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MS 584 Expansions for the Conditional Distribution and Density of a Standard Estimate

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MS 584 Expansions for the Conditional Distribution and Density of a Standard Estimate

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Abstract: A good approximation for the distribution of an estimate, is vital for statistical inference. Here we give Edgeworth expansions for the conditional density and the conditional distribution of *any* multivariate standard estimate.

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1. Introduction and Summary

Suppose that we have a non-lattice estimate $\hat{w}$ of an unknown parameter $w \in R^q$ of a statistical model, based on a sample of size n. The distribution of a *standard* estimate is determined by the coefficients obtained by expanding its cumulants in powers of n^{-1}. In §2 we summarise the extended Edgeworth-Cornish-Fisher expansions of Withers (1984) for $\hat{w}$ when $q = 1$. Then we give the multivariate Edgeworth expansions to $O(n^{-2})$. We show that the distribution of $X_n = n^{1/2}(\hat{w} - w)$ has the form $Prob.(X_n \leq x) \approx \sum_{r=0}^{\infty} n^{-r/2} P_r(x)$ where $P_0(x) = \Phi_V(x)$, the normal distribution with $V = (k_1^{i_1 i_2})$, and for $r \geq 1$, $P_r(x) = \sum_{k=1}^{3r} [P_{rk}(x) : \ k - r \text{ even}]$ where $P_{rk}(x)$ has q^k terms, reducible using symmetry. Its density has a similarly form. We argue that these expansions may be valid even if $q = q_n \to \infty$ if $q_n / n^{1/6}$ is bounded.

§3 gives these expansions in complete detail when $q = 2$.

In §4 we suppose that $q \geq 2$ and partition X_n as $\binom{\mathbf{X}_{n1}}{\mathbf{X}_{n2}}$ of dimensions $q_1 \geq 1$ and $q_2 \geq 1$. We derive expansions for the conditional density and distribution of $\mathbf{X}_{n1}$ given $\mathbf{X}_{n2}$ to $O(n^{-2})$. §5 specialises to bivariate estimates.

§6 gives the extended Cornish-Fisher expansions for the quantiles of the conditional distribution when $q_1 = 1$. An example is the distribution of a sample mean given the sample variance.

2. Extended Edgeworth-Cornish-Fisher theory

Univariate estimates. Suppose that $\hat{w}$ is a *standard estimate* of $w \in R$ with respect to n, typically the sample size. That is, $E\,\hat{w} \to w$ as $n \to \infty$, and its rth cumulant can be expanded as

$$\kappa_r(\hat{w}) \approx \sum_{j=r-1}^{\infty} n^{-j}\, a_{rj} \text{ for } r \geq 1, \tag{1}$$

where the *cumulant coefficients* a_{rj} may depend on n but are bounded as $n \to \infty$, and a_{21} is bounded away from 0. Here and below $\approx$ indicates an asymptotic expansion that need not converge. So (1) holds in the sense that

$$\kappa_r(\hat{w}) = \sum_{j=r-1}^{I-1} n^{-j}\, a_{rj} + O(n^{-I}) \text{ for } I \geq r \geq 1,$$

where $y_n = O(x_n)$ means that y_n / x_n is bounded in n. Withers (1984) extended Cornish and Fisher (1937) and Fisher and Cornish (1960) to give the distribution and quantiles of

$$Y_n = (n/a_{21})^{1/2}(\hat{w} - w)$$

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have asymptotic expansions in powers of $n^{-1/2}$:

$$P_n(x) = Prob(Y_n \leq x) \approx \Phi(x) - \phi(x) \sum_{r=1}^{\infty} n^{-r/2}\, h_r(x), \tag{2}$$

$$p_n(x) = dP_n(x)/dx \approx \phi(x)\ [1 + \sum_{r=1}^{\infty} n^{-r/2}\, \overline{h}_r(x)], \tag{3}$$

$$\Phi^{-1}(P_n(x)) \approx x - \sum_{r=1}^{\infty} n^{-r/2}\, f_r(x),\ P_n^{-1}(\Phi(x)) \approx x + \sum_{r=1}^{\infty} n^{-r/2}\, g_r(x), \tag{4}$$

where $\Phi(x) = Prob(N \leq x)$, $N \sim \mathcal{N}(0,1)$ is a unit normal random variable with density $\phi(x) = (2\pi)^{-1/2}e^{-x^2/2}$, and $h_r(x), \overline{h}_r(x), f_r(x), g_r(x)$ are polynomials in x and the standardized cumulant coefficients

$$\begin{aligned}
&A_{ri} = a_{ri}/a_{21}^{r/2} : \\
&h_1(x) = f_1(x) = g_1(x) = A_{11} + A_{32}H_2/6,\ \overline{h}_1(x) = A_{11}H_1 + A_{32}H_3/6, \\
&h_2(x) = (A_{11}^2 + A_{22})H_1/2 + (A_{11}A_{32} + A_{43}/4)H_3/6 + A_{32}^2H_5/72, \\
&f_2(x) = (A_{22}/2 - A_{11}A_{32}/3)H_1 + A_{43}H_3/24 - A_{32}^2(4x^3 - 7x)/36, \\
&g_2(x) = A_{22}H_1/2 + A_{43}H_3/24 - A_{32}^2(2x^3 - 5x)/36, \\
&\overline{h}_2(x) = (A_{11}^2 + A_{22})H_2/2 + (A_{11}A_{32} + A_{43}/4)H_4/6 + A_{32}^2H_6/72,
\end{aligned} \tag{5}$$

and so on, where H_k is the kth Hermite polynomial,

$$\begin{aligned}
H_k &= H_k(x) = \phi(x)^{-1}(-d/dx)^k\phi(x) \\
&= E(x+iN)^k \text{ for } k \geq 0,\ i = \sqrt{-1},\ N \sim \mathcal{N}(0,1) : \\
H_0 &= 1,\ H_1 = x,\ H_2 = x^2 - 1,\ H_3 = x^3 - 3x,\ H_4 = x^4 - 6x^2 + 3, \\
H_5 &= x^5 - 10x^3 + 15x, H_6 = x^6 - 15x^4 + 45x^2 - 15.\ \cdots
\end{aligned} \tag{6}$$

See Withers (1984) for $g_r,\ r \leq 6$,, Withers (2000) for (6) and §6 for relations between h_r, f_r, g_r. Also,

$$\begin{aligned}
&\ln\,[p_n(x)/\phi(x)] \approx \sum_{r=1}^{\infty} n^{-r/2}\, b_r(x) \text{ where } b_1(x) = h_1(x), \\
&b_2(x) = -A_{11}^2/2 + (A_{22} - A_{32}A_{11})H_2/2 - A_{32}^2(3x^4 - 12x^2 + 5)/24 + A_{43}H_4/24.
\end{aligned}$$

For $r > 1$, $b_r(x)$ is a polynomial of order only $r+2$, while $\overline{h}_r(x)$ is of order $3r$.

The original Edgeworth expansion was for $\hat{w}$ the mean of n independent identically distributed random variables from a distribution with rth cumulant κ_r. So (1) holds with $a_{ri} = \kappa_r I(i = r-1)$, and other $a_{ri} = 0$. An explicit formula for its general term was given in Withers and Nadarajah (2009) using Bell polynomials.

Ordinary Bell polynomials. For a sequence $e = (e_1, e_2, \ldots)$, *the partial ordinary Bell polynomial* $\tilde{B}_{rs} = \tilde{B}_{rs}(e)$, is defined by the identity

$$\text{for } s \geq 0,\ S^s = \sum_{r=s}^{\infty} z^r \tilde{B}_{rs}(e) \text{ where } S = \sum_{r=1}^{\infty} z^r e_r,\ z \in R. \tag{7}$$

$$\text{So, } \tilde{B}_{r0} = \delta_{r0},\ \tilde{B}_{r1} = e_r,\ \tilde{B}_{rr} = e_1^r,\ \tilde{B}_{32} = 2e_1e_2, \tag{8}$$

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where $\delta_{00} = 1,\ \delta_{r0} = 0$ for $r \neq 0$. They are tabled on p309 of Comtet (1974). *The complete ordinary Bell polynomial*, $\tilde{B}_r(e)$ is defined in terms of S by

$$e^S = \sum_{r=0}^{\infty} z^r \tilde{B}_r(e). \text{ So } \tilde{B}_0(e) = 1 \text{ and for } r \geq 1,\ \tilde{B}_r(e) = \sum_{s=1}^{r} \tilde{B}_{rs}(e)/s! : \tag{9}$$

$$\tilde{B}_1(e) = e_1,\ \tilde{B}_2(e) = e_2 + e_1^2/2,\ \tilde{B}_3(e) = e_3 + e_1e_2 + e_1^3/6. \tag{10}$$

Multivariate estimates. Suppose that $\hat{w}$ is a *standard estimate* of $w \in R^p$ with respect to n. That is, $E\ \hat{w} \to w$ as $n \to \infty$, and for $r \geq 1$, $1 \leq i_1,\ \ldots,\ i_r \leq p$, the rth order cumulants of $\hat{w}$ can be expanded as

$$\bar{k}^{1-r} = \kappa(\hat{w}^{i_1}, \ldots, \hat{w}^{i_r}) = \sum_{d=r-1}^{\infty} \bar{k}_d^{1-r}\ n^{-d},\ \bar{k}_d^{1-r} = k_d^{i_1-i_r}, \tag{11}$$

where the *cumulant coefficients* $\bar{k}_d^{1-r} = k_d^{i_1-i_r}$ may depend on n but are bounded as $n \to \infty$. So the bar replaces i_j by j: $\bar{k}_0^1 = w^{i_1},\ \bar{k}_1^{12} = k_1^{i_1i_2}$.

$$X_n = n^{1/2}(\hat{w} - w) \xrightarrow{\mathcal{L}} \mathcal{N}_p(0, V) \text{ for } V = (k_1^{i_1i_2}),\ p \times p, \tag{12}$$

with density and distribution

$$\phi_V(x) = (2\pi)^{-q/2}(det(V))^{-1/2}\exp(-x'V^{-1}x/2),\ \Phi_V(x) = \int_{-\infty}^{x} \phi_V(x)dx.$$

V may depend on n, but we assume that $det(V)$ is bounded away from 0. Set

$$\bar{t}_r = t_{i_r},\ \bar{b}_{2d}^{1-r} = \bar{k}_d^{1-r},\ \bar{b}_{2d+1}^{1-r} = 0,\ e_j(t) = \sum_{r=1}^{j+2} \bar{b}_{r+j}^{1-r}\ \bar{t}_1 \ldots \bar{t}_r/r!. \tag{13}$$

$$\text{So } e_1 = \bar{k}_1^1\bar{t}_1 + \bar{k}_2^{1-3}\bar{t}_1\bar{t}_2\bar{t}_3/6,\ e_2 = \bar{k}_2^{12}\bar{t}_1\bar{t}_2/2 + \bar{k}_3^{1-4}\bar{t}_1 \ldots \bar{t}_4/24.$$

$$\text{For } r \geq 1,\ \tilde{B}_r(e(t)) = \sum_{k=1}^{3r} [\bar{P}_r^{1-k}\ \bar{t}_1 \ldots \bar{t}_k :\ k - r \text{ even}], \tag{14}$$

where for $r \leq 3$, $\bar{P}_r^{1-k}$ is a function of $\bar{k}_e^{1-r}$ given for the 1st time in the appendix. In (13), (14) and below, we use *the tensor summation convention* of implicitly summing $i_1, \ldots, i_r$ over their range $1, \ldots, q$. We make $\bar{P}_r^{1-k}$ symmetric in $i_1, \ldots, i_k$ using the operator $\mathcal{S}$ that symmetrizes over $i_1, \ldots, i_k$:

$$\bar{P}_1^1 = \bar{k}_1^1,\ \bar{P}_1^{1-3} = \bar{k}_2^{1-3}/6,\ \bar{P}_2^{12} = \bar{k}_2^{12}/2 + k_1^1\bar{k}_1^2/2, \tag{15}$$

$$\bar{P}_2^{1-4} = \bar{k}_3^{1-4}/24 + \mathcal{S}\ \bar{k}_1^1\bar{k}_2^{234}/6,\ \bar{P}_2^{1-6} = \mathcal{S}\ \bar{k}_2^{1-3}\bar{k}_2^{4-6}/72, \tag{16}$$

$$\bar{P}_3^1 = \bar{k}_2^1,\ \bar{P}_3^{1-3} = \bar{k}_3^{1-3}/6 + \mathcal{S}\ \bar{k}_2^{12}\bar{k}_1^3/2 + \bar{k}_1^1\bar{k}_1^2\bar{k}_1^3/6,$$

$$\bar{P}_3^{1-5} = \bar{k}_4^{1-5}/120 + \mathcal{S}\ \bar{k}_3^{1-4}\bar{k}_1^5/24 + \mathcal{S}\ \bar{k}_2^{12}\bar{k}_2^{3-5}/12 + \mathcal{S}\ \bar{k}_2^{123}\bar{k}_1^4\bar{k}_1^5/12,$$
$$\bar{P}_3^{1-7} = \mathcal{S}\ \bar{k}_2^{123}\bar{k}_3^{4-7}/144 + \mathcal{S}\ \bar{k}_2^{123}\bar{k}_2^{4-6}\bar{k}_1^7/72,$$
$$\bar{P}_3^{1-9} = \mathcal{S}\ \bar{k}_2^{1-3}\bar{k}_2^{4-6}\bar{k}_2^{7-9}/6^3. \tag{17}$$

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The terms involving $\mathcal{S}$ are given in the Appendix A. By Withers and Nadarajah (2010b) or Withers (2024), X_n has distribution and density

$$Prob.(X_n \leq x) \approx \sum_{r=0}^{\infty} n^{-r/2} P_r(x),\ p_{X_n}(x) \approx \sum_{r=0}^{\infty} n^{-r/2} p_r(x),\ x \in R^p, \tag{18}$$

$$\text{where } P_0(x) = \Phi_V(x),\ p_0(x) = \phi_V(x), \text{ and for } r \geq 1, \tag{19}$$

$$P_r(x) = \tilde{B}_r(e(-\partial/\partial x))\, \Phi_V(x),\ p_r(x) = \tilde{B}_r(e(-\partial/\partial x))\, \phi_V(x), \tag{20}$$

So for $\partial_i = \partial/\partial x_i,\ \bar{\partial}_k = \partial_{i_k}$ and $\bar{O}^{1-k} = (-\bar{\partial}_1)\ldots(-\bar{\partial}_k)$,

$$P_r(x) = \sum_{k=1}^{3r} [P_{rk}(x) :\ k - r \text{ even}] \text{ for } P_{rk}(x) = \bar{P}_r^{1-k}\, \bar{O}^{1-k}\, \Phi_V(x), \tag{21}$$

$$p_r(x)/\phi_V(x) = \sum_{k=1}^{3r} [\tilde{p}_{rk} :\ k - r \text{ even}] = \tilde{p}_r(x) \text{ say}, \tag{22}$$

$$\tilde{p}_{rk} = \bar{P}_r^{1-k} \bar{H}^{1-k} \text{ where } \bar{H}^{1-k} = \bar{H}^{1-k}(x, V) = \phi_V(x)^{-1} \bar{O}^{1-k}\, \phi_V(x) \tag{23}$$

is *the multivariate Hermite polynomial.* For their dual form see Withers and Nadarajah (2014). By Withers (2020), for $i = \sqrt{-1}$,

$$\bar{H}^{1-k} = E\, \Pi_{j=1}^{k} (\bar{y}_j + i\bar{Y}_j) \text{ where } \bar{y}_j = y_{i_j},\ \bar{Y}_j = Y_{i_j},\ y = V^{-1}x,$$

$$Y \sim \mathcal{N}_p(0, V^{-1}). \text{ So, } H^1 = y_1,\ \bar{H}^1 = \bar{y}_1,\ H^{12} = y_1 y_2 - V^{12},\ \bar{H}^{12} = \bar{y}_1 \bar{y}_2 - \bar{V}^{12},$$

$$H^{1-3} = y_1 y_2 y_3 - \sum^3 V^{12} y_3,\ \sum^3 V^{12} y_3 = V^{12} y_3 + V^{13} y_2 + V^{23} y_1,$$

$$H^{1-4} = y_1 \ldots y_4 - \sum^6 V^{12} y_3 y_4 + \sum^3 V^{12} V^{34},$$

$$H^{1-5} = y_1 \ldots y_5 - \sum^{10} V^{12} y_3 \ldots y_5 + \sum^5 y_5 \sum^3 V^{12} V^{34},$$

$$H^{1-6} = y_1 \ldots y_6 - \sum^{15} V^{12} y_3 \ldots y_6 + \sum^{15} y_5 y_6 \sum^3 V^{12} V^{34} - \sum^{45} V^{12} V^{34} V^{56},$$

where $V^{i_1 i_2}$ is the (i_1, i_2) element of V^{-1}, and $\bar{V}^{j_1 j_2}$ is the (i_{j_1}, i_{j_2}) element of V^{-1}. This gives $\bar{H}^{1-k}$ in terms of the moments of Y. For example

$$P_1(x) = e_1(-\partial/\partial x))\, \Phi_V(x) = \sum_{r=1}^{3} \bar{b}_{r+1}^{1-r}\, \bar{O}^{1-r}\, \Phi_V(x)/r! = \sum_{k=1,3} P_{1k}(x),$$

$$P_{11}(x) = \bar{k}_1^1\, (-\bar{\partial}_1)\, \Phi_V(x),\ P_{13}(x) = \bar{k}_2^{1-3}\, \bar{O}^{1-3}\, \Phi_V(x)/6,$$

$$p_1(x) = \bar{k}_1^1\, (-\bar{\partial}_1)\, \phi_V(x) + \bar{k}_2^{1-3}\, \bar{O}^{1-3}\, \phi_V(x)/6,$$

$$\tilde{p}_1(x) = p_1(x)/\phi_V(x) = \sum_{k=1,3} \tilde{p}_{1k},\ \tilde{p}_{11} = \bar{k}_1^1\, \bar{H}^1,\ \tilde{p}_{13} = \bar{k}_2^{1-3}\, \bar{H}^{1-3}/6, \tag{24}$$

$$P_2(x) = \sum_{k=2,4,6} P_{2k},\ \tilde{p}_2(x) = \sum_{k=2,4,6} \tilde{p}_{2k}, \tag{25}$$

$$P_3(x) = \sum_{k=1,3,5,7,9} P_{3k}(x),\ \tilde{p}_3(x) = \sum_{k=1,3,5,7,9} \tilde{p}_{3k}. \tag{26}$$

This gives the Edgeworth expansion for the distribution of Y_n to $O(n^{-2})$. See Withers (2024) for more terms.

For large q, $P_{rk}, \tilde{p}_{rk}$ of (21) and (23) have $\sim q^{3r}$ terms. So if $q = q_n \to \infty$ and $M_{rn} = max_k |\bar{P}_r^{1-k}|$, then $n^{-r/2}(P_r(x), p_r(x) \sim M_{rn} \nu_n^r$ where $\nu_n = n^{-1/2} q_n^3$. So if for example M_{rn} is bounded, then the Edgeworth series should converge if $q_n/n^{1/6} \to 0$.

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The log density can be expanded as

$$\ln\left[p_{Y_n}(x)/\phi_V(x)\right] \approx \sum_{r=1}^{\infty} n^{-r/2}\, b_r(x). \tag{27}$$

$$\text{So by (18), } \tilde{p}_r(x) = \tilde{B}_r(b(x)) \text{ where } b = (b_2, b_2, \dots). \tag{28}$$

See Withers and Nadarajah (2016). Also for $H_k(x)$ of (6),

$$H^{j^k} = E\,(y_j + iY_j)^k = \tau_j^k H_k(\tau_j^{-1} y_j) \text{ where } \tau_j = (V^{jj})^{1/2}: \tag{29}$$

$$H^j = y_j,\ H^{j^2} = y_j^2 - V^{jj},\ H^{j^3} = y_j^3 - 3V^{jj}y_j,\ H^{j^4} = y_j^4 - 6V^{jj}y_j^2 + 3(V^{jj})^2.$$

$$\text{So if } q = 1, H^{1^k}(x, V) = \sigma^{-k} H_k(\sigma x) \text{ where } \sigma = V^{1/2}.. \tag{30}$$

Example 1. *Let $\hat{w}$ be a sample mean. Then E $\hat{w} = w$, and only the leading coefficient in (11) are non-zero. So $\bar{k}_1^1 = \bar{P}_1^1 = \tilde{p}_{11} = 0$. In order needed, the non-zero P_r^{1-k} are*

$$\bar{P}_1^{1-3} = \bar{k}_2^{1-3}/3!,\ \bar{P}_2^{1-4} = \bar{k}_3^{1-4}/4!,\ \bar{P}_2^{1-6} \text{ of (16)},$$
$$\bar{P}_3^{1-5} = \bar{k}_4^{1-5}/5!,\ \bar{P}_3^{1-7} = \mathcal{S}\,\bar{k}_2^{123}\bar{k}_3^{4-7}/144,\ \bar{P}_3^{1-9} \text{ of (17)}.$$

$\tilde{p}_{rk}$ and P_{rk} have q^k terms but many are duplicates. We now show how symmetry reduces this to $\sim \binom{q}{k}$ terms. We use the multinomial coefficient $\binom{k}{a\dots b} = k!/a!\dots b!$. For example $\binom{3}{111} = 6$.

Set $T_r^{i_1\dots i_k} = \bar{P}_r^{1-k}\,\bar{H}^{1-k}$ where tensor summation is *not* used. By (22),

$$\tilde{p}_{rk} = \sum_{i_1,\dots,i_k=1}^{q} T_r^{i_1\dots i_k}. \text{ So,}$$

$$\tilde{p}_{r1} = \sum_{i_1=1}^{q} T_r^{i_1},\ \tilde{p}_{r2} = \sum_{i_1=1}^{q} T_r^{i_1 i_1} + 2\sum_{i_1>i_2}^{\binom{q}{2}} T_r^{i_1 i_2},$$

$$\tilde{p}_{r3} = \sum_{i_1=1}^{q} T_r^{i_1 i_1 i_1} + \binom{3}{1}\sum_{i_1\neq i_2}^{q(q-1)} T_r^{i_1 i_1 i_2} + \binom{3}{111}\sum_{i_1>i_2>i_3}^{\binom{q}{3}} T_r^{i_1 i_2 i_3},$$

$$\tilde{p}_{r4} = \sum_{i_1=1}^{q} T_r^{i_1 i_1 i_1 i_1} + \binom{4}{1}\sum_{i_1\neq i_2}^{q(q-1)} T_r^{i_1 i_1 i_1 i_2} + \binom{4}{2}\sum_{i_1>i_2}^{\binom{q}{2}} T_r^{i_1 i_1 i_2 i_2} + \binom{4}{211}\sum_{i_1>i_2}^{q\binom{q}{2}} T_r^{i_1 i_1 i_2 i_3}$$
$$+\,4!\sum_{i_1>i_2>i_3>i_4}^{\binom{q}{4}} T_r^{i_1 i_2 i_3 i_4},$$

$$\tilde{p}_{r6} = \sum_{i_1=1}^{q} T_r^{i_1^6} + \binom{6}{1}\sum_{i_1\neq i_2}^{q(q-1)} T_r^{i_1^5 i_2} + \binom{6}{2}\sum_{i_1\neq i_2}^{q(q-1)} T_r^{i_1^4 i_2^2} + \binom{6}{3}\sum_{i_1>i_2}^{\binom{q}{2}} T_r^{i_1^3 i_2^3}$$
$$+\binom{6}{411}\sum^{3\binom{q}{3}} T_r^{i_1^4 i_2 i_3} + \binom{6}{321}\sum^{6\binom{q}{3}} T_r^{i_1^3 i_2^2 i_3} + \binom{6}{222}\sum_{i_1>i_2>i_3}^{\binom{q}{3}} T_r^{i_1^2 i_2^2 i_3^2}$$
$$+\binom{6}{3111}\sum_{i_2>i_3>i_4}^{4\binom{q}{4}} T_r^{i_1^3 i_2 i_3 i_4} + \binom{6}{2211}\sum_{i_1>i_2, i_3>i_4}^{6\binom{q}{4}} T_r^{i_1^2 i_2^2 i_3 i_4}$$
$$+\binom{6}{21111}\sum_{i_2>i_3>i_4>i_5}^{4\binom{q}{4}} T_r^{i_1^2 i_2 i_3 i_4 i_5} + 6!\sum_{i_1>\cdots>i_6}^{\binom{q}{6}} T_r^{i_1\dots i_6},$$

where all i_j are distinct. Similarly we can write out $\tilde{p}_{rk}$ for $k = 5, 7, 9$. This reduces the number of terms in $\tilde{p}_{rk}$ from q^k to $q + \binom{q}{2}$ for $k = 2$, to $q + q(q-1) + \binom{q}{3}$ for $k = 3$, to $q + q(q-1) + \binom{q}{2} + q\binom{q}{2} + \binom{q}{4}$ for $k = 4$, and to $q + 5\binom{q}{2} + 10\binom{q}{3} + 14\binom{q}{4} + \binom{q}{6}$ for $k = 6$.

If we reinterpret $T_r^{i_1 \dots i_k}$ as $\bar{P}_r^{1-k}\,(-\bar{\partial}_1)\dots(-\bar{\partial}_k)\,\Phi_V(x)$, where again tensor summation is *not* used, then we can reinterpret the above expression for $\tilde{p}_{rk}$, as an expression for $P_{rk}(x)$. For example,

$$P_{r2}(x) = \sum_{i_1=1}^{q} P_r^{i_1 i_1}(-\partial_{i_1})^2\,\Phi_V(x) + 2\sum_{i_1>i_2}^{\binom{q}{2}} P_r^{i_1 i_2}(-\partial_{i_1})(-\partial_{i_2})\,\Phi_V(x).$$

These results can be extended to *Type B estimates*, that is to $\hat{w}$ with cumulant expansions not of type (11), but

$$\kappa(\hat{w}^{i_1}, \dots, \hat{w}^{i_r}) \approx \sum_{d=2r-2}^{\infty} b_d^{i_1 - i_r}\, n^{-d/2}.$$

3. The Distribution of $n^{1/2}(\hat{w} - w)$ for $q = 2$

We first give $\tilde{p}_{rk}$ of (22) for $r \le 3$, and then $P_{rk}(x)$ of (21).

$$\tilde{p}_{rk} = \sum_{b=0}^{k} P_r(1^{k-b}2^b)\,H_{k-b,b} \text{ where } P_r(1^a 2^b) = \binom{a+b}{a} P_r^{1^a 2^b}, \tag{31}$$

for $\bar{P}_r^{1-k}$ of (16), (17), where we use the dual notation,

$$\begin{aligned} H_{ab} = H^{1^a 2^b} &= E\,(y_1 + iY_1)^a\,(y_2 + iY_2)^b \\ &= \sum_{j_1=0}^{a}\binom{a}{j_1} y_1^{a-j_1} \sum_{j_2=0}^{b}\binom{b}{j_2} y_2^{b-j_2} E\,(iY_1)^{j_1}(iY_2)^{j_2}. \end{aligned} \tag{32}$$

So H_{k0} and H_{0k} are given by (29) with $j = 1$ and 2,

$$H_{11} = y_1 y_2 - V^{12}, H_{21} = y_1^2 y_2 - V^{11} y_2 - 2V^{12} y_1,\ H_{12} = y_1 y_2^2 - V^{22} y_1 - 2V^{12} y_2.$$

For more examples see Withers (2000). $P_r(1^b 2^a)$ is just $P_r(1^a 2^b)$ with 1 and 2 reversed. The other $P_r(1^a 2^b)$ needed in (31) for $\tilde{p}_{rk}$ are as follows.

For $\tilde{p}_{11},\ P_1(j) = k_1^j$. For $\tilde{p}_{13},\ P_1(j^3) = k_2^{jjj}/6,\ P_1(1^2 2) = k_2^{112}/2$.

For $\tilde{p}_{31},\ P_3(j) = k_2^j$. For $\tilde{p}_{33},\ P_3(j^3) = k_3^{j^3}/6 + k_2^{jj} k_1^j/2 + (k_1^j)^3/6$,
$P_3(1^2 2) = [k_3^{112} + 2k_1^1 k_2^{12} + k_1^2 k_2^{11} + (k_1^1)^2 k_1^2]/6$.

For $\tilde{p}_{22},\ P_2(j^2) = k_2^{jj}/2 + (k_1^j)^2/2,\ P_2(12) = k_2^{12} + k_1^1 k_1^2$.

For $\tilde{p}_{24},\ P_2(j^4) = k_3^{j^4}/24 + k_1^j k_2^{j^3}/6,\ P_2(1^3 2) = k_3^{1112}/6 + k_1^1 k_2^{112}/3 + k_1^2 k_2^{111}/3$,
$P_2(1^2 2^2) = k_3^{1122}/4 + k_1^1 k_2^{112}/2 + k_1^2 k_2^{122}/2$,

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For $\tilde{p}_{26}$, $P_2(j^6) = (k_2^{jjj})^2/72$, $P_2(1^5 2) = k_2^{111} k_2^{112}/12$,
$P_2(1^4 2^2) = k_2^{111} k_2^{122}/12 + (k_2^{112})^2/8$, $P_2(1^3 2^3) = k_2^{111} k_2^{222}/36 + k_2^{112} k_2^{122}/4$,

For $\tilde{p}_{35}$, $P_3(j^5) = k_4^{j^5}/120 + k_3^{j^4} k_1^j/24 + k_2^{jjj}[k_2^{jj} + (k_1^j)^2]/12$,
$P_3(1^4 2)/5 = P_3^{1^4 2} = k_4^{1^4 2}/120 + S_1/24 + S_2/12 + S_3/12$ where
$S_1 = (4k_1^1 k_3^{1112} + k_1^2 k_3^{1111})/5$ by (A3), $S_2 = (3k_2^{11} k_2^{112} + 2k_2^{12} k_2^{111})/5$ by (A4),
$S_3 = [2k_1^1 k_1^2 k_2^{111} + 3(k_1^1)^2 k_2^{112}]/5$ by (A4),
$P_3(1^3 2^2) = 10 P_3^{1^3 2^2} = k_4^{1^3 2^2}/12 + k_3^{1^3 2} k_1^2/3 + k_3^{1122} k_1^1/4 + k_3^{1^3} (k_1^2)^2/12$
$+ k_3^{112} k_1^2 k_1^2/2 + k_3^{122} (k_1^1)^2/4$.

For $\tilde{p}_{37}$, $P_3(j^7) = k_3^{j^4} k_2^{j^3}/144 + (k_2^{j^3})^2 k_1^j/72$,
$P_3(1^6 2) = 7(S_4/144 + S_5/72)$ where by (A7), $S_4 = (3k_2^{112} k_3^{1111} + 4k_2^{111} k_3^{1112})/7$,
$S_5 = [k_1^2 (k_2^{111})^2 + 6k_1^1 k_2^{111} k_2^{112}]/7$, so that
$P_3(1^6 2) = k_2^{112} k_3^{1111}/48 + k_2^{111} k_3^{1112}/36 + k_1^2 (k_2^{111})^2/72 + k_1^1 k_2^{111} k_2^{112}/12$.
$P_3(1^5 2^2) = k_3^{1^4} k_2^{122}/48 + k_3^{1^3 2} k_2^{112}/12 + k_3^{1122} k_2^{111}/24 + 5k_2^{111} k_2^{112} k_1^2/24$
$+ k_2^{111} k_2^{122} k_1^1/12 + (k_2^{112})^2 k_1^1/12$,
$P_3(1^4 2^3) = 35 P_3^{1^4 2^3} = A/144 + B/72$ for $A = k_3^{1^4} k_2^{222} + 13 k_3^{1^3 2} k_2^{122} + 17 k_3^{1122} k_2^{112}$
$+ k_3^{1222} k_2^{111}$, $B = (k_2^{111} k_2^{222} + 15 k_2^{112} k_2^{122}) k_1^1 + (9 k_2^{111} k_2^{122} + 10 k_2^{112} k_2^{112}) k_1^2$.

For $\tilde{p}_{39}$, $P_3(j^9) = (k_2^{j^3})^3/6^3$, $P_3(1^8 2) = 9(k_2^{111})^2 k_2^{112}/6^3$,
and for $\bar{a}^{123} = \bar{k}_2^{123}$, $6^3\ P_3(1^7 2^2) = 9(a^{111})^2 a^{122} + 3a^{111}(a^{112})^2$,
$6^3\ P_3(1^6 2^3) = 3[(a^{111})^2 a^{222} + 18 a^{111} a^{112} a^{122} + 9(a^{112})^3]$,
$6^3\ P_3(1^5 2^4) = 9[a^{111} a^{112} a^{222} + 15 a^{111} (a^{122})^2 + 27 (a^{112})^2 a^{122}]$.

(18) and (22) now give the distribution and density of $X_n = n^{1/2}(\hat{w} - w)$ to $O(n^{-2})$. Set

$$H_{ab}^* = (-\partial_1)^a (-\partial_2)^b\ \Phi_V(x). \text{ So } H_{ab}^* = H_{a-1,b-1}\ \phi_V(x) \text{ if } a > 0, b > 0,$$
$$H_{a0}^* = \int_{-\infty}^{x_2} H_{a-1,0} \phi_V(x) dx_2 \text{ if } a > 0,\ H_{0b}^* = \int_{-\infty}^{x_1} H_{0,b-1} \phi_V(x) dx_1 \text{ if } b > 0.$$

Then $P_{rk}(x)$ of (21) is given by replacing H_{ab} by H_{ab}^* in the expressions above for $\tilde{p}_{rk}$. That is,

$$P_{rk}(x) = \sum_{b=0}^{k} P_r(1^{k-b} 2^b)\ H_{k-b,b}^*. \tag{33}$$

(18) and (21) now give $Prob(X_n \le x)$ to $O(n^{-2})$ for $X_n = n^{1/2}(\hat{w} - w)$.

4. The Conditional Density and Distribution

For $q = q_1 + q_2$, $q_1 \ge 1$, and $q_2 \ge 1$, partition x, $y = V^{-1}x$, $X_n = n^{1/2}(\hat{w} - w)$ and $X \sim \mathcal{N}_q(0, V)$, as $\binom{\mathbf{x}_1}{\mathbf{x}_2}$, $\binom{\mathbf{y}_1}{\mathbf{y}_2}$, $\binom{\mathbf{X}_{n1}}{\mathbf{X}_{n2}}$, $\binom{\mathbf{X}_1}{\mathbf{X}_2}$, where $\mathbf{x}_i, \mathbf{y}_i, \mathbf{X}_{ni}$ are vectors of length q_i. Partition V, V^{-1} as $(\mathbf{V}_{ij}), (\mathbf{V}^{ij})$ where

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$\mathbf{V}_{ij}, \mathbf{V}^{ij}$ are $q_i \times q_j$.
The conditional density of $\mathbf{X}_{n1}$ given $(\mathbf{X}_{n2} = \mathbf{x}_2)$, is

$$p_{1.2}(\mathbf{x}_1) = p_{X_n}(x)/p_{X_{n2}}(\mathbf{x}_2) = \phi_{1.2}(\mathbf{x}_1)\ (1+S)/(1+S_2) \tag{34}$$

$$\text{where } S = p_{X_n}(x)/\phi_V(x) - 1 \approx \sum_{r=1}^{\infty} n^{-r/2}\tilde{p}_r(x), \tag{35}$$

$$S_2 = p_{X_{n2}}(\mathbf{x}_2)/\phi_{\mathbf{V}_{22}}(\mathbf{x}_2) - 1 \approx \sum_{r=1}^{\infty} n^{-r/2} f_r,\ f_r = p_r^*(\mathbf{x}_2), \tag{36}$$

$p_r^*(\mathbf{x}_2)$ is $\tilde{p}_r(x)$ of (22) for $\mathbf{X}_{n2}$, and $\phi_{1.2}(\mathbf{x}_1)$ is the density of $\mathbf{X}_1|(\mathbf{X}_2 = \mathbf{x}_2)$. By (37)–(39), §2.5 of Anderson (1958),

$$\phi_{1.2}(\mathbf{x}_1) = \phi_V(x)/\phi_{\mathbf{V}_{22}}(\mathbf{x}_2) = \phi_{V_{1.2}}(\mathbf{x}_1 - \mu_{1.2}), \tag{37}$$

$$\text{where } \mu_{1.2} = \mathbf{V}_{12}\mathbf{V}_{22}^{-1}\mathbf{x}_2,\ V_{1.2} = \mathbf{V}_{11} - \mathbf{V}_{12}\mathbf{V}_{22}^{-1}\mathbf{V}_{21}. \tag{38}$$

The distribution of $\mathbf{X}_1|(\mathbf{X}_2 = \mathbf{x}_2)$ is

$$\Phi_{1.2}(\mathbf{x}_1) = \Phi_{V_{1.2}}(\mathbf{x}_1 - \mu_{1.2}). \tag{39}$$

$$\text{Set } \bar{H}_q^{1-k} = \bar{H}^{1-k} = \bar{H}^{1-k}(x, V),\ \bar{H}_{q2}^{1-k} = \bar{H}^{1-k}(\mathbf{x}_2, \mathbf{V}_{22}). \tag{40}$$

By (22), for $r \geq 1$ and $\bar{P}_r^{1-k}$ of (14)–(16),

$$p_r^*(\mathbf{x}_2) = \sum_{k=1}^{3r}[p_{rk}^* :\ k - r \text{ even}],\ \text{where } p_{rk}^* = \bar{P}_r^{1-k}\bar{H}_{q2}^{1-k} \tag{41}$$

and $\bar{H}_{q2}^{1-k}$ is given by replacing $y = V^{-1}x$ and $(V^{ij}) = V^{-1}$ in $\bar{H}_q^{1-k}$ by

$$z = \mathbf{V}_{22}^{-1}\mathbf{x}_2 \text{ and } (U^{ij}) = \mathbf{V}_{22}^{-1}, \tag{42}$$

and now implicit summation in (41) is for $i_1, \ldots, i_k$ over $q_1+1, \ldots, q$. So,

$$p_1^*(\mathbf{x}_2) = \sum_{k=1,3} p_{1k}^*,\ p_{11}^* = \sum_{i_1=q_1+1}^{q} \bar{k}_1^1\ \bar{H}_{q2}^1,\ p_{13}^* = \sum_{i_1,i_2,i_3=q_1+1}^{q} \bar{k}_2^{1-3}\ \bar{H}_{q2}^{1-3}/6, \tag{43}$$

$$\bar{H}_{q2}^1 = \bar{z}_1,\ \bar{H}_{q2}^{1-3} = \bar{z}_1\bar{z}_2\bar{z}_3 - \sum^3 \bar{U}^{12}\bar{z}_3,$$

$$p_2^*(\mathbf{x}_2) = \sum_{k=2,4,6} p_{2k}^* \text{ for } p_{2k}^* \text{ of (41) and } \bar{P}_2^{1-k} \text{ of (15), (16)}, \tag{44}$$

$p_3^*(\mathbf{x}_2) = \sum_{k=1,3,5,7,9} p_{3k}^*$, and so on. For $\tilde{B}_{rs}(e)$ of (7), set

$$B_r^* = B_r^*(e) = \sum_{s=0}^{r} \tilde{B}_{rs}(e). \tag{45}$$

So, $B_0^* = 1\ B_1^* = e_1,\ B_2^* = e_2 + e_1^2,\ B_3^* = e_3 + 2e_1e_2 + e_1^3,$

and $(1+S_2)^{-1} = \sum_{r=0}^{\infty} n^{-r/2}C_r$ where $C_r = B_r^*(-f)$ for f_r of (36) :

$$C_0 = 1,\ C_1 = -f_1,\ C_2 = f_1^2 - f_2,\ C_3 = -f_1^3 + 2f_1f_2 - f_3. \tag{46}$$

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So the conditional density $p_{1.2}(\mathbf{x}_1)$ of (34), relative to $\phi_{V_{1.2}}(\mathbf{x}_1)$ of (37), is

$$p_{1.2}(\mathbf{x}_1)/\phi_{1.2}(\mathbf{x}_1) \approx \sum_{r=0}^{\infty} n^{-r/2} D_r \text{ where}$$
$$D_r = C_r \otimes \tilde{p}_r(x) = \sum_{i=0}^{r} C_{r-i}\, \tilde{p}_i(x) : \tag{47}$$
$$D_0 = \tilde{p}_0(x) = 1,\ D_1 = C_1 + \tilde{p}_1(x),\ D_2 = C_2 + C_1\tilde{p}_1(x) + \tilde{p}_2(x), \tag{48}$$
$$D_3 = C_3 + C_2\tilde{p}_1(x) + C_1\tilde{p}_2(x) + \tilde{p}_3(x). \tag{49}$$

So now we have the conditional density to $O(n^{-2})$. The expansion for *the conditional distribution* about $\Phi_{1.2}(\mathbf{x}_1)$ of (39), is

$$P_{1.2}(\mathbf{x}_1) = Prob.(\mathbf{X}_{n1} \leq \mathbf{x}_1 | \mathbf{X}_{n2} = \mathbf{x}_2) \approx \Phi_{1.2}(\mathbf{x}_1) + \sum_{r=1}^{\infty} n^{-r/2} G_r \tag{50}$$
$$\text{where } G_r = \int_{-\infty}^{\mathbf{x}_1} D_r\, d\Phi_{1.2}(\mathbf{x}_1) = \int_{-\infty}^{\mathbf{x}_1} D_r \text{ for short}$$
$$= C_r \otimes g_r = \sum_{i=0}^{r} C_{r-i}\, g_i \text{ for } g_r = \int_{-\infty}^{\mathbf{x}_1} \tilde{p}_r(x) :\ g_0 = \Phi_{1.2}(\mathbf{x}_1), \tag{51}$$
$$\text{and for } r \geq 1,\ g_r = \sum_{k=1}^{3r} [G_{rk} :\ k - r \text{ even}] \text{ where by (23)}, \tag{52}$$
$$G_{rk} = \int_{-\infty}^{\mathbf{x}_1} \tilde{p}_{rk} = \bar{P}_r^{1-k} \bar{I}^{1-k} \text{ and for } \theta = \phi_{\mathbf{V}_{22}}(\mathbf{x}_2),\ \partial^\dagger = \Pi_{i=q_1+1}^{q} \partial_i, \tag{53}$$
$$\bar{I}^{1-k} = \theta^{-1} \int_{-\infty}^{\mathbf{x}_1} \bar{H}_q^{1-k} \phi_V(x) d\mathbf{x}_1 = \theta^{-1} (-\bar{\partial}_1) \ldots (-\bar{\partial}_k) \partial^\dagger\, \Phi_V(x). \tag{54}$$

This gives g_r in terms of $\bar{I}^{1-k}$, given by (54) in terms of θ and derivatives of $\Phi_V(x)$. (51) now gives G_r in terms of f_r of (36). So G_1, G_2, G_3 and (50) give the conditional distribution to $O(n^{-2})$. Alternatively, as $\bar{H}_q^{1-k}$ is a polynomial in $\mathbf{x}_1 = (x_{11}, \ldots, x_{1q_1})'$, by (37), $\bar{I}^{1-k} = \int_{-\infty}^{\mathbf{x}_1} \bar{H}_q^{1-k}\, d\Phi_{1.2}(\mathbf{x}_1)$ is linear in

$$\int_{-\infty}^{\mathbf{x}_1} x_{1j_1} \ldots x_{1j_s}\, d\Phi_{1.2}(\mathbf{x}_1) = \int_{-\infty}^{\mathbf{u}} (\mu_{1.2} + u)_{1j_1} \ldots (\mu_{1.2} + u)_{1j_s}\, d\Phi_{V_{1.2}}(\mathbf{u})$$

for $0 \leq s \leq k$ where $\mathbf{u} = \mathbf{x}_1 - \mu_{1.2}$. We now illustrate this.

The case $q_1 = 1$. So

$$\mathbf{x}_1 = x_1,\ \mathbf{X}_1 = X_1,\ \mathbf{X}_{n1} = X_{n1},\ \mathbf{V}_{11} = V_{11}. \text{ Set } \sigma_{1.2} = V_{1.2}^{1/2},$$
$$u = \sigma_{1.2}^{-1}(x_1 - \mu_{1.2}),\ U = \sigma_{1.2}^{-1}(X_1 - \mu_{1.2}),\ U_n = \sigma_{1.2}^{-1}(X_{n1} - \mu_{1.2}). \tag{55}$$

By (39), $U_n | (\mathbf{X}_{n2} = \mathbf{x}_2) \xrightarrow{\mathcal{L}} U \sim \mathcal{N}(0,1)$. By (50), for C_r of (46),

$$P_n(u) = Prob(U_n \leq u | (\mathbf{X}_{n2} = \mathbf{x}_2)) \approx \sum_{r=0}^{\infty} n^{-r/2} G_r(u) \text{ where} \tag{56}$$
$$G_r(u) = C_r \otimes g_r(u),\ G_0(u) = g_0(u) = \Phi(u), \text{ and for } r \geq 1,\ g_r(u) = g_r$$

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given by (52) in terms of $G_{rk}(u) = G_{rk}$. For θ of (53), by (37), $\theta^{-1}\phi_V(x)dx_1 = \phi(u)du$. Set

$$H^{1-k}[y] = h^{1-k}(u) = \sum_{s=0}^{k} h_s^{1-k}\, u^s = H_q^{1-k} \text{ at } y = V^{-1}x, \tag{57}$$

$$y_{i0} = \sum_{j=2}^{q} V^{ij}x_j,\ a_i = V^{i1}\mu_{1.2} + y_{i0},\ \text{ and } b_i = V^{i1}\sigma_{1.2}. \tag{58}$$

As $x_1 = \mu_{1.2} + \sigma_{1.2}\, u,\ y_i = V^{ij}x_j = a_i + b_i\, u.$

So $y = a + bu,\ h_s^{1-k} = H^{1-k}_{\bullet j_1 \ldots j_s}[a]\, b_{j_1} \ldots b_{j_s}.$

By (53),

$$G_{rk} = \bar{P}_r^{1-k}\bar{I}^{1-k} \text{ where } \bar{I}^{1-k} = \sum_{s=0}^{k} \bar{h}_s^{1-k}\, \gamma_s \text{ and } \gamma_s = \int_{-\infty}^{u} u^s\, \phi(u)du : \tag{59}$$

$$\gamma_0 = \Phi(u),\ \gamma_1 = -\phi(u),\ \gamma_s = (s-1)\gamma_{s-2} - u^{s-1}\phi(u),\ \text{for } s \geq 2. \tag{60}$$

For example $\bar{H}_q^i = \bar{y}_i,\ \bar{h}_0^1 = \bar{a}_1,\ \bar{h}_1^1 = \bar{b}_1.$

$$\bar{H}_q^{1-3} = \Pi_{j=1}^3 \bar{y}_j - \sum^3 \bar{y}_1\, \bar{V}^{23} = \Pi_{j=1}^3(\bar{a}_j + \bar{b}_j u) - \sum^3 (\bar{a}_1 + \bar{b}_1 u)\, \bar{V}^{23}. \tag{61}$$

So, $G_{r1} = \bar{P}_r^1 \bar{I}^1$ for $\bar{I}^1 = \sum_{s=0}^{1} \bar{h}_s^1 \gamma_s = \bar{a}_1\gamma_0 + \bar{b}_1\gamma_1,$

$$G_{r3} = \bar{P}_r^{1-3}\bar{I}^{1-3} \text{ for } \bar{I}^{1-3} = \sum_{s=0}^{3} \bar{h}_s^{1-3}\gamma_s, \tag{62}$$

where $\bar{h}_0^{1-3} = \Pi_{j=1}^3 \bar{a}_j - \sum^3 \bar{a}_1\, \bar{V}^{23},\ \bar{h}_1^{1-3} = \sum^3 \bar{b}_1(\bar{a}_2\bar{a}_3 - \bar{V}^{23}),$

$$\bar{h}_2^{1-3} = \sum^3 \bar{a}_1\bar{b}_2\bar{b}_3,\ \bar{h}_3^{1-3} = \Pi_{j=1}^3 \bar{b}_j. \tag{63}$$

5. The Case $q_1 = q_2 = 1$

In this case $q = 2$ and for $z,\ U^{11}$ of (42) and u of (55),

$\mathbf{x}_j = x_j,\ \mathbf{V}_{i_1 i_2} = V_{i_1 i_2},\ \mathbf{V}^{i_1 i_2} = V^{i_1 i_2},\ V_{1.2} = V_{11} - V_{12}^2 V_{22}^{-1},$

$z = V_{22}^{-1}x_2,\ U^{11} = V_{22}^{-1},\ \Phi_{1.2}(x_1) = \Phi_{V_{1.2}}(x_1 - \mu_{1.2}) = \Phi(u).$

By (41), for $r \geq 1,\ p_r^*(x_2) = \sum_{k=1}^{3r} [p_{rk}^* :\ k - r \text{ even}],$ where $p_{rk}^* = P_r(2^k)\, H_k^*,$

$H_k^* = H_k(x_2, V_{22}) = \sigma_2^{-k}\, H_k(\sigma_2 x_2)$ for $\sigma_2 = V_{22}^{1/2},$

and $H_k(x)$ of (6). For example by (30), $H_3^* = z^3 - 3V_{22}^{-1}z \neq H_3(x)$. $\tilde{p}_{rk}$ and $P_{rk}(x)$ are given in §3 in terms of $P_r(1^a 2^b) = \binom{a+b}{a}\, P_r^{1^a 2^b}$.

(47) gives D_r in terms of $\tilde{p}_r$ and p_r^*, which are given for $r \leq 3$ by (22) in terms of $\tilde{p}_{rk}$ of §3. For H_{ab} of (32), set

$$H_{ab}(u) = \sum_{s=0}^{a+b} H_{abs}\, u^s = H_{ab} \text{ at } y = V^{-1}x,\ x_1 = \mu_{1.2} + \sigma_{1.2}\, u. \tag{64}$$

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For example,

$$H_{10}(u) = y_1,\ H_{01}(u) = y_2,\ y_i = h_0^i + h_1^i u \text{ of (58). So } H_{10s} = h_s^1,\ H_{01s} = h_s^2.$$

$$\text{So by (31), } G_{rk} = \sum_{a=0}^{k} P_r(1^a 2^{k-a})\ I_{a,k-a}^k \text{ where } I_{ab}^k = \sum_{s=0}^{k} H_{abs}\ \gamma_s, \tag{65}$$

for γ_s of (59) and (60). For example

$$G_{r1} = P_r(1)I_{10}^1 + P_r(2)I_{01}^1 \text{ for } I_{10}^1 = I^1,\ I_{01}^1 = I^2 \text{ of (62).}$$

$$G_{r3} = \sum_{a=0}^{3} P_r(1^a 2^{3-a}) I_{a,3-a}^3. \text{ So we need } I_{ab}^3 = \sum_{s=0}^{3} H_{abs}\gamma_s \text{ for } ab = 30, 21, 12, 03.$$

$$H_{30} = H_2^{111} = \sum_{s=0}^{3} h_s^{111} u^s \text{ of (57) where by (63),}$$

$$h_0^{111} = a_1^3 - 3a_1 V^{11},\ h_1^{111} = 3b_1[a_1^2 - V^{11}],\ h_2^{111} = 3a_1 b_1^2,\ h_3^{111} = b_1^3.$$

$$H_{21} = H_2^{112} \sum_{s=0}^{3} h_s^{112} u^s \text{ where } h_0^{112} = a_1^2 a_2 - 2a_1 V^{12} - a_2 V^{11},$$

$$h_1^{112} = 2h_0^1[h_0^1 h_0^2 - V^{12}] + h_0^2[(h_0^1)^2 - V^{11}],\ h_2^{112} = 2a_1 b_1 b_2 + a_2 b_1^2,\ h_3^{112} = b_1^2 b_2,$$

and H_{bas} is giving by reversing 1 and 2 in H_{abs}. Alternatively, we can use

Theorem 1. *Set $\theta = \phi_{V_{22}}(x_2)$. For $k - r$ even, G_{rk} of (53) is given by*

$$G_{rk} = \theta^{-1}\ [P_r(2^k) \int_{-\infty}^{x_1} H_{0k}\ \phi_V(x) dx_1 - b_{rk}\ \phi_V(x)], \text{ for } r \geq 1,\ k \geq 1,$$

$$\text{where } b_{rk} = \sum_{a=1}^{k} P_r(1^a 2^{k-a})\ H_{a-1,k-a}. \tag{66}$$

$$\text{So for } r \geq 1,\ g_r = \theta^{-1}\ [\int_{-\infty}^{x_1} A_r\ \phi_V(x) dx_1 - B_r\ \phi_V(x)], \tag{67}$$

$$\text{where } (A_r, B_r) = \sum_{k=1}^{3r} \{\ (P_r(2^k) H_{0k},\ b_{rk}) : k - r \text{ even}\}. \tag{68}$$

PROOF For $\bar{I}_{1-k}$ of (54),

$$G_{rk} = \bar{P}_r^{1-k} \bar{I}_{1-k} = P_r(2^k) I(2^k) + \sum_{a=1}^{k} P_r(1^a 2^{k-a})\ I(1^a 2^{k-a}), \text{ where}$$

$$\theta\ I(2^k) = \int_{-\infty}^{x_1} H_{0k}\ \phi_V(x) dx_1, \text{ and for } a \geq 1,\ \theta\ I(1^a 2^b) = -H_{a-1,b}\ \phi_V(x). \quad \square$$

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This gives G_{rk}, and so g_r of (52) and so $P_{1.2}$ to $O(n^{-(r+1)/2})$, in terms of the coefficients $P_r(1^a2^b)$. So (67) gives g_1, g_2, g_3 in terms of $P_r(1^a2^b)$ of §3 via

$$B_1 = \sum_{k=1,3} b_{1k} \text{ where } b_{11} = P_1(1) = k_1^1,$$
$$b_{13} = P_1(1^3)H_{20} + P_1(112)H_{11} + P_1(122)H_{02}$$
$$= (k_2^{111}H_{20} + 3k_2^{112}H_{11} + 3k_2^{122}H_{02})/6,$$
$$B_2 = \sum_{k=2,4,6} b_{2k} \text{ where } b_{22} = P_2(11)\, H_{10} + P_2(12)\, H_{01},$$
$$b_{24} = P_2(1^4)\, H_{30} + P_2(1^32)\, H_{21} + P_2(1^22^2)\, H_{12} + P_2(12^3)\, H_{03}.$$
$$b_{26} = P_2(1^6)H_{50} + P_2(1^52)H_{41} + P_2(12^5)\, H_{05} + \sum_{12}^{2} P_2(1^42^2)\, H_{32} + P_2(1^32^3)\, H_{23},$$

$$B_3 = \sum_{k=1,3,5,7,9} (P_3(2^k)H_{0k},\, b_{3k}) \text{ for } b_{3k} \text{ of (66)}.$$

The explicit form for (66), despite the work needed to obtain H_{abs} of (64).

Example 2. *If the distribution of $\hat{w}$ is symmetric about w, then for r odd, $p_r(x) = P_r(x) = 0$, and the non-zero $\bar{P}_2^{1-k}$ are*
$\bar{P}_2^{12} = \bar{k}_2^{12},\ \bar{P}_2^{1-4} = \bar{k}_3^{1-4}/24.$

Example 3. *Let $\hat{w}$ be a sample mean. Then E $\hat{w} = w$, and only the leading coefficients in (11) are non-zero. So $\bar{k}_1^1 = \bar{P}_1^1 = \tilde{p}_{11} = p_{11}^* = 0$. The non-zero P_r^{1-k} were given in Example 2.1. For $q = 2$ $\tilde{p}_{rk}, P_{rk}(x)$ are given by §3 with these non-zero P_r^{1-k}, and $\tilde{p}_{11} = \tilde{p}_{22} = \tilde{p}_{31} = 0$.*

$$\text{For } \tilde{p}_{13},\ P_1(j^3) = k_2^{j^3}/3!,\ P_1(1^22) = k_2^{112}/2,\ P_1(12^2) = k_2^{122}/2.$$
$$\text{For } \tilde{p}_{24},\ P_2(j^4) = k_3^{j^4}/4!,\ P_2(1^32) = k_3^{1112}/6,\ P_2(12^3) = k_3^{1222}/6,$$
$$P_2(1^22^2) = k_3^{1122}/4.$$
$$\text{For } \tilde{p}_{35},\ P_3(j^5) = k_4^{j^5}/5!,\ P_3(1^42)/5 = k_4^{1^42}/5!,\ P_3(1^32^2) = k_4^{1^32^2}/12.$$
$$\text{For } \tilde{p}_{37},\ P_3(j^7) = k_3^{j^4}k_2^{j^3}/144,\ P_3(1^62) = k_2^{112}k_3^{1111}/48 + k_2^{111}k_3^{1112}/36,$$
$$P_3(1^52^2) = k_3^{1^4}k_2^{122}/48 + k_3^{1^32}k_2^{112}/12 + k_3^{1122}k_2^{111}/24,$$
$$P_3(1^42^3) = A/144 \text{ for } A = k_3^{1^4}k_2^{222} + 13k_3^{1^32}k_2^{122} + 17k_3^{1122}k_2^{112} + k_3^{1222}k_2^{111}.$$

$\bar{P}_2^{1-6}$ needed for $\tilde{p}_{26}$ of §3 does not simplify. Nor does $\bar{P}_3^{1-9}$ of §3 needed for $\tilde{p}_{35}$.

Example 4. *Consider the classical problem of the distribution of a sample mean, given the sample variance. So $q = 2$. Let $\hat{w}_2, \hat{w}_2$ be the usual unbiased estimates of the 1st 2 cumulants w_1, w_2 from a univariate random sample of size n from a distribution with rth cumulant κ_r. So $w_1 = \kappa_1,\ w_2 = \kappa_2$. By the last 2 equations of §12.15 and (12.35)–(12.38) of Stuart and Ord (1991), the cumulant coefficients needed for $\bar{P}_r^{1-k}$ of (14) for $r \leq 3$, that is, the coefficients needed for the conditional density to $O(n^{-2})$ are*

$$k_1^{11} = \kappa_2,\ k_1^{12} = \kappa_3,\ k_1^{22} = \kappa_4 + 2\kappa_2^2,\ \Rightarrow V = \begin{pmatrix} \kappa_2 & \kappa_3 \\ \kappa_3 & \kappa_4 + 2\kappa_2^2 \end{pmatrix},$$
$$k_1^1 = k_1^2 = 0,\ k_2^{111} = \kappa_3,\ k_2^{112} = k_2^{122} = 0,\ k_2^{222} = (6) + 12(24) + 4(3^2) + 8(2^3),$$

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$$
\begin{aligned}
&k_2^{11} = k_2^{12} = 0,\ k_2^{22} = 2(2^2),\ k_3^{1111} = (4),\ k_3^{1112} = (5),\\
&k_3^{1122} = k_3^{1222} = 0,\ k_3^{2222} = (8) + 24(26) + 32(35) + 32(4^2) + 144(2^2 4) + 96(23^2)\\
&+48(2^4),\ k_2^1 = k_2^2 = k_3^{111} = k_3^{112} = k_3^{122} = 0,\ k_3^{222} = 12(24) + 16(2^3),\\
&k_4^{1^5} = k_4^{1^3 2^2} = k_4^{1^2 2^3} = k_4^{12^4} = 0,\ k_4^{1^4 2} = (6), k_4^{2^5} = (10) + 40(28) + 80(37) + 200(46)\\
&+96(5^2) + 480(2^2 6) + 1280(235) + 1280(24^2) + 960(3^2 4) + 1920(2^3 4)\\
&+1920(2^2 3^2) + 384(2^5),\ \textit{where}\ (i_1^{j_1} i_2^{j_2} \dots) = \kappa_{i_1}^{j_1} \kappa_{i_2}^{j_2} \dots .
\end{aligned}
$$

(47) gives D_r in terms of $\tilde{p}_r$ and p_r^*, that is, in terms of $\tilde{p}_{rk}$ and p_{rk}^* of §3 in terms of $P_r(1^a 2^b)$ of (31). In this example, many of these are 0. By (15)–(17) and the Appendix A, the non-zero $P_r(1^a 2^b)$ are in order needed,

$$
\begin{aligned}
&P_1(j^3) = k_2^{jjj}/6,\ P_2(2^2) = \kappa_2^2,\ P_2(j^4) = k_3^{jjjj}/24,\ P_2(1^3 2) = k_3^{1112}/6 = \kappa_5/6,\\
&P_2(j^6) = (k_2^{jjj})^2/72,\ P_2(1^3 2^3) = k_2^{111} k_2^{222}/36.\ P_3(j^3) = k_3^{jjj}/6.\\
&P_3(2^5) = k_4^{2^5}/120 + k_2^{22} k_2^{222}/12.\ P_3(j^7) = k_2^{jjj} k_3^{j^4}/144,\ P_3(1^6 2) = k_2^{111} k_3^{1112}/36,\\
&P_3(1^4 2^3) = k_2^{222} k_3^{1^4}/144,\ P_3(1^3 2^4) = (k_2^{111} k_3^{2^4} + k_2^{222} k_3^{1^3 2})/144,\\
&P_3(j^9) = (k_2^{jjj})^3/6^3,\ P_3(1^6 2^3) = 3(k_2^{111})^2 k_2^{222}/6^3,\ P_3(1^3 2^6) = 3k_2^{111}(k_2^{222})^2/6^3.\\
&\textit{So},\ \tilde{p}_{11} = 0,\ \tilde{p}_{13} = P_1(1^3)\ H_{30} + P_1(2^3)\ H_{03},\ ,\ \tilde{p}_{22} = P_2(2^2) H_{02},\\
&\tilde{p}_{24} = P_2(1^4) H_{40} + P_2(2^4) H_{04} + P_2(1^3 2) H_{31},\ \tilde{p}_{26} = P_2(1^6) H_{60} + P_2(2^6) H_{06}\\
&\ +P_2(1^3 2^3) H_{33},\ \tilde{p}_{31} = 0,\ \tilde{p}_{33} = P_3(1^3) H_{30} + P_3(2^3) H_{03},\ \tilde{p}_{35} = P_3(2^5) H_{05},\\
&\tilde{p}_{37} = P_3(1^7) H_{70} + P_3(2^7) H_{07} + P_3(1^6 2) H_{61} + P_3(1^4 2^3) H_{43} + P_3(1^3 2^4) H_{34}\\
&\ +P_3(1^2 2^5) H_{25} + P_3(12^6) H_{16} + P_3(2^7) H_{07},\\
&\tilde{p}_{39} = P_3(1^9) H_{90} + P_3(1^6 2^3) H_{63} + P_3(1^3 2^6) H_{36} + P_3(2^9) H_{09}.
\end{aligned}
$$

(24)–(26) now give $\tilde{p}_r(x)$ and $p_r(x)^*$ for $r \leq 3$. By (18) and (47), this gives the conditional density $p_{1.2}(\mathbf{x}_1)$ to $O(n^{-2})$. (67) gives g_r needed for the conditional distribution $P_{1.2}(\mathbf{x}_1)$ to $O(n^{-2})$ in terms of (A_r, B_r) of (68). So

$$
\begin{aligned}
&B_1 = b_{13} = P_1(1^3) H_{20},\ B_2 = \sum_{k=2,4,6} b_{2k}\ \textit{where}\ b_{22} = P_2(11)\ H_{10} + P_2(12)\ H_{01},\\
&b_{24} = P_2(1^4)\ H_{30} + P_2(1^3 2)\ H_{21},\ b_{26} = P_2(1^6) H_{50} + P_2(1^3 2^3)\ H_{23},\\
&B_3 = \sum_{k=1,3,5,7,9} b_{3k}\ \textit{where}\ b_{31} = 0,\ b_{33} = P_3(1^3)\ H_{20},\ b_{35} = 0,\\
&b_{37} = P_3(1^7)\ H_{60} + P_3(1^6 2)\ H_{51} + P_3(1^4 2^3)\ H_{33} + P_3(1^3 2^4)\ H_{24},
\end{aligned}
$$

$$
b_{39} = P_3(1^9)\ H_{80} + P_3(1^6 2^3)\ H_{53} + P_3(1^3 2^6)\ H_{26}.
$$

6. Conditional Cornish-Fisher Expansions

Suppose that $q_1 = 1$. Here we invert the conditional distribution (56), to obtain its its extended Cornish-Fisher expansions similar to (4). For any function $f(u)$ with finite derivatives, set $f_{.j} = (d/du)^j f(u)$.

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Lemma 1. *Suppose that $G_0(u) : R \to (0,1)$ is 1 to 1 increasing with jth derivative $G_{0.j}$, and for some $t \in R$,*

$$P(u) = G_0(u) + S_1 \text{ where } S_1 \approx \sum_{r=1}^{\infty} t^r G_r(u). \tag{69}$$

Set $G_r = G_r(u),\ Q(u) = G_0^{-1}(P(u))$.

$$\text{Then } Q(u) = u + S_2 \text{ where } S_2 \approx \sum_{r=1}^{\infty} t^r J_r, \tag{70}$$

$$J_r = G_{0.1}^{-1}[G_r - \sum_{j=2}^{r} G_{0.j}\ \tilde{B}_{rj}(J)/j!] \text{ for } \tilde{B}_{rj} \text{ of (7)}. \tag{71}$$

For example $J_1 = G_{0.1}^{-1}\ G_1,\ J_2 = G_{0.1}^{-1}\ G_2 - G_{0.1}^{-3} G_{0.2}\ G_1^2/2$.
$J_3 = G_{0.1}^{-1}\ G_3 - G_{0.1}^{-3} G_{0.2}\ G_1 G_2 - G_{0.1}^{-4} G_{0.3}\ G_1^3/6 + G_{0.1}^{-5} G_{0.2}^2\ G_1^2/2$.

Set $B'_{rj} = \tilde{B}_{rj}(K),\ C_{rs} = \sum_{j=0}^{r} J_{s.j}\ B'_{rj}/j!$.

$$\text{Then } Q^{-1}(u) = P^{-1}(G_0(u)) = u + S_3 \text{ where } S_3 \approx \sum_{k=1}^{\infty} t^k K_k, \tag{72}$$

$$\text{and } K_k = -\sum_{s=1}^{k} C_{k-s,s}. \text{ For example, } K_1 = -J_1,\ K_2 = J_1 J_{1.1} - J_2, \tag{73}$$

$K_3 = -J_3 + J_{1.1} J_2 + (J_{2.1} - J_{1.1}^2) J_1 - J_{1.2} J_1^2/2$.

PROOF Set $B_{rj} = \tilde{B}_{rj}(J)$.

For $r \geq 1$, $S_2^j \approx \sum_{r=j}^{\infty} t^r B_{rj}$. $P(u) = G_0(u + S_2) \approx \sum_{j=0}^{\infty} G_{0.j}\ S_2^j$.

So by (8), $G_r(u) = \sum_{j=1}^{r} G_{0.j}\ B_{rj}/j! = G_{0.1}\ J_r + \sum_{j=2}^{r} G_{0.j}\ B_{rj}/j!$.

$J_2 = G_{0.1}^{-1}[G_2 - G_{0.2} J_1^2/2],\ J_3 = G_{0.1}^{-1}[G_3 - (G_{0.2} J_1 J_2 - G_{0.3} J_1^3/6]$.

$u = Q(u + S_3) \approx \sum_{j=0}^{\infty} S_3^j\ Q_{.j}/j!$ where $Q_{.j} \approx \sum_{r=0}^{\infty} t^r\ J_{r.j}$.

$S_3^j \approx \sum_{r=j}^{\infty} t^r B'_{rj}$. So $u = \sum_{r=0}^{\infty} t^r \sum_{j=0}^{r} Q_j B'_{rj}/j! = \sum_{k=0}^{\infty} t^k A_k$ where

$A_k = \sum_{r+s=k} C_{rs}$. Also $C_{0s} = J_s,\ C_{r0} = K_r$ for $r \geq 1$.

So $0 = A_k = K_k + \sum_{s=1}^{k} C_{k-s,s}:\ A_0 = C_{00} = J_0,\ A_1 = C_{01} + C_{10}$.

$0 = A_1 = J_1 + K_1,\ K_1 = -C_{01} = -J_1$.

$0 = A_2 = C_{02} + C_{11} + C_{20}$ where $C_{11} = \sum_{j=0}^{1} J_{1.j}\ B'_{rj} = K_1 J_{1.1}$.

So $K_2 = -J_2 - K_1 J_{1.1} = J_1 J_{1.1} - J_2$.

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One obtains K_3 similarly. □ A different form for (73) was given in Theorem A2 of Withers (1983). So

$$K_4 = -J_4 - \sum_{j=1}^{3} J_{4-j.1}K_j - J_{2.2}J_1^2/2 + J_{1.2}J_1K_2 + J_{1.3}J_1^3/6,$$
$$K_5 = -J_5 - \sum_{j=1}^{4} J_{5-j.1}K_j - J_{3.2}J_1^2/2 + J_{2.2}J_1K_2 - J_{1.2}(K_2^2/2 + K_1K_3)$$
$$+ J_{2.3}J_1^3/6 - J_{1.3}J_1^2K_2/2 - J_{1.4}J_1^4/24.$$

We can now give the quantiles of the conditional distribution (56).

Theorem 2.

$$\textit{Suppose that } P_n(u) \approx \sum_{r=0}^{\infty} n^{-r/2}G_r(u). \textit{ Set } G_r = G_r(u).$$
$$\textit{Then } \Phi^{-1}(P_n(u)) \approx \sum_{r=0}^{\infty} n^{-r/2}J_r, \textit{ and } P_n^{-1}(\Phi(u)) \approx \sum_{r=0}^{\infty} n^{-r/2}K_r,$$
$$\textit{where } J_k = \sum_{r=1}^{k} \phi(u)^{-r}\, J_{kr},\ K_k = \sum_{r=1}^{k} \phi(u)^{-r}\, K_{kr},$$

$$J_{k1} = -K_{k1} = G_k,\ J_{22} = uG_1^2/2,\ J_{32} = uG_1G_2,\ J_{33} = (2u^2+1)G_1^2/6,$$
$$J_{11,1} = uG_1 + G_{1.1},\ J_{12,1} = (u^2+1)G_1 + (u+1)G_{1.1} + G_{1.2},$$
$$K_{22} = uG_1/2 + G_{1.1},\ K_{32} = G_{1.1}(G_2 + G_{2.1}) + uG_1G_{2.1},$$
$$K_{33} = \sum_{j=0}^{3} d_j G_1^j \textit{ for } d_0 = uG_2G_{1.1},\ d_1 = u^2G_2 + (u-1)G_{1.1}^2,$$
$$d_2 = -u^2/2 + 2(u^2-u)G_{1.1} - G_{1.2}/2,\ d_3 = (6u^3 - 5u^2 + 3u - 4)/6.$$

A simpler formula for J_k is

$$J_k = -\sum_{j=1}^{k} c_j(u)\, \tilde{B}_{kj}(\lambda)/j! \textit{ where } \lambda_r = -\phi(u)^{-1}G_r(u),$$
$$c_j = c_j(u) = \phi(u)^j\, (d/dv)^j\, \Phi^{-1}(v) \textit{ at } v = \Phi(u). \tag{74}$$
$$\textit{So, } c_1 = 1,\ c_2 = u,\ c_3 = 2u^2+1,\ c_4 = 6u^3+5u,\ c_5 = 24u^4 + 46u^2 + 7,$$
$$J_1 = \phi(u)^{-1}G_1,\ J_2 = \phi(u)^{-1}G_2 + \phi(u)^{-2}uG_1^2/2,$$
$$J_3 = \phi(u)^{-1}G_3 + \phi(u)^{-2}uG_1G_2 + \phi(u)^{-3}(2u^2+1)G_1^3/6,$$
$$K_1 = -\phi(u)^{-1}G_1,\ K_2 = -\phi(u)^{-1}G_2 + \phi(u)^{-2}K_{22},$$
$$K_3 = -\phi(u)^{-1}G_3 + \phi(u)^{-2}K_{32} + \phi(u)^{-3}K_{33}.$$

PROOF Apply Lemma 6.1 to (56) with $t = n^{-1/2}$. Take $G_0(u) = \Phi(u)$ and $G_r = C_r \otimes g_r$ for g_r given by (52) in terms of G_{rk} of (59). So

$$G_{0.1} = \phi(u) \text{ and for } j \geq 1,\ G_{0.j} = (-1)^{j-1}H_{j-1}(u)\phi(u),$$

and for $P(u) = P_n(u)$ of (56), $\Phi^{-1}(P_n(u))$ and $P_n^{-1}(\Phi(u))$ are given by (70) and (72) in terms of J_k, K_k and their derivatives. These are given by

$$J_{k.i} = \sum_{r=1}^{k} \phi(u)^{-r} J_{ki,r}, \; K_{k.i} = \sum_{r=1}^{k} \phi(u)^{-r} K_{ki,r},$$
$$\text{where } J_{21,1} = G_{2.1}, \; J_{21,2} = uG_2 + (u^2+1/2)G_1^2 + uG_1G_{1.1},$$
$$K_{32} = -J_{32} + J_{11,1}(J_{21} + J_{21,1}),$$
$$K_{33} = -J_{33} + J_{11,1}(J_{22} + J_{22,1}) - J_{11}J_{11,1}^2 - J_{11}^2 J_{12,1}/2.$$

(74) follows from (3.2) of Withers (1984). □

We had hoped to read the conditional a_{ri} off the conditional density. But the expansion (47) cannot be put into the form (3) if $\bar{k}_2^{1-3} \neq 0$, as the coefficient of x^2 in $\bar{h}_1(x)$ of (5) is 0. So the conditional estimate is generally not a standard estimate. (An exception is when $\hat{w} \sim \mathcal{N}_q(w, n^{-1}V)$ since then $\bar{k}_2^{1-3} = 0$ and by (50), $P_{1.2}(\mathbf{x}_1) = \Phi_{1.2}(\mathbf{x}_1)$ of (39). We have yet to see what exponential families this extends to.) It might be possible to remedy this by extending the results here to Type B estimates. But there seems little point in doing so.

7. Conclusions

§2 and the Appendix A give the density and distribution of $X_n = n^{1/2}(\hat{w} - w)$ to $O(n^{-2})$, for $\hat{w}$ any standard estimate, in terms of certain functions of the cumulants coefficients $\bar{k}_j^{1-r}$ of (11), the coefficients $\bar{P}_r^{1-k}$ of (14)–(17). Most estimates of interest are standard estimates, including functions of sample moments, like the sample correlation, and any multivariate function of k-statistics,. §3 gave the density and distribution of $X_n = n^{1/2}(\hat{w} - w)$ in more detail when $q = 2$ using the dual notation $P_r(1^a 2^b)$. §4 gave the conditional density and distribution of $\mathbf{X}_{n1}$ given $\mathbf{X}_{n2}$ to $O(n^{-2})$ where $\binom{\mathbf{X}_{n1}}{\mathbf{X}_{n2}}$ is any partition of X_n. The expansion (47) gives the conditional density of a standard estimate in terms of D_r of (47). The conditional distribution (50) to $O(n^{-2})$ requires the function $\bar{I}_{1-k}$ of (54), or its expansion (59) or (65). §6 gave the extended Cornish-Fisher expansions for the quantiles of the conditional distribution when $q_1 = 1$.

8. Discussion

A good approximation for the distribution of an estimate, is vital for statistical inference. It enables one to explore the distribution's dependence on underlying parameters, such as correlation. Our analytic method avoids the need for simulation or jack-knife or bootstrap methods while providing greater accuracy than them. Hall (1992) uses the Edgeworth expansion to show that the bootstrap gives accuracy to $O(n^{-1})$. Hall (1988) says that "2nd order correctness usually cannot be bettered". Fortunately this is not true for our analytic method. Simulation, while popular, can at best shine a light on behaviour when there is only a small number of parameters.

Estimates based on a sample of independent but not identically distributed random vectors, are also generally standard estimates. For example for a univariate sample mean $\bar{w} = n^{-1}\sum_{j=1}^n X_{jn}$ where X_{jn} has rth cumulant κ_{rjn}, then $\kappa_r(\bar{w}) = n^{1-r}\kappa_r$ where $\kappa_r = n^{-1}\sum_{j=1}^n \kappa_{rjn}$ is the average rth cumulant. For some examples, see Skovgaard (1981a, 1981b) and Withers and Nadarajah (2010a, 2020b). The last is for a function of a weighted mean of complex random matrices.

A promising approach is the use of conditional cumulants. §6.2 of McCullagh (1984) uses conditional cumulants to give the conditional density of a sample mean to $O(n^{-3/2})$. §5.6 of McCullagh (1987) gave formulas for the 1st 4 cumulants conditional on $\mathbf{X}_2 = \mathbf{x}_2$ when $\mathbf{X}_1$ and $\mathbf{X}_2$ are uncorrelated. He says that assumption can be removed but gives no details how. That might give an alternative to our approach, but seems unlikely as the conditional estimate is generally not a standard estimate.

(7.5) of Barndoff-Nielsen and Cox (1989) gave the 3rd order expansion for the conditional density of a sample mean to $O(n^{-3/2})$, but did not attempt to integrate it.

Here we have only considered expansions about the normal. However expansions about other distributions can greatly reduce the number of terms by matching the leading bias coefficient. The framework for this is Withers and Nadarajah (2010a). For expansions about a matching gamma, see Withers and Nadarajah (2011, 2014).

The results here can be extended to tilted (saddlepoint) expansions by applying the results of Withers and Nadarajah (2010a). Tilting was 1st used in statistics by Daniels (1954). He gave an approximation to the density of a sample mean. A conditional distribution by tilting was first given by Skovgaard (1987) up to $O(n^{-1})$ for the distribution of a sample mean conditional on correlated sample means. For some examples, see Barndoff-Nielsen and Cox (1989). For other some results on conditional distributions, see Pfanzagl (1979), Booth et al. (1992), DiCiccio et al. (1993), Hansen (1994), Moreira (2003), Chapter 4 of Butler (2007), and Kluppelberg and Seifert (2020). The results given here form the basis for constructing confidence intervals and confidence regions. See Withers (1989).

Appendix A. The Coefficients $\bar{P}_r^{1-k}$ Needed for (14)

Here we give the coefficients $\bar{P}_r^{1-k}$ needed for (14) for $r \le 3$ using the symmetrising operator $\mathcal{S}$. They are given for $r = 1$ by (15), and for $r = 2,3$ by (16)–(17) and the following.

$\bar{P}_2^{1-4}$ needs $\mathcal{S}\ \bar{k}_1^1\bar{k}_2^{234}$ where $\mathcal{S}\ a^1b^{234} = (a^1b^{234} + a^2b^{341} + a^3b^{412} + a^4b^{123})/4.$
$\bar{P}_2^{1-6}$ needs $\mathcal{S}\ \bar{k}_2^{1-3}\bar{k}_2^{4-6}$ where

$$\mathcal{S}\ a^{1-3}a^{4-6} = (a^{1-3}a^{4-6} + a^{124}a^{356} + a^{125}a^{346} + a^{126}a^{346} + a^{134}a^{256} + a^{135}a^{246} + a^{136}a^{245} + a^{145}a^{236} + a^{146}a^{235} + a^{156}a^{234})/10. \quad \text{(A1)}$$

$\bar{P}_3^{1-3}$ needs $\mathcal{S}\ \bar{k}_2^{12}\bar{k}_1^3$ where $\mathcal{S}\ a^{12}b^3 = (a^{12}b^3 + a^{13}b^2 + a^{23}b^1)/3.$

$\bar{P}_3^{1-5}$ needs $\mathcal{S}\ \bar{k}_3^{1-4}\bar{k}_1^5,\ \mathcal{S}\ \bar{k}_2^{12}\bar{k}_2^{3-5},\ \mathcal{S}\ \bar{k}_2^{123}\bar{k}_1^4\bar{k}_1^5,$ (A2)

where $\mathcal{S}\ a^1b^{2-5} = (a^1b^{2-5} + \dots a^5b^{1-4})/5,$ (A3)

$$\mathcal{S}\ a^{12}b^{345} = (12.345 + 13.245 + 14.235 + 15.234 + 23.145 + 24.135 + 25.134 + 34.125 + 35.124 + 45.123)/10 \text{ for } 12.345 = a^{12}b^{345}, \quad \text{(A4)}$$

for $1.2. = 1.2.345 = a^1a^2b^{345}$, $\mathcal{S}\ a^1a^2b^{345}$
$= (1.2. + 1.3. + 1.4. + 1.5. + 2.3. + 2.4. + 2.5. + 3.4. + 3.5. + 4.5.)/10.$ (A5)

$\bar{P}_3^{1-7}$ needs $\mathcal{S}\ \bar{k}_2^{123}\bar{k}_3^{4-7},\ \mathcal{S}\ \bar{k}_2^{123}\bar{k}_2^{456}\bar{k}_1^7,$ where (A6)

for $123. = a^{123}b^{4-7}$, $\mathcal{S}\ a^{123}b^{4-7} = (123. + 124. + \cdots + 567.\ )/\binom{7}{3},$ (A7)

$\mathcal{S}\ a^{123}a^{456}b^7 = \mathcal{S}\ 123.456.7 = (b^7\mathcal{S}\ 123.456 + \cdots + b^1\mathcal{S}\ 234.567)/7$ (A8)

say where $\mathcal{S}\ 123.456 = \mathcal{S}\ a^{1-3}a^{4-6}$ of (A1),

$\bar{P}_3^{1-9} = \mathcal{S}\ \bar{k}_2^{1-3}\bar{k}_2^{4-6}\bar{k}_2^{7-9}/6^3,$

where for $123.456.789 = a^{1-3}a^{4-6}a^{7-9}$ and $123- = 123.\ \mathcal{S}\ 456.789$ of (A1),

$$\mathcal{S}\ a^{1-3}a^{4-6}a^{7-9} = [123- + 124- + 125- + 126- + 127- + 128- + 129- + 134- + 135- + 136- + 137- + 138- + 139- + 145- + 146- + 147- + 148- + 149- + 156- + 157- + 158- + 159- + 167- + 168- + 169- + 178- + 179- + 189-]/28.$$

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