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On the Omnidimensional Convex Polytopes and *n*-Balls in Negative, Fractional and Complex Dimensions

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Abstract: The study shows that the recurrence relations defining volumes and surfaces of omnidimensional convex polytopes and *n*-balls are continuous and defined for complex *n*, whereas in the indefinite points their values are given in the sense of a limit of a function. The volume of an *n*-simplex is a bivalued function for $n < 0$, and thus the surfaces of *n*-simplices and *n*-orthoplices are also bivalued functions for $n < 1$. Applications of these formulas to the omnidimensional polytopes inscribed in and circumscribed about *n*-balls reveal previously unknown properties of these geometric objects in negative, real dimensions. In particular for $0 < n < 1$ the volumes of the omnidimensional polytopes are larger than the volumes of circumscribing *n*-balls, while their volumes and surfaces are smaller than the volumes of inscribed *n*-balls. Specific products and quotients of volumes and surfaces of the omnidimensional polytopes and *n*-balls are shown to be independent of the gamma function.

Keywords: regular basic convex polytopes; circumscribed and inscribed polytopes; negative dimensions; fractal dimensions; complex dimensions

1. Introduction

In *n*-dimensional space, *n*-dimensional objects have ($n-1$)-dimensional surfaces which have a dimension of volume in ($n-1$)-dimensional space. However, this sequence has a singularity at $n = -1$. A 0-dimensional point in 0-dimensional space has a vanishing (−1)-surface being a vanishing volume of the (−1)-dimensional void. But the surface of the (−1)-dimensional void is not (−2)-dimensional. It is undefined. This discontinuity, along with the recently discovered [1] reflection relations around zero for volumes and surfaces of *n*-cubes inscribed inside *n*-balls, hint that thinking about dimension in terms of a point on a number axis, with negative dimensions being simply analytic continuations from positive dimensions [2], may be misleading. Thinking in terms of dimension as a point on a number semiaxis, similarly to a point on a radius, seems more appropriate. Thus *n*-dimension corresponds to ($-n$)-dimension. Considering dimension of a set as the length exponent at which that set can be measured [3] makes the negative dimensions to refer to densities as positive ones refer to quantities [3]. Thus, (−2)-dimensional pressure, for example, considered in terms of a density (e.g. in units of kg/m^2) corresponds to 2-dimensional area (e.g. in units of m^2) that it acts upon. Following the same logic, gravitational force $F = GMm/R^2$ acting towards a center enclosed by a 2-dimensional surface is (−2)-dimensional, whereas centripetal force $F = mV^2/R^1$ acting towards a center enclosed by a 1-dimensional perimeter is (−1)-dimensional.

This study extends the prior research [1] presenting novel recurrence relations for volumes and surfaces of *n*-balls, regular *n*-simplices, and *n*-orthoplices. It was signaled in the prior research that these recurrence relations are continuous on their domains of definitions for $n \in \mathbb{R}$, whereas the starting points for fractional dimensions can be provided, e.g., using spline interpolation between two (or three in the case of n-balls) subsequent integer dimensions. It was also conjectured that for $0 < n < 1$ volumes of n-cubes inscribed inside n-balls are larger than volumes of those n-balls.

This study shows that the recurrence relations of the prior research [1] are continuous, whereas their values at the singular points can be given in the sense of a limit of a function. The properties of the three omnidimensional, regular, convex polytopes, present for all $n \in \mathbb{N}_0$ [4], including inscribed in and circumscribed about n-balls are presented. It is shown that for $0 < n < 1$ their volumes are larger than volumes of circumscribing n-balls, while their volumes and surfaces are smaller than volumes of inscribed n-balls.

The paper is structured as follows. Section 2 summarizes known formulas for omnidimensional, regular, convex polytopes in natural dimensions that are employed in the further sections of the paper. In Section 3 it is shown that these recurrence relations can be naturally extended to complex, continuous dimensions, yielding complex values, as illustrated in Section 4 in the example of n-balls. Section 5 examines the properties of the omnidimensional, regular, convex polytopes inscribed in and circumscribed about n-balls for $n \in \mathbb{R}$. Section 6 hints possible applications and concludes the findings of this paper.

2. Known Formulas for Omnidimensional, Convex Polytopes and *n*-Balls

It is known that the volume of an n-ball (B) is

$$V_n(R)_B = \frac{\pi^{n/2}}{\Gamma(n/2+1)} R^n , \tag{1}$$

where $\Gamma(\mathbb{C} \to \mathbb{C})$ is the Euler's gamma function and R denotes the n-ball radius. This implies that volumes of n-balls are complex in complex dimensions (cf. Section 4). The volume of an n-ball can be expressed [5] in terms of the volume of an $(n-2)$-ball of the same radius as a recurrence relation

$$V_n(R)_B = \frac{2\pi R^2}{n} V_{n-2}(R)_B , \tag{2}$$

where $V_0(R)_B := 1$ and $V_1(R)_B := 2R$. It was shown in the prior research [1] that the relation (2) can be extended into negative dimensions as

$$V_n(R)_B = \frac{n+2}{2\pi R^2} V_{n+2}(R)_B , \tag{3}$$

solving (2) for V_{n-2} and assigning new $n \in \mathbb{Z}$ as the previous $n-2$. A radius recurrence relation

$$f_n \doteq \frac{2}{n} f_{n-2} , \tag{4}$$

defined [1] for $n \in \mathbb{N}$, where $f_0 := 1$ and $f_1 := 2$, allows for expressing the volume n-ball as

$$V_n(R)_B \doteq f_n \pi^{\lfloor n/2 \rfloor} R^n , \tag{5}$$

where “⌊x⌋“ denotes the floor function giving the greatest integer less than or equal to its argument x. The relation (4) can be analogously as formula (2) extended [1] into negative dimensions as

$$f_n = \frac{n+2}{2} f_{n+2}, \tag{6}$$

which allows to define $f_{-1} := 1$, $f_0 := 1$ to initiate (4) or (6). Known [5] surface of an n-ball is

$$S_n(R)_B = \frac{n}{R} V_n(R)_B. \tag{7}$$

Known [6, 7] volume of a regular n-simplex (S) having the edge length A is

$$V_n(A)_S = \frac{\sqrt{n+1}}{n!\sqrt{2^n}} A^n. \tag{8}$$

The formula (8) can be written [1] as a recurrence relation

$$V_n(A)_S \doteq A V_{n-1}(A)_S \sqrt{\frac{n+1}{2n^3}}, \tag{9}$$

with $V_0(A)_S := 1$, to remove the indefiniteness of the factorial for $n < 1$. Formula (9) can be solved for V_{n-1}. Assigning new $n \in \mathbb{Z}$ as the previous $n - 1$, yields [1]

$$V_n(A)_S = \frac{V_{n+1}(A)_S}{A} \sqrt{\frac{2(n+1)^3}{n+2}}, \tag{10}$$

which also removes the singularity for $n = 0$ present in known formula (8). Aby n-simplex has $n + 1$ ($n - 1$)-facets [5]. Therefore, its surface is

$$S_n(A)_S = (n+1) V_{n-1}(A)_S. \tag{11}$$

Known [5] volume of n-orthoplex (O) is

$$V_n(A)_O = \frac{\sqrt{2^n}}{n!} A^n. \tag{12}$$

Formula (12) can be written [1] as a recurrence relation

$$V_n(A)_O \doteq A V_{n-1}(A)_O \frac{\sqrt{2}}{n}, \tag{13}$$

with $V_0(A)_O := 1$, to remove the indefiniteness of the factorial for $n < 1$. Solving (13) for V_{n-1} and assigning new $n \in \mathbb{Z}$ as the previous $n - 1$, yields [1]

$$V_n(A)_O = V_{n+1}(A)_O \frac{n+1}{A\sqrt{2}}, \tag{14}$$

which also removes singularity for $n = 0$ present in formula (12). Any n-orthoplex has 2^n facets [5], which are regular $(n-1)$-simplices. Therefore, its surface is

$$S_n(A)_O = 2^n V_{n-1}(A)_S. \tag{15}$$

3. Continuous Recurrence Relations in Complex Dimensions

The recurrence relations presented in the preceding section can be naturally extended to complex, continuous dimensions.

Theorem 1.

Recurrence relations (2), (3), (5) (n-balls) are continuous for $n \in \mathbb{C}$, wherein for $n = -2k - 2$, $k \in \mathbb{N}_0$ their values are given in the sense of a limit of a function.

Proof 1.

Comparing (1) with (3) and setting $m = n + 2$ and $k = m/2$, yields the n-ball volume

$$\begin{gathered} V_n(R)_B = \frac{\pi^{n/2}}{\Gamma(n/2+1)} R^n = \frac{n+2}{2\pi R^2} V_{n+2}(R)_B \\ V_{n+2}(R)_B = \frac{\pi^{n/2} 2\pi^{2/2}}{(n+2)\Gamma(n/2+1)} R^{n+2} \quad V_m(R)_B = \frac{\pi^{m/2} 2}{m\Gamma(m/2)} R^m = \frac{\pi^k}{k\Gamma(k)} R^{2k}, \\ V_n(R)_B = \frac{\pi^{n/2}}{\Gamma(n/2+1)} R^n \quad V_n(D)_B = \frac{\pi^{n/2}}{2^n\Gamma(n/2+1)} D^n \end{gathered} \tag{16}$$

which recovers (1), as $n\Gamma(n/2)/2 = \Gamma(n/2 + 1)$ for $n > 0$, $n \in \mathbb{C}$. On the other hand, (3) corresponds to (5)

$$\begin{gathered} V_n(R)_B = \frac{n+2}{2\pi R^2} V_{n+2}(R)_B = \frac{n+2}{2} f_{n+2} \pi^{\lfloor n/2 \rfloor} R^n \\ V_{n+2}(R)_B = \pi^1 f_{n+2} \pi^{\lfloor n/2 \rfloor} R^{n+2} \quad V_m(R)_B = f_m \pi^{1+\lfloor (m-2)/2 \rfloor} R^m = f_m \pi^{\lfloor m/2 \rfloor} R^m \end{gathered}, \tag{17}$$

for $n \in \mathbb{C}$, which completes the proof. □

Also

$$\lim_{n \to -2k-2,\, k \in \mathbb{N}_0} \pi^{n/2} D^n 2^{-n} \frac{1}{\Gamma(n/2+1)} = a \cdot 0 = 0, \tag{18}$$

where $a \neq 0$, $a \in \mathbb{C}$.

Using (7) and (16) the surface of an n-balls is given by

$$S_n(D)_B = \frac{2^{1-n} n \pi^{n/2}}{\Gamma(n/2+1)} D^{n-1}. \tag{19}$$

Theorem 2.

Recurrence relations (9), (10) (regular n-simplices) are continuous for $n \in \mathbb{C}$, wherein for $n = -k - 1$, $k \in \mathbb{N}_0$ their values are given in the sense of a limit of a function.

Proof 2.

Expressing the factorial in (8) by the gamma function, comparing (8) with (10), and setting $m = n + 1$, yields the regular n-simplex volume

$$\begin{gathered} V_n(A)_S = \frac{\sqrt{n+1}}{n!\sqrt{2^n}} A^n = \frac{\sqrt{n+1}}{\Gamma(n+1) 2^{n/2}} A^n = \frac{V_{n+1}(A)_S}{A} \sqrt{\frac{2(n+1)^3}{n+2}} \\ V_{n+1}(A)_S = \frac{\sqrt{n+1}\sqrt{n+2}}{\Gamma(n+1) 2^{(n+1)/2} \sqrt{(n+1)^3}} A^{n+1} \\ V_m(A)_S = \frac{m\sqrt{m}\sqrt{m+1}}{\Gamma(m+1) 2^{m/2}\sqrt{m^3}} A^m \quad V_n(A)_S = \frac{\sqrt{n+1}}{\Gamma(n+1) 2^{n/2}} A^n \begin{cases} 1 & n \geq 0 \\ \pm 1 & n < 0 \end{cases} \end{gathered}, \tag{20}$$

which recovers (8), as $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$, and completes the proof. □

Also

$$\lim_{n \to -k-1,\, k \in \mathbb{N}_0} 2^{-n/2} A^n \sqrt{n+1} \frac{1}{\Gamma(n+1)} = a \cdot 0 = 0, \tag{21}$$

where $a \in \mathbb{C}$.

For $n < -1$ n-simplex volume formula (20) is imaginary and for $n < 0$ it is a bivalued function, as $n\sqrt{n}/\sqrt{n^3} = 1$ only for $n \in \mathbb{R}$, $n > 0$. Thus, its general form, involving principal branch for $n \geq 0$ and the 2nd branch for $n < 0$ is

$$V_n(A)_S = \frac{\sqrt{n+1}}{\Gamma(n+1) 2^{n/2}} A^m \frac{n\sqrt{n}}{\sqrt{n^3}}. \tag{22}$$

Using (11) and (20) the surface of a regular n-simplex is given by

$$S_n(A)_S = \frac{n^{3/2}(n+1)}{\Gamma(n+1) 2^{(n-1)/2}} A^{n-1} \begin{cases} 1 & n \geq 1 \\ \pm 1 & n < 1 \end{cases}. \tag{23}$$

For $n < 0$ n-simplex surface formula (23) is imaginary and for $n < 1$ it is a bivalued function, as $(n-1)\sqrt{(n-1)}/\sqrt{(n-1)^3} = 1$ only for $n \in \mathbb{R}$, $n > 1$. Thus, its general form is

$$S_n(A)_S = \frac{n^{3/2}(n+1)}{\Gamma(n+1)\,2^{(n-1)/2}} A^{n-1} \frac{(n-1)\sqrt{n-1}}{\sqrt{(n-1)^3}}. \tag{24}$$

Theorem 3.

Recurrence relations (13), (14) (n-orthoplices) are continuous for $n \in \mathbb{C}$, wherein for $n = -k - 1$, $k \in \mathbb{N}_0$ their values are given in the sense of a limit of a function.

Proof 3.

Expressing the factorial in (12) by the gamma function, comparing (12) with (14), and setting $m = n + 1$, yields the n-orthoplex volume

$$\begin{aligned} V_n(A)_O &= \frac{\sqrt{2^n}}{n!} A^n = \frac{\sqrt{2^n}}{\Gamma(n+1)} A^n = V_{n+1}(A)_O \frac{n+1}{A\sqrt{2}} \\ V_{n+1}(A)_O &= \frac{2^{(n+1)/2}}{(n+1)\Gamma(n+1)} A^{n+1} \\ V_m(A)_O &= \frac{\sqrt{2^m}}{m\Gamma(m)} A^m \quad V_n(A)_O = \frac{2^{n/2}}{\Gamma(n+1)} A^n \end{aligned}, \tag{25}$$

which recovers (12), as $n\Gamma(n) = \Gamma(n + 1)$ for $n \in \mathbb{C}\backslash\{n \in \mathbb{Z}, n \leq -1\}$ and $\Gamma(n + 1) = n!$ for $n \in \mathbb{N}_0$, and completes the proof. □

Using (15) and (25) the surface of an n-orthoplex is given by

$$S_n(A)_O = \frac{n^{3/2}\,2^{(n+1)/2}}{\Gamma(n+1)} A^{n-1} \begin{cases} 1 & n \geq 1 \\ \pm 1 & n < 1 \end{cases}. \tag{26}$$

For $n < 0$, $n \notin \mathbb{Z}$ n-orthoplex surface formula (26) is imaginary and for $n < 1$ it is a bivalued function, as $(n - 1)\sqrt{(n - 1)}/\sqrt{(n - 1)^3} = 1$ only for $n \in \mathbb{R}$, $n > 1$. Thus, its general form, involving two branches, is

$$S_n(A)_O = \frac{n^{3/2}\,2^{(n+1)/2}}{\Gamma(n+1)} A^{n-1} \frac{(n-1)\sqrt{n-1}}{\sqrt{(n-1)^3}}. \tag{27}$$

Continuous recurrence relations (16)-(27) are shown in Figure 1, along with the integer recurrence relations (2)-(15).

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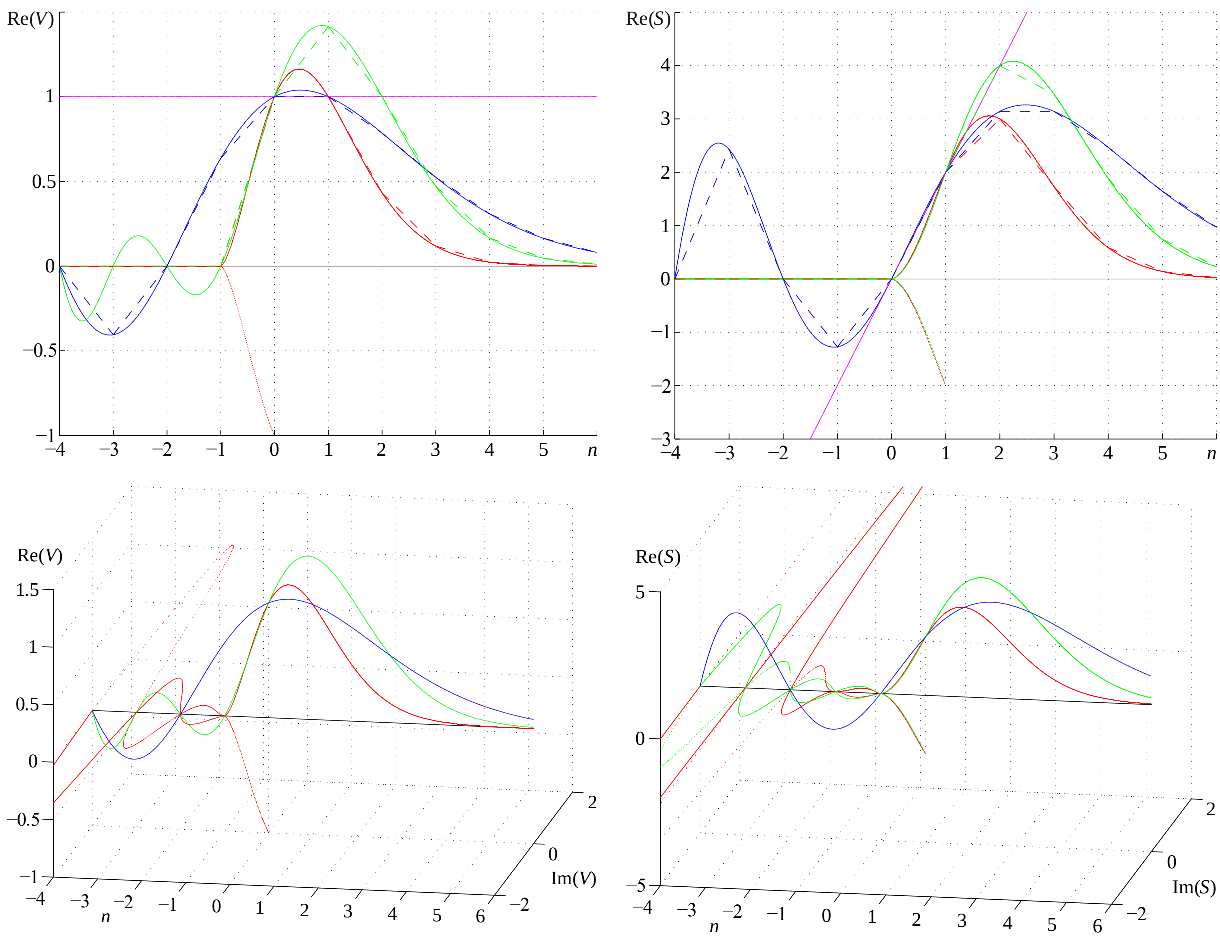

Figure 1. Graphs of volumes (V) and surfaces (S) of unit edge length regular n-simplices (red), n-orthoplices (green), n-cubes (pink), and unit diameter n-balls (blue), along with the integer recurrence relations (dashed lines) and the 2nd branches (dotted lines) for n = [−4, 6].

4. The volume of an *n*-Ball in Complex Dimensions

The gamma function is defined for all complex numbers except the non-positive integers. Therefore the volumes and surfaces (16)-(27) of n-balls and omnidimensional polytopes containing the gamma function are also defined for all $n = a + ib \in \mathbb{C}$. For example, in the case of n-balls [9]

$$\pi^{n/2} = \pi^{(a+ib)/2} = \pi^{a/2}\left[\cos\left(\frac{b}{2}\ln(\pi)\right) + i\sin\left(\frac{b}{2}\ln(\pi)\right)\right], \tag{28}$$

$$R^n = R^{a+ib} = R^a\left[\cos\left(b\ln(R)\right) + i\sin\left(b\ln(R)\right)\right], \tag{29}$$

and the volume (1) and surface (7) become

$$V_n(R)_B = \pi^{\frac{a}{2}} R^a \frac{\left\{\cos\left[b\ln\left(R\sqrt{\pi}\right)\right] + i\sin\left[b\ln\left(R\sqrt{\pi}\right)\right]\right\}}{\Gamma\left(\frac{a+ib}{2}+1\right)}, \tag{30}$$

$$S_n\left(R\right)_B=\left(a+ib\right)\pi^{\frac{a}{2}}R^{a-1}\frac{\left\{\cos\left[b\ln\left(R\sqrt{\pi}\right)\right]+i\sin\left[b\ln\left(R\sqrt{\pi}\right)\right]\right\}}{\Gamma\left(\frac{a+ib}{2}+1\right)}, \qquad (31)$$

where we have used cos(a)cos(b) − sin(a)sin(b) = cos(a + b) and cos(a)sin(b) + sin(a)cos(b) = sin(a + b), as shown in Figure 2 for unit radius n-balls.

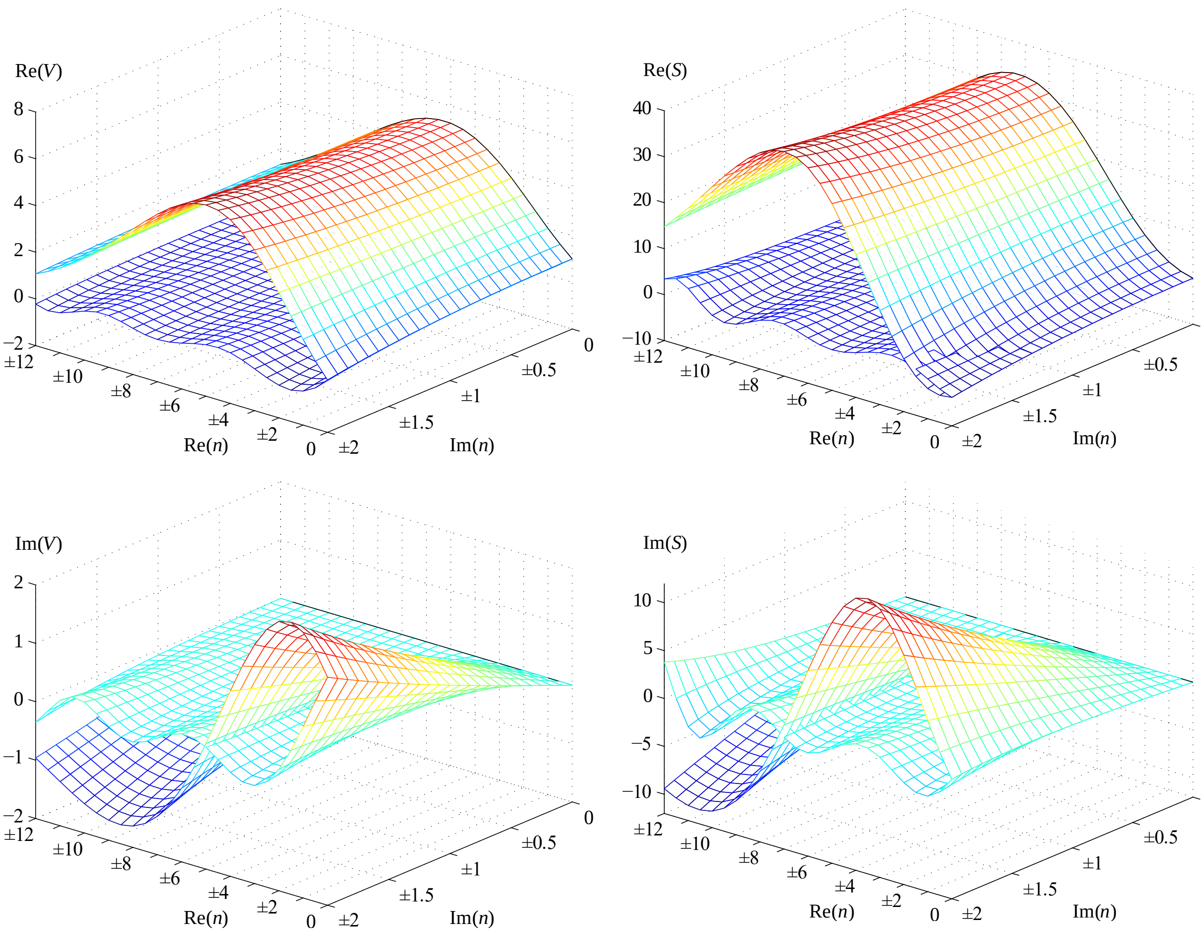

Figure 2. Graphs of complex volumes (V) and surfaces (S) of unit radius n-balls in complex dimensions $n = a + ib$ for a = [0, ±12], b = [0, ±2].

In particular for $n = 3 + ib$, $b \in \mathbb{R}$ (spacetime dimensionality) equation (30) becomes

$$V_n\left(R\right)_B=\pi^{\frac{3}{2}}R^{3}\frac{\left\{\cos\left[b\ln\left(R\sqrt{\pi}\right)\right]+i\sin\left[b\ln\left(R\sqrt{\pi}\right)\right]\right\}}{\Gamma\left(\frac{3+ib}{2}+1\right)}, \qquad (32)$$

which reduces to familiar $V_3(R)_B = 4\pi R^3/3$ for $n = 3 + 0i$, i.e. at the present moment. Note that the anti-symmetry of the imaginary part of the volume (30), in a way, establishes the arrow of time and is independent on Re(n) for Im(n) = 0.

5. Basic Regular Polytopes Inscribed in and Circumscribed About *n*-Balls

Each of the three regular polytopes can be inscribed in and circumscribed about an *n*-ball, and this is considered in this section on the basis of the continuous relations presented in Section 3.

5.1 Regular n-Simplices Inscribed in and Circumscribed About n-Balls

The diameter D_{BCS} of an *n*-ball circumscribed about a regular *n*-simplex (*BCS*) is known [7] to be

$$D_{BCS} = \frac{\sqrt{2n}}{\sqrt{n+1}} A, \tag{33}$$

where *A* is the edge length. Hence, the edge length A_{SIB} of a regular *n*-simplex inscribed (*SIB*) inside an *n*-ball (*B*) with diameter *D* is

$$A_{SIB} = \frac{\sqrt{n+1}}{\sqrt{2n}} D, \tag{34}$$

so that the regular *n*-simplex volume (20) becomes

$$V_{SIB} = \frac{n^{-n/2}\left(n+1\right)^{(n+1)/2}}{\Gamma\left(n+1\right)2^n} D^n \begin{cases} 1 & n \geq 0 \\ \pm 1 & n < 0 \end{cases}. \tag{35}$$

For $n < -1$ the inscribed *n*-simplex volume (35) is imaginary (as $n^{-n/2}$ introduces the imaginary unit for $n < 0$) and divergent with *n* approaching negative infinity, for $n < 0$ it is a bivalued function of *n*, and is complex for $-1 < n < 0$, with the real part being equal to the imaginary part for $n = -1/2$. It is zero for $n = -k$, $k \in \mathbb{N}$ and for $0 < n < 1$ it is larger than the volume of the circumscribing *n*-ball.

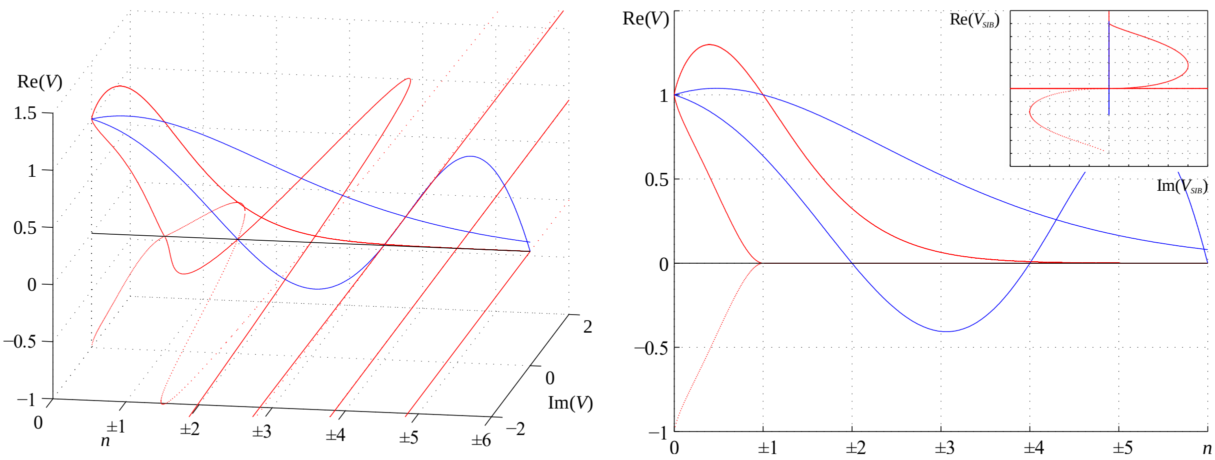

Figure 3. Graphs of volumes (*V*) of regular *n*-simplices (red) inscribed in unit diameter *n*-balls and volumes of unit diameter *n*-balls (blue) for *n* = [0, ±6] (inset for *n* = [−1, 0]).

Similarly, the surface (23) of a regular inscribed *n*-simplex with edge length *A* given by (34) is

$$S_{SIB} = \frac{n^{(4-n)/2}\left(n+1\right)^{(n+1)/2}}{\Gamma\left(n+1\right)2^{n-1}} D^{n-1} \begin{cases} 1 & n \geq 1 \\ \pm 1 & n < 1 \end{cases}, \tag{36}$$

as shown in in Figure 4. For $n < -1$ the inscribed n-simplex surface (36) is imaginary and divergent with n approaching negative infinity, for $n < 1$ it is a bivalued function of n, and is complex for $-1 < n < 0$, with the real part being equal to the imaginary part for $n = -1/2$. It is zero for $n = -k$, $k \in \mathbb{N}_0$.

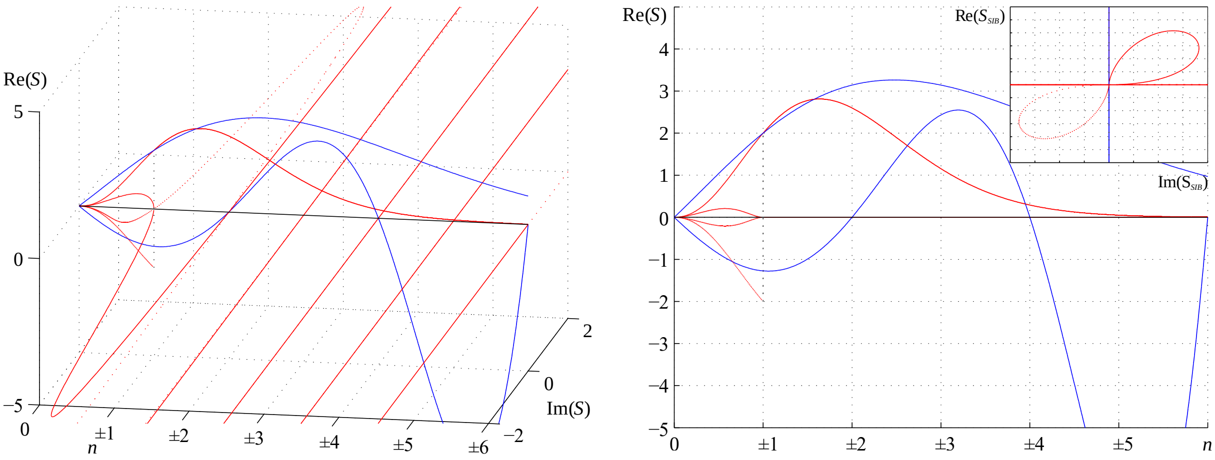

Figure 4. Graphs of surfaces (S) of regular n-simplices (red) inscribed in unit diameter n-balls and surfaces of unit diameter n-balls (blue) for $n = [0, \pm 6]$ (inset for $n = [-1, 0]$).

The diameter D_{BIS} of an n-ball inscribed inside a regular n-simplex (BIS) is known [7] to be

$$D_{BIS} = \frac{\sqrt{2}}{\sqrt{n}\sqrt{n+1}} A, \tag{37}$$

where A is the edge length. Hence, the edge length A_{SCB} of a regular n-simplex circumscribed (SCB) about an n-ball (B) with diameter D is

$$A_{SCB} = \frac{\sqrt{n}\sqrt{n+1}}{\sqrt{2}} D, \tag{38}$$

so that its volume (20) becomes

$$V_{SCB} = \frac{n^{n/2}\left(n+1\right)^{(n+1)/2}}{\Gamma\left(n+1\right)2^{n}} D^{n} \begin{cases} 1 & n \geq 0 \\ \pm 1 & n < 0 \end{cases}, \tag{39}$$

as shown in in Figure 5. For $n < 0$ the circumscribed n-simplex volume (39) is a complex, bivalued function of n, whereas both branches are left-handed and convergent to zero with n approaching negative infinity. For $0 < n < 1$ it is smaller than the volume of the inscribed n-ball (cf. Table 2). It is zero for $n = -k$, $k \in \mathbb{N}$ and real for $n = -(2k + 1)/2$, $k \in \mathbb{N}$. For the principal branch and $D = 1$ it amounts

$$V_{(-(2k+1)/2)SCB} = \frac{i^{2k}\, 2^{(4k+1)/2}\,(2k-1)^{(1-2k)/4}}{\Gamma\left(\frac{1-2k}{2}\right)(2k+1)^{(1+2k)/4}} \approx \approx 0.7, 0.5618, 0.4251, 0.3172, 0.2353, \ldots \qquad (40)$$

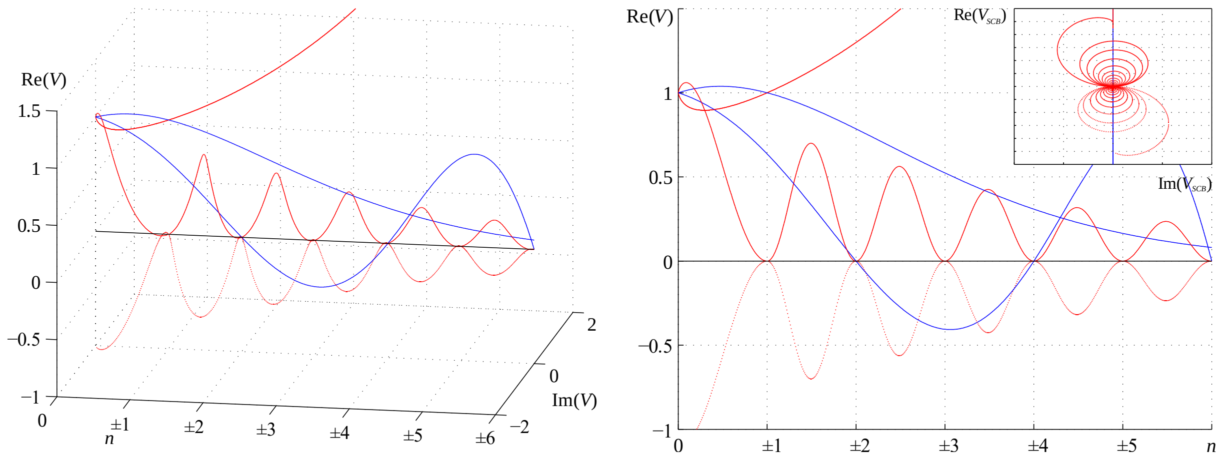

Figure 5. Graphs of volumes (V) of regular n-simplices (red) circumscribed about unit diameter n-balls and volumes of unit diameter n-balls (blue) for n = [0, ±6] (inset for n = [−30, 0]).

Furthermore for $n = -1/2$ and for $n = -(2k + 3)/4$, $k \in \mathbb{N}$, the real part the volume (39) is equal to the imaginary part up to a modulus. For $n = -1/2$ $V_{(-1/2)SCB} = (1 - i)/\sqrt{\pi} \approx 0.5642(1 - i)$. Otherwise, for the principal branch and $D = 1$ it amounts

$$V_{(-(2k+3)/4)SCB} = \frac{(1+i)(-i)^{k-3}\, 2^{(6k+3)/4}\,(2k-1)^{(1-2k)/8}}{\Gamma\left(\frac{1-2k}{4}\right)(2k+3)^{(2k+3)/8}} \approx \approx \{-0.3549, -0.3359, 0.2996, 0.2626, -0.2283, \ldots\}(1+i)(-i)^{k-3} \qquad (41)$$

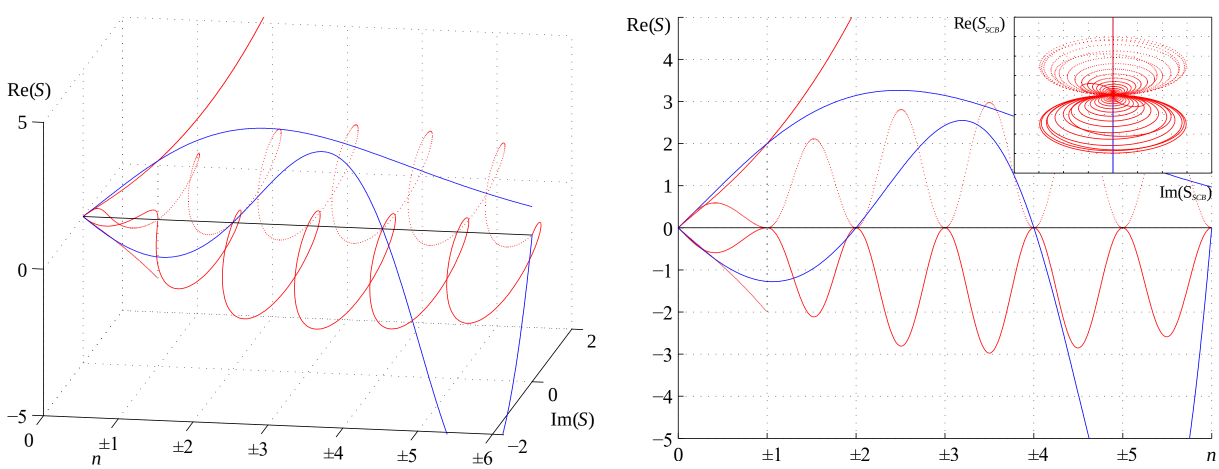

Figure 6. Graphs of surfaces (S) of regular n-simplices (red) circumscribed about unit diameter n-balls and surfaces of unit diameter n-balls (blue) for n = [0, ±6] (inset for n = [−30, 0].

Similarly, the surface (23) of a regular circumscribed n-simplex with edge length A_{SCB} (38) is

$$S_{SCB} = \frac{n^{(n+2)/2}(n+1)^{(n+1)/2}}{\Gamma(n+1)2^{n-1}} D^{n-1} \begin{cases} 1 & n \geq 1 \\ \pm 1 & n < 1 \end{cases}, \tag{42}$$

as shown in in Figure 6. For $n < 1$ the circumscribed n-simplex surface formula (42) branches, and for $0 < n < 1$ it is smaller than the surface of the inscribed n-ball. For $n < 0$ is complex with both branches being right-handed towards negative infinity or the branch point. It is zero for $n = -k$, $k \in \mathbb{N}_0$ and real for $n = -(2k + 1)/2$, $k \in \mathbb{N}$, achieving local maximum at $n \approx -7/2$. For the principal branch and $D = 1$ it amounts

$$S_{(-(2k+1)/2)SCB} = \frac{-i^{2k} 2^{(4k+1)/2} (2k-1)^{(1-2k)/4}}{\Gamma\left(\frac{1-2k}{2}\right)(2k+1)^{(2k-3)/4}} \approx \\ \approx -2.1, -2.809, -\underline{2.976}, -2.854, -2.588, \ldots \tag{43}$$

Its real part is equal to the imaginary part up to a modulus for $n = -1/2$ and for $n = -(2k + 3)/4$, $k \in \mathbb{N}$. For $n = -1/2$ $S_{(-1/2)SCB} = (-1 + i)/\sqrt{\pi}$. Otherwise, for the principal branch and $D = 1$ it amounts

$$S_{(-(2k+3)/4)SCB} = \frac{(1+i)(-i)^{k-1} 2^{(6k-1)/4} (2k-1)^{(1-2k)/8}}{\Gamma\left(\frac{1-2k}{4}\right)(2k+3)^{(2k-5)/8}} \approx \\ \approx \{-0.8873, -1.1755, 1.3484, 1.4443, -\underline{1.4842}, -1.4828, \ldots\}(1+i)(-i)^{k-1} \tag{44}$$

The surface (42) is initially divergent to achieve local modulus maximum of about 2.9757 at $n \approx -3.4997$ (numerical), and local real maximum of about -2.976 at $n = -7/2$, and then becomes convergent to zero with n approaching negative infinity.

5.2 *n*-Orthoplices Inscribed in and Circumscribed About *n*-Balls

The diameter D_{BCO} of an n-ball circumscribed about an n-orthoplex (BCO) is known [8] to be

$$D_{BCO} = \sqrt{2}A, \tag{45}$$

where A is the edge length. Hence, the edge length A_{OIB} of an n-orthoplex inscribed inside an n-ball (OIB) with diameter D is

$$A_{OIB} = \frac{1}{\sqrt{2}} D, \tag{46}$$

so that its volume (25) becomes

$$V_{OIB} = \frac{1}{\Gamma(n+1)} D^n, \tag{47}$$

as shown in in Figure 7. The inscribed n-orthoplex volume formula (47) is real for $n \in \mathbb{R}$, where for $n = -k$, $k \in \mathbb{N}$, its zero values are given in the sense of a limit of a function (cf. (21)), and for $0 < n < 1$ it is larger than the volume of the circumscribing n-ball (cf. Table 2).

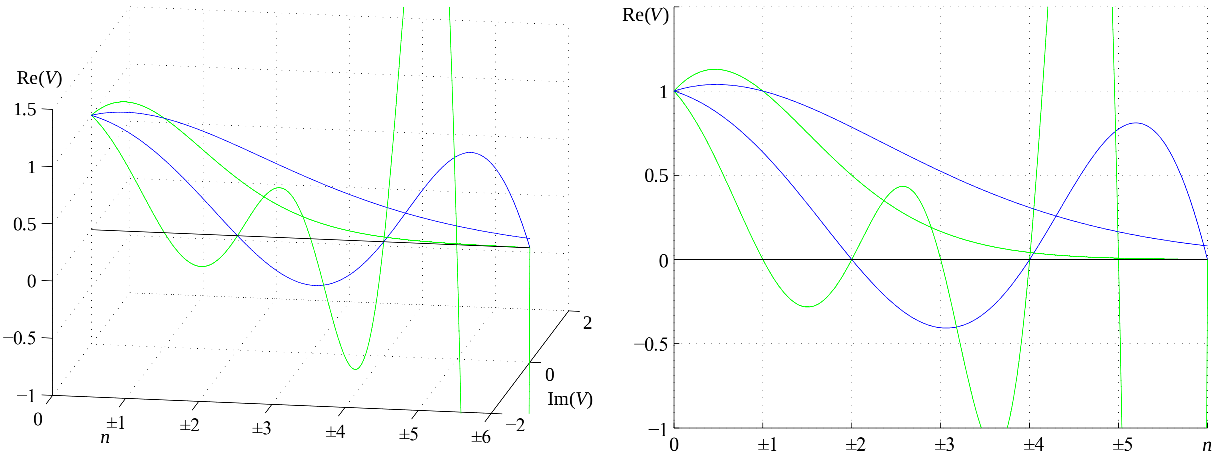

Figure 7. Graphs of volumes (V) of n-orthoplices (green) inscribed in unit diameter n-balls and volumes of unit diameter n-balls (blue) for $n = [0, \pm 6]$.

Similarly, the surface (26) of the inscribed n-orthoplex with edge length A given by (46) becomes

$$S_{OIB} = \frac{2n^{3/2}}{\Gamma(n+1)} D^{n-1} \begin{cases} 1 & n \geq 1 \\ \pm 1 & n < 1 \end{cases}, \qquad (48)$$

as shown in in Figure 8. For $n < 1$ inscribed n-orthoplex surface (48) branches, for $n < 0$, $n \notin \mathbb{Z}$ it is imaginary and oscillatory divergent with n approaching negative infinity, and for $n \leq -1$, $n \in \mathbb{Z}$, its zero values are given in the sense of a limit of a function.

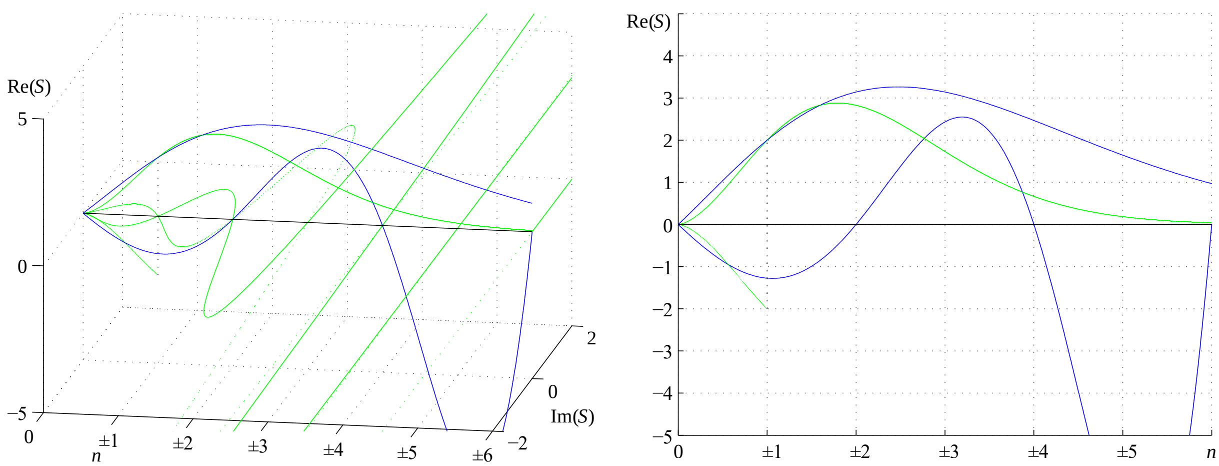

Figure 8. Graphs of surfaces (S) of n-orthoplices (green) inscribed in unit diameter n-balls and surfaces of unit diameter n-balls (blue) for $n = [0, \pm 6]$.

The diameter D_{BIO} of an n-ball inscribed inside an n-orthoplex (BIO) is known [8] to be

$$D_{BIO} = \sqrt{\frac{2}{n}} A, \tag{49}$$

where A is the edge length. Hence, the edge length A_{OCB} of an n-orthoplex circumscribed about an n-ball (OCB) with diameter D is

$$A_{OCB} = \sqrt{\frac{n}{2}} D, \tag{50}$$

so that its volume (25) becomes

$$V_{OCB} = \frac{n^{n/2}}{\Gamma(n+1)} D^n, \tag{51}$$

as shown in Figure 9. Circumscribed n-orthoplex volume (51) is a singlevalued function, is complex for $n < 0$, crossing the quadrants of the complex plane in the order $\{\mathrm{Re}(V_{OCB}) > 0,\ \mathrm{Im}(V_{OCB}) < 0\}$, $\{\mathrm{Re}(V_{OCB}) > 0,\ \mathrm{Im}(V_{OCB}) > 0\}$, $\{\mathrm{Re}(V_{OCB}) < 0,\ \mathrm{Im}(V_{OCB}) > 0\}$, and $\{\mathrm{Re}(V_{OCB}) < 0,\ \mathrm{Im}(V_{OCB}) < 0\}$. It oscillates and is initially convergent to achieve local modulus maximum of about 0.1181 at $n \approx -3.4976$ (numerical) and then becomes divergent with n approaching negative infinity. For $n = -k$, $k \in \mathbb{N}$, its zero values are given in the sense of a limit of a function. For $0 < n < 1$ it is smaller than the volume of the inscribed n-ball (cf. Table 2). For $n = -(2k + 1)/2$, $k \in \mathbb{N}_0$ the real part of the volume (51) equals the imaginary part up to a modulus, achieving local maximum at $n \approx -7/2$ and for $D = 1$ amounts

$$V_{(-(2k+1)/2)OCB} = \frac{(1+i)(-i)^{k-3}\, 2^{(2k-1)/4}}{\Gamma\left(\frac{1-2k}{2}\right)(2k+1)^{(2k+1)/4}} \approx$$
$$\approx \{0.4744, -0.1472, 0.0952, -\underline{0.0835}, 0.0888, -0.1084, \ldots\}(1+i)(-i)^{k-3} \tag{52}$$

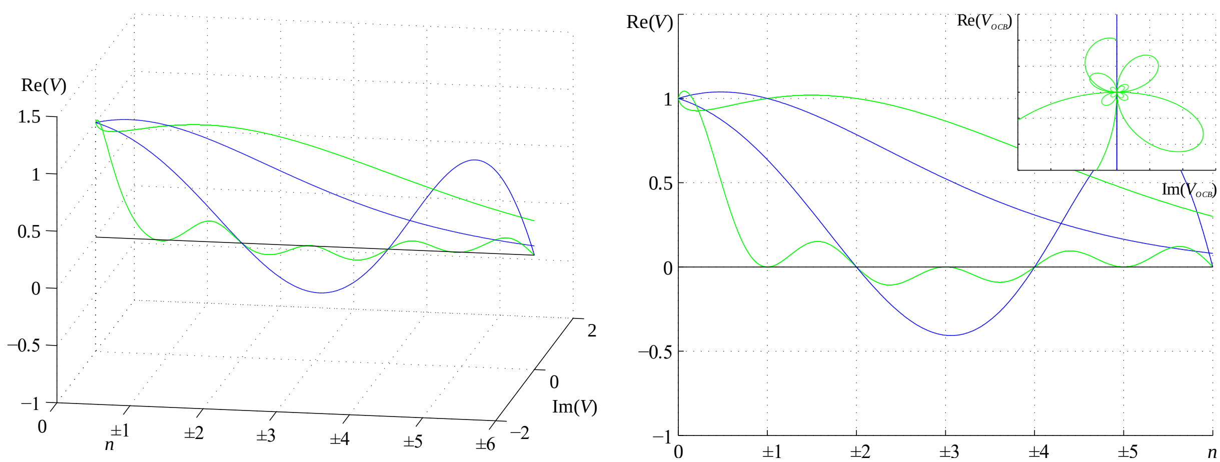

Figure 9. Graphs of volumes (V) of n-orthoplices (green) circumscribed about unit diameter n-balls and volumes of unit diameter n-balls (blue) for $n = [0, \pm 6]$.

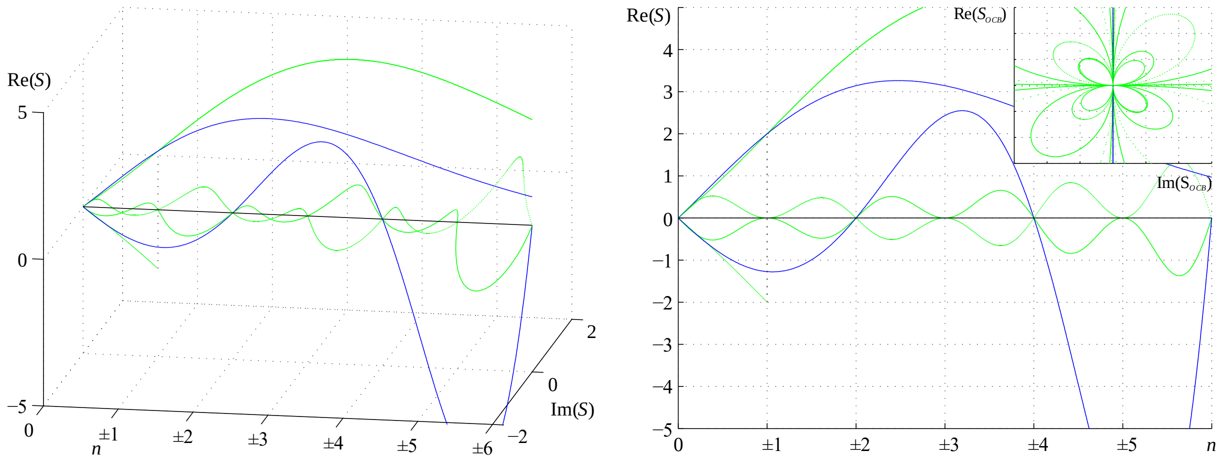

Figure 10. Graphs of surfaces (*S*) of *n*-orthoplices (green) circumscribed about unit diameter *n*-balls and surfaces of unit diameter *n*-balls (blue) for *n* = [0, ±6].

Similarly, the surface (26) of the circumscribed *n*-orthoplex with edge length *A* given by (50) becomes

$$S_{OCB} = \frac{2n^{n/2+1}}{\Gamma(n+1)} D^{n-1} \begin{cases} 1 & n \geq 1 \\ \pm 1 & n < 1 \end{cases}, \tag{53}$$

as shown in Figure 10. Circumscribed *n*-orthoplex surface (53) is a bivalued function for $n < 1$, is complex for $n < 0$, crossing the quadrants of the complex plane in the order $\{\mathrm{Re}(S_{OCB}) < 0, \mathrm{Im}(S_{OCB}) > 0\}$, $\{\mathrm{Re}(S_{OCB}) < 0, \mathrm{Im}(S_{OCB}) < 0\}$, $\{\mathrm{Re}(S_{OCB}) > 0, \mathrm{Im}(S_{OCB}) < 0\}$, and $\{\mathrm{Re}(S_{OCB}) > 0, \mathrm{Im}(S_{OCB}) > 0\}$. It oscillates and is initially convergent to achieve local modulus maximum of about 0.6244 at $n \approx -1.5$ (numerical), and then becomes divergent with *n* approaching negative infinity. For $n \leq -1$, $n \in \mathbb{Z}$, its zero values are given in the sense of a limit of a function (cf. (21)). For $0 < n < 1$ it is smaller than the surface of the inscribed *n*-ball (cf. Table 3). Furthermore its real part is equal to the imaginary part up to a modulus for $n = -(2k + 1)/2$, $k \in \mathbb{N}_0$. It achieves local maximum at $n = -3/2$ and for the principal branch and $D = 1$ and amounts

$$\begin{aligned} S_{(-(2k+1)/2)OCB} &= \frac{(1+i)(-i)^{k-1}\, 2^{(2k-1)/4} (2k+1)^{(3-2k)/4}}{\Gamma\left(\frac{1-2k}{2}\right)} \approx \\ &\approx \{0.4744, -\underline{0.4415}, 0.4759, -0.5846, 0.7989, \ldots\}(1+i)(-i)^{k-1} \end{aligned} . \tag{54}$$

5.3 *n*-Cubes Inscribed in and Circumscribed About *n*-Balls

The edge length A_{CCB} of an *n*-cube circumscribed about an *n*-ball (*CCB*) corresponds to the diameter *D* of this n-ball. Thus, the volume of this cube is simply

$$V_{CCB} = D^n, \tag{55}$$

and the surface is

$$S_{CCB} = 2nD^{n-1}. \tag{56}$$

However, the edge length A_{CIB} of an n-cube inscribed inside an n-ball (*CIB*) of diameter D is $A_{CIB} = D/\sqrt{n}$, which is singular for $n = 0$ and complex for $n < 0$, rendering [1] the following volume and the surface of an n-cube inscribed in an n-ball

$$V_{nCIB} = n^{-n/2} D^n, \tag{57}$$

$$S_{nCIB} = 2n^{(3-n)/2} D^{n-1}. \tag{58}$$

The reflection relation can be obtained setting $m = -n$ in (57), yielding [1] the volume and the surface

$$V_{mCIB} = i^m m^{m/2} D^{-m}, \tag{59}$$

$$S_{mCIB} = -2i^{m+1} m^{(3+m)/2} D^{-m-1}, \tag{60}$$

which are complex for $m \in \mathbb{R}$. Volumes (57) and (59) correspond to each other [1] for $n \le 0$, $n \in \mathbb{R}$ and for $n = 2k$, $k \in \mathbb{Z}$, as shown in Figure 11. Surfaces (58) and (60) correspond to each other [1] for $n \in \mathbb{R}$, $n \le 0$, and for $n = 2k - 1$, $k \in \mathbb{Z}$, as shown in Figure 12.

For $n \ge 0$ (by convention $0^0 := 1$) the inscribed n-cube volume (57) is real, complex if $n < 0$, becoming real if n is negative and even and imaginary if n is negative and odd, and divergent with n approaching negative infinity. For $0 < n < 1$ it is larger than the volume of the circumscribing n-ball. For $n \ge 0$ the inscribed n-cube surface (58) is real, complex if $n < 0$, becoming real if n is negative and odd and imaginary if n is negative and even, and divergent with n approaching negative infinity.

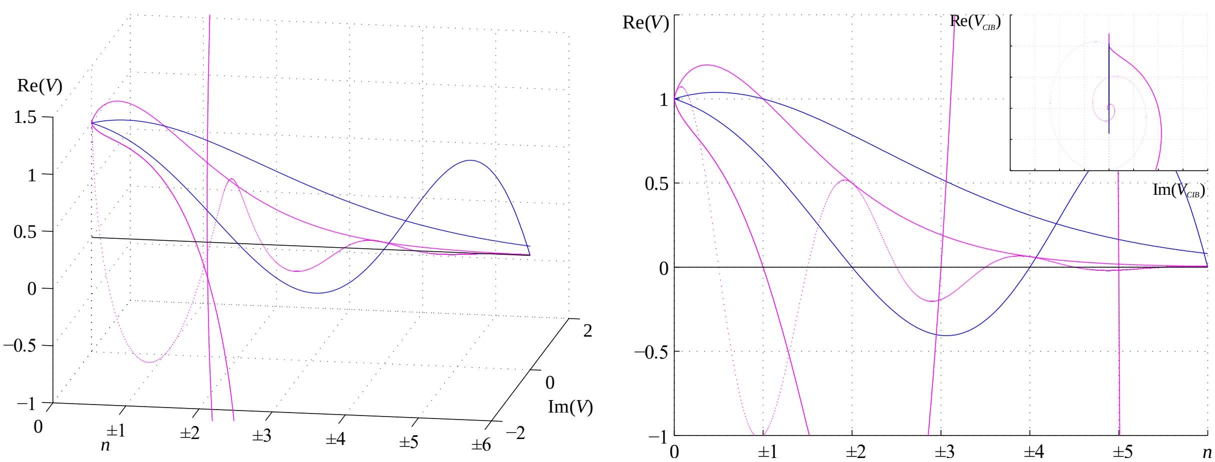

Figure 11. Graphs of volumes (V) of n-cubes (pink) inscribed in unit diameter n-balls with the reflection relation (dotted) and volumes of unit diameter n-balls (blue) for $n = [0, \pm 6]$.

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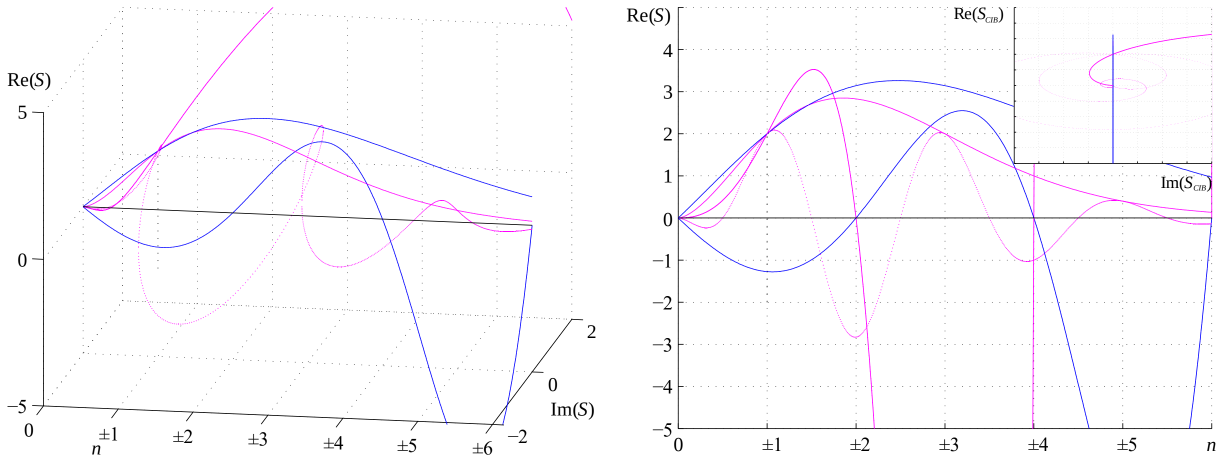


Figure 12. Graphs of surfaces (S) of n-cubes (pink) inscribed in unit diameter n-balls with the reflection relation (dotted) and volumes of unit diameter n-balls (blue) for n = [0, ±6].

5.4 Summary

The principal branches of the volumes and surfaces of the omnidimensional polytopes, discussed in this section, are summarized in Table 1, where n-balls are defined in terms of their diameters, which concept is closer to the concept of the edge length of a polytope. Further properties of these polytopes are listed in Tables 2-5.

Table 1. Volumes and surfaces of regular n-simplices, n-orthoplices, and n-cubes inscribed in and circumscribed about an n-balls.

	inscribed in n-ball (IB)		circumscribed about n-ball (CB)	
	(V) volume/D^n	(S) surface/D^{n-1}	(V) volume/D^n	(S) surface/D^{n-1}
(S)	$\frac{n^{-n/2}(n+1)^{(n+1)/2}}{\Gamma(n+1)2^n}$ (2)	$\frac{n^{(4-n)/2}(n+1)^{(n+1)/2}}{\Gamma(n+1)2^{n-1}}$ (2)	$\frac{n^{n/2}(n+1)^{(n+1)/2}}{\Gamma(n+1)2^n}$ (2)	$\frac{n^{(n+2)/2}(n+1)^{(n+1)/2}}{\Gamma(n+1)2^{n-1}}$ (2)
(O)	$\frac{1}{\Gamma(n+1)}$ (1)	$\frac{2n^{3/2}}{\Gamma(n+1)}$ (2)	$\frac{n^{n/2}}{\Gamma(n+1)}$ (1)	$\frac{2n^{n/2+1}}{\Gamma(n+1)}$ (2)
(C)	$n^{-n/2}$ (1)	$2n^{(3-n)/2}$ (1)	1 (1)	$2n$ (1)

(1) one branch, (2) two branches.

It was shown [1] that the following metric independent relation holds between volumes (57) of n-cubes inscribed in an n-ball

$$V_{nCIB}V_{(-n)CIB} = D^n n^{-n/2} i^n D^{-n} n^{n/2} = i^n \,. \tag{61}$$

Similar metric independent relations can be derived for volumes and surfaces of the remaining omnidimensional convex polytopes

$$V_{nS}V_{(-n)S} = \frac{-(1+n)^{1/2}(1-n)^{1/2}\sin[\pi(n+1)]}{\pi n}, \tag{62}$$

$$V_{nSIB}V_{(-n)SIB} = \frac{-i^{n}(1+n)^{(1+n)/2}(1-n)^{(1-n)/2}\sin[\pi(n+1)]}{\pi n}, \tag{63}$$

$$V_{nSCB}V_{(-n)SCB} = \frac{-i^{-n}(1+n)^{(1+n)/2}(1-n)^{(1-n)/2}\sin[\pi(n+1)]}{\pi n}, \tag{64}$$

$$V_{nO}V_{(-n)O} = V_{nOIB}V_{(-n)OIB} = \frac{-\sin[\pi(n+1)]}{\pi n}, \tag{65}$$

$$V_{nOCB}V_{(-n)OCB} = \frac{-i^{-n}\sin[\pi(n+1)]}{\pi n}, \tag{66}$$

where we used $m = n + 1$ and Euler's reflection formula

$$\Gamma(n+1)\Gamma(-n+1) = \Gamma(m)\Gamma(2-m) = \Gamma(m)\Gamma(1-m)(1-m) = \frac{-\pi n}{\sin[\pi(n+1)]}, \tag{67}$$

and

$$S_{nCIB}S_{(2-n)CIB} = 4n^{(3-n)/2}(2-n)^{(1+n)/2}, \tag{68}$$

$$S_{nS}S_{(2-n)S} = \frac{-n^{1/2}(n+1)(2-n)^{1/2}(3-n)\sin[\pi(n+1)]}{\pi(1-n)}, \tag{69}$$

$$S_{nSIB}S_{(2-n)SIB} = \frac{-n^{(2-n)/2}(n+1)^{(n+1)/2}(2-n)^{n/2}(3-n)^{(3-n)/2}\sin[\pi(n+1)]}{\pi(1-n)}, \tag{70}$$

$$S_{nSCB}S_{(2-n)SCB} = \frac{-n^{n/2}(n+1)^{(n+1)/2}(2-n)^{(2-n)/2}(3-n)^{(3-n)/2}\sin[\pi(n+1)]}{\pi(1-n)}, \tag{71}$$

$$S_{nO}S_{(2-n)O} = S_{nOIB}S_{(2-n)OIB} = \frac{-4n^{1/2}(2-n)^{1/2}\sin[\pi(n+1)]}{\pi(1-n)}, \tag{72}$$

$$S_{nOCB}S_{(2-n)OCB} = \frac{-4n^{n/2}(2-n)^{(2-n)/2}\sin[\pi(n+1)]}{\pi(1-n)}, \tag{73}$$

where we have also used $m = n + 1$ and Euler's reflection formula

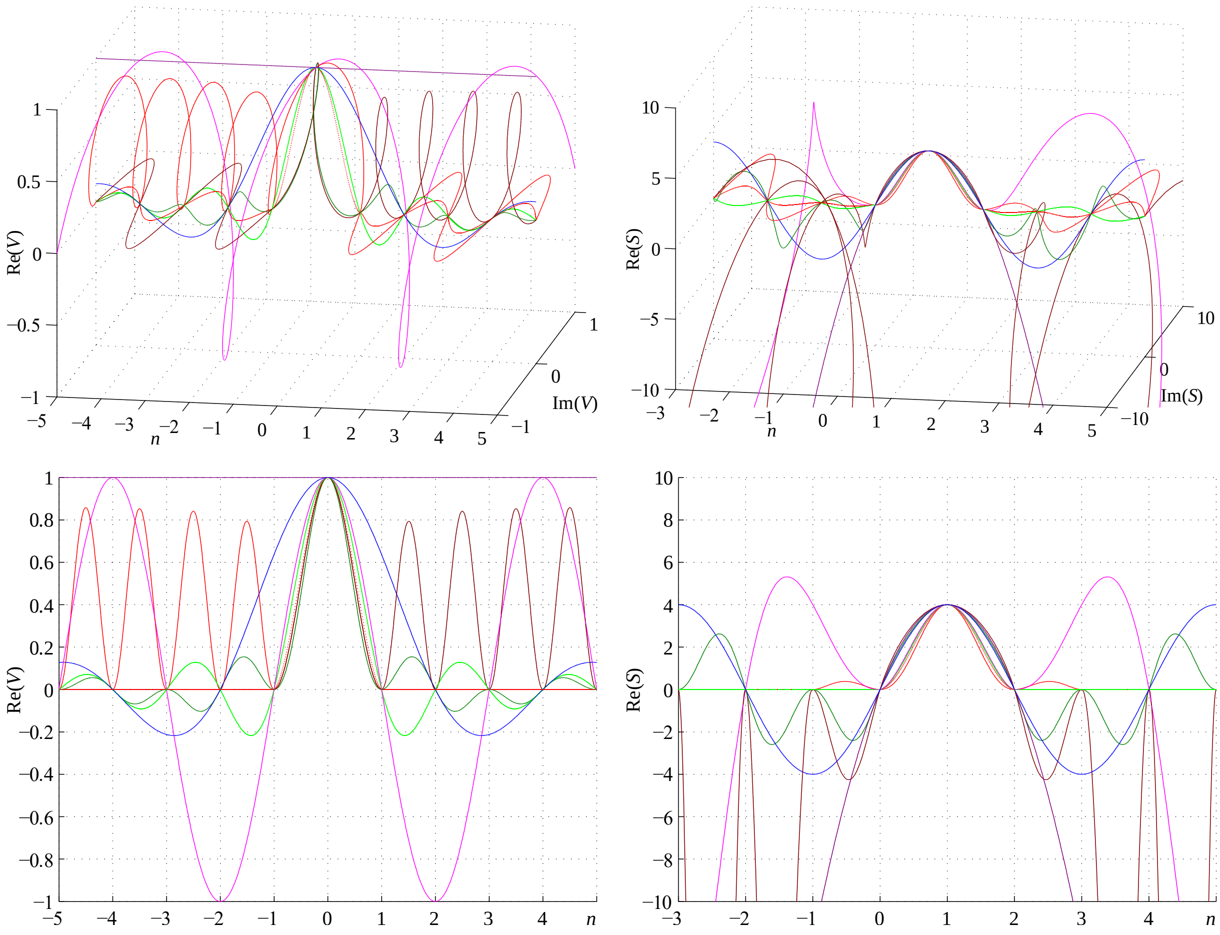
$$\Gamma(n+1)\Gamma(3-n)=\Gamma(m)\Gamma(1-m)(1-m)(2-m)(3-m)=\frac{-\pi n(1-n)(2-n)}{\sin\left[\pi(n+1)\right]}. \tag{74}$$

Knowing the volume (30) and surface (31) of n-balls in complex dimensions we can extend (in a way correct) the relations $S_{nB}S_{(2-n)B} = 4\mathrm{Re}(i^{n-1})$ (24) and $2\pi n V_{nB}V_{(-n)B} = 4\mathrm{Re}(i^{n-1})$ (27) between n-ball surfaces and volumes in in integer dimensions, disclosed in the prior research [1]. Simple two products of respectively (30) for $-n = -a - ib$ and (31) for $2 - n = 2 - a - ib$ and yield

$$V_{nB}V_{(-n)B}=\frac{2\sin(\pi n/2)}{\pi n}=\frac{-2\sin\left[\pi(n/2+1)\right]}{\pi n}, \tag{75}$$

$$S_{nB}S_{(2-n)B}=4\sin(\pi n/2)=-4\sin\left[\pi(n/2+1)\right]. \tag{76}$$

Relations (61)-(76) are shown in Figure. 13. Curiously, the imaginary part of the surface relations (68)-(73), and (76) vanishes for $0 < n < 2$.



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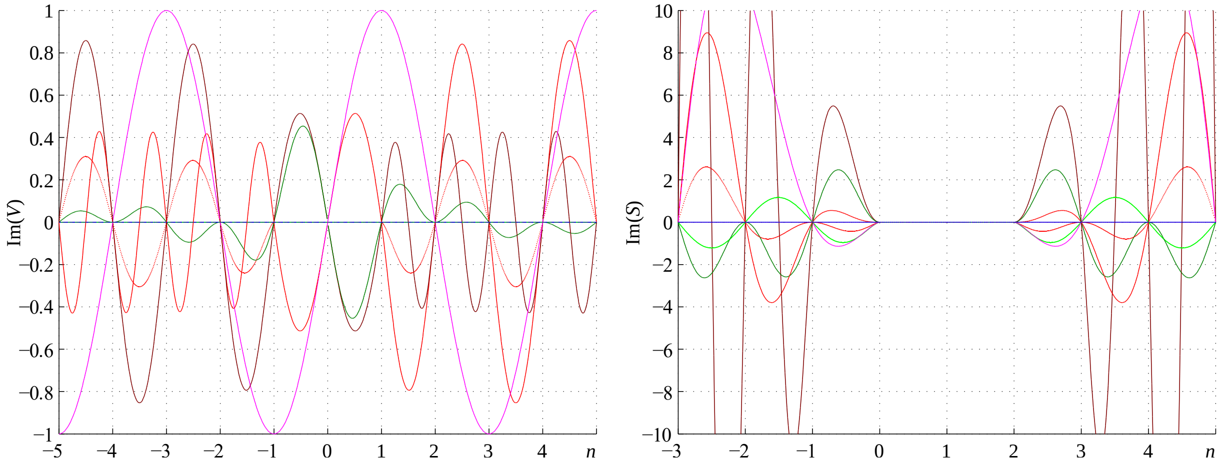

Figure 13. Metric independent relations between volumes (*V*) and surfaces (*S*) of omnidimensional convex polytopes and *n*-balls (*SIB* - red, *SCB* - dark red, *S* - red doted, *OIB* - green, *OCB* - dark green, *O* - green dotted, *CIB* - magenta, *CCB* - dark magenta, *B* - blue).

Also the following relations (principal branch, where applicable)

$$\frac{V_{nSCB}}{V_{nSIB}} = n^n, \tag{77}$$

$$\frac{S_{nSCB}}{S_{nSIB}} = n^{n-1}, \tag{78}$$

$$\frac{V_{nOCB}}{V_{nOIB}} = \frac{V_{nCCB}}{V_{nCIB}} = n^{n/2}, \tag{79}$$

$$\frac{S_{nOCB}}{S_{nOIB}} = \frac{S_{nCCB}}{S_{nCIB}} = n^{(n-1)/2}, \tag{80}$$

relating formulas (39), (35); (42), (36); (51), (47); (55), (57); (53), (48); and (56), (58) with each other, can be easily obtained. Notably, as *n*-cube is dual to *n*-orthoplex, the ratios of their volumes and surfaces circumscribed about *n*-balls to, respectively, volumes and surfaces inscribed in *n*-balls are the same.

Also the following particular symmetries between $n = -1/2$ and $n = 1/2$ hold for (35), (36); (39), (42); (47), (48); (51), (53); (57), (58); and (55), (56) (principal branch, where applicable)

$$V_{(-1/2)SIB} = 2S_{(-1/2)SIB}D, \quad V_{(1/2)SIB} = 2S_{(1/2)SIB}D, \tag{81}$$

$$V_{(-1/2)SCB} = -S_{(-1/2)SCB}D, \quad V_{(1/2)SCB} = S_{(1/2)SCB}D, \tag{82}$$

$$V_{(-1/2)OIB} = i\sqrt{2}S_{(-1/2)OIB}D, \quad V_{(1/2)OIB} = \sqrt{2}S_{(1/2)OIB}D, \tag{83}$$

$$V_{(-1/2)OCB} = -S_{(-1/2)OCB}D, \quad V_{(1/2)OCB} = S_{(1/2)OCB}D, \tag{84}$$

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$$V_{(-1/2)CIB} = \sqrt{2}S^{*}_{(-1/2)CIB}D, \quad V_{(1/2)CIB} = \sqrt{2}S_{(1/2)CIB}D, \tag{85}$$

$$V_{(-1/2)CCB} = -S_{(-1/2)CCB}D, \quad V_{(1/2)CCB} = S_{(1/2)CCB}D, \tag{86}$$

where "*" denotes a complex conjugate.

Behavior of volumes of regular *n*-simplices inscribed in and circumscribed about *n*-balls, *n*-orthoplices circumscribed about *n*-balls, and *n*-cubes inscribed in *n*-balls, around *n* = 0, illustrated in Figure 14 supports the semiaxis hypothesis: the singularity is alleviated.

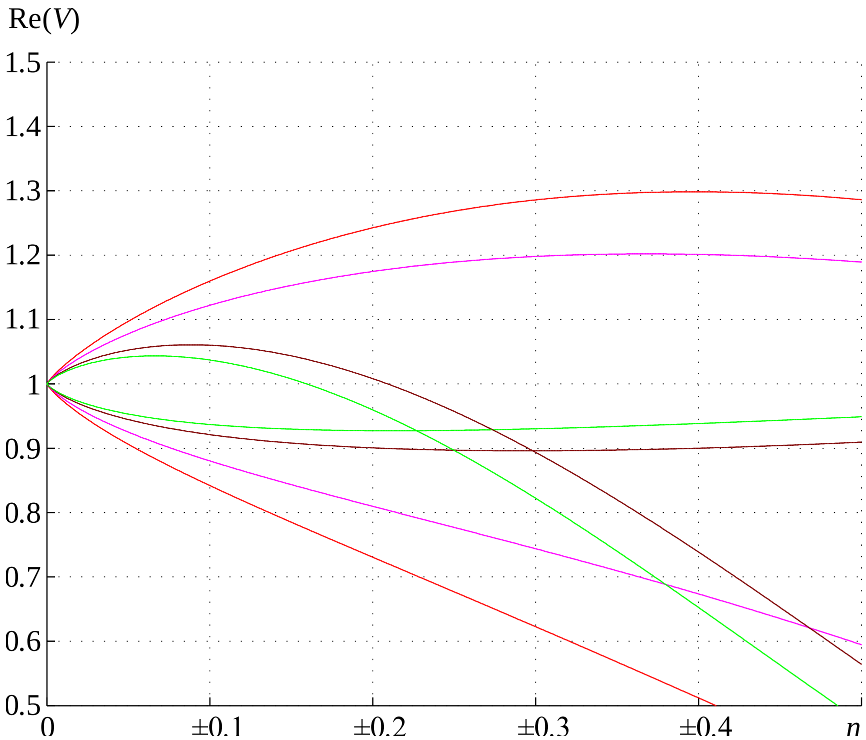

Figure 14. Graphs of the real part of volumes (*V*) of unit edge length regular *n*-simplices inscribed in (red) and circumscribed about (black) *n*-balls, *n*-orthoplices circumscribed about *n*-balls (green), and *n*-cubes inscribed in *n*-balls (pink) in the vicinity of *n* = 0.

Table 2. Particular values of volumes of omnidimensional polytopes inscribed in and circumscribed about unit diameter *n*-balls (principal branch only).

n	−3/2	−1	−1/2	0	½	1	3/2
V_B	0.331	$2/\pi \approx 0.637$	0.867	1	1.039	1	0.908
V_{SIB}	$\frac{-i3^{3/4}}{\sqrt{\pi}} \approx -i1.286$	0	$\frac{1+i}{\sqrt{2\pi}} \approx$ $0.399(1+i)$	1	$\frac{3^{3/4}}{\sqrt{\pi}} \approx 1.286$	1	$\frac{5^{5/4}}{3^{7/4}\sqrt{\pi}} \approx 0.617$
V_{SCB}	$\frac{2^{3/2}}{3^{3/4}\sqrt{\pi}} \approx 0.7$	0	$\frac{1-i}{\sqrt{\pi}} \approx$ $0.564(1-i)$	1	$\frac{3^{3/4}}{\sqrt{2\pi}} \approx 0.909$	1	$\frac{5^{5/4}}{3^{1/4}2^{3/2}\sqrt{\pi}} \approx 1.133$
V_{OIB}	$\frac{-1}{2\sqrt{\pi}} \approx -0.282$	0	$\frac{1}{\sqrt{\pi}} \approx 0.564$	1	$\frac{2}{\sqrt{\pi}} \approx 1.128$	1	$\frac{4}{3\sqrt{\pi}} \approx 0.752$

n	$-3/2$	-1	$-1/2$	0	½	1	3/2
V_{OCB}	$\frac{1+i}{2^{3/4}3^{3/4}\sqrt{\pi}} \approx 0.147(1+i)$	0	$\frac{1-i}{2^{1/4}\sqrt{\pi}} \approx 0.474(1-i)$	1	$\frac{2^{3/4}}{\sqrt{\pi}} \approx 0.949$	1	$\frac{2^{5/4}}{2^{1/4}\sqrt{\pi}} \approx 1.02$
V_{CIB}	$\frac{(i-1)3^{3/4}}{2^{-5/4}} \approx 0.958(i-1)$	i	$\frac{(1+i)}{2^{3/4}} \approx 0.595(i+1)$	1	$2^{1/4} \approx 1.189$	1	$\frac{2^{3/4}}{3^{3/4}} \approx 0.738$
V_{CCB}	1	1	1	1	1	1	1

Table 3. Particular values of surfaces of omnidimensional polytopes inscribed in and circumscribed about unit diameter *n*-balls (principal branch only).

n	$-3/2$	-1	$-1/2$	0	1/2	1	3/2
S_B	-0.992	$-4/\pi \approx -1.273$	-0.867	0	1.039	2	2.723
S_{SIB}	$\frac{-i3^{11/4}}{2\sqrt{\pi}} \approx -5.787i$	0	$\frac{1+i}{\sqrt{\pi}\,2^{3/2}} \approx 0.199(1+i)$	0	$\frac{3^{3/4}}{2\sqrt{\pi}} \approx 0.643$	2	$\frac{3^{1/4}5^{5/4}}{2\sqrt{\pi}} \approx 2.776$
S_{SCB}	$\frac{-2^{3/2}}{3^{1/4}\sqrt{\pi}} \approx -2.1$	0	$\frac{-1+i}{\sqrt{\pi}} \approx 0.564(-1+i)$	0	$\frac{3^{3/4}}{\sqrt{2\pi}} \approx 0.909$	2	$\frac{3^{3/4}5^{5/4}}{2^{3/2}\sqrt{\pi}} \approx 3.4$
S_{OIB}	$\frac{3^{3/2}i}{2^{3/2}\sqrt{\pi}} \approx 1.036i$	0	$\frac{-i}{\sqrt{2\pi}} \approx -0.399i$	0	$\frac{\sqrt{2}}{\sqrt{\pi}} \approx 0.798$	2	$\frac{2^{3/2}\sqrt{3}}{\sqrt{\pi}} \approx 2.764$
S_{OCB}	$\frac{-(1+i)3^{1/4}}{2^{1/4}\sqrt{\pi}} \approx -0.442(1+i)$	0	$\frac{(-1+i)}{2^{1/4}\sqrt{\pi}} \approx -0.474(-1+i)$	0	$\frac{2^{3/4}}{\sqrt{\pi}} \approx 0.949$	2	$\frac{2^{5/4}3^{3/4}}{\sqrt{\pi}} \approx 3.059$
S_{CIB}	$\frac{(1+i)3^{9/4}}{2^{7/4}} \approx 3.521(1+i)$	2	$\frac{1-i}{2^{5/4}} \approx 0.42(1-i)$	0	$2^{-1/4} \approx 0.841$	2	$3^{3/4}2^{1/4} \approx 2.711$
S_{CCB}	-3	-2	-1	0	1	2	3

Table 4. Volumes of omnidimensional polytopes inscribed in and circumscribed about unit diameter *n*-balls ($n \in \mathbb{R}$ unless stated otherwise; "no complex" means that the relation is real or imaginary; for $n \to -\infty$ all relations are oscillatory divergent).

<table>
<tr><th>n</th><th>bival.</th><th>complex</th><th>real</th><th>imaginary</th><th>zero</th><th>div./conv.</th></tr>
<tr><td>V_B (16)</td><td>no</td><td rowspan="2">no</td><td>$n \in \mathbb{R}$</td><td>no</td><td>$n = -2k,\ k \in \mathbb{N}_0$</td><td rowspan="3">$0,\ n \to \infty$
$-\infty,\ n \to -\infty$</td></tr>
<tr><td>V_S (20)</td><td rowspan="3">$n < 0$</td><td>$n = -k,\ k \in \mathbb{N}$
$n \geq -1,\ n \in \mathbb{R}$</td><td rowspan="2">$n < -1,\ n \notin \mathbb{Z}$</td><td rowspan="5">$n = -k,\ k \in \mathbb{N}$</td></tr>
<tr><td>V_{SIB} (35)</td><td>$-1 < n < 0$
(RH)</td><td>$n = -k,\ k \in \mathbb{N}$
$n \geq 0,\ n \in \mathbb{R}$</td></tr>
<tr><td>V_{SCB} (39)</td><td>$n < 0$
(LH)</td><td>$n = -(k+1)/2,\ k \in \mathbb{N}$
$n \geq 0,\ n \in \mathbb{R}$</td><td rowspan="4">no</td><td>$\infty,\ n \to \infty$
$0,\ n \to -\infty$</td></tr>
<tr><td>V_O (25)</td><td rowspan="3">no</td><td rowspan="2">no</td><td rowspan="2">$n \in \mathbb{R}$</td><td rowspan="3">$0,\ n \to \infty$
$-\infty,\ n \to -\infty$</td></tr>
<tr><td>V_{OIB} (47)</td></tr>
<tr><td>V_{OCB} (51)</td><td>$n < 0$
(LH)</td><td>$n = -k,\ k \in \mathbb{N}$
$n \geq 0,\ n \in \mathbb{R}$</td></tr>
</table>

n	bival.	complex	real	imaginary	zero	div./conv.
V_C		no	$n \in \mathbb{R}$		no	const
V_{CIB} (57)		$n < 0$	$n = -2k$, $k \in \mathbb{N}$ $n \geq 0$, $n \in \mathbb{R}$	$n = -2k - 1$, $k \in \mathbb{N}_0$		0, $n \to \infty$ $-\infty$, $n \to -\infty$
V_{CCB} (55)		no	$n \in \mathbb{R}$	no		const

Table 5. Surfaces of omnidimensional polytopes inscribed in and circumscribed about unit diameter n-balls ($n \in \mathbb{R}$ unless stated otherwise; "no complex" means that the relation is real or imaginary; for $n \to -\infty$ all relations are oscillatory divergent, with the exception of S_C and S_{CCB}).

n	bival.	complex	real	imaginary	zero	div./conv.
S_B (19)	no	no	$n \in \mathbb{R}$	no	$n = -2k$, $k \in \mathbb{N}_0$	0, $n \to \infty$ $-\infty$, $n \to -\infty$
S_S (23)	$n < 1$		$n = -k$, $k \in \mathbb{N}$ $n \geq 0$, $n \in \mathbb{R}$	$n < 0$, $n \notin \mathbb{Z}$	$n = -k$, $k \in \mathbb{N}_0$	
S_{SIB} (36)		$-1 < n < 0$ (RH)		$n < -1$, $n \notin \mathbb{Z}$		
S_{SCB} (42)		$n < 0$ (LH)	$n = -(k + 1)/2$, $k \in \mathbb{N}$ $n \geq 0$, $n \in \mathbb{R}$	no		∞, $n \to \infty$ 0, $n \to -\infty$
S_O (26)		no	$n = -k$, $k \in \mathbb{N}$ $n \geq 0$, $n \in \mathbb{R}$	$n < 0$, $n \notin \mathbb{Z}$		0, $n \to \infty$ $-\infty$, $n \to -\infty$
S_{OIB} (48)						
S_{OCB} (53)		$n < 0$ (LH)		no		
S_C	no	no	$n \in \mathbb{R}$	no	$n = 0$	∞, $n \to \infty$ $-\infty$, $n \to -\infty$
S_{CIB} (58)		$n < 0$	$n = -4k - 1$, $k \in \mathbb{N}_0$ $n \geq 0$, $n \in \mathbb{R}$	$n = -4k - 3$ $k \in \mathbb{N}_0$	no	0, $n \to \infty$ $-\infty$, $n \to -\infty$
S_{CCB} (56)		no	$n \in \mathbb{R}$	no	$n = 0$	∞, $n \to \infty$ $-\infty$, $n \to -\infty$

6. Conclusions

The recurrence relations (2), (3), (5), and (9), (10), (13), (14) defining volumes and surfaces of the omnidimensional polytopes and n-balls can be expressed by the gamma function (16), (19); (20), (23); (25), (26) and thus are continuous for $n \in \mathbb{C}$. For $n = -2k - 2$, $k \in \mathbb{N}_0$ in the case of n-balls, and for $n = -k - 1$, $k \in \mathbb{N}_0$ in the case of n-simplices and n-orthoplices their values are given in the sense of a limit of a function. The starting points for fractional dimensions are given due to the continuity of the gamma function.

In the negative dimensions the volume of an n-simplex is a bivalued function. Thus, the surfaces of n-simplices and n-orthoplices are also bivalued functions for $n < 1$. Moreover, as the gamma function is a function of a complex argument and value, these volumes and surfaces inherit this gamma function property.

Applications of these formulas to the omnidimensional polytopes inscribed in and circumscribed about an n-balls revealed the properties of these geometric objects in negative, real dimensions. In particular for $0 < n < 1$ the volumes of the omnidimensional polytopes are larger than volumes of circumscribing n-balls, while their volumes and surfaces are smaller than volumes of inscribed n-balls. It was shown that certain products (61)-(76) and quotients (77)-(80) of volumes and surfaces of the omnidimensional polytopes and n-balls are independent on the gamma function.

The results of this study could perhaps be applied in linguistic statistics, where the dimension in the distribution for frequency dictionaries is chosen to be negative [10], in fog computing, where *n*-simplex is related to a full mesh pattern, *n*-orthoplex is linked to a quasi-full mesh structure, and *n*-cube is referred to as a certain type of partial mesh layout [11], and in molecular physics and crystallography. Perhaps the menagerie of rational numbers discovered in this study is related to the 2-dimensional quantum hall effect.

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