \left[(M_V^2 - k \cdot P_2) (\epsilon_\gamma \cdot \epsilon_\rho) (Q \cdot (k + P_2) + m_\rho^2) \right. \\
& - (M_V^2 - k \cdot P_2) (Q \cdot \epsilon_\gamma) (\epsilon_\rho \cdot (P_2 + Q)) \\
& \left. + (\epsilon_\gamma \cdot P_2) \left((Q \cdot P_2) (\epsilon_\rho \cdot Q) - (k \cdot Q + m_\rho^2) (\epsilon_\rho \cdot P_2) \right) \right] \\
& + M_V^2 (k \cdot Q) (\epsilon_\gamma \cdot P_1) (\epsilon_\rho \cdot (P_2 + Q)) \\
& + (k \cdot P_2) (Q \cdot P_1) (\epsilon_\gamma \cdot \epsilon_\rho) (Q \cdot (k + P_2)) \\
& - (k \cdot P_2) (Q \cdot \epsilon_\rho) (\epsilon_\gamma \cdot P_1) (Q \cdot (k + P_2)) \\
& + m_\rho^2 (k \cdot P_2) (\epsilon_\gamma \cdot P_1) (\epsilon_\rho \cdot P_2) \\
& + (k \cdot \epsilon_\rho) \left[m_\rho^2 (k \cdot P_2) (\epsilon_\gamma \cdot P_1) + (P_1 \cdot P_2) (\epsilon_\gamma \cdot P_2) (Q \cdot P_2 + m_\rho^2) \right. \\
& - m_\rho^2 M_V^2 (\epsilon_\gamma \cdot P_1) - M_V^2 (Q \cdot P_2) (\epsilon_\gamma \cdot P_1) \\
& + (k \cdot P_1) \left((k \cdot P_2 - M_V^2) (Q \cdot \epsilon_\gamma) + (Q \cdot P_2) (\epsilon_\gamma \cdot P_2) \right) \\
& + (Q \cdot \epsilon_\gamma) \left((Q \cdot P_1) (M_V^2 - k \cdot P_2) + (P_1 \cdot P_2) (k \cdot P_2) \right) \\
& \left. - (Q \cdot P_1) (Q \cdot P_2) (\epsilon_\gamma \cdot P_2) \right] \\
& + M_V^2 (-k \cdot Q) (Q \cdot P_1) (\epsilon_\gamma \cdot \epsilon_\rho) \\
& - (k \cdot P_2) (Q \cdot P_1) (Q \cdot \epsilon_\gamma) (\epsilon_\rho \cdot P_2) \\
& + (k \cdot Q) (Q \cdot P_1) (\epsilon_\gamma \cdot P_2) (\epsilon_\rho \cdot P_2).
\end{aligned} \tag{A7}$$

Finally, the explicit expressions for the functions l_1 , l_2 , l_3 , and l_4 are:

$$\begin{aligned}
l_1 = & \left((\epsilon_\gamma \cdot P_1) [(k \cdot P_2) (\epsilon_\rho \cdot P_2) - (k \cdot P_2 + m_\pi^2) (k \cdot \epsilon_\rho)] \right. \\
& + (k \cdot P_1) [(k \cdot P_2 + m_\pi^2) (\epsilon_\gamma \cdot \epsilon_\rho) - (\epsilon_\gamma \cdot P_2) (\epsilon_\rho \cdot P_2)] \\
& \left. + (P_1 \cdot P_2) [(\epsilon_\gamma \cdot P_2) (k \cdot \epsilon_\rho) - (k \cdot P_2) (\epsilon_\gamma \cdot \epsilon_\rho)] \right) \\
& \times \left(2d_3 (k \cdot P_2) + d_{12} m_\pi^2 + d_3 m_\rho^2 \right),
\end{aligned} \tag{A8}$$

$$\begin{aligned}
l_2 = & -2(k \cdot P_2)^2 \left[(\epsilon_\gamma \cdot P_1) \left(d_3(\epsilon_\rho \cdot (k + P_2)) + (d_3 - d_4)(\epsilon_\rho \cdot P_1) \right) \right. \\
& + (\epsilon_\gamma \cdot \epsilon_\rho) \left(d_4(P_1 \cdot P_2) + m_\pi^2(d_{12} - d_3 + d_4) \right) \Big] \\
& + (k \cdot P_2) \left[(k \cdot \epsilon_\rho) \left(2(\epsilon_\gamma \cdot P_2) \left(d_4(P_1 \cdot P_2) + m_\pi^2(d_{12} - d_3 + d_4) \right) \right. \right. \\
& + \left(m_\pi^2(d_{12} - 2d_3) + d_3 m_\rho^2 \right) (\epsilon_\gamma \cdot P_1) \Big) \\
& + (\epsilon_\gamma \cdot P_1) \left((\epsilon_\rho \cdot P_2) \left(d_{12} m_\pi^2 + d_3(-2m_\pi^2 + m_\rho^2 + 2M_V^2) \right) \right. \\
& + (d_3 - d_4)(M_V - m_\pi)(M_V + m_\pi)(\epsilon_\rho \cdot P_1) \Big) \\
& + (\epsilon_\gamma \cdot \epsilon_\rho) \left(m_\pi^2(M_V - m_\pi)(M_V + m_\pi)(d_{12} - d_3 + d_4) \right. \\
& \left. \left. - (P_1 \cdot P_2) \left(d_{12} m_\pi^2 + (d_3 - d_4)(M_V - m_\pi)(M_V + m_\pi) + d_3 m_\rho^2 \right) \right) \right] \\
& + (k \cdot P_1) \left[(\epsilon_\gamma \cdot \epsilon_\rho) \left((k \cdot P_2) \left(-2d_4(k \cdot P_2) - m_\pi^2(d_{12} + d_4) \right. \right. \right. \\
& + \left. d_3(m_\pi^2 - m_\rho^2 + M_V^2) + d_4 M_V^2 \right) + M_V^2(d_{12} m_\pi^2 + d_3 m_\rho^2) \Big) \\
& + (\epsilon_\gamma \cdot P_2) \left(-(\epsilon_\rho \cdot P_2) \left(-2d_3(k \cdot P_2) + d_{12} m_\pi^2 \right. \right. \\
& + \left. d_3(-2m_\pi^2 + m_\rho^2 + 2M_V^2) \right) \\
& \left. \left. - \left((-2(k \cdot P_2) - m_\pi^2 + M_V^2) \left((d_3 + d_4)(k \cdot \epsilon_\rho) + (d_3 - d_4)(\epsilon_\rho \cdot P_1) \right) \right) \right) \right] \\
& + (k \cdot \epsilon_\rho) \left[(\epsilon_\gamma \cdot P_2) \left((P_1 \cdot P_2) \left(d_{12} m_\pi^2 + (d_3 - d_4)(M_V - m_\pi)(M_V + m_\pi) + d_3 m_\rho^2 \right) \right. \right. \\
& + \left. m_\pi^2(m_\pi - M_V)(m_\pi + M_V)(d_{12} - d_3 + d_4) \right) \\
& \left. \left. - M_V^2(d_{12} m_\pi^2 + d_3 m_\rho^2) (\epsilon_\gamma \cdot P_1) \right] , \tag{A9}
\end{aligned}$$

$$\begin{aligned}
l_3 = & -4(k \cdot P_2)^3 \left[(\epsilon_\gamma \cdot P_1) \left((d_3 - d_4)(\epsilon_\rho \cdot P_1) + d_3(\epsilon_\rho \cdot (k + P_2)) \right) \right. \\
& \left. + (\epsilon_\gamma \cdot \epsilon_\rho) \left((d_{12} - d_3 + d_4)m_\pi^2 + d_4(P_1 \cdot (k + P_2)) \right) \right] \\
& + 2 \left[(d_3 - d_4) \left((M_V^2 - 2m_\pi^2)(\epsilon_\gamma \cdot P_1) + 2(k \cdot P_1)(\epsilon_\gamma \cdot P_2) \right) (\epsilon_\rho \cdot P_1) \right. \\
& + \left((d_{12}m_\pi^2 + d_3(M_V^2 - 3m_\pi^2 + m_\rho^2))(\epsilon_\gamma \cdot P_1) + 2d_3(k \cdot P_1)(\epsilon_\gamma \cdot P_2) \right) (\epsilon_\rho \cdot P_2) \\
& + (k \cdot \epsilon_\rho) \left((d_{12}m_\pi^2 + d_3(M_V^2 - 3m_\pi^2 + m_\rho^2))(\epsilon_\gamma \cdot P_1) \right. \\
& \left. + 2(\epsilon_\gamma \cdot P_2) \left((d_{12} - d_3 + d_4)m_\pi^2 + (d_3 + d_4)(k \cdot P_1) + d_4(P_1 \cdot P_2) \right) \right) \\
& + (\epsilon_\gamma \cdot \epsilon_\rho) \left((d_{12} - d_3 + d_4)(M_V^2 - 2m_\pi^2)m_\pi^2 \right. \\
& \left. + \left(-d_{12}m_\pi^2 + d_4(M_V^2 - 2m_\pi^2) + d_3(m_\pi - m_\rho)(m_\pi + m_\rho) \right) (P_1 \cdot (k + P_2)) \right) \left. \right] (k \cdot P_2)^2 \\
& + \left[(d_3 - d_4) \left((M_V - m_\pi)m_\pi^2(M_V + m_\pi)(\epsilon_\gamma \cdot P_1) \right. \right. \\
& - 2(M_V^2 - 2m_\pi^2)(k \cdot P_1)(\epsilon_\gamma \cdot P_2) \left. \right) (\epsilon_\rho \cdot P_1) \\
& - \left((M_V - m_\pi)(M_V + m_\pi) \left((d_{12} - 2d_3)m_\pi^2 + d_3m_\rho^2 \right) (\epsilon_\gamma \cdot P_1) \right. \\
& + 2(d_{12}m_\pi^2 + d_3(M_V^2 - 3m_\pi^2 + m_\rho^2))(k \cdot P_1)(\epsilon_\gamma \cdot P_2) \left. \right) (\epsilon_\rho \cdot P_2) \\
& + (k \cdot \epsilon_\rho) \left(2(\epsilon_\gamma \cdot P_2) \left((d_{12} - d_3 + d_4)(2m_\pi^2 - M_V^2)m_\pi^2 \right. \right. \\
& - (d_3 + d_4)(M_V^2 - 2m_\pi^2)(k \cdot P_1) \\
& + \left. \left. \left((d_{12} - d_3)m_\pi^2 + d_3m_\rho^2 - d_4(M_V^2 - 2m_\pi^2) \right) (P_1 \cdot P_2) \right) \right. \\
& - (M_V - m_\pi)(M_V + m_\pi) \left((d_{12} - 2d_3)m_\pi^2 + d_3m_\rho^2 \right) (\epsilon_\gamma \cdot P_1) \left. \right) \\
& + (M_V - m_\pi)(M_V + m_\pi)(\epsilon_\gamma \cdot \epsilon_\rho) \left((d_{12} - d_3 + d_4)m_\pi^4 \right. \\
& + \left. \left. \left((d_{12} - d_3 + d_4)m_\pi^2 + d_3m_\rho^2 \right) (P_1 \cdot (k + P_2)) \right) \right] (k \cdot P_2) \\
& - (M_V - m_\pi)(M_V + m_\pi)(\epsilon_\gamma \cdot P_2) \\
& \times \left[(k \cdot P_1) \left((d_3 - d_4)m_\pi^2(\epsilon_\rho \cdot P_1) \right. \right. \\
& - \left. \left. \left((d_{12} - 2d_3)m_\pi^2 + d_3m_\rho^2 \right) (\epsilon_\rho \cdot P_2) \right) \right. \\
& + (k \cdot \epsilon_\rho) \left((d_{12} - d_3 + d_4)m_\pi^4 + (d_3 + d_4)(k \cdot P_1)m_\pi^2 \right. \\
& \left. \left. + \left((d_{12} - d_3 + d_4)m_\pi^2 + d_3m_\rho^2 \right) (P_1 \cdot P_2) \right) \right], \tag{A10}
\end{aligned}$$

$$\begin{aligned}
l_4 = & 2(k \cdot P_2)^3 \left[(\epsilon_\gamma \cdot P_1) \left((d_3 - d_4)(\epsilon_\rho \cdot P_1) + d_3(\epsilon_\rho \cdot (k + P_2)) \right) \right. \\
& + (\epsilon_\gamma \cdot \epsilon_\rho) \left((d_{12} - d_3 + d_4)m_\pi^2 + d_4(P_1 \cdot P_2) \right) \Big] \\
& - (k \cdot P_2)^2 \left[(\epsilon_\gamma \cdot P_1) \left((d_3 - d_4)(M_V - m_\pi)(M_V + m_\pi)(\epsilon_\rho \cdot P_1) \right. \right. \\
& + \left. \left((d_{12} - 2d_3)m_\pi^2 + d_3m_\rho^2 \right)(\epsilon_\rho \cdot P_2) \right) \\
& + (\epsilon_\gamma \cdot \epsilon_\rho) \left((d_{12} - d_3 + d_4)(M_V - m_\pi)(M_V + m_\pi)m_\pi^2 \right. \\
& + \left. \left(d_4M_V^2 - (d_{12} + d_4)m_\pi^2 + d_3(M_V^2 + m_\pi^2 - m_\rho^2) \right)(P_1 \cdot P_2) \right) \\
& + (k \cdot \epsilon_\rho) \left(\left(d_{12}m_\pi^2 + d_3(2M_V^2 - 2m_\pi^2 + m_\rho^2) \right)(\epsilon_\gamma \cdot P_1) \right. \\
& + \left. \left. 2(\epsilon_\gamma \cdot P_2) \left((d_{12} - d_3 + d_4)m_\pi^2 + d_4(P_1 \cdot P_2) \right) \right) \right] \\
& + (k \cdot P_2) \left[\left(d_{12}m_\pi^2 + d_3m_\rho^2 \right) \left((\epsilon_\gamma \cdot P_1)(\epsilon_\rho \cdot P_2) - (\epsilon_\gamma \cdot \epsilon_\rho)(P_1 \cdot P_2) \right) M_V^2 \right. \\
& + (k \cdot \epsilon_\rho) \left((\epsilon_\gamma \cdot P_2) \left((d_{12} - d_3 + d_4)(M_V - m_\pi)(M_V + m_\pi)m_\pi^2 \right. \right. \\
& + \left. \left. \left(d_4M_V^2 - (d_{12} + d_4)m_\pi^2 + d_3(M_V^2 + m_\pi^2 - m_\rho^2) \right)(P_1 \cdot P_2) \right) \right. \\
& + \left. \left. - 2d_3M_V^2m_\pi^2(\epsilon_\gamma \cdot P_1) \right) \right] \\
& + (k \cdot P_1) \left[\left(M_V^2 \left(d_{12}m_\pi^2 + d_3m_\rho^2 \right) m_\pi^2 \right. \right. \\
& + (k \cdot P_2) \left(2d_3M_V^2m_\pi^2 \right. \\
& + (k \cdot P_2) \left(d_{12}m_\pi^2 + d_3m_\rho^2 + (d_3 - d_4)(M_V - m_\pi)(M_V + m_\pi) + 2d_4(k \cdot P_2) \right) \Big) \Big) (\epsilon_\gamma \cdot \epsilon_\rho) \\
& + (\epsilon_\gamma \cdot P_2) \left((M_V^2 - m_\pi^2 - 2(k \cdot P_2))(k \cdot P_2) \right. \\
& \times \left. \left((d_3 + d_4)(k \cdot \epsilon_\rho) + (d_3 - d_4)(\epsilon_\rho \cdot P_1) \right) \right. \\
& + \left. \left((d_{12}m_\pi^2 + d_3m_\rho^2)M_V^2 + 2d_3(k \cdot P_2)^2 \right. \right. \\
& + \left. \left. - \left((d_{12} - 2d_3)m_\pi^2 + d_3m_\rho^2 \right)(k \cdot P_2) \right)(\epsilon_\rho \cdot P_2) \right) \Big] \\
& - M_V^2(d_{12}m_\pi^2 + d_3m_\rho^2)(k \cdot \epsilon_\rho) \left(m_\pi^2(\epsilon_\gamma \cdot P_1) - (\epsilon_\gamma \cdot P_2)(P_1 \cdot P_2) \right). \tag{A11}
\end{aligned}$$

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