Article

Not peer-reviewed version

Fuzzy Double Yang Transform with Application to Fuzzy Parabolic Volterra Integro-Differential Equation

Atanaska Georgieva * , Slav I. Cholakov , Maria Vasileva , Yordanka Gudalova

Posted Date: 21 March 2025

doi: 10.20944/preprints202503.1587.v1

Keywords: fuzzy double Yang transform; fuzzy parabolic Volterra integro-differential equation; generalized Hukuhara partial differentiable

Article

Fuzzy Double Yang Transform with Application to Fuzzy Parabolic Volterra Integro-Differential Equation

Atanaska Georgieva *, Slav I. Cholakov, Maria Vasileva and Yordanka Gudalova

Faculty of Mathematics and Informatics, University of Plovdiv, 4000 Plovdiv, Bulgaria
* Correspondence: atanaska@uni-plovdiv.bg

Abstract: This article introduces a new fuzzy double integral transformation called the fuzzy double Yang transformation. We review some of the main properties of the transformation and find the conditions for its existence. We prove the theorems for partial derivatives and fuzzy unitary convolution. All new results are applied to find an exact solution to the fuzzy parabolic Volterra integro-differential equation with a memory kernel. In addition, a numerical example is provided to illustrate the accuracy and superiority of the proposed method with the help of symmetric triangular fuzzy numbers.

Keywords: fuzzy double Yang transform; fuzzy parabolic Volterra integro-differential equation; generalized Hukuhara partial differentiable

MSC: 35A22; 44A15; 44A35

---

1. Introduction

Partial differential equations are used for dynamic modeling of complex processes in various fields such as physics, chemistry, fluid and quantum mechanics, biology, and economics. They are predominantly applied to the so-called instantaneous phenomena whose behavior depends on their momentary state. A large part of the processes require the model to account for their behavior over a previous time interval. As a result, it is necessary to use partial integro-differential equations, as they show the cumulative behavior of the process. The different types of partial differential equations are related to the various types of differential and integral operators. One of them is the parabolic Volterra integro-differential equation. It has important physical applications in modeling dynamical systems, where one can explore the effects of the "memory" of the system. Such systems are developed, for example, in compression of viscoelastic media [1], nuclear reactor dynamics [2], expansion problems [3], reaction diffusion problems [4], and thermally conductive materials with functional memory [5].

Over the last few years, we have noticed an incredible interest in fuzzy mathematics due to the many applications in various fields, especially physics, engineering, medicine, and economics [6–11]. This trend defines the need for studying fuzzy ordinary differential equations [12–14], fuzzy partial differential equations [15–19] and fuzzy integro-differential equations [20–22] through publishing many articles related to these fuzzy equations.

Numerical solutions to the fuzzy parabolic Volterra integro-differential equation using the reproducing kernel Hilbert space method can be found in [23]. Recently, in order to find the exact solution to linear fuzzy integro-differential equation, fuzzy integral transforms have been used. In [24], using the fuzzy Laplace transform, the analytical solution of the fuzzy parabolic Volterra partial integro-differential equations under generalized Hukuhara partial differentiability was found. The fuzzy single and fuzzy double Sumudu transformation [25,26] as well as the fuzzy double Natural transformation [27] have been applied to the fuzzy Volterra partial integro-differential equation.

The Yang transform is introduced by Yang [28] and is applied to differential equation in the steady heat-transfer problem. Recently, Ullah et al. [14] are proposed fuzzy single Yang transform for finding the solution of second order fuzzy differential equations of integer and fractional order.

The article's main goal is to extend the fuzzy single Yang transform to the fuzzy double Yang transform which allows us to find the exact solution of a fuzzy parabolic Volterra integro-differential equation under generalized Hukuhara differentiability. More precisely, we look at the following fuzzy nonhomogeneous parabolic Volterra integro-differential equation with simmetric memory kernel $k(t)$ in the infinite domain

$$g'_{t,gH}(x,t) \oplus \int_0^t k(t-s) \odot g(x,t)dxdt = \sigma \odot g''_{xx,gH}(x,t) \oplus f(x,t),\ x \geq 0,\ t \geq 0, \tag{1}$$

where σ is any positive constant, $g(x,t)$ is the unknown fuzzy function, $f(x,t)$ is a given fuzzy function.

The remainder of this work is structured as follows: In Section 2 we briefly introduce the basic notations, definitions, and theorems that will be used in the main part of the paper. In Section 3 the single fuzzy Yang transform is defined and some basic properties are demonstrated for this transform. In Section 4 a fuzzy double Yang transform for a fuzzy function is defined and some properties and theorems, several relations related to existence, gH-partial derivatives, and single convolution are presented. A fuzzy parabolic Volterra integro-differential equation with memory kernel is defined under generalized partial Hukuhara differentiability and a solution of this equation by a fuzzy double Yang transform method is investigated in Section 5. Moreover, a numerical example is constructed to clarify the details and efficiency of the method in Section 6. Conclusions are given in Section 7.

2. Premilinaries

The following section consists of the necessary notations, definitions, and theorems which are useful in this research.

Let E^1 denote the set of fuzzy subsets of the real axis, i.e. $\nu : \mathbb{R} \to [0,1]$ that possesses the following properties:

(i) ν is upper semi-continuous on $\mathbb{R}$ for all $\nu \in E^1$;
(ii) ν is normal for all $\nu \in E^1$;
(iii) ν is fuzzy convex for all $\nu \in E^1$;
(iv) $cl\{\eta \in \mathbb{R} :\ \nu(\eta) > 0\}$ is compact, where *cl denotes the closure of a subset.*

Then we say that E^1 is a space of fuzzy numbers. It is clear that any real number a can be interpreted as a fuzzy number $\tilde{a} = \chi(a)$ and therefore $\mathbb{R} \subset E^1$. The r-level set of the fuzzy number ν we denote

$$[\nu]^r = \begin{cases} \{\eta \in \mathbb{R} : \nu(\eta) \geq r\}, & 0 < r \leq 1, \\ cl\{\eta \in \mathbb{R} :\ \nu(\eta) > 0\}, & r = 0. \end{cases}$$

Then from (i) to (iv), it follows that for each $0 \leq r \leq 1$, the r-level sets of fuzzy number ν are nonempty closed intervals of the form

$$[\nu]^r = [\underline{\nu}(r), \overline{\nu}(r)].$$

A triangular fuzzy number ν is defined as an ordered triple $\nu = (\nu_1, \nu_2, \nu_3)$, where $\nu_1 \leq \nu_2 \leq \nu_3$ has r-cuts

$$[\nu]^r = [\nu_1 + (\nu_2 - \nu_1)r, \nu_3 - (\nu_3 - \nu_2)r],\ \ 0 \leq r \leq 1.$$

Let μ and ν be two fuzzy numbers and $k \in \mathbb{R}$. Then the addition $\mu \oplus \nu \in E^1$ and the scalar multiplication $k \odot \mu \in E^1$ are defined as having the level cuts

$$[\mu \oplus \nu]^r = [\mu]^r + [\nu]^r = \{\xi + \eta :\ \xi \in [\mu]^r,\ \eta \in [\nu]^r\}$$

$$[k \odot \mu]^r = k.[\mu]^r = \{k\xi :\ \xi \in [\mu]^r\},\ \ [0]^r = \{0\}\ \ for\ all\ \ 0 \leq r \leq 1.$$

Denote $\mathbb{R}_+ = [0, +\infty)$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 March 2025 doi:10.20944/preprints202503.1587.v1

Definition 2.1. *[29] The Hausdorff distance between fuzzy numbers is given by* $D : E^1 \times E^1 \to \mathbb{R}_+$ *as*

$$D(\mu,\nu) = \sup_{0\leq r\leq 1} \max\{|\underline{\mu}(r) - \underline{\nu}(r)|, |\overline{\mu}(r) - \overline{\nu}(r)|\},$$

where $[\mu]^r = [\underline{\mu}(r), \overline{\mu}(r)]$ *and* $[\nu]^r = [\underline{\nu}(r), \overline{\nu}(r)]$.

The metric space (E^1, D) is complete separable and locally compact, and the following properties of the metric D are well known:

(i) $D(\lambda \oplus \nu, \mu \oplus \nu) = D(\lambda, \mu)$ for all $\lambda,\ \mu,\ \nu \in E^1$;
(ii) $D(k \odot \mu, k \odot \nu) = |k| D(\mu, \nu)$ for all $\mu,\ \nu \in E^1$ and $k \in \mathbb{R}$;
(iii) $D(\lambda \oplus \mu, \nu \oplus \kappa) \leq D(\lambda, \nu) + D(\mu, \kappa)$ for all $\lambda,\ \mu,\ \nu,\ \kappa \in E^1$.

Definition 2.2. *[29] Let* $\mu,\ \nu \in E^1$*. If there exists a fuzzy number* λ *such that* $\mu = \nu \oplus \lambda$*, then* λ *is called the Hukuhara difference (H-difference) of* μ *and* ν*, and it is denoted by* $\mu \ominus_H \nu$.

The r-cuts of H-difference are

$$[\mu \ominus_H \nu]^r = [\underline{\mu}(r) - \underline{\nu}(r), \overline{\mu}(r) - \overline{\nu}(r)],$$

where $[\mu]^r = [\underline{\mu}(r), \overline{\mu}(r)]$ and $[\nu]^r = [\underline{\nu}(r), \overline{\nu}(r)]$.

Clearly, $\mu \ominus_H \mu = \{\tilde{0}\}$; if $\mu \ominus_H \nu$ exists, it is unique.

Definition 2.3. *[29] Given* $\mu, \nu \in E^1$*, the generalized Hukuhara difference (gH-difference) is the fuzzy quantity* $\lambda \in E^1$*, if it exists, such that*

$$\mu \ominus_{gH} \nu = \lambda \Leftrightarrow \begin{cases} (i) & \mu = \nu \oplus \lambda, \\ or\ (ii) & \nu = \mu \oplus (-1) \odot \lambda. \end{cases} \tag{2}$$

It is easy to show that (i) and (ii) valid if and only if λ is a crisp number.

In terms of the r-cuts, we have

$$[\mu \ominus_{gH} \nu]^r = [\min\{\underline{\mu}(r) - \underline{\nu}(r), \overline{\mu}(r) - \overline{\nu}(r)\}, \max\{\underline{\mu}(r) - \underline{\nu}(r), \overline{\mu}(r) - \overline{\nu}(r)\}]$$

and, if the H-difference exists, then $\mu \ominus_H \nu = \mu \ominus_{gH} \nu$. The conditions for the existence of $\mu \ominus_{gH} \nu = w \in E^1$ are given in [30].

Proposition 2.1. *[29] Let* $\mu,\ \nu \in E^1$*, then*

$$D(\mu \ominus_{gH} \nu, \tilde{0}) = D(\mu, \nu).$$

Proposition 2.2. *[30] Let* $\mu, \nu \in E^1$*. If* $\mu \ominus_{gH} \nu$ *exists, it is unique and has the following properties*

(i) $\mu \ominus_{gH} \mu = \tilde{0}$;
(ii) $(\mu \oplus \nu) \ominus_{gH} \nu = \mu,\ \ \mu \ominus_{gH} (\mu \ominus_H \nu) = \nu$;
(iii) if $\mu \ominus_{gH} \nu$ *exists then also* $(-\nu) \ominus_{gH} (-\mu)$ *does and* $\tilde{0} \ominus_{gH} (\mu \ominus_{gH} \nu) = (-\nu) \ominus_{gH} (-\mu)$;
(iv) $\mu \ominus_{gH} \nu = \nu \ominus_{gH} \mu = \lambda$ *if and only if* $\lambda = -\lambda$*: furthermore,* $\lambda = \tilde{0}$ *if and only if* $\mu = \nu$;
(v) If $\nu \ominus_{gH} \mu$ *exists then either* $\mu \oplus (\nu \ominus_{gH} \mu) = \mu$ *or* $\nu \ominus_H (\nu \ominus_{gH} \mu) = \mu$ *and if both equalities hold then* $\nu \ominus_{gH} \mu$ *is a crisp set.*

2.1. The One-Variable Fuzzy Calculus

In this section, we present basic definitions and theorems for a fuzzy-valued function of one-variable which will be used throughout the paper.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 March 2025 doi:10.20944/preprints202503.1587.v1

A function $g : [c,d] \subset \mathbb{R} \to E^1$ is called a fuzzy-valued function. The r-level representation of this fuzzy function g given by $g(t,r) = [\underline{g}(t,r), \overline{g}(t,r)]$, $t \in [c,d]$ for all $0 \le r \le 1$.

Definition 2.4. *[31] We say that fuzzy-valued function $g : [c,d] \to E^1$ is continuous at $t_0 \in [c,d]$, if*

$$\lim_{t \to t_0} D(g(t), g(t_0)) = 0$$

provided that limits exists.

The function g is fuzzy continuous on $[c,d]$ if g is continuous in each $t_0 \in [c,d]$.

Definition 2.5. *[30] Let $t_0 \in (c,d)$ and k be such that $t_0 + k \in (c,d)$. Then the generalized Hukuhara derivative (gH-derivative) of a function $g : (c,d) \to E^1$ at t_0 are called the fuzzy number $g'_{gH}(t_0)$ wich defined as*

$$g'_{gH}(t_0) = \lim_{k \to 0} \frac{1}{k} [g(t_0 + k) \ominus_{gH} g(t_0)], \tag{3}$$

if limit exists.

Theorem 2.1. *[29] Let $g : (c,d) \to E^1$ be gH-differentiable at $t_0 \in [c,d)$. Then g is fuzzy continuous at t_0.*

The next theorem gives the expression of the fuzzy gH-derivative in terms of the derivatives of the endpoints of the level sets.

Theorem 2.2. *[30] Let $g : [c,d] \to E^1$ be a fuzzy-valued function with r-levels $g(t,r) = [\underline{g}(t,r), \overline{g}(t,r)]$ and the real-valued functions $\underline{g}(.,r)$ and $\overline{g}(.,r)$ be differentiable at t_0 for all $0 \le r \le 1$. Then the function $g(t)$ is gH-differentiable at $t_0 \in (c,d)$ if and only if one of the following two cases holds:*

(i) $\underline{g}(t_0,r)$ is increasing, $\overline{g}(t_0,r)$ is decreasing and $\underline{g}(t_0,r) \le \overline{g}(t_0,r)$;
(ii) $\underline{g}(t_0,r)$ is decreasing, $\overline{g}(t_0,r)$ is increasing and $\overline{g}(t_0,r) \le \underline{g}(t_0,r)$.

Moreover, we have

$$g'_{gH}(t_0,r) = [min\{\underline{g}(t_0,r),\ \overline{g}(t_0,r)\}, max\{\underline{g}(t_0,r),\ \overline{g}(t_0,r)\}]$$

for all $0 \le r \le 1$.

If $\underline{g}(.,r)$ and $\overline{g}(.,r)$ are both differentiable and according to Theorem 2.2, then for the definition of gH-differentiability we distinguish two cases, corresponding to (i) and (ii) of Equation (2).

Definition 2.6. *[30] Let $g : [c,d] \to E^1$ and $t_0 \in (c,d)$, with $\underline{g}(.,r)$ and $\overline{g}(.,r)$ both differentiable at t_0. The fuzzy-valued function g is called:*

1. *(i)-gH-differentiable at t_0 if*

$$g'_{gH}(t_0,r) = \left[\underline{g}'(t_0,r), \overline{g}'(t_0,r)\right] \quad for\ all\ r \in [0,1]; \tag{4}$$

2. *(ii)-gH-differentiable at t_0 if*

$$g'_{gH}(t_0,r) = \left[\overline{g}'(t_0,r), \underline{g}'(t_0,r)\right] \quad for\ all\ r \in [0,1]. \tag{5}$$

Theorem 2.3. *[30] Let $f,\ g : (c,d) \to E^1$ be gH-differentiable. Then $f(t) \oplus g(t)$ is gH-differentiable and*

$$(f \oplus g)'_{gH}(t) = f'_{gH}(t) \oplus g'_{gH}(t).$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 March 2025 doi:10.20944/preprints202503.1587.v1

Theorem 2.4. *[29] Let $f : [c,d] \to E^1$ and $g : [c,d] \to \mathbb{R}$ be two differentiable functions. Then*

$$\int_c^d g(t) \odot f'_{gH}(t)dt = (g(d) \odot f(d)) \ominus_{gH} (g(c) \odot f(c)) \ominus_{gH} \int_c^d g'(t) \odot f(t)dt.$$

Theorem 2.5. *[29] Let $g : \mathbb{R} \to E^1$ be a fuzzy-valued function with r-levels $g(t,r) = [\underline{g}(t,r), \overline{g}(t,r)]$. Suppose that the functions $\underline{g}(t,r)$ and $\overline{g}(t,r)$ are Riemann integrable on $\mathbb{R}$ for all $0 \le r \le 1$. Then $g(t)$ is improper fuzzy Riemann-integrable on $\mathbb{R}$. Moreover, we have*

$$\left[\int_{-\infty}^{\infty} g(t)dt\right]^r = \left[\int_{-\infty}^{\infty} \underline{g}(t,r)dt, \int_{-\infty}^{\infty} \overline{g}(t,r)dt\right]$$

for all $0 \le r \le 1$.

2.2. The Two-Variable Fuzzy Calculus

Let $g : Q \subset \mathbb{R} \times \mathbb{R} \to E^1$ be a fuzzy-valued function of two variable with r-levels $g(x,t,r) = [\underline{g}(x,t,r), \overline{g}(x,t,r)]$ for all $(x,t) \in Q$ and $0 \le r \le 1$.

Definition 2.7. *[32] Let $(x_0, t_0) \in Q$ the constants h and k be such that $(x_0 + h, t_0) \in Q$ and $(x_0, t_0 + k) \in Q$. Then the first generalized Hukuhara partial derivative (gH-p-derivative) of a fuzzy-valued function $g : Q \to E^1$ at $(x_0, t_0) \in Q$ with respect to x and t are called the fuzzy numbers $g'_x(x_0, t_0)$ and $g'_t(x_0, t_0)$ with defined as*

$$g'_{x,gH}(x_0, t_0) = \lim_{h \to 0} \frac{1}{h}[g(x_0 + h, t_0) \ominus_{gH} g(x_0, t_0)],$$

$$g'_{t,gH}(x_0, t_0) = \lim_{k \to 0} \frac{1}{k}[g(x_0, t_0 + k) \ominus_{gH} g(x_0, t_0)].$$

Definition 2.8. *[32] Let $g : Q \to E^1$ be fuzzy-valued function and $(x_0, t_0) \in Q$. Suppose that the functions $\underline{g}(x,t,r)$ and $\overline{g}(x,t,r)$ are partially differentiable in (x_0, t_0) with respect to variable t. Also, we say that the function $g(x,t)$ is:*

1. *(i) − p-gH-differentiable at (x_0, t_0) with respect to t if*

$$g'_{t,gH}(x_0, t_0, r) = \left[\underline{g}'_t(x_0, t_0, r), \overline{g}'_t(x_0, t_0, r)\right] \quad for\ all\ r \in [0,1], \tag{6}$$

2. *(ii) − p-gH-differentiable at (x_0, y_0) with respect to variable x if*

$$g'_{t,gH}(x_0, t_0, r) = \left[\overline{g}'_t(x_0, t_0, r), \underline{g}'_t(x_0, t_0, r)\right] \quad for\ all\ r \in [0,1]. \tag{7}$$

Theorem 2.6. *[33] Let $g : \mathbb{R}_+ \times \mathbb{R}_+ \to E^1$ be a fuzzy-valued function. Assume that $\int_0^\infty g(x,t)dt$ is convergent for each $\mathbb{R}_+$ and $\int_0^\infty g(x,t)dx$ as a function t is convergent on $\mathbb{R}_+$. Then*

$$\int_0^\infty \int_0^\infty g(x,t)dtdx = \int_0^\infty \int_0^\infty g(x,t)dxdt.$$

3. Fuzzy Yang Transform

In this section, we introduce the definition and basic properties of the fuzzy Yang transform (FYT) [14].

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 March 2025 doi:10.20944/preprints202503.1587.v1

Definition 3.1. *The fuzzy Yang integral transform for a fuzzy function $g(t)$ is defined as*

$$Y_t[g(t)] = G(\beta) = \int_0^\infty e^{-\frac{t}{\beta}} \odot g(t)dt, \tag{8}$$

provided that the improper fuzzy integral exists and where t and β are transform variables.

Definition 3.2. *The inverse fuzzy Yang integral transform is given by*

$$Y_\beta^{-1}[G(\beta)] = g(t) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} e^{\frac{t}{\beta}} \odot G(\beta)d\beta, \tag{9}$$

where the function $G(\beta)$ is analytic for all β such that $Re\beta > b$.

Theorem 3.1. *If $g(t)$ is a continuous fuzzy function in every finite interval $0 \le t \le T$ and $g(t)$ is of exponential order e^{dt}, if it satisfies*

$$D(g(t), \tilde{0}) \le Le^{dt}, \ \ t \in [0, T], \ L > 0.$$

Then, the fuzzy Yang transform of $g(t)$ exists for all β such that $Re(\frac{1}{\beta}) > d$.

Proof. Using the Definition 3.1, we obtain

$$D(Y_t[g(t)], \tilde{0}) = D(G(\beta), \tilde{0}) = D\left(\int_0^\infty e^{-\frac{t}{\beta}} \odot g(t)dt, \tilde{0}\right).$$

Using the property of improper fuzzy integral, we get

$$\begin{aligned} D(G(\beta), \tilde{0}) \ &= D(\int_0^\infty e^{-\frac{t}{\beta}} \odot g(t)dt, \tilde{0}) \le \int_0^\infty e^{-\frac{t}{\beta}} D(g(t), \tilde{0})dt \le \\ &\le L\int_0^\infty e^{-(\frac{1}{\beta}-d)t}dt = \frac{L\alpha}{1-d\beta}. \end{aligned}$$

Thus, the improper fuzzy integral converges for all $Re(\frac{1}{\beta}) > d$ and $Y_t[g(t)]$ exists. □

The classical Yang transform is applied to some special functions in [28].

(i) $Y_t[1] = \beta$;

(ii) $Y_t[t^n] = \begin{cases} \beta^{n+1} n! & n = 1, 2, 3, \ldots \\ \beta^{n+1}\Gamma(n+1) & n > 0 \end{cases}$

(iii) $Y_t[e^{dt}] = \frac{\beta}{1-d\beta}$ for all $d \in \mathbb{R}$;

(iv) $Y_t[\sin dt] = \frac{d\beta}{1+d^2\beta^2}$ for all $d \in \mathbb{R}$;

(v) $Y_t[\cos dt] = \frac{\beta^2}{1+d^2\beta^2}$ for all $d \in \mathbb{R}$;

(vi) $Y_t[\sinh dt] = \frac{d\beta^2}{1-d^2\beta^2}$ for all $d \in \mathbb{R}$;

(vii) $Y_t[\cosh dt] = \frac{\beta}{1-d^2\beta^2}$ for all $d \in \mathbb{R}$.

We will give some of the basic properties of the fuzzy Yang transform.

Theorem 3.2. *(Linearity) If $G_1(\beta) = Y_t[g_1(t)]$ and $G_2(\beta) = Y_t[g_2(t)]$. Then*

$$Y_t[b_1 \odot g_1(t) \oplus b_2 \odot g_2(t)] = b_1 \odot Y_t[g_1(t)] \oplus b_2 \odot Y_t[g_2(t)],$$

where $b_1, \ b_2 \in \mathbb{R}$ such that $b_1, \ b_2 \ge 0$ or $b_1, \ b_2 \le 0$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 March 2025 doi:10.20944/preprints202503.1587.v1

Proof. Using the Definition 3.1 and the property of improper fuzzy integral, we get

$$\begin{aligned} Y_t[b_1 \odot g_1(t) \oplus b_2 \odot g_2(t)] &= \int_0^\infty e^{-\frac{t}{\beta}}(b_1 \odot g_1(t) \oplus b_2 \odot g_2(t))dt = \\ &= \int_0^\infty b_1 e^{-\frac{t}{\beta}} \odot g_1(t)dt \oplus \int_0^\infty b_2 e^{-\frac{t}{\beta}} \odot g_2(t))dt = \\ &= b_1 \odot Y_t[g_1(t)] \oplus b_2 \odot Y_t[g_2(t)]. \end{aligned}$$

□

Remark 3.1. *Using the Definition 3.2, we can show that Y_t^{-1} a linear transformation, i.e.*

$$Y_t^{-1}[b_1 \odot G_1(\beta) \oplus b_2 \odot G_2(\beta)] = b_1 \odot Y_t^{-1}[G_1(\beta)] \oplus b_2 \odot Y_t^{-1}[G_2(\beta)]$$

Theorem 3.3. *(Change of Scale) If $G(\beta) = Y_t[g(t)]$, then for some constant b it follows*

$$Y_t[g(bt)] = \frac{1}{b} \odot G\left(\frac{1}{b}\beta\right).$$

Proof. Using the Definition 3.1, we have

$$Y_t[g(bt)] = \int_0^\infty e^{-\frac{t}{\beta}} \odot g(bt)dt.$$

Put $bt = t_1$ and $dt = \frac{1}{b}dt_1$ in above equation, we have

$$Y_t[g(bt)] = \int_0^\infty \frac{1}{b} e^{-\frac{t_1}{b\beta}} \odot g(t_1)d(t_1) = \frac{1}{b} \odot G\left(\frac{1}{b}\beta\right).$$

□

Theorem 3.4. *(Duality) If $Y_t[g(t)] = G(\beta)$ is the fuzzy Yang transform and $L_t[g(t)] = F(\beta)$ is the fuzzy Laplace transform of $g(t)$, then*

$$G(\beta) = F\left(\frac{1}{\beta}\right) \text{ and } F(\beta) = G\left(\frac{1}{\beta}\right)$$

Proof. Using the Definition 3.1, we have

$$G(\beta) = Y_t[g(t)] = \int_0^\infty e^{-\frac{t}{\beta}} \odot g(t)dt = \int_0^\infty e^{-\frac{1}{\beta}t} \odot g(t)dt = F\left(\frac{1}{\beta}\right).$$

and

$$F(\beta) = \int_0^\infty e^{-\beta t} \odot g(t)dt = \int_0^\infty e^{-\frac{t}{\frac{1}{\beta}}} \odot g(t)dt = G\left(\frac{1}{\beta}\right).$$

□

Theorem 3.5. *Let us consider*

(i) $g(t)$ *be a continuous fuzzy function for all* $t \geq 0$;

(ii) $g(t)$ *be of exponential order* e^{dt} *i.e.*

$$D(g(t), \tilde{0}) \leq Le^{dt}, \quad t \in [0, T], \; L > 0;$$

(iii) $g'_{gH}(t)$ *be continuous in every finite closed interval* $0 \leq t \leq T$.

Then,

1. $Y_t[g'_{gH}(t)] = (-1) \odot g(0) \ominus_{gH} (-1)\frac{1}{\beta} \odot Y_t[g(t)]$;
2. $Y_t[g''_{gH}(t)] = (-1) \odot g'_{gH}(0) \ominus_{gH} \left(\frac{1}{\beta} \odot g(0) \ominus_{gH} \frac{1}{\beta^2} \odot Y_t[g(t)]\right)$,

for all $Re(\frac{1}{\beta}) > d$.

Proof. We prove case 1. Using definition of improper fuzzy integral and Theorem 2.2, we get

$$Y_t[g'_{gH}(t)] = \int_0^\infty e^{-\frac{t}{\beta}} \odot g'_{gH}(t)dt = \lim_{\xi\to\infty} \int_0^\xi e^{-\frac{t}{\beta}} \odot g'_{gH}(t)dt =$$
$$= \lim_{\xi\to\infty} (e^{-\frac{x}{\alpha}} \odot g(t)|_0^\xi) \ominus_{gH} (-1)\frac{1}{\alpha} \lim_{\xi\to\infty} \int_0^\xi e^{-\frac{t}{\beta}} \odot g(t)dt$$
$$= \lim_{\xi\to\infty} (e^{-\frac{\xi}{\beta}} \odot g(\xi)) \ominus_{gH} g(0) \ominus_{gH} (-1)\frac{1}{\beta} \int_0^\infty e^{-\frac{t}{\beta}} \odot g(t)dt.$$

From condition (ii), we obtain

$$\lim_{\xi\to\infty} D(e^{-\frac{\xi}{\beta}} \odot g(\xi), \tilde{0}) = \lim_{\xi\to\infty} e^{-\frac{\xi}{\beta}} D(g(\xi), \tilde{0}) \leq \lim_{\xi\to\infty} Le^{-(\frac{1}{\beta}-d)\xi} = 0. \tag{10}$$

Hence, by Proposition 2.1 and Equation (10), we have

$$Y_t[g'_{gH}(t)] = (-1) \odot g(0) \ominus_{gH} (-1)\frac{1}{\beta} \odot Y_t[g(t)]. \tag{11}$$

Similarly, from Equation (11) and Definition 3.1, we get

$$\begin{aligned} Y_t[g''_{gH}(t)] &= (-1) \odot g'_{gH}(0) \ominus_{gH} (-1)\tfrac{1}{\beta} \odot Y_t[g'_{gH}(t)] = \\ &= (-1) \odot g'_{gH}(0) \ominus_{gH} (-1)\tfrac{1}{\beta}\left((-1) \odot g(0) \ominus_{gH} (-1)\tfrac{1}{\beta} \odot Y_t[g(t)]\right) = \\ &= (-1) \odot g'_{gH}(0) \ominus_{gH} \left(\tfrac{1}{\beta} \odot g(0) \ominus_{gH} \tfrac{1}{\beta^2} \odot Y_t[g(t)]\right). \end{aligned}$$

□

Corollary 3.1. *Let $g(x,t)$ be a fuzzy function of two variables. Then, we have*

(i) $Y_t[g'_{t,gH}(x,t)] = (-1) \odot g(x,0) \ominus_{gH} (-1)\frac{1}{\beta} \odot Y_t[g(x,t)]$;

(ii) $Y_t[g''_{tt,gH}(x,t)] = (-1) \odot g'_{t,gH}(x,0) \ominus_{gH} \left(\frac{1}{\beta} \odot g(x,0) \ominus_{gH} \frac{1}{\beta^2} \odot Y_t[g(x,t)]\right)$.

4. Fuzzy Double Yang Transform

In the following section, we introduce the fuzzy double Yang transform (FDYT), that is, two fuzzy Yang transforms of order one. We give the fundamental properties and theorems related to the existence and fuzzy partial derivatives. Moreover, the fuzzy single convolution theorem is illustrated.

Definition 4.1. *The fuzzy double Yang transform of a fuzzy function $g : \mathbb{R}_+ \times \mathbb{R}_+ \to E^1$ is defined by*

$$G(\alpha, \beta) = Y_2[g(x,t)] = Y_x[Y_t[g(x,t)]] = \int_0^\infty \int_0^\infty e^{-\frac{x}{\alpha}-\frac{t}{\beta}} \odot g(x,t)dtdx, \tag{12}$$

provided that the improper fuzzy double integral exists. Here, α and β are complex numbers.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 March 2025 doi:10.20944/preprints202503.1587.v1

Definition 4.2. *The inverse fuzzy double Yang transform is given by*

$$Y_2^{-1}[G(\alpha,\beta)] = g(x,t) = \frac{1}{2\pi i}^2 \int_{a-i\infty}^{a+i\infty}\int_{b-i\infty}^{b+i\infty} e^{\frac{x}{\alpha}} e^{\frac{t}{\beta}} \odot G(\alpha,\beta) d\alpha d\beta, \tag{13}$$

where the function $G(\alpha,\beta)$ is analytic for all α and β such that $Re\alpha > a$ and $Re\beta > b$.

Theorem 4.1. *Let $g(x,t)$ be a continuous fuzzy function in $(0,X)\times(0,T)$ and $g(x,t)$ be of exponential order e^{cx+dt}, if it satisfies*

$$D(g(x,t),\tilde{0}) \leq Le^{cx+dt}, \ \ (x,t) \in [0,X]\times[0,T], \ L > 0.$$

Then, the fuzzy double Yang transform of the function $g(x,t)$ exists for all α and β such that $Re(\frac{1}{\alpha}) > c$ and $Re(\frac{1}{\beta}) > d$.

Proof. Using Definition 4.1 and the property of improper fuzzy double integral, we obtain

$$\begin{aligned} D(Y_2[g(x,t)],\tilde{0}) &= D(G(\alpha,\beta),\tilde{0}) = D(\int_0^\infty\int_0^\infty e^{-\frac{x}{\alpha}-\frac{t}{\beta}} \odot g(x,t)dxdt,\tilde{0}) \leq \\ &\leq \int_0^\infty\int_0^\infty e^{-\frac{x}{\alpha}-\frac{t}{\beta}} D(g(x,t),\tilde{0})dxdt \leq \\ &\leq L\int_0^\infty\int_0^\infty e^{-(\frac{1}{\alpha}-c)x}e^{-(\frac{1}{\beta}-d)t}dxdt = \frac{L\alpha\beta}{(1-c\beta)(1-d\beta)}. \end{aligned}$$

Thus, the improper fuzzy double integral converges for all $Re(\frac{1}{\alpha}) > c$ and $Re(\frac{1}{\beta}) > d$ and $Y_2[g(x,t)]$ exist. □

Double Yang transform of some important functions.

(i) $Y_2[1] = \alpha\beta$;

(ii) $Y_2[x^m t^n] = \begin{cases} \alpha^{m+1}\beta^{n+1} m!n! & m,n = 1,2,3,\ldots \\ \alpha^{m+1}\beta^{n+1}\Gamma(m+1)\Gamma(n+1) & m>0, n>0 \end{cases}$

(iii) $Y_2[e^{cx+dt}] = \frac{\alpha\beta}{(1-c\alpha)(1-d\beta)}$ for all $c,d\in\mathbb{R}$;

(iv) $Y_2[\sin(cx+dt)] = \frac{\alpha\beta}{(1+c^2\alpha^2)(1+d^2\beta^2)}$ for all $c,d\in\mathbb{R}$;

(v) $Y_2[\cos(cx+dt)] = \frac{cd\alpha^2\beta^2}{(1+c^2\alpha^2)(1+d^2\beta^2)}$ for all $c,d\in\mathbb{R}$.

Now, we present some properties for the FDYT.

Remark 4.1. *According to Theorem 3.2, we can proof that if $g_1(x,t)$ and $g_2(x,t)$ are fuzzy functions, then*

$$Y_2[\gamma_1 \odot g_1(x,t) \oplus \gamma_2 \odot g_2(x,t)] = \gamma_1 \odot Y_2[g(x,t)] \oplus \gamma_2 \odot Y_2[g(x,t)],$$

where $\gamma_1,\ \gamma_2 \in \mathbb{R}$ such that $\gamma_1,\ \gamma_2 \geq 0$ or $\gamma_1,\ \gamma_2 \leq 0$.

Theorem 4.2. *(Shifting) Let c and d be any constants and $g(x,t)$ be a continuous fuzzy function of two variables x and t. Then,*

$$Y_2[e^{-(cx+dt)} \odot g(x,t)] = G\left(\frac{\alpha}{1+c\alpha}, \frac{\beta}{1+b\beta}\right). \tag{14}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 March 2025 doi:10.20944/preprints202503.1587.v1

Proof. Using Definition 4.1, we have

$$\begin{aligned} Y_2[e^{-(cx+dt)} \odot g(x,t)] &= \int_0^\infty \int_0^\infty e^{-(cx+dt)} e^{-(\frac{x}{\alpha} - \frac{t}{\beta})} \odot g(x,t) dx dt = \\ &= \int_0^\infty \int_0^\infty e^{-x(\frac{1}{\alpha}+c) - t(\frac{1}{\beta}+d)} \odot g(x,t) dx dt = \\ &= \int_0^\infty \int_0^\infty e^{-x(\frac{1+c\alpha}{\alpha}) - t(\frac{1+b\beta}{\beta})} \odot g(x,t) dx dt = \\ &= G\left(\frac{\alpha}{1+c\alpha}, \frac{\beta}{1+b\beta}\right). \end{aligned}$$

□

Theorem 4.3. *(Heaviside Function) Let $g(x,t)$ be a continuous fuzzy function and*

$$H(x-\delta, t-\varepsilon) = \begin{cases} 1, & x > \delta, t > \varepsilon \\ 0, & x < \delta, t < \varepsilon, \end{cases}$$

where $H(x-\delta, t-\varepsilon)$ is the Heaviside function and $\delta, \varepsilon \in \mathbb{R}$. If $Y_2[g(x,t)] = G(\alpha, \beta)$, then

$$Y_2[H(x-\delta, t-\varepsilon) \odot g(x-\delta, t-\varepsilon)] = e^{-\frac{\delta}{\alpha} - \frac{\varepsilon}{\beta}} \odot G(\alpha, \beta).$$

Proof. Using Definition 4.1, we find

$$\begin{aligned} &Y_2[H(x-\delta, t-\varepsilon) \odot g(x-\delta, t-\varepsilon)] = \\ &= \int_0^\infty \int_0^\infty e^{-\frac{x}{\alpha} - \frac{t}{\beta}} H(x-\delta, t-\varepsilon) \odot g(x-\delta, t-\varepsilon) dx dt = \\ &= \int_\varepsilon^\infty \int_\delta^\infty e^{-\frac{x}{\alpha} - \frac{t}{\beta}} \odot g(x-\delta, t-\varepsilon) dx dt. \end{aligned}$$

We make a change of variable

$$\nu = x - \delta, \quad \mu = t - \varepsilon.$$

Then

$$x = \nu + \delta, \quad t = \mu + \varepsilon, \quad d\nu = dx, \quad d\mu = dt.$$

Hence

$$\begin{aligned} &Y_2[H(x-\delta, t-\varepsilon) \odot g(x-\delta, t-\varepsilon)] = \int_0^\infty \int_0^\infty e^{-\frac{\nu+\delta}{\alpha} - \frac{\mu+\varepsilon}{\beta}} \odot g(\nu, \mu) d\nu d\mu = \\ &= e^{-\frac{\delta}{\alpha} - \frac{\varepsilon}{\beta}} \int_0^\infty \int_0^\infty e^{-\frac{\nu}{\alpha} - \frac{\mu}{\beta}} \odot g(\nu, \mu) d\nu d\mu = e^{-\frac{\delta}{\alpha} - \frac{\varepsilon}{\beta}} \odot G(\alpha, \beta). \end{aligned}$$

□

Definition 4.3. *[27] If $k(t)$ and $g(x,t)$ are fuzzy Riemann integrable functions defined for all $x,\ t \geq 0$, then fuzzy convolution of $k(t)$ and $g(x,t)$ respect to t is given by*

$$(k * g)(x,t) = \int_0^t k(t-s) g(x,s) ds$$

and the symbol $*$ denotes the fuzzy convolution respect to t.

 doi:10.20944/preprints202503.1587.v1

Theorem 4.4. *(Convolution Theorem) Let $k : \mathbb{R}_+ \to E^1$ and $g : \mathbb{R}_+ \times \mathbb{R}_+ \to E^1$ be continuous fuzzy functions. Then the fuzzy double Yang transform of the convolution of these two functions is as*

$$Y_2[(k*g)(x,t)] = Y_t[k(t)]Y_2[g(x,t)]. \tag{15}$$

Proof. Using the definition of fuzzy double Yang transform and convolution, we find

$$\begin{aligned} Y_2[(k*g)(x,t)] &= \int_0^\infty\int_0^\infty e^{-\frac{x}{\alpha}-\frac{t}{\beta}} \odot (k*g)(x,t)dxdt = \\ &= \int_0^\infty\int_0^\infty e^{-\frac{x}{\alpha}-\frac{t}{\beta}} \odot \left(\int_0^t k(t-s)g(x,s)ds\right)dxdt. \end{aligned}$$

Let $\xi = t-s,\;\; d\xi = dt$. Then

$$\begin{aligned} Y_2[(k*g)(x,t)] &= \int_0^\infty\int_0^\infty e^{-\frac{x}{\alpha}-\frac{\xi+s}{\beta}} \odot \left(\int_0^\infty k(\xi)g(x,s)ds\right)dxd\xi = \\ &= \int_0^\infty e^{-\frac{\xi}{\beta}} \odot k(\xi)d\xi \int_0^\infty\int_0^\infty e^{\frac{x}{\alpha}-\frac{s}{\beta}} \odot g(x,s)dxds = \\ &= Y_t[k(t)]Y_2[g(x,t)]. \end{aligned}$$

□

Theorem 4.5. *Let $g(x,t)$ be a continuous fuzzy function and $Y_2[g(x,t)] = G(\alpha,\beta)$, then*

(i) $Y_2[g'_{x,gH}(x,t)] = (-1) \odot Y_t[g(0,t)] \ominus_{gH} (-1)\frac{1}{\alpha} \odot G(\alpha,\beta)$;

(ii) $Y_2[g'_{t,gH}(x,t)] = (-1) \odot Y_x[g(x,0)] \ominus_{gH} (-1)\frac{1}{\beta} \odot G(\alpha,\beta)$;

(iii) $Y_2[g''_{xx,gH}(x,t)] = (-1) \odot Y_t[g'_x(0,t)] \ominus_{gH} \left(\frac{1}{\alpha} \odot Y_t[g(0,t)] \ominus_{gH} \frac{1}{\alpha^2} \odot G(\alpha,\beta)\right)$;

(iv) $Y_2[g''_{tt,gH}(x,t)] = (-1) \odot Y_x[g'_t(x,0)] \ominus_{gH} \left(\frac{1}{\beta} \odot Y_x[g(x,0)] \ominus \frac{1}{\beta^2} \odot G(\alpha,\beta)\right)$;

(v) $Y_2[g''_{xt,gH}(x,y)] = (-1) \odot Y_x[g'_{x,gH}(x,0)] \ominus_{gH} \left(\frac{1}{\beta} \odot Y_t[g(0,t)] \ominus_{gH} \frac{1}{\alpha\beta} \odot G(\alpha,\beta)\right)$;

(vi) $S_2[g''_{tx,gH}(x,y)] = (-1) \odot Y_t[g'_{t,gH}(0,t)] \ominus_{gH} \left(\frac{1}{\alpha} \odot Y_x[g(x,0)] \ominus_{gH} \frac{1}{\alpha\beta} \odot G(\alpha,\beta)\right)$.

Proof. Using Theorem 3.5, we find

$$\begin{aligned} Y_2[g'_{x,gH}(x,t)] &= Y_t[Y_x[g'_{x,gH}(x,t)]] = Y_t\Big[(-1) \odot g(0,t) \ominus_{gH} (-1)\tfrac{1}{\alpha} \odot Y_x[g(x,t)]\Big] = \\ &= (-1) \odot Y_t[g(0,t)] \ominus_{gH} (-1)\tfrac{1}{\alpha} \odot Y_t[Y_x[g(x,t)]] = \\ &= (-1) \odot Y_t[g(0,t)] \ominus_{gH} (-1)\tfrac{1}{\alpha} \odot G(\alpha,\beta). \end{aligned}$$

In the same manner, we can obtain the case (iii).

$$\begin{aligned} &Y_2[g''_{xx,gH}(x,t)] = Y_t[Y_x[g''_{xx,gH}(x,t)]] = \\ &= Y_t\Big[(-1) \odot g'_{x,gH}(0,t) \ominus_{gH} \left(\tfrac{1}{\alpha} \odot g(0,t) \ominus_{gH} \tfrac{1}{\alpha^2} \odot Y_x[g(x,t)]\right)\Big] = \\ &= (-1) \odot Y_t[g'_{x,gH}(0,t)] \ominus_{gH} \left(\tfrac{1}{\alpha} \odot Y_t[g(0,t)] \ominus_{gH} \tfrac{1}{\alpha^2} \odot Y_t[Y_x[g(x,t)]]\right) = \\ &= (-1) \odot Y_t[g'_{x,gH}(0,t)] \ominus_{gH} \left(\tfrac{1}{\alpha} \odot Y_t[g(0,t)] \ominus_{gH} \tfrac{1}{\alpha^2} \odot G(\alpha,\beta)\right). \end{aligned}$$

The proof of case (v) is analogous to the proof of case (iii).

$$\Upsilon_2[g''_{xt,gH}(x,t)] = \Upsilon_x[\Upsilon_t[g''_{xt,gH}(x,t)]] =$$
$$= \Upsilon_x\Big[(-1)\odot g'_{x,gH}(x,0)\ominus_{gH}(-1)\tfrac{1}{\beta}\odot\Upsilon_t[g'_{x,gH}(x,t)]\Big] =$$
$$= (-1)\odot\Upsilon_x[g'_{x,gH}(x,0)]\ominus_{gH}(-1)\tfrac{1}{\beta}\odot\Upsilon_t[\Upsilon_x[g'_{x,gH}(x,t)]] =$$
$$= (-1)\odot\Upsilon_x[g'_{x,gH}(x,0)]\ominus_{gH}(-1)\tfrac{1}{\beta}\odot\Upsilon_t\Big[(-1)\odot g(0,t)\ominus_{gH}(-1)\tfrac{1}{\alpha}\odot\Upsilon_x[g(x,y)]\Big] =$$
$$= (-1)\odot\Upsilon_x[g'_{x,gH}(x,0)]\ominus_{gH}\Big(\tfrac{1}{\beta}\odot\Upsilon_t[g(0,t)]\ominus_{gH}\tfrac{1}{\alpha\beta}\odot S_y[S_x[f(x,y)]]\Big) =$$
$$= (-1)\odot\Upsilon_x[g'_{x,gH}(x,0)]\ominus_{gH}\Big(\tfrac{1}{\beta}\odot\Upsilon_t[g(0,t)]\ominus_{gH}\tfrac{1}{\alpha\beta}\odot G(\alpha,\beta)\Big).$$

□

5. Method of Fuzzy Double Yang Transform

To illustrate the use of FDYT, we solve the fuzzy parabolic Voltera integro-differential equation with a memory kernel $k:\mathbb{R}_+\to\mathbb{R}_+$. This equation is defined as

$$g'_{t,gH}(x,t)\oplus\int_0^t k(t-s)\odot g(x,t)dxdt = \sigma\odot g''_{xx,gH}(x,t)\oplus f(x,t),\quad x\ge 0,\quad t\ge 0, \tag{16}$$

where σ is any positive constant, $g(x,t)$ is the unknown fuzzy function, $f(x,t)$ is a given fuzzy function.

Assume initial conditions of

$$g(x,0)=\psi_0(x) \tag{17}$$

and boundary conditions of

$$g(0,t)=\varphi_0(t),\quad g'_x(0,t)=\varphi_1(t). \tag{18}$$

First, we apply FDYT to (16) as

$$\Upsilon_2\Big[g'_{t,gH}(x,t)\Big]\oplus\Upsilon_2\left[\int_0^t k(t-s)\odot g(x,t)dxdt\right] = \Upsilon_2\Big[\sigma\odot g''_{xx,gH}(x,t)\Big]\oplus\Upsilon_2[f(x,t)].$$

Using convolution theorem, we have

$$\Upsilon_2\Big[g'_{t,gH}(x,t)\Big]\oplus\Upsilon_t[k(t)]\odot\Upsilon_2[g(x,t)] = \sigma\odot\Upsilon_2\Big[g''_{xx,gH}(x,t)\Big]\oplus\Upsilon_2[f(x,t)].$$

The derivative properties of FDYT (Theorem 4.5) and the above equation yield

$$\begin{aligned}&(-1)\odot\Upsilon_x[g(x,0)]\ominus_{gH}(-1)\tfrac{1}{\beta}\odot G(\alpha,\beta)\oplus K(\beta)\odot G(\alpha,\beta) =\\ &= \sigma\odot\Big[(-1)\odot\Upsilon_t[g'_x(0,t)]\ominus_{gH}\Big(\tfrac{1}{\alpha}\odot\Upsilon_t[g(0,t)]\ominus_{gH}\tfrac{1}{\alpha^2}\odot G(\alpha,\beta)\Big)\Big]\oplus F(\alpha,\beta).\end{aligned} \tag{19}$$

where

$$K(\beta)=\Upsilon_t[k(t)],\quad G(\alpha,\beta)=\Upsilon_2[g(x,t)],\quad F(\alpha,\beta)=\Upsilon_2[f(x,t)].$$

Next apply the fuzzy Yang transform to the initial and boundary conditions

$$\Psi_0(\alpha)=\Upsilon_x[\psi_0(x)],\quad \Phi_0(\beta)=\Upsilon_t[\varphi_0(t)],\quad \Phi_1(\beta)=\Upsilon_t[\varphi_1(t)].$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 March 2025 doi:10.20944/preprints202503.1587.v1

and then put in (19) as

$$(-1) \odot \Psi_0(\alpha) \ominus_{gH} (-1)\tfrac{1}{\beta} \odot G(\alpha,\beta) \oplus K(\beta) \odot G(\alpha,\beta) =$$
$$= (-\sigma) \odot \Phi_1(\beta) \ominus_{gH} \left(\tfrac{\sigma}{\alpha} \odot \Phi_0(\beta) \ominus_{gH} \tfrac{\sigma}{\alpha^2} \odot G(\alpha,\beta)\right) \oplus F(\alpha,\beta).$$

Using Proposition 2.1, we have

$$\left(\tfrac{1}{\beta} + K(\beta) - \tfrac{\sigma}{\alpha^2}\right) \odot G(\alpha,\beta) =$$
$$= \Psi_0(\alpha) \oplus (-\sigma) \odot \Phi_1(\beta) \oplus \tfrac{(-\sigma)}{\alpha} \odot \Phi_0(\beta) \oplus F(\alpha,\beta).$$

Hence

$$G(\alpha,\beta) = B \odot \left[\Psi_0(\alpha) \oplus (-\sigma) \odot \Phi_1(\beta) \oplus \frac{(-\sigma)}{\alpha} \odot \Phi_0(\beta) \oplus F(\alpha,\beta)\right], \tag{20}$$

where

$$B = \frac{\alpha^2\beta}{\alpha^2 + \alpha^2\beta K(\beta) - \sigma\beta}. \tag{21}$$

Finally, take the inverse FDYT of (20) as

$$g(x,t) = Y_2^{-1}\left[B \odot \left(\Psi_0(\alpha) \oplus (-\sigma) \odot \Phi_1(\beta) \oplus \frac{(-\sigma)}{\alpha} \odot \Phi_0(\beta) \oplus F(\alpha,\beta)\right)\right].$$

6. Examples

Example 6.1. *Consider the following fuzzy parabolic Volterra integro-differential equation*

$$g'_{t,gH}(x,t) \oplus \int_0^t k(t-s) \odot g(x,t)dxdt = g''_{xx,gH}(x,t) \oplus f(x,t), \quad x \geq 0,\ t \geq 0 \tag{22}$$

with initial conditions

$$g(x,0,r) = x \odot (1,2,3) \tag{23}$$

and boundary conditions

$$g(0,t,r) = 0 \odot (1,2,3), \quad g'_x(0,t,r) = e^{-t} \odot (1,2,3). \tag{24}$$

In this case

$$\sigma = 1, \quad k(t) = 2e^t, \quad f(x,t) = \left(xe^t - 2xe^{-t}\right) \odot (1,2,3).$$

Then, we have

$$\psi_0(x) = x \odot (1,2,3), \quad \varphi_0(t) = 0 \odot (1,2,3), \quad \varphi_1(t) = e^{-t} \odot (1,2,3).$$

Hence

$$Y_t[k(t)] = K(\beta) = \frac{2\beta}{1-\beta} \odot (1,2,3), \quad Y_x[\psi_0(x)] = \Psi_0(\alpha) = \alpha^2 \odot (1,2,3),$$

$$Y_t[\varphi_0(t)] = \Phi_0(\beta) = 0 \odot (1,2,3), \quad Y_t[\varphi_1(t)] = \Phi_1(\beta) = \frac{\beta}{1+\beta} \odot (1,2,3),$$

$$Y_2[f(x,t) = F(\alpha,\beta) = \alpha^2\beta\left(\frac{1}{1-\beta} + \frac{2}{1+\beta}\right) \odot (1,2,3).$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 March 2025 doi:10.20944/preprints202503.1587.v1

By substituting the values of the fuzzy functions $K(\beta)$, $\Psi_0(\alpha)$, $\Phi_0(\beta)$ and $\Phi_1(\beta)$ in Equations (20) and (21), we obtain

$$G(\alpha,\beta) = B \odot \left[\alpha^2 \odot (1,2,3) \oplus \tfrac{(-\beta)}{1+\beta} \odot (1,2,3) \oplus \alpha^2\beta\left(\tfrac{1}{1-\beta} + \tfrac{2}{1+\beta}\right) \odot (1,2,3)\right],$$

where

$$B = \frac{\alpha^2\beta(1-\beta)}{\alpha^2(2\beta^2-\beta+1) - \beta(1-\beta)}.$$

Hence

$$G(\alpha,\beta) = B\left[\alpha^2 - \frac{\beta}{1+\beta} + \frac{\alpha^2\beta}{1-\beta} - \frac{2\alpha^2\beta}{1+\beta}\right] \odot (1,2,3) = \frac{\alpha^2\beta}{1+\beta} \odot (1,2,3).$$

Taking inverse fuzzy double Yang transform we find the solution of the equation (22) - (24) is

$$g(x,t) = Y_2^{-1}[G(\alpha,\beta)] = Y_2^{-1}\left[\frac{\alpha^2\beta}{1+\beta} \odot (1,2,3)\right] = xe^{-t} \odot (1,2,3).$$

7. Conclusions

In this research paper, we introduce a new fuzzy integral transformation called the fuzzy double Yang transform, which is defined with the help of the fuzzy unitary Yang transform. We find conditions for its existence and establish some of its basic properties. We proved theorems about partial derivatives and fuzzy unit convolution. Using these new results, we successfully obtained the exact solution of a fuzzy parabolic Volterra integro-differential equation with symmetric memory kernel. We constructed a numerical example to verify the application of the new method. As a result, we propose that this method is further expanded in future work, so that it can be applied to the solution of various nonlinear fuzzy partial integro-differential equations related to physical and engineering problems.

Author Contributions: Conceptualization, A.G., S.C., M.V.; methodology, A.G., S.C., M.V; validation, A.G., S.C. M.V.,; formal analysis, A.G., S.C., M.V., Y.G.; writing–original draft preparation, A.G., S.C., M.V., Y.G.; writing–review and editing A.G., S.C., M.V., Y.G.; funding acquisition A.G., S.C., M.V., Y.G.

Funding: This study is financed by the European Union-Next Generation EU through the National Recovery and Resilience Plan of the Republic of Bulgaria, project DUECOS BG-RRP-2.004-0001-C01.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declare no conflict of interest.

References

1. Habetler, G. J.; Schiffman, R. L. A finite difference method for analyzing the compression of poro-viscoelastic media. *Computing* **1970**, *6*, 342–348.
2. Yanik, E. G.; Fairweather, G. Finite element methods for parabolic and hyperbolic partial integro-differential equations. *Nonlinear Anal. Theory Methods Appl.* **1988**, *12*, 785–809.
3. Souplet, P. Blow-up in nonlocal reaction-diffusion equations. *SIAM J. Math. Anal.* **1998**, *29*, 1301–1334.
4. Dehghan, M.; Shokri, A. A numerical method for two-dimensional Schrödinger equation using collocation and radial basis functions. *Comput. Math. Appl.* **2007**, *54*, 136–146.
5. Dehghan, M.; Shakeri, F. Solution of parabolic integro-differential equations arising in heat conduction in materials with memory via He's variational iteration technique. *Int. J. Numer. Methods Biomed. Eng.* **2010**, *26*, 705–715.
6. Almeida, C. G.; Meyer, J. F. and Vasconcelos, H. L. Fuzzy parameter in a partial differential equation model for population dispersal of Leaf-Cutting Ants, Nonlinear Analysis: Real World Applications, **2011**, *12*, 3397–3412.
7. Leite, J.; Cecconello, M.; Leite, J.; Bassanezi, R. C. On Fuzzy Solutions for Diffusion Equation, Journal of Applied Mathematics, **2015**, Article ID 874931, 1–10.
8. Di Martino, F.; Perfilieva, I.; Sessa, S. A Summary of F-Transform Techniques in Data Analysis. *Electronics* **2021**, *10*, 1771.

 doi:10.20944/preprints202503.1587.v1

9. Kadham, S.; Alkiffai, A. Model Tumor Response to Cancer Treatment Using Fuzzy Partial SH-Transform: An Analytic Study. *Int. J. of Math. and Comput. Science*, **2022**, *18*, 23–28.
10. Napole, C.; Derbeli, M.; Barambones, O. Experimental Analysis of a Fuzzy Scheme against a Robust Controller for a Proton Exchange Membrane Fuel Cell System. *Symmetry* **2022**, *14*, 139.
11. Routaray, M.; Sahu, P. K.; Chalishajar, D. N. The Fuzzy Differential Transform Method for the Solution of the System of Fuzzy Integro-Differential Equations Arising in Biological Model. *Mathematics*, **2023**, *11*, 3840.
12. Bede, B.; Rudas, I. J. and Bencsik, A. L. First order linear fuzzy differential equations under generalized differentiability, *Information Sciences*, **2007**, *177*, 1648–1662.
13. Nieto, J.; Khastan, A. and Ivaz, K. Numerical solution of fuzzy differential equations under generalized differentiability, *Nonlinear Analysis: Hybrid Systems*, **2009**, *3*, 7000–7007.
14. Ullah, Abd; Ullah, Aman; Ahmad, Shabir and Van Hoa, Ngo. Fuzzy Yang transform for second order fuzzy differential equations of integer and fractional order. *Physica Scripta* **2023**, *98*, 044003.
15. Osman, M.; Gong, Z.; Mustafa, A. Comparison of fuzzy Adomian decomposition method with fuzzy VIM for solving fuzzy heat-like and wave-like equations with variable coefficients. *Advances in Difference Equations* **2020**, 327.
16. Stabestari, R. M.; Ezzati, R. The Fuzzy Double Laplace Transforms and their Properties with Applications to Fuzzy Wave Equation. *New Math. and Natural Comp.* **2021**, *17*, 319-338.
17. Kshirsagar, K. A.; Nikam, V. R.; Gaikwad, S. B.; Tarate, S. A. The double fuzzy Elzaki transform for solving fuzzy partial differential equations. *Journal of the Chungcheong mathematical society* **2022**, *35*, 2.
18. Osman, M.; Xia, Y.; Omer, O. A.; Hamoud, A. On the Fuzzy Solution of Linear-Nonlinear Partial Differential Equations. *Mathematics* **2022**, *10*, 2295.
19. Abdeljawad, T.; Younus, A.; Alqudah, M. A.; Atta, U. On Fuzzy Conformable Double Laplace Transform with Applications to Partial Differential Equations. *Computer Modeling in Engineering and Sciences* **2023**, *134*, 2163-2191.
20. Gumah, G.; Naser, M.; Al-Smadi, M. and Al-Omari, S. Application of reproducing kernel Hilbert space method for solving second order fuzzy Volterra integro-differential equations. *Advances in Difference Equations*, **2018**, *475*, 1-15.
21. Ullaha, Z.; Ahmada, S.; Ullaha, A. and Akgul, A. On solution of fuzzy Volterra integro-differential equations. *Arab Journal of Basic and Applied Sciences*, **2021**, *28*, 330–339.
22. Araour, M.; Mennouni, A. A New Procedures for Solving Two Classes of Fuzzy Singular Integro-Differential Equations: Airfoil Collocation Methods. *Int. J. Appl. Comput. Math.* **2022**, *8*, 35.
23. Gumah, G. Reproducing kernel Hilbert space method to solve fuzzy partial Volterra integro-differential equations. *Filomat*, **2024**, *38:24*, 8553–8564.
24. Qahremania, E.; Allahviranlooa, T.; Abbasbandyc, S. and Ahmadyd, N. A study on the fuzzy parabolic Volterra partial integro-differential equations. *Journal of Intelligent and Fuzzy Systems* **2021**, *40(1)*, 1639-1654.
25. Georgieva, A.; Spasova, M. Solving partial fuzzy integro-differential equations using fuzzy Sumudu transform method. *AIP Conf. Proc.* **2021**, *2321*, 030010-1–030010-8.
26. Georgieva, A. Double Fuzzy Sumudu transform to solve partial Volterra fuzzy integro-differential equations. *Mathematics* **2020**, *8*, 692.
27. Georgieva, A.; Spasova, M. Solution of partial fuzzy integro-differential equations by double natural transform. *AIP Conf. Proc.* **2022**, *2459*, 030012-1–030012-8.
28. Yang, X. J. A new integral transform method for solving steady heat-transfer problem. *Therm. Sci.* **2016**, *20*, 639–642.
29. Armand, A.; Allanviranloo, T. and Gouyandeh, Z. Some fundamental results on fuzzy calculus. *Iranian Journal of Fuzzy Systems* **2018**, *15 (3)*, 27-46.
30. Bede, B.; Stefanini, L. Generalized differentiability of fuzzy-valued functions. *Fuzzy Sets Syst.* **2013**, *230*, 119-141.
31. Anastassiou, G. A. Fuzzy Mathematics: Approximation Theory. *Studies in Fuzziness and Soft Computing*, Springer–Verlag Berlin Heidelberg, **2010**.
32. Allahviranloo, T.; Gouyndeh, Z.; Ahmand, A. and Hasanoglu, A. On fuzzy solutions of heat equation based on generalized Hukuhara differentiability. *Fuzzy Sets Syst.* **2015**, *265*, 1-23.
33. Gouyandeh, Z.; Allahviranloo, T.; Abbasbandy, S.; Armand, A. A fuzzy solution of heat equation under generalized Hukuhara differentiability by fuzzy Fourier transform. *Fuzzy Sets Syst.* **2017**, *309*, 81-97.

 doi:10.20944/preprints202503.1587.v1