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Not peer-reviewed version

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Posted Date: 24 June 2025

doi: 10.20944/preprints202506.1807.v1

Keywords: algebra quasi-variety lattice

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Criticality Versus Relative Sub-Direct Irreducibility

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Abstract

Here, we study connections between (quasi-equationally) critical and relatively sub-directly irreducible members of [finitely-generated] quasi-varieties [subsuming those between former ones and critical sets and enabling one to reduce the task of finding latter ones to that of former ones. We then demonstrate the usefulness of our universal elaboration by applying it to quasi-varieties of {De} Morgan lattices, the finite lattice of which was found in an earlier work of ours.]

Keywords: quasi-variety; quasi-equationally critical algebra; relatively sub-directly irreducible algebra; critical set; De Morgan lattice

JEL Classification: 06D30; 08A30; 08B05; 08B26; 08C15

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1. Introduction

As it is well-known due to the quasi-equational relativization of Birkgoff Theorem, being, in its turn, an immediate consequence of Zorn Lemma and the inductivity of the closure system of congruences of an abstract algebra with quotient in a quasi-variety, such, being constituted exactly by isomorphic copies of sub-direct products of its relatively sub-directly irreducible algebras, is generated by latter ones. Likewise, according to [8], any finitely-generated quasi-variety is generated by a critical set, being a finite skeleton without infinite members (more specifically, an anti-chain under embedability) and determined uniquely up to isomorphism between members of different critical sets. In particular, as it is well-known, any locally-finite one, being generated by its finitely-generated (i.e., finite, in that case) algebras, is so by its critical ones. These generic observations inevitably raise the issue of systematizing connections between the notions involved, becoming the primary objective of the universal part of the present study, yielding, as a by-product, a new and quite transparent insight/proof to main results of [8], due to our using the conception of relative sub-direct irreducibility, avoided therein at all. It appears that any finite relatively sub-directly irreducible algebra of a quasi-variety is critical. In particular, any relatively sub-directly irreducible algebra of a finitely-generated quasi-variety, being finite due to its being embedable into a member of any generating finite set without infinite members, is critical. Conversely, any member of a critical set is relatively sub-directly irreducible in the quasi-variety generated by the set. In particular, knowing critical sets generating arbitrary finitely-generated sub-quasi-varieties of a given [quasi-]variety, we immediately get a skeleton of the class of critical algebras of the latter, thus marking the framework of those of relatively sub-directly irreducible algebras of former ones. Perhaps, a most representative and illustrative instance of application of this universal technique is the variety of *(De) Morgan lattices* in the sense of [5], generating critical sets of sub-quasi-varieties of which — all appearing finitely-generated — have been factually found in [7].

The rest of the work is as follows. Section 2 is a brief summary of basis issues underlying the work. Section 3 is devoted to main generic issues. These are then applied to quasi-varieties of De Morgan lattices in Section 4. Finally, Section 5 is a concise summary of principal contributions of the work presented here and a brief outline of further related work.

 doi:10.20944/preprints202506.1807.v1

2. General Background

2.1. Set-Theoretical Background

Non-negative integers are identified with sets/ ordinals of lesser ones, their set/ordinal being denoted by ω. Unless any confusion is possible, one-element sets are identified with their elements.

Given any sets A, B, D and $\theta \subseteq A^2$, let $\wp_{(\omega)}([B,]A)$ be the set of all (finite) subsets of A [including B], $(\Delta_A|\nu_\theta) \triangleq \{\langle a, a|\theta[\{a\}]\rangle \mid a \in A\}$ and $(A/\theta) \triangleq \nu_\theta[A]$, A-tuples {viz., functions with domain A} being written in the sequence form $\bar{t}$ with t_a, where $a \in A$, standing for $\pi_a(\bar{t})$. Then, given any $\bar{S} \in \wp(D)^B$ and $\bar{f} \in \prod_{b \in B} S_b^A$, we have its *functional product* $(\prod^{\mathrm{F}} \bar{f}) : A \to (\prod_{b \in B} S_b), a \mapsto \langle f_b(a) \rangle_{b \in B}$ such that

$$\ker(\prod^{\mathrm{F}} \bar{f}) = (A^2 \cap (\bigcap_{b \in B} (\ker f_b))), \tag{1}$$

$$\forall b \in B : f_b = ((\prod^{\mathrm{F}} \bar{f}) \circ \pi_b), \tag{2}$$

$f_0 \odot f_1$ standing for $(\prod^{\mathrm{F}} \bar{f})$, whenever $B = 2$.

A *lower/upper cone of* a poset $\mathcal{P} = \langle P, \leqq \rangle$ is any $C \subseteq P$ such that, for all $a \in C$ and $b \in P$, $(a \geqq / \leqq b) \Rightarrow (b \in C)$. Then, an $a \in S \subseteq P$ is said to be *minimal/maximal in* S, if $\{a\}$ is a lower/upper cone of S, their set being denoted by $(\min / \max)_{\mathcal{P}|\leqq}(S)$, in case of the equality of which to S, this is called an *anti-chain of* $\mathcal{P}$.

An $X \in Y \subseteq \wp(A)$ is said to be *meet-irreducible in* Y, if $\forall Z \in \wp(Y) : ((A \cap (\bigcap Z)) = X) \Rightarrow (X \in Z)$, their set being denoted by $\mathrm{MI}(Y)$. A $\mathcal{U} \subseteq \wp(A)$ is said to be *upward-directed*, if $\forall \mathcal{S} \in \wp_\omega(\mathcal{U}) : \exists T \in (\mathcal{U} \cap \wp(\bigcup \mathcal{S}, A))$, subsets of $\wp(A)$ closed under unions of upward directed subsets being called *inductive*. A *[finitary] closure operator over* A is any unary operation on $\wp(A)$ such that $\forall X \in \wp(A), \forall Y \in \wp(X) : (X \cup C(C(X)) \cup C(Y)) \subseteq C(X)[= (\bigcup C[\wp_\omega(X)])]$. A *closure system over* A is any $\mathcal{C} \subseteq \wp(A)$ containing A and closed under intersections of subsets containing A, any $\mathcal{B} \subseteq \wp(A)$ {such that $\mathcal{C} = \{A \cap (\bigcap \mathcal{S}) \mid \mathcal{S} \subseteq \mathcal{B}\}$ being called a *(closure) basis of* $\mathcal{C}$ and} determining the closure operator $C_{\mathcal{B}} \triangleq \{\langle Z, A \cap (\bigcap (\mathcal{B} \cap \wp(Z, A)))\rangle \mid Z \in \wp(A)\}$ over A such that $\mathcal{B} \subseteq (\mathrm{img}\, C_{\mathcal{B}})\{= \mathcal{C}\}$. Conversely, $\mathrm{img}\, C$ is a closure system over A such that $C_{\mathrm{img}\, C} = C$, being inductive iff C is finitary, and forming a complete lattice under the partial ordering by inclusion with meet/join $(\Delta_{\wp(A)}/C)(A \cap ((\bigcap / \bigcup)\mathcal{S}))$ of any $\mathcal{S} \subseteq (\mathrm{img}\, C)$, C and $\mathrm{img}\, C$ being called *dual to* one another.

Remark 1. *Due to Zorn Lemma, according to which any non-empty inductive set has a maximal element,* $\mathrm{MI}(\mathcal{C})$ *is a basis of any inductive closure system* $\mathcal{C}$. □

2.2. Algebraic Background

Unless otherwise specified, we deal with a fixed but arbitrary finitary algebraic (viz., functional) signature Σ, "[non-one-element] Σ-algebras"/"their carriers|class" being denoted by "/respective capital Fraktur/Italic letters {with /same indices}"|$\mathsf{A}_\Sigma^{[>1]}$. Let Tm_Σ be the set of Σ-terms with variables in $\{x_i\}_{i \in \omega}$ and $\mathrm{Eq}_\Sigma \triangleq \mathrm{Tm}_\Sigma^2$, any $([\langle \Gamma,]\langle \phi, \psi \rangle[\rangle]) \in ([\wp_\omega(\mathrm{Eq}_\Sigma) \times] \mathrm{Eq}_\Sigma)$ being viewed as the Σ*[-quasi]-equation/-identity* $[\Gamma \to](\phi \approx \psi)$ /"identified with the universal closure of $[\bigwedge \Gamma \to](\phi \approx \psi)$, $[\mathcal{Q}]\mathcal{I}(\mathsf{K})$ denoting the set of those true in a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$".

As usual, given a $\mathsf{K} \subseteq \mathsf{A}_\Sigma$, $\mathbf{H}|\mathbf{I}|\mathbf{S}_{(>1)}|\mathbf{P}^{[\mathrm{SD/U}]}\mathsf{K}$ denotes the class of all "homomorphic images"|"isomorphic copies"|"(non-one-element) subalgebras"|"["sub-direct "/ultra-]products of tuples" of members of K, and, in case of its being subsumed by K, this is said to be *imaginative|abstract|(non-trivially-)hereditary|[sub-/ultra-]multiplicative*. Then, a *frame of* a(n abstract) $\mathsf{K} \subseteq \mathsf{A}_\Sigma$ is any $\mathsf{F} \subseteq \mathsf{A}_\Sigma$ such that $\mathsf{F} \subseteq \mathsf{K} \subseteq \mathbf{I}\mathsf{F}$ (i.e., $\mathsf{K} = \mathbf{I}\mathsf{F}$). Likewise, a *skeleton [of* K*]* is any [frame] $\mathsf{S} \subseteq \mathsf{A}_\Sigma$ [of K] without pair-wise distinct isomorphic members. Given a(n abstract {sub-multiplicative} [not necessarily] hereditary)

$\mathsf{K} \subseteq \mathsf{A}_\Sigma \ni \mathfrak{A}$, let $\mathrm{Co}_\mathsf{K}(\mathfrak{A}) \triangleq \{\theta \in \mathrm{Co}(\mathfrak{A}) \mid (\mathfrak{A}/\theta) \in \mathsf{K}\}$, in which case, (by the Homomorphism Theorem) for all $\mathfrak{B} \in \mathsf{A}_\Sigma$ and all $h \in \hom^{[\mathrm{S}]}_{\langle \mathrm{I}\rangle}(\mathfrak{A},\mathfrak{B}) \triangleq \{g \in \hom(\mathfrak{A},\mathfrak{B}) \mid [h[A] = B]\langle(\ker g) = \Delta_A\rangle]\}$:

$$\forall \theta \in \mathrm{Co}_{(\mathsf{K})}(\mathfrak{B}) : (\ker h) \subseteq h^{-1}[\theta] \triangleq \{\bar{a} \in A^2 \mid (\bar{a} \circ h) \in \theta\} \in \mathrm{Co}_{(\mathsf{K})}(\mathfrak{A}),$$
$$[\forall \vartheta \in (\mathrm{Co}(\mathfrak{A}) \cap \wp(\ker h, A^2)) : h[\vartheta] \triangleq \{\bar{b} \circ h \mid \bar{b} \in \vartheta\} \in \mathrm{Co}_{(\mathsf{K})}(\mathfrak{B}),$$
$$\vartheta = h^{-1}[h[\vartheta]], \theta = h[h^{-1}[\theta]], \quad (3)$$

and so the posets $\mathrm{Co}_{(\mathsf{K})}(\mathfrak{A}) \cap \wp(\ker h, A^2)$ and $\mathrm{Co}_{(\mathsf{K})}(\mathfrak{B})$ partially ordered by inclusion are isomorphic, while] for all $\Theta \subseteq \mathrm{Co}(\mathfrak{A})$, $\eta_\Theta \triangleq (A^2 \cap (\bigcap\Theta)) \in \mathrm{Co}(\mathfrak{A})$, whereas, since, for any set I, $\overline{\mathfrak{B}} \in \mathsf{A}_\Sigma^I$ and $\bar{f} \in (\prod_{i\in I}\hom(\mathfrak{A},\mathfrak{B}_i))$:

$$(\prod\nolimits^{\mathrm{F}} \bar{f}) \in \hom(\mathfrak{A}, \prod_{i\in I}\mathfrak{B}_i), \quad (4)$$

by (1), (2) and the Homomorphism Theorem:

$$\epsilon_\Theta \triangleq (\prod\nolimits^{\mathrm{F}}_{\zeta\in\Theta}(\nu_{\eta_\Theta}^{-1} \circ \nu_\zeta)) \in \hom_{\mathrm{I}}(\mathfrak{A}/\eta_\Theta, \prod\nolimits_{\zeta\in\Theta}(\mathfrak{A}/\zeta)),$$
$$\forall \zeta \in \Theta : \pi_\zeta[\epsilon_\Theta[A/\eta_\Theta]] = (A/\zeta), \quad (5)$$

so $\mathrm{Co}_{([\{\mathsf{K}\}])}(\mathfrak{A})$ is a closure system over A^2.

Model classes of sets of Σ-[quasi-]identities are called *[quasi-]"varieties/equational"* (these are exactly hereditary multiplicative [abstract ultra-multiplicative not necessarily] imaginative classes), the least one, including a {finite} $\mathsf{K} \subseteq \mathsf{A}_\Sigma$ {without infinite members}, $[\mathbf{Q}]\mathbf{V}(\mathsf{K}) = \mathrm{Mod}([\mathfrak{Q}]\mathfrak{I}(\mathsf{K}))[= \mathbf{ISPP}^{\mathrm{U}}\mathsf{K}\{= \mathbf{ISP}\mathsf{K}\}]$ being called *{finitely-}generated by* K [cf. [4] {and [2, Corollary 2.3]}]. Classes of Σ-algebras without infinite finitely-generated members are called *locally-finite*, finitely-generated [quasi-]varieties being so.

Given a(n equational) quasi-variety $\mathsf{Q} \subseteq (\mathsf{A}_\Sigma\{\cap\mathbf{ISP}\mathsf{K}\})$ {where $\mathsf{K} \subseteq (\mathsf{A}_\Sigma[\cap(\mathsf{Q}(\cup \mathsf{A}_\Sigma))])$}, an $\mathfrak{A} \in \mathsf{Q}$ is called *[relatively] simple/"sub-directly irreducible"*, if $\Delta_A \in (\max_\subseteq / \mathrm{MI})(\mathrm{Co}_{[\mathsf{K}]}(\mathfrak{A}) \setminus (\{A^2\}/\varnothing))$, with their class

$$[\mathrm{R}](\mathrm{Si}/\mathrm{SI})(\mathsf{Q})(= (\mathrm{Si}\,/\,\mathrm{SI})(\mathsf{Q}))$$
$$\subseteq|\supseteq ((\mathsf{A}_\Sigma^{\gtrless 1}\{\cap\mathbf{IS}\mathsf{K}\})|([\mathrm{R}]\mathrm{Si}(\mathsf{Q}) \cup (\langle\mathbf{I}\rangle[\langle\mathrm{R}\rangle](\mathrm{Si}/\mathrm{SI})(\mathsf{Q})))), \quad (6)$$

in view of 3.

Remark 2. *Given any Σ-algebra $\mathfrak{A}$ and any $\theta \in \mathrm{Co}(\mathfrak{A})$, a Σ-quasi-identity $\Gamma \to (\phi \approx \psi)$ is true in $\mathfrak{A}/\theta$ iff, for all $\bar{a} \in A^\omega$, $\varphi^\mathfrak{A}[x_i/a_i]_{i\in\omega}\ \theta\ \psi^\mathfrak{A}[x_i/a_i]_{i\in\omega}$, whenever, for each $(\xi \approx \varpi) \in \Gamma$, $\xi^\mathfrak{A}[x_i/a_i]_{i\in\omega}\ \theta\ \varpi^\mathfrak{A}[x_i/a_i]_{i\in\omega}$, in which case, for every quasi-variety $\mathsf{Q} \subseteq \mathsf{A}_\Sigma$, the closure system $\mathrm{Co}_\mathsf{Q}(\mathfrak{A})$ over A^2 is inductive, and so, by Remark 1, (3) and (5), $\mathsf{Q} = \mathbf{IP}^{\mathrm{SD}}\,\mathrm{RSI}(\mathsf{Q})$.* □

A Σ-algebra $\mathfrak{A}$ is said to be *congruence-distributive*, if the complete lattice $\mathrm{Co}(\mathfrak{A})$ ordered by inclusion is distributive (such is the case, whenever, e.g., $\Sigma_+ \triangleq \{\wedge,\vee\} \subseteq \Sigma$ and $\mathfrak{A}{\restriction}\Sigma_+$ is a lattice; cf. [6]), in which case, by (3), for any finite set I, any $\overline{\mathfrak{B}} \in [\mathrm{Si}](\mathsf{A}_\Sigma)^I$ and any $\bar{h} \in (\prod_{i\in I}\hom^{\mathrm{S}}(\mathfrak{A},\mathfrak{B}_i))$ such that $(A^2 \cap (\bigcap_{i\in I}(\ker h_i))) = \Delta_A$, $\mathrm{Co}(\mathfrak{A}) = \{A^2 \cap (\bigcap_{i\in I} h_i^{-1}[\theta_i]) \mid \bar{\theta} \in (\prod_{i\in I}\mathrm{Co}(\mathfrak{B}_i))\}$ [and so, for any congruence-distributive sub-direct product $\mathfrak{C}$ of $\overline{\mathfrak{B}}$,

$$\mathrm{Co}(\mathfrak{C}) = \{C^2 \cap (\bigcap_{j\in J}\ker(\pi_j{\restriction}C)) \mid J \subseteq I\}]. \quad (7)$$

3. Main Universal Issues

A finite $(\mathfrak{A}/\mathsf{C}) \in/\subseteq [\mathsf{K} \subseteq]\mathsf{A}_\Sigma$ /"without infinite members" is said to be *quasi-equationally) critical* /[8], if every $\mathsf{S} \subseteq \mathbf{S}(\mathfrak{A}/\mathsf{C})$ such that $(\mathfrak{A}/\mathsf{C}) \in/\subseteq \mathbf{QV}(\mathsf{S})$ contains/includes $\mathfrak{A}/\mathsf{C}$ [their class

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being denoted by $\mathrm{Cr}(\mathsf{K})$]/"in which case any subset of C is so, and so is any $\mathfrak{B} \in \mathsf{C}$, this being in $\mathsf{A}_{\Sigma}^{\geq 1} \setminus \mathbf{IS}(\mathsf{C} \setminus \{\mathfrak{B}\})$". First, by (6) and Remark 2, we immediately have:

Lemma 1 ([8, Theorem 1] in the non-optional case)**.** *Let $\mathsf{K} \subseteq \mathsf{A}_{\Sigma}$ be a finite set without infinite members and $\mathsf{Q} \triangleq \mathbf{QV}(\mathsf{K})$ (as well as $\mathsf{F} \triangleq (\mathrm{RSI}(\mathsf{Q}) \cap \mathbf{S}\mathsf{K})$, S a skeleton of F and $\mathsf{C} \triangleq \{\mathfrak{A} \in \mathsf{S} \mid \mathfrak{A} \notin \mathbf{IS}(\mathsf{S} \setminus \{\mathfrak{A}\})\}$). Then, (both F and $\mathsf{S} \subseteq \mathbf{IS}\mathsf{C}$ are frames of $\mathrm{RSI}(\mathsf{Q})$, in which case $\mathsf{Q} = \mathbf{QV}(\mathsf{F}|\mathsf{S}|\mathsf{C})$, C being critical, and so) Q is generated by a critical (sub)set (of $\mathbf{S}_{>1}\mathsf{K}$).*

This immediately yields the following well-known observation:

Corollary 1. *Any locally-finite quasi-variety Q is generated by $\mathrm{Cr}(\mathsf{Q})$.*

But what is main, by Lemma 1 and (6), we immediately get:

Theorem 1 ([8, Theorem 2 (and Corollary 1)] in the non-{}-optional case)**.** *Let $(\mathsf{K}/(\mathsf{C}([']))) \subseteq \mathsf{A}_{\Sigma}$ be a finite/critical set "without infinite members"/ and $\mathsf{Q} \triangleq \mathbf{QV}(\mathsf{K})$. Suppose $\mathbf{QV}((\mathsf{C}([']))) = \mathsf{Q}$. Then, $\mathsf{C}\{\subseteq \mathrm{RSI}(\mathsf{Q})\} \subseteq \mathbf{IS}\mathsf{K}$ (in which case $\mathbf{I}\mathsf{C} = \mathbf{I}\mathsf{C}'$, and so $\{\langle \mathfrak{A}, \mathfrak{A}' \rangle \in (\mathsf{C} \times \mathsf{C}') \mid \mathfrak{A} \in \mathbf{I}\mathfrak{A}'\}$ is a bijection between C and C').*

It is in the sense of Lemma 1 (existence) and Theorem 1 (uniqueness) that critical sets are intrinsic characterizations of finitely-generated quasi-varieties. By (6) and Theorem 1, we eventually get:

Corollary 2. *Let $\mathfrak{A}$ be a finite Σ-algebra, $\mathsf{Q}' \triangleq \mathbf{QV}(\mathfrak{A})$ and $\mathsf{Q} \subseteq \mathsf{A}_{\Sigma}$ a (finitely-generated) quasi-variety. Then, $\mathfrak{A}$ is critical if[f] $\mathfrak{A} \in \mathrm{RSI}(\mathsf{Q}['])$. (In particular, $\mathrm{RSI}(\mathsf{Q}) \subseteq \mathrm{Cr}(\mathsf{Q})$).*

Though, because of existence of critical non-subdirectly-irreducible algebras — cf. the next section, the "only if" part does not hold in the non-[]-optional case, it is the "if" part implying the ()-optional inclusion that enables one to restrict the framework of relative sub-direct irreducibility by criticality, simplifying the task of determining the former much collectively with Theorem 1 marking the framework of relative sub-direct irreducibility from below. In the next section, we demonstrate it upon the basis of [7].

4. Applications to Quasi-varieties of De Morgan Lattices

From now on, unless otherwise specified, we entirely follow [7] under identification of $x|y$ with $x_{0|1}$, dealing with the signature $\Sigma \triangleq (\Sigma_{+} \cup \{\neg\})$. For any $n \in (\omega \setminus 2)$, $\mathfrak{L}_n$ denotes the chain lattice over $L_n \triangleq n$, $\mathfrak{K}_n$ denoting the chain Kleene lattice with Σ_{+}-reduct $\mathfrak{L}_n$ and unary operation $\{\langle i, n-1-i \rangle \mid i \in n\}$. Let $\mathfrak{D}_4$ be the diamond Morgan lattice with Σ_{+}-reduct $\mathfrak{L}_2^2$ and unary operation $\{\langle \langle j,k \rangle, \langle 1-k, 1-j \rangle \rangle \mid j,k \in 2\}$, $\mathfrak{B}_2 \triangleq \mathfrak{K}_2$ the Boolean lattice over 2, $(\mathfrak{C}\wr\mathfrak{S})_{8\wr 6} \triangleq ((\mathfrak{M}\wr\mathfrak{K})_{4\wr 3} \times \mathfrak{B}_2)$ the cube≀stair Morgan≀Kleene lattice and $(\mathsf{Q}\|\mathsf{C}\|(\mathsf{I}/\mathsf{R}))_{0|1|2|3|4|5|6|7} \triangleq ((\mathsf{DML}\|\{\mathfrak{D}_4\}\|(\{\mathfrak{K}_3, \mathfrak{B}_2\} \cup (\varnothing/\{\mathfrak{C}_8, \mathfrak{S}_6, \mathfrak{K}_4\})))|((\mathsf{NIDML} \cup \mathsf{KL})\|\{\mathfrak{C}_8, \mathfrak{K}_3\}\|(\{\mathfrak{B}_2\} \cup (\varnothing/\{\mathfrak{S}_6, \mathfrak{K}_4\})))|(\mathsf{NIDML}\|\{\mathfrak{C}_8\}\|\{\mathfrak{S}_6, \mathfrak{K}_4, \mathfrak{B}_2\})|(\mathsf{KL}\|\{\mathfrak{K}_3\}\|(\{\mathfrak{B}_2\} \cup (\varnothing/\{\mathfrak{S}_6, \mathfrak{K}_4\})))|(\mathsf{NIKL}\|\{\mathfrak{S}_6\}\|\{\mathfrak{K}_4, \mathfrak{B}_2\})|(\mathsf{RKL}\|\{\mathfrak{K}_4\}\|\{\mathfrak{B}_2\})|(\mathsf{BL}\|\{\mathfrak{B}_2\}\|\varnothing)|(\mathsf{S}\|\varnothing\|\varnothing))$.

4.1. *Criticality of generating sets versus criticality of De Morgan lattices*

According to [7], Q_i, where $i \in 8$, exhaust quasi-varieties of De Morgan lattices and are generated by C_i, the criticality of which was implicitly proved therein. We start from making this point explicit. More precisely, summing *proofs* of Lemmas 4.9, 4.10 as well as Cases 2, 3, 4 and 9 of Theorem 4.8 of [7] up, we, first, have:

Lemma 2 (Generic Embaddability Lemma)**.** *Let $\langle \mathfrak{A}, j, k \rangle \in \{\langle \mathfrak{D}_4, 0, 1 \rangle, \langle \mathfrak{C}_8, 1, 3 \rangle, \langle \mathfrak{K}_3, 1, 2 \rangle, \langle \mathfrak{S}_6, 4, 5 \rangle, \langle \mathfrak{K}_4, 5, 6 \rangle, \langle \mathfrak{B}_2, 6, 7 \rangle\}$ and $\mathfrak{B} \in (\mathsf{Q}_j \setminus \mathsf{Q}_k)$. Then, $(\mathfrak{A} \in \mathsf{C}_1) \Rightarrow ((\mathsf{C}_1 \setminus \{\mathfrak{A}\}) \subseteq \mathsf{Q}_k)$ and $\mathfrak{A} \in ((\mathsf{Q}_j \setminus \mathsf{Q}_k) \cap \mathbf{IS}\{\mathfrak{B}\})$.*

 doi:10.20944/preprints202506.1807.v1

This, by Theorems [7, 4.8], 1 and the embedability of no finite algebra into any proper subalgebra, immediately yields:

Corollary 3. *For any $i \in 8$, C_i is critical, in which case* $(\bigcup_{l \in (8[\setminus\{1\}])} \mathsf{C}_l) = \mathsf{R}_0$ *is a skeleton of* $\mathrm{Cr}(\mathsf{Q}_0)$, *and so* $\mathsf{C}_i \cup \mathsf{R}_i$ *is that of* $\mathrm{Cr}(\mathsf{Q}_i)$.

4.2. Relative Subdirect Irreducibility Versus Criticality

In this way, taking Corollaries 2, 3 and (6) into account, the task of finding relative sub-directly irreducibles of any quasi-variety Q of De Morgan lattices is reduced to determining which members of $\mathsf{Q} \cap \mathsf{R}_0$ are relatively sub-directly irreducible. On the other hand, any $\mathfrak{A} \in \mathsf{R}_0$ is embedable into either $\mathfrak{D}_4$ or $\mathfrak{C}_8$. In the former case, the well-known simplicity of non-one-element subalgebras of $\mathfrak{D}_4$ (cf. [1,3]) implies the simplicity and so [relative] subdirect irreducibility of $\mathfrak{A}$. Otherwise, $\mathfrak{A} \notin \mathsf{BL}$, so the opposite case is due to the following auxiliary observation equally covering the former case just for exposition to be self-contained:

Lemma 3. *Let* ($\mathsf{Q} \subseteq \mathsf{A}_\Sigma$ *be a quasi-variety,* $\mathfrak{A} \in ((\mathsf{Q} \setminus \mathsf{BL}) \cap \mathbf{S}\mathfrak{C}_8)$ *and)* $\mathfrak{B} \in \mathbf{S}\mathfrak{D}_4(\ni \mathfrak{C} \triangleq (\mathfrak{D}_4 \restriction \pi_0[A]))$. *Then,* $\mathrm{Co}(\mathfrak{B}) = \{\Delta_B, B^2\}$. *(In particular,* $(\mathfrak{A} \in \mathrm{RSI}(\mathsf{Q})) \Leftrightarrow (\mathfrak{C} \notin \mathsf{Q})$.*)*

Proof. By contradiction. For suppose there is some $\theta \in (\mathrm{Co}(\mathfrak{B}) \setminus \{\Delta_B, B^2\})$, in which case $|B| > 2$, and so, first, for each $m \in 2$, as $B \subseteq (\pi_0[B] \times \pi_1[B])$ and, for all $\Bbbk \in 2$, $\pi_\Bbbk[B] \subseteq 2$, we have $|\pi_m[B]| \neq 1$, i.e., $\pi_m[B] = 2$. Second, neither $B \subseteq \Delta_2$, i.e., there is some $a \in (2^2 \setminus \Delta_2)$, nor $B \subseteq (2^2 \setminus \Delta_2)$, i.e., $(B \cap \Delta_2) \neq \varnothing$, in which case there is some $b \in (B \cap \Delta_2)$, and so $B \supseteq \{b, \neg^{\mathfrak{B}} b\} = \Delta_2$. Then, $\mathfrak{B}' \triangleq (\mathfrak{B} \restriction \Sigma_+)$ is a congruence-distributive sub-direct square of $\mathfrak{L}_2$, in which case, by (7) and the simplicity of two-element algebras, there is some $\ell \in 2$ such that $\mathrm{Co}(\mathfrak{B}') \ni \theta = \ker(\pi_\ell \restriction B)$, and so, as $c \triangleq (2 \times \{a_\ell\}) \in \Delta_2 \subseteq B$, $\langle a, c \rangle \in \theta$. In particular, $a = \neg^{\mathfrak{B}} a \;\theta\; \neg^{\mathfrak{B}} c = (2 \times \{1 - a_\ell\})$, in which case $2 \ni a_\ell = (1 - a_\ell)$, and so this contradiction completes the argument of the non-optional case. (First, as $\mathfrak{A} \in \mathbf{S}(\mathfrak{C} \times \mathfrak{B}_2)$ and $\mathfrak{B}_2 \in \mathsf{BL}$, we have $\mathfrak{C} \notin \mathsf{BL}$, in which case $C \nsubseteq \Delta_2$, and so there is some $d \in (C \setminus \Delta_2)$. Take any $e \in A$ such that $\pi_0(e) = d$, in which case $\{\langle 0, e \wedge^{\mathfrak{A}} \neg^{\mathfrak{A}} e \rangle, \langle 1, e \vee^{\mathfrak{A}} \neg^{\mathfrak{A}} e \rangle\} \in \hom_{\mathrm{I}}(\mathfrak{B}_2, \mathfrak{A})$, and so $\mathfrak{B}_2 \in \mathsf{Q}$. Furthermore, since $\mathfrak{B}_2 \in \mathsf{BL} \not\ni \mathfrak{A}$ has no proper subalgebra, $f \triangleq (\pi_1 \restriction A) \in \hom^{\mathrm{S}}(\mathfrak{A}, \mathfrak{B}_2)$ is not injective. Likewise, $g \triangleq (\pi_0 \restriction A) \in \hom^{\mathrm{S}}(\mathfrak{A}, \mathfrak{C})$ is not injective, because, otherwise, we would have $h \triangleq (g^{-1} \circ f) \in \hom(\mathfrak{C}, \mathfrak{B}_2)$, and so would get $2 \ni h(d) = h(\neg^{\mathfrak{C}} d) = (1 - h(d))$, for $\neg^{\mathfrak{C}} d = d$. Then, by the Homomorphism Theorem, $\mathrm{Co}_{\mathsf{Q}}(\mathfrak{A}) \ni \vartheta \triangleq (\ker f) \neq \Delta_A$ contains $\eta \triangleq (\ker g) \neq \Delta_A = (\vartheta \cap \eta)$ iff $\mathfrak{C} \in \mathsf{Q}$. On the other hand, $\mathfrak{A}$ is a congruence-distributive sub-direct product of $\mathfrak{C}$ and $\mathfrak{B}_2$, in which case, by (7), the non-optional case and the simplicity of two-element algebras, $\mathrm{Co}_{\mathsf{Q}}(\mathfrak{A}) \subseteq \mathrm{Co}(\mathfrak{A}) = \{\Delta_A, \vartheta, \eta, A^2\}$, and so $(\mathfrak{A} \in \mathrm{RSI}(\mathsf{Q})) \Leftrightarrow (\eta \notin \mathrm{Co}_{\mathsf{Q}}(\mathfrak{A}))$, as required.) □

Finally, for any $i \in 2$, $\epsilon^4_{3,i} \triangleq (+\restriction(2^2 \setminus \{\langle 1 - i, i \rangle\}))^{-1} \in \hom_{\mathrm{I}}(\mathfrak{K}_3, \mathfrak{D}_4)$, while, for any $n \in (\omega \setminus 2)$, $\epsilon^n_2 \triangleq \{\langle j, j \cdot (n-1) \rangle \mid j \in 2\} \in \hom_{\mathrm{I}}(\mathfrak{B}_2, \mathfrak{K}_n)$, whereas $\epsilon^6_4 \triangleq (+\restriction((3 \times 2) \setminus \{\langle 2-k, k \rangle \mid k \in 2\}))^{-1} \in \hom_{\mathrm{I}}(\mathfrak{K}_4, \mathfrak{S}_6)$ with $\pi_0[\epsilon^6_4[4]] = 3$, in which case, by (1), (2) and (4), $\epsilon^8_{6,i} \triangleq (((\pi_0 \restriction (3 \times 2)) \circ \epsilon^4_{3,i}) \odot (\pi_1 \restriction (3 \times 2))) \in \hom_{\mathrm{I}}(\mathfrak{S}_6, \mathfrak{C}_8)$ with $\pi_0[\epsilon^8_{6,i}[3 \times 2]] = \epsilon^4_{3,i}[3]$, and so $\epsilon^8_{4,i} \triangleq (\epsilon^6_4 \circ \epsilon^8_{6,i}) \in \hom_{\mathrm{I}}(\mathfrak{K}_4, \mathfrak{C}_8)$ with $\pi_0[\epsilon^8_{4,i}[4]] = \epsilon^4_{3,i}[3]$, as well as $(\Delta_2 \odot \Delta_2) = (\epsilon^3_2 \circ \epsilon^4_{3,i}) \in \hom_{\mathrm{I}}(\mathfrak{B}_2, \mathfrak{D}_4)$. Then, by (6), Corollaries 2, 3, Lemmas 2, 3 and Theorem 1, we eventually get:

Theorem 2. *For any $i \in 8$, $\mathsf{C}_i \cup \mathsf{I}_i$ is a skeleton of* $\mathrm{RSI}(\mathsf{Q}_i)$.

5. Conclusions

Thus, the universal algebraic technique elaborated in Section 3 and going back to [8] has been successfully applied within the framework of De Morgan lattices in Section 4 upon the basis of [7]. It can equally be applied within wider/another context. This essentially goes beyond the scopes of the present study and is going to be presented elsewhere.

doi:10.20944/preprints202506.1807.v1

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