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Article

A New Way to Unify All Fermion and Boson Fields, Including Gravity

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Abstract

The description of the internal spaces of fermion and boson fields with "basis vectors", which are the superposition of odd and even products of the operators γ^a , offers in $d = 2(2n + 1)$ -dimensions, such as $d = (13 + 1)$, a unified picture of all so far observed fermions (quarks, leptons, antiquarks and antileptons that appear in families) and bosons (gravitons, photons, weak bosons, gluons and scalars), under the condition that all fields have non-zero angular momenta only in the $d = (3 + 1)$, $SO(3, 1)$, of ordinary space-time. Bosons, which also carry the spatial index α (which is for tensors and vectors $\mu = (0, 1, 2, 3)$ and for scalars $\sigma \geq 5$) appear in two orthogonal groups. In any $d = 2(2n + 1)$ -dimensional space the number of internal states of fermions in all families and their Hermitian conjugate partners is equal to the number of internal states of boson states. The article presents general properties of massless fermion and boson fields and their mutual interactions in this theory, which determine the Lagrangian density of both fields and their interactions. It particularly illustrates "basis vectors" and their properties in $d = (13 + 1)$ and $d = (5 + 1)$. The article presents new results and discusses open problems in this theory.

Keywords: Fermion fields; Boson fields; second quantization; Lagrangian density; internal spaces; spin-charge-family theory

1. Introduction

The author, with collaborators, succeeded in demonstrating in a long series of works [1–16] that the model, named the *spin-charge-family* theory, offers an elegant description of the second-quantised fermion fields, appearing in families, written as the tensor products of the basis in ordinary space-time and the basis, named "basis vectors", in internal spaces, presented as superpositions of odd products of operators γ^a , arranged in nilpotents and projectors, which are eigenvectors of the (chosen) Cartan subalgebra members [1–8].

Three years ago [17–22] the author started to use an equivalent description for boson fields, as so far used for fermion fields, recognising the possibility from 30 years ago [1–3, 17–20]: The internal space of boson second quantised fields can be described by the "basis vectors", presented as superpositions of even products of operators γ^a , arranged in nilpotents and projectors, which are eigenvectors of the Cartan subalgebra members.

The theory of massless fermions and bosons, with the non-zero momentum only in $d = (3 + 1)$, describing internal spaces by an odd number of nilpotents, the rest of the projectors (for fermions) and an even number of nilpotents, the rest of the projectors (for bosons, which also carry the ordinary space-time index α), determines all the properties of fermion and boson fields and of their mutual interactions, realised by the algebraic multiplication:

- a. Explains the Dirac's postulates for the second quantised fermion and boson fields.
- b. Determines Lorentz (and correspondingly the gauge) symmetry of fermion and boson fields.
- c. Determines couplings among fermion and boson fields, and consequently the Lagrange densities.
- d. Determines families of fermions, any family of which includes in $d = 2(2n + 1)$ -dimensional internal space fermions and antifermions. Consequently, the vacuum does not have the negative energy of the Dirac vacuum; it is just the quantum vacuum.
- e. Determines two orthogonal kinds of boson fields; One applies to fermions from the left-hand side,

the other from the right-hand side. One transforms fermions within each family, the other transforms a member of a family into the same member of other families. Both are expressible as algebraic products of fermion fields and their Hermitian conjugate partners ¹.

f. Fermion states are algebraically orthogonal.

g. Although the internal spaces of fermions and bosons demonstrate so many different properties (anticommuting fermions appear in families, and have half-integer spins and charges in the fundamental representations, commuting bosons appear in two orthogonal groups, have no families, and have integer spins and charges in adjoint representations), the simple algebraic multiplication with the γ^a relates both kinds of “basis vectors” ([20], App. B).

h. Properties presented from d. to g. influence the Feynman diagrams, which should reproduce the experimental data.

i. Second quantised fermion and boson fields are described as a tensor product of bases in ordinary space-time and of “basis vectors” describing the internal spaces of fields.

. The analysis of the fermion and boson internal spaces with respect to the subgroups $SO(1,3)$, $SU(2)$, $SU(2)$, $SU(3)$, $U(1)$ of the group $SO(13,1)$, offers the description of the observed families of quarks and leptons, appearing in families, and of tensor (gravitons), vector (photons, weak bosons, gluons), and scalar (Higgs) boson fields, explaining also other observed properties of fermions and bosons (like the appearance of the dark matter [12], the matter-antimatter asymmetry in the universe [13], and several other predictions [14,31]).

. The Pauli matrices for spins and charges in any even d for either fundamental or adjoint representations can easily be found by applying the corresponding operators, S^{ab} or $\bar{S}^{ab}$, on the “basis vectors” of fermions or of bosons.

. There are in internal space with $d = 2(2n + 1)$, $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ “basis vectors” of fermion fields, arranged in an odd number of nilpotents, and the same number of their Hermitian conjugate partners. And there are twice $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ “basis vectors” with an even number of nilpotents describing two orthogonal boson fields.

. In odd-dimensional spaces, $d = (2n + 1)$, the fermion and boson fields have very peculiar properties: Half of the “basis vectors”, $2^{\frac{2n}{2}-1} \times 2^{\frac{2n}{2}-1}$, have the properties of fields in the $2n$ -dimensional space (the anticommuting “basis vectors” appear in families and have their Hermitian conjugate partners in a separate group, the commuting “basis vectors” appear in two orthogonal groups). Among the rest of the “basis vectors”, that is, in $2^{\frac{2n}{2}-1} \times 2^{\frac{2n}{2}-1}$ cases, the anticommuting appear in two orthogonal groups, and commuting appear in families and have their Hermitian conjugate partners in a separate group [18–20].

. In this contribution, all fields, fermions and bosons (tensors, vectors and scalars) are massless. This contribution does not discuss the breaking of symmetries and the appearance of massive fermion fields. There are condensates which make several scalar fields, as well as some of the fermion and vector boson fields, massive, discussed in Ref. [8]. ²

In Sect. 2 the internal spaces of fermion and boson fields are presented as “basis vectors” which are algebraic products, $*_A$, of an odd number of nilpotents (for fermions) and an even number of nilpotents (for bosons), the rest are projectors. Nilpotents and projectors are chosen to be eigenvectors of the Cartan subalgebra members of the Lorentz algebra, as demonstrated in Subsect. 2.1.

Subsect. 2.1 also presents algebraic relations among the “basis vectors” of fermion and two kinds of boson fields, what determines the Lagrange densities of interacting fermion and boson fields.

The creation operators, presented in Subsect. 2.2, are tensor products, $*_T$, of the “basis vectors” and the basis in ordinary space-time.

¹ Knowing the “basic vectors” of fermion fields, we know also the “basic vectors” of the boson fields, although the properties of fermion fields are very different from the properties of the boson fields.

² The breaks of symmetries are expected to follow similarly to the case when we describe the one kind of boson fields with ω^{ab}_α and another kind with $\bar{\omega}^{ab}_\alpha$, presented in Ref. [8].

In Subsect. 2.3, the states active only in $d = (3 + 1)$ -dimensions of ordinary space-time, while the internal space is active in $d = 2(2n + 1)$ dimensions, are discussed. In Subsubsect. 2.3.1 of this section, the concrete algebraic relations among fermion and boson fields for the two cases that the internal space has $d = (5 + 1)$ and $d = (13 + 1)$, are presented.

In Sect. 3.1, the general algebraic structure of the second quantised fermion and boson fields, following from the properties presented in Sect. 2, is presented.

In Sect. 4, we present shortly what we have learned in the last three years.

In Subsect. 4.1, the problems which remain to be solved in this theory, to find out whether the theory offers the right description of the observed fermion and boson second quantised fields which determine the history (and the future) of our universe, are discussed.

In Appendix A “basis vectors” of one family of quarks and leptons, and antiquarks and antileptons are presented.

In Appendix B, the relations among the Grassmann algebra and the two kinds of the Clifford algebras are discussed.

In Appendix C, the relations are presented that are needed in all the sections.

In Appendix D odd and even “basis vectors” for $d = (5 + 1)$ cases are presented in detail, meant to be used like an exercise for $d = 13 + 1$.

2. Internal Spaces of Second Quantised Fermion and Boson Fields

We shall need all the requirements of algebraic relations among fermion and boson fields derived in Sect. 2 in the next section 3, which generalizes these requirements.

Sect. 2 overviews briefly (following several papers [20] and the references therein) the description of the internal spaces of the second-quantised fermion and boson fields as algebraic products of nilpotents and projectors, which are the superposition of odd and even products of γ^a 's. However, the new recognitions, some of them presented in [20] and the others, which the author has later clarified, are presented.

As explained in Sect. B, Eq. (A63), the Grassmann algebra offers two kinds of operators, γ^a 's and $\tilde{\gamma}^a$'s with the properties, Eq. (1)

$$\begin{aligned}\{\gamma^a, \gamma^b\}_+ &= 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+, \\ \{\gamma^a, \tilde{\gamma}^b\}_+ &= 0, \quad (a, b) = (0, 1, 2, 3, 5, \dots, d), \\ (\gamma^a)^\dagger &= \eta^{aa} \gamma^a, \quad (\tilde{\gamma}^a)^\dagger = \eta^{aa} \tilde{\gamma}^a.\end{aligned}\quad (1)$$

We use one of the two kinds, γ^a 's, to generate the “basis vectors” describing internal spaces of fermions and bosons. They are arranged in products of nilpotents and projectors.

$$\begin{aligned}{}^{ab}(k): &= \frac{1}{2}(\gamma^a + \frac{\eta^{aa}}{ik}\gamma^b), \quad ({}^{ab}(k))^2 = 0, \\ {}^{ab}[k]: &= \frac{1}{2}(1 + \frac{i}{k}\gamma^a\gamma^b), \quad ({}^{ab}[k])^2 = {}^{ab}[k],\end{aligned}\quad (2)$$

so that each nilpotent and each projector is the eigenstate of one of the Cartan (chosen) subalgebra members of the Lorentz algebra

$$\begin{aligned}S^{03}, S^{12}, S^{56}, \dots, S^{d-1\ d}, \\ \tilde{S}^{03}, \tilde{S}^{12}, \tilde{S}^{56}, \dots, \tilde{S}^{d-1\ d}, \\ S^{ab} = S^{ab} + \tilde{S}^{ab},\end{aligned}\quad (3)$$

where $S^{ab} = \frac{i}{4}\{\gamma^a, \gamma^b\}_+$, while $\tilde{S}^{ab} = \frac{i}{4}\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+$ are used to determine additional quantum numbers, in the case of fermions are called the family quantum numbers.

Being eigenstates of both operators, of S^{ab} and $\tilde{S}^{ab}$, nilpotents and projectors carry both quantum numbers S^{ab} and $\tilde{S}^{ab}$

$$\begin{aligned} S^{ab} \begin{pmatrix} ab \\ k \end{pmatrix} &= \frac{k}{2} \begin{pmatrix} ab \\ k \end{pmatrix}, & \tilde{S}^{ab} \begin{pmatrix} ab \\ k \end{pmatrix} &= \frac{k}{2} \begin{pmatrix} ab \\ k \end{pmatrix}, \\ S^{ab} \begin{pmatrix} ab \\ [k] \end{pmatrix} &= \frac{k}{2} \begin{pmatrix} ab \\ [k] \end{pmatrix}, & \tilde{S}^{ab} \begin{pmatrix} ab \\ [k] \end{pmatrix} &= -\frac{k}{2} \begin{pmatrix} ab \\ [k] \end{pmatrix}, \end{aligned} \quad (4)$$

with $k^2 = \eta^{aa}\eta^{bb}$.

In even-dimensional spaces, the states in internal spaces are defined by the “basis vectors” which are products of $\frac{d}{2}$ nilpotents and projectors, and are the eigenstates of all the Cartan subalgebra members.

Fermions are products of an odd number of nilpotents (at least one), the rest are projectors; Bosons are products of an even number of nilpotents (or none), the rest are projectors. We call them odd and even “basis vectors”.

The odd “basis vectors” have the eigenvalues of the Cartan subalgebra members, Eq. (3, 4), either of S^{ab} or $\tilde{S}^{ab}$, equal to half integer, $\pm\frac{i}{2}$ or $\pm\frac{1}{2}$.

The even “basis vectors” have the eigenvalues of the Cartan subalgebra members, Eq. (3, 4), $S^{ab} = S^{ab} + \tilde{S}^{ab}$, equal to $\pm i$ or ± 1 or zero.

2.1. “Basis Vectors” Describing Internal Spaces of Fermion and Boson Fields

The odd products of nilpotents (at least one, the rest are projectors), called *odd “basis vectors”*, differ essentially from the even products of nilpotents (none or at least two), called *even “basis vectors”* (the rest are projectors). Either odd or even “basis vectors” are chosen to be eigenvectors of all the Cartan subalgebra members, Eq. (3).

The algebraic product of any two members of the odd or even “basis vectors” can easily be calculated when taking into account the relations following from Eq. (1). The most needed relations are presented in Eq. (5).

$$\begin{aligned} \gamma^a \begin{pmatrix} ab \\ k \end{pmatrix} &= \eta^{aa} \begin{pmatrix} ab \\ -k \end{pmatrix}, & \gamma^b \begin{pmatrix} ab \\ k \end{pmatrix} &= -ik \begin{pmatrix} ab \\ -k \end{pmatrix}, & \gamma^a \begin{pmatrix} ab \\ [k] \end{pmatrix} &= \begin{pmatrix} ab \\ -k \end{pmatrix}, & \gamma^b \begin{pmatrix} ab \\ [k] \end{pmatrix} &= -ik\eta^{aa} \begin{pmatrix} ab \\ -k \end{pmatrix}, \\ \tilde{\gamma}^a \begin{pmatrix} ab \\ k \end{pmatrix} &= -i\eta^{aa} \begin{pmatrix} ab \\ [k] \end{pmatrix}, & \tilde{\gamma}^b \begin{pmatrix} ab \\ k \end{pmatrix} &= -k \begin{pmatrix} ab \\ [k] \end{pmatrix}, & \tilde{\gamma}^a \begin{pmatrix} ab \\ [k] \end{pmatrix} &= i \begin{pmatrix} ab \\ k \end{pmatrix}, & \tilde{\gamma}^b \begin{pmatrix} ab \\ [k] \end{pmatrix} &= -k\eta^{aa} \begin{pmatrix} ab \\ k \end{pmatrix}, \\ \begin{pmatrix} ab \\ k \end{pmatrix} \begin{pmatrix} ab \\ -k \end{pmatrix} &= \eta^{aa} \begin{pmatrix} ab \\ [k] \end{pmatrix}, & \begin{pmatrix} ab \\ -k \end{pmatrix} \begin{pmatrix} ab \\ k \end{pmatrix} &= \eta^{aa} \begin{pmatrix} ab \\ [-k] \end{pmatrix}, & \begin{pmatrix} ab \\ k \end{pmatrix} \begin{pmatrix} ab \\ [k] \end{pmatrix} &= 0, & \begin{pmatrix} ab \\ k \end{pmatrix} \begin{pmatrix} ab \\ [-k] \end{pmatrix} &= \begin{pmatrix} ab \\ k \end{pmatrix}, \\ \begin{pmatrix} ab \\ -k \end{pmatrix} \begin{pmatrix} ab \\ [k] \end{pmatrix} &= \begin{pmatrix} ab \\ -k \end{pmatrix}, & \begin{pmatrix} ab \\ [k] \end{pmatrix} \begin{pmatrix} ab \\ k \end{pmatrix} &= \begin{pmatrix} ab \\ k \end{pmatrix}, & \begin{pmatrix} ab \\ [k] \end{pmatrix} \begin{pmatrix} ab \\ -k \end{pmatrix} &= 0, & \begin{pmatrix} ab \\ [k] \end{pmatrix} \begin{pmatrix} ab \\ [-k] \end{pmatrix} &= 0, \\ \begin{pmatrix} ab \\ k \end{pmatrix}^\dagger &= \eta^{aa} \begin{pmatrix} ab \\ -k \end{pmatrix}, & ((k))^2 &= 0, & \begin{pmatrix} ab \\ k \end{pmatrix} \begin{pmatrix} ab \\ -k \end{pmatrix} &= \eta^{aa} \begin{pmatrix} ab \\ [k] \end{pmatrix}, \\ \begin{pmatrix} ab \\ [k] \end{pmatrix}^\dagger &= \begin{pmatrix} ab \\ [k] \end{pmatrix}, & ([k])^2 &= [k], & \begin{pmatrix} ab \\ [k] \end{pmatrix} \begin{pmatrix} ab \\ [-k] \end{pmatrix} &= 0. \end{aligned} \quad (5)$$

More algebraic relations can be found in Appendix C.

The odd “basis vectors”, named $\hat{b}_f^{m\dagger}$, m determines the family member quantum number, f determines the quantum number of a family, appear in $2^{\frac{d}{2}-1}$ irreducible representations - families, all with the same properties with respect to S^{ab} , while distinguishing with respect $\tilde{S}^{ab}$. Each family has $2^{\frac{d}{2}-1}$ members. Their Hermitian conjugate partners $(\hat{b}_f^{m\dagger})^\dagger = \hat{b}_f^m$, appearing in a separate group, have $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members. As already written, the odd “basis vectors” have the eigenvalues of the Cartan subalgebra members, Eq. (3), either of S^{ab} or $\tilde{S}^{ab}$ half integer, $\pm\frac{i}{2}$ or $\pm\frac{1}{2}$.

The algebraic product of any two members of the odd “basis vectors” are equal to zero. The same is true for any two members of the Hermitian conjugated partner.

$$\begin{aligned} \hat{b}_f^{m\dagger} * \hat{b}_{f'}^{m'\dagger} &= 0, & \hat{b}_f^m * \hat{b}_{f'}^m &= 0, \\ \forall m, m', & & f, f'. \end{aligned} \quad (6)$$

Choosing the vacuum state equal to

$$|\psi_{oc}\rangle = \sum_{f=1}^{2^{\frac{d}{2}-1}} \hat{b}_f^m *_A \hat{b}_f^{m\dagger} |1\rangle, \quad (7)$$

for one of the members m , which can be anyone of the odd irreducible representations f , it follows that the odd “basis vectors” obey the relations

$$\begin{aligned} \hat{b}_f^m *_A |\psi_{oc}\rangle &= 0. |\psi_{oc}\rangle, \\ \hat{b}_f^{m\dagger} *_A |\psi_{oc}\rangle &= |\psi_{oc}\rangle, \\ \{\hat{b}_f^m, \hat{b}_{f'}^{m'}\} *_A |\psi_{oc}\rangle &= 0. |\psi_{oc}\rangle, \\ \{\hat{b}_f^{m\dagger}, \hat{b}_{f'}^{m'\dagger}\} *_A |\psi_{oc}\rangle &= 0. |\psi_{oc}\rangle, \\ \{\hat{b}_f^m, \hat{b}_{f'}^{m'\dagger}\} *_A |\psi_{oc}\rangle &= \delta^{mm'} \delta_{ff'} |\psi_{oc}\rangle, \end{aligned} \quad (8)$$

as postulated by Dirac for the second quantised fermion fields. Here the odd “basis vectors” anti-commute, since the odd products of γ^a 's anti-commute.

The odd “basis vectors” and their Hermitian conjugate partners are normalised as follows

$$\langle \psi_{oc} | (\hat{b}_f^{m\dagger})^\dagger *_A \hat{b}_{f'}^{m'\dagger} | \psi_{oc} \rangle = \delta^{mm'} \delta_{ff'} \langle \psi_{oc} | \psi_{oc} \rangle, \quad (9)$$

the vacuum state $\langle \psi_{oc} | \psi_{oc} \rangle$ is normalised to identity.³

The even “basis vectors”, named $I \hat{\mathcal{A}}_f^{m\dagger}$ and $II \hat{\mathcal{A}}_f^{m\dagger}$, appear in two orthogonal groups

$$I \hat{\mathcal{A}}_f^{m\dagger} *_A II \hat{\mathcal{A}}_f^{m\dagger} = 0 = II \hat{\mathcal{A}}_f^{m\dagger} *_A I \hat{\mathcal{A}}_f^{m\dagger}. \quad (12)$$

Each group has $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members with the Hermitian conjugate partners within the group.

The even “basis vectors” have the eigenvalues of the Cartan subalgebra members, Eq. (3), $S^{ab} = S^{ab} + \tilde{S}^{ab}$, equal to $\pm i$ or ± 1 or zero.

The algebraic products, $*_A$, of two members of each of these two groups have the property

$$^i \hat{\mathcal{A}}_f^{m\dagger} *_A ^i \hat{\mathcal{A}}_{f'}^{m'\dagger} \rightarrow \begin{cases} ^i \hat{\mathcal{A}}_{f'}^{m\dagger}, i = (I, II) \\ \text{or zero.} \end{cases} \quad (13)$$

³ Let us, as an exercise, present the odd “basis vectors” and their Hermitian conjugate partners for $d = (3 + 1)$, taken from Ref. [19]. The odd “basis vectors” appear in two families, each family has two members.

$$\begin{array}{ccc} f=1 & f=2 & \\ \tilde{S}^{03} = \frac{i}{2}, \tilde{S}^{12} = -\frac{1}{2} & \tilde{S}^{03} = -\frac{i}{2}, \tilde{S}^{12} = \frac{1}{2} & S^{03} \quad S^{12} \\ \hat{b}_1^{1\dagger} = \begin{pmatrix} 03 & 12 \\ +i & + \end{pmatrix} & \hat{b}_2^{1\dagger} = \begin{pmatrix} 03 & 12 \\ +i & + \end{pmatrix} & \frac{i}{2} \quad \frac{1}{2} \\ \hat{b}_1^{2\dagger} = \begin{pmatrix} 03 & 12 \\ -i & - \end{pmatrix} & \hat{b}_2^{2\dagger} = \begin{pmatrix} 03 & 12 \\ -i & - \end{pmatrix} & -\frac{i}{2} \quad -\frac{1}{2}. \end{array} \quad (10)$$

Their Hermitian conjugate partners have the properties

$$\begin{array}{ccc} S^{03} = -\frac{i}{2}, S^{12} = \frac{1}{2} & S^{03} = \frac{i}{2}, S^{12} = -\frac{1}{2} & \tilde{S}^{03} \quad \tilde{S}^{12} \\ \hat{b}_1^1 = \begin{pmatrix} 03 & 12 \\ -i & + \end{pmatrix} & \hat{b}_2^1 = \begin{pmatrix} 03 & 12 \\ +i & - \end{pmatrix} & -\frac{i}{2} \quad -\frac{1}{2} \\ \hat{b}_1^2 = \begin{pmatrix} 03 & 12 \\ -i & + \end{pmatrix} & \hat{b}_2^2 = \begin{pmatrix} 03 & 12 \\ +i & - \end{pmatrix} & \frac{i}{2} \quad \frac{1}{2}. \end{array} \quad (11)$$

The vacuum state $|\psi_{oc}\rangle$, Eq. (7), is equal to: $|\psi_{oc}\rangle = \frac{1}{\sqrt{2}} ([-i] \begin{pmatrix} 03 & 12 \\ + & + \end{pmatrix} + [+i] \begin{pmatrix} 03 & 12 \\ - & - \end{pmatrix})$.

For a chosen (m, f, f') , there is (out of $2^{\frac{d}{2}-1}$) only one m' giving a non-zero contribution.⁴

To be able to propose the action for fermion and boson second quantized fields, we need to know the algebraic application, $*_A$, of boson fields on fermion fields and fermion fields on boson fields.

The algebraic application, $*_A$, of the even “basis vectors” ${}^I\hat{\mathcal{A}}_f^{m\dagger}$ on the odd “basis vectors” $\hat{b}_{f'}^{m'\dagger}$, we call this left multiplication, gives

$${}^I\hat{\mathcal{A}}_f^{m\dagger} *_A \hat{b}_{f'}^{m'\dagger} \rightarrow \begin{cases} \hat{b}_{f'}^{m\dagger}, \\ \text{or zero.} \end{cases} \quad (16)$$

Eq. (16) demonstrates that ${}^I\hat{\mathcal{A}}_f^{m\dagger}$, applying on $\hat{b}_{f'}^{m'\dagger}$, transforms the odd “basis vector” into another odd “basis vector” of the same family, transferring to the odd “basis vector” integer spins or gives zero.

We find for the second group of boson fields, ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$,

$$\hat{b}_f^{m\dagger} *_A {}^{II}\hat{\mathcal{A}}_{f'}^{m'\dagger} \rightarrow \begin{cases} \hat{b}_{f'}^{m\dagger}, \\ \text{or zero,} \end{cases} \quad (17)$$

demonstrating that the application of the odd “basis vector” $\hat{b}_f^{m\dagger}$ on ${}^{II}\hat{\mathcal{A}}_{f'}^{m'\dagger}$ leads to another odd “basis vector” $\hat{b}_{f'}^{m\dagger}$ belonging to the same family member m of a different family f' . We call this the right multiplication of ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$ on the odd “basis vector” $\hat{b}_f^{m\dagger}$.

The rest of possibilities give zero.

$$\hat{b}_f^{m\dagger} *_A {}^I\hat{\mathcal{A}}_{f'}^{m'\dagger} = 0, \quad {}^{II}\hat{\mathcal{A}}_f^{m\dagger} *_A \hat{b}_{f'}^{m'\dagger} = 0, \quad \forall(m, m', f, f'). \quad (18)$$

Let us add that the internal spaces of boson second quantized fields can be written as the algebraic products of the odd “basis vectors” and their Hermitian conjugate partners: $\hat{b}_f^{m\dagger}$ and $(\hat{b}_{f'}^{m''\dagger})^\dagger$.

$${}^I\hat{\mathcal{A}}_f^{m\dagger} = \hat{b}_{f'}^{m'\dagger} *_A (\hat{b}_{f'}^{m''\dagger})^\dagger, \quad (19)$$

$${}^{II}\hat{\mathcal{A}}_f^{m\dagger} = (\hat{b}_{f'}^{m'\dagger})^\dagger *_A \hat{b}_{f'}^{m'\dagger}. \quad (20)$$

Family members $\hat{b}_{f'}^{m'\dagger}$ of any family f' generates in the algebraic product $\hat{b}_{f'}^{m'\dagger} *_A (\hat{b}_{f'}^{m''\dagger})^\dagger$ the same $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ even “basis vectors” ${}^I\hat{\mathcal{A}}_f^{m\dagger}$, each family member m' generates in $(\hat{b}_{f'}^{m'\dagger})^\dagger *_A \hat{b}_{f'}^{m'\dagger}$ the same $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ even “basis vectors” ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$ ⁵.

⁴ Let us, for an exercise, present the $2^{\frac{4}{2}-1} \times 2^{\frac{4}{2}-1}$ “basis vectors” for $d = (3 + 1)$, the members of the group ${}^I\mathcal{A}_f^{m\dagger}$, taken from Ref. [19]

$$\begin{array}{ccccc} & \mathcal{S}^{03} & \mathcal{S}^{12} & & \mathcal{S}^{03} & \mathcal{S}^{12} \\ {}^I\mathcal{A}_1^{1\dagger} = \begin{pmatrix} 03 & 12 \\ +i & + \end{pmatrix} & 0 & 0 & , {}^I\mathcal{A}_2^{1\dagger} = \begin{pmatrix} 03 & 12 \\ +i & + \end{pmatrix} & i & 1 \\ {}^I\mathcal{A}_1^{2\dagger} = \begin{pmatrix} 03 & 12 \\ -i & - \end{pmatrix} & -i & -1 & , {}^I\mathcal{A}_2^{2\dagger} = \begin{pmatrix} 03 & 12 \\ -i & - \end{pmatrix} & 0 & 0, \end{array} \quad (14)$$

and $2^{\frac{4}{2}-1} \times 2^{\frac{4}{2}-1}$ even “basis vectors” ${}^{II}\mathcal{A}_f^{m\dagger}$, $m = (1, 2), f = (1, 2)$,

$$\begin{array}{ccccc} & \mathcal{S}^{03} & \mathcal{S}^{12} & & \mathcal{S}^{03} & \mathcal{S}^{12} \\ {}^{II}\mathcal{A}_1^{1\dagger} = \begin{pmatrix} 03 & 12 \\ +i & - \end{pmatrix} & 0 & 0 & , {}^{II}\mathcal{A}_2^{1\dagger} = \begin{pmatrix} 03 & 12 \\ +i & - \end{pmatrix} & i & -1 \\ {}^{II}\mathcal{A}_1^{2\dagger} = \begin{pmatrix} 03 & 12 \\ -i & + \end{pmatrix} & -i & 1 & , {}^{II}\mathcal{A}_2^{2\dagger} = \begin{pmatrix} 03 & 12 \\ -i & + \end{pmatrix} & 0 & 0. \end{array} \quad (15)$$

One can easily check the above relations from Eqs. (10 - 20) if taking into account Eq. 1, or Eq.(5).

⁵ It follows that ${}^I\hat{\mathcal{A}}_f^{m\dagger}$, expressed by $\hat{b}_{f'}^{m'\dagger} *_A (\hat{b}_{f'}^{m''\dagger})^\dagger$, applying on $\hat{b}_{f'}^{m''\dagger}$, obey Eq. (16), and $\hat{b}_{f'}^{m''\dagger}$ applying on ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$, expressed by $(\hat{b}_{f'}^{m'\dagger})^\dagger *_A \hat{b}_{f'}^{m'\dagger}$, obey Eq. (17).

2.2. Fermions and Bosons Creation Operators

The creation operators for either fermions or bosons must be defined as the tensor products, $*_T$, of both contributions, the “basis vectors” describing the internal space of fermions or bosons and the basis in ordinary space-time in the momentum or coordinate representation ⁶.

Let us start with the definition of the single particle states in ordinary space-time in momentum representation, briefly overviewing Refs. [20], ([8], Subsect. 3.3 and App. J).

$$\begin{aligned} |\vec{p}\rangle &= \hat{b}_{\vec{p}}^{\dagger} |0_p\rangle, \quad \langle \vec{p}| = \langle 0_p| \hat{b}_{\vec{p}}, \\ \langle \vec{p}|\vec{p}'\rangle &= \delta(\vec{p} - \vec{p}') = \langle 0_p| \hat{b}_{\vec{p}} \hat{b}_{\vec{p}'}^{\dagger} |0_p\rangle, \\ \langle 0_p| \hat{b}_{\vec{p}} \hat{b}_{\vec{p}'}^{\dagger} |0_p\rangle &= \delta(\vec{p}' - \vec{p}), \end{aligned} \quad (21)$$

with $\langle 0_p|0_p\rangle = 1$. The operator $\hat{b}_{\vec{p}}^{\dagger}$ pushes a single particle state with zero momentum by an amount $\vec{p}$. Taking into account that $\{\hat{p}^i, \hat{p}^j\}_- = 0$ and $\{\hat{x}^k, \hat{x}^l\}_- = 0$, while $\{\hat{p}^i, \hat{x}^j\}_- = i\eta^{ij}$, it follows

$$\begin{aligned} \langle \vec{p}|\vec{x}\rangle &= \langle 0_{\vec{p}}| \hat{b}_{\vec{p}} \hat{b}_{\vec{x}}^{\dagger} |0_{\vec{x}}\rangle = (\langle 0_{\vec{x}}| \hat{b}_{\vec{x}} \hat{b}_{\vec{p}}^{\dagger} |0_{\vec{p}}\rangle)^{\dagger} \\ \langle 0_{\vec{p}}|\{\hat{b}_{\vec{p}}^{\dagger}, \hat{b}_{\vec{p}'}^{\dagger}\}_-|0_{\vec{p}}\rangle &= 0, \quad \langle 0_{\vec{p}}|\{\hat{b}_{\vec{p}}, \hat{b}_{\vec{p}'}\}_-|0_{\vec{p}}\rangle = 0, \quad \langle 0_{\vec{p}}|\{\hat{b}_{\vec{p}}, \hat{b}_{\vec{p}'}^{\dagger}\}_-|0_{\vec{p}}\rangle = 0, \\ \langle 0_{\vec{x}}|\{\hat{b}_{\vec{x}}^{\dagger}, \hat{b}_{\vec{x}'}^{\dagger}\}_-|0_{\vec{x}}\rangle &= 0, \quad \langle 0_{\vec{x}}|\{\hat{b}_{\vec{x}}, \hat{b}_{\vec{x}'}\}_-|0_{\vec{x}}\rangle = 0, \quad \langle 0_{\vec{x}}|\{\hat{b}_{\vec{x}}, \hat{b}_{\vec{x}'}^{\dagger}\}_-|0_{\vec{x}}\rangle = 0, \\ \langle 0_{\vec{p}}|\{\hat{b}_{\vec{p}}, \hat{b}_{\vec{x}}^{\dagger}\}_-|0_{\vec{x}}\rangle &= e^{i\vec{p}\cdot\vec{x}} \frac{1}{\sqrt{(2\pi)^{d-1}}}, \quad \langle 0_{\vec{x}}|\{\hat{b}_{\vec{x}}, \hat{b}_{\vec{p}}^{\dagger}\}_-|0_{\vec{p}}\rangle = e^{-i\vec{p}\cdot\vec{x}} \frac{1}{\sqrt{(2\pi)^{d-1}}}. \end{aligned} \quad (22)$$

The momentum basis is continuously infinite, while the internal space of either fermion or boson fields has a finite number of “basis vectors” - twice $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ for fermions and twice $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ for bosons, provided that the internal space includes d -dimensions. We assume that fermions and bosons are active only $d = (3 + 1)$.

The creation operator for a free massless fermion field of the energy $p^0 = |\vec{p}|$, belonging to the family f and to a superposition of family members m applying on the vacuum state ($|\psi_{oc}\rangle *_T |0_{\vec{p}}\rangle$) can be written as (we follow [8], Subsect.3.3.2, and the references therein)

$$\hat{\mathbf{b}}_f^{s\dagger}(\vec{p}) = \sum_m c_f^{sm}(\vec{p}) \hat{b}_{\vec{p}}^{\dagger} *_T \hat{b}_f^{m\dagger}. \quad (23)$$

The vacuum state for fermions, $|\psi_{oc}\rangle *_T |0_{\vec{p}}\rangle$, includes both spaces, the internal part, Eq.(7), and the momentum part, Eq. (21). The creation operators in the coordinate representation can be written as $\hat{\mathbf{b}}_f^{s\dagger}(\vec{x}, x^0) = \sum_m \hat{b}_f^{m\dagger} *_T \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}} c_f^{sm}(\vec{p}) \hat{b}_{\vec{p}}^{\dagger} e^{-i(p^0 x^0 - \vec{p}\cdot\vec{x})}$, [18], ([8], subsect. 3.3.2. and the references therein).

The creation operators, $\hat{\mathbf{b}}_f^{s\dagger}(\vec{p})$, and their Hermitian conjugate partners annihilation operators, $(\hat{\mathbf{b}}_f^{s\dagger}(\vec{p}))^{\dagger} = \hat{\mathbf{b}}_f^s(\vec{p})$, creating and annihilating the single fermion states, respectively, fulfil when applying the vacuum state, ($|\psi_{oc}\rangle *_T |0_{\vec{p}}\rangle$), the anti-commutation relations for the second quantized fermions, postulated by Dirac (Ref. [8], Subsect. 3.3.1, Sect. 5). The anticommuting properties of

⁶ To the boson second quantised fields, we need to add the space index α . Since our space-time is flat, we shall use a instead of α , and n instead of μ and for the scalar index, $a \geq 5$, (our fermions and bosons have non-zero momentum only in $d = (3 + 1)$ of ordinary space-time) index s .

the creation operators for fermions are determined by the odd “basis vectors”, the basis in ordinary space-time, namely, commute ⁷.

The creation operator for a free massless boson field of the energy $p^0 = |\vec{p}|$, with the “basis vectors” belonging to one of the two groups, ${}^i\hat{\mathcal{A}}_f^{m\dagger}, i = (I, II)$, applying on the vacuum state, $|1\rangle \rightarrow {}^*T|0_{\vec{p}}\rangle$, must carry the space index a , describing the a component of the boson field in the ordinary space ⁸. We, therefore, add the space index a , as well as the dependence on the momentum [20]

$${}^i\hat{\mathcal{A}}_{fa}^{m\dagger}(\vec{p}) = {}^i\mathcal{C}_{fa}^m(\vec{p}) \cdot {}^*T {}^i\hat{\mathcal{A}}_f^{m\dagger}, i = (I, II), \quad (25)$$

with ${}^i\mathcal{C}_{fa}^m(\vec{p}) = {}^i\mathcal{C}_{fa}^m \hat{b}_{\vec{p}}^{\dagger}$, with $\hat{b}_{\vec{p}}^{\dagger}$ defined in Eqs. (21, 22) ⁹.

The creation operators for boson fields in the coordinate representation one finds using Eqs. (21, 22), ${}^i\hat{\mathcal{A}}_{fa}^{m\dagger}(\vec{x}, x^0) = {}^i\hat{\mathcal{A}}_f^{m\dagger} \cdot {}^*T \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}} {}^i\mathcal{C}_{fa}^m \hat{b}_{\vec{p}}^{\dagger} e^{-i(p^0 x^0 - \vec{p} \cdot \vec{x})} |_{p^0=|\vec{p}|}, i = (I, II)$.

Assuming that the internal space has $d = (13 + 1)$, while fermions and bosons have non-zero momenta only in $d = (3 + 1)$ of the ordinary space-time, both creation operators manifest (after analysing $SO(13, 1)$ with respect to the subgroups $SO(3, 1)$, $SU(2) \times SU(2)$, $SU(3)$ and $U(1)$ of the Lorentz group $SO(13, 1)$) the properties of fermions and bosons as assumed by the *standard model* before the electroweak phase transitions: Clifford odd creation operators defining the fermion fields manifest all the properties of quarks and leptons and antiquarks and antileptons, appearing in families, and the Clifford even boson creation operators, ${}^I\hat{\mathcal{A}}_{fa}^{m\dagger}$, defining the boson fields manifest for a equal to $n = (0, 1, 2, 3)$ all the properties of gauge fields (photons, weak bosons, and gluons - predicting the second weak fields and explaining the gravitons).

For a equal to $s \geq 5$, the Clifford even boson creation operators, ${}^{II}\hat{\mathcal{A}}_{fs}^{m\dagger}$, manifest properties of the scalar Higgs, causing after the electroweak phase transitions masses of quarks and leptons and antiquarks and antileptons, and some of the gauge fields.

The assumption that the internal spaces of fermion and boson fields are describable by the odd and even “basis vectors”, respectively, leads to the conclusion that the internal spaces of all the boson fields - gravitons (the gauge fields of the spins $SO(3, 1)$), photons (the gauge fields of $U(1)$), weak bosons (the gauge fields of one of the $SU(2)$) and gluons (the gauge fields of $SU(3)$ (together with $U(1)$ origin in $SO(6)$) - must also be described by the even “basis vectors”, all must carry the index $a = n = (0, 1, 2, 3)$.

Both groups of even “basis vectors” manifest as the gauge fields of the corresponding fermion fields: One concerning the family members quantum numbers, determined by S^{ab} , the other concerning the family quantum numbers, determined by $\tilde{S}^{ab}$.

Let us point out that although it looks like that this theory postulates two kinds of boson fields, not yet observed so far, this is not the case: All the theories so far postulate the families of fermions and the scalar fields giving masses to fermions and weak bosons in addition to the internal spaces of fermions and bosons. In our case, the families are present without being postulated. Our boson fields of the second kind have, in theories so far, realization in Higgs.

7

$$\begin{aligned} \langle 0_{\vec{p}} | \{ \hat{\mathbf{b}}_{f'}^{s'}(\vec{p}'), \hat{\mathbf{b}}_f^{s\dagger}(\vec{p}) \} + | \psi_{oc} \rangle | 0_{\vec{p}} \rangle &= \delta^{ss'} \delta_{ff'} \delta(\vec{p}' - \vec{p}) \cdot | \psi_{oc} \rangle, \\ \{ \hat{\mathbf{b}}_{f'}^{s'}(\vec{p}'), \hat{\mathbf{b}}_f^{s\dagger}(\vec{p}) \} + | \psi_{oc} \rangle | 0_{\vec{p}} \rangle &= 0 \cdot | \psi_{oc} \rangle | 0_{\vec{p}} \rangle, \\ \{ \hat{\mathbf{b}}_{f'}^{s'+\dagger}(\vec{p}'), \hat{\mathbf{b}}_f^{s\dagger}(\vec{p}) \} + | \psi_{oc} \rangle | 0_{\vec{p}} \rangle &= 0 \cdot | \psi_{oc} \rangle | 0_{\vec{p}} \rangle, \\ \hat{\mathbf{b}}_f^{s\dagger}(\vec{p}) | \psi_{oc} \rangle | 0_{\vec{p}} \rangle &= | \psi_f^s(\vec{p}) \rangle, \\ \hat{\mathbf{b}}_f^s(\vec{p}) | \psi_{oc} \rangle | 0_{\vec{p}} \rangle &= 0 \cdot | \psi_{oc} \rangle | 0_{\vec{p}} \rangle, \\ | p^0 | &= | \vec{p} |. \end{aligned} \quad (24)$$

⁸ The vacuum state for bosons is chosen to be identity.

⁹ In the general case, the energy eigenstates of bosons are in a superposition of ${}^i\hat{\mathcal{A}}_f^{m\dagger}$, for either $i = I$ or $i = II$.

The proposed description of the internal spaces offers families of fermions, scalar fields and gauge fields: $^I \hat{\mathcal{A}}_f^{m\dagger}$, transferring the integer quantum numbers to the odd “basis vectors”, $\hat{b}_f^{m\dagger}$, changes the family members’ quantum numbers, leaving the family quantum numbers unchanged, manifesting the properties of the gauge fields; The second group, $^{II} \hat{\mathcal{A}}_f^{m\dagger}$, transferring the integer quantum numbers to the “basis vector” $\hat{b}_f^{m\dagger}$, changes the family quantum numbers leaving the family members quantum numbers unchanged, manifesting properties of the scalar fields, which give masses to quarks and leptons, and to the weak bosons.

The propose description of the internal spaces of fermions and bosoms predicts more families than observed so far, and more gauge fields. We expect that the breaks of the starting symmetry will show, why we have not observed them [8,18](yet).

2.3. States of Fermions and Bosons Active Only in $d = (3 + 1)$

We take the states of fermion and boson fields to have non-zero momentum only in $d = (3 + 1)$. This refers to the Poincaré group (with the infinitesimal generators $M^{ab}(= L^{ab} + S^{ab}), p^c$) applying only in $d = (3 + 1)$, while in the internal space, the Lorentz group (with the infinitesimal generators $S^{ab}, \tilde{S}^{ab} = S^{ab} + \tilde{S}^{ab}$) applies to the whole internal space $d = 2(2n + 1)$. We discuss in this section the algebraic relations among fermion and boson fields (2, 2.1) in the case that the internal space has $d = (5 + 1)$ and $d = (13 + 1)$, Subsect. 2.3.1.

The odd and even “basis vectors” are presented in the case that $d = (5 + 1)$ in App. D in Tables (A2, A3, A4, A5, A6). This toy model is discussed as an exercise.

In Table A1 the odd “basis vectors” are presented in the case that $d = (13 + 1)$ for one family of fermions - quarks and leptons and antiquarks and antileptons - as products of an odd number of nilpotents (at least one, up to seven). The “basis vectors” are eigenstates of all the Cartan subalgebra memebbers, Eq. (3), of the Lorentz algebra.

The creation and annihilation operators are for odd and even “basis vectors” the tensor products, $*_T$, of the basis in ordinary space-time in $d = (3 + 1)$, and the “basis vectors” in internal space, with $d = (5 + 1)$ or $d = (13 + 1)$: For anti-commuting creation operators we have $\hat{\mathbf{b}}_f^{s\dagger}(\vec{p}) = \sum_m c^{sm}_f(\vec{p}) \hat{b}_p^{s\dagger} *_T \hat{b}_f^{m\dagger}$, Eq. (23).

For the commuting creation operators with the “basis vectors” belonging to one of the two groups, $^i \hat{\mathcal{A}}_f^{m\dagger}, i = (I, II)$, carrying the space index a , we have $^i \hat{\mathcal{A}}_{fa}^{m\dagger}(\vec{p}) = {}^i C^m_{fa}(\vec{p}) *_T {}^i \hat{\mathcal{A}}_f^{m\dagger}, i = (I, II)$, Eq. (25).

2.3.1. Internal Spaces of Fermions and Bosons in $d = (5 + 1)$ and $d = (13 + 1)$

a. Let us start with *the toy model for electrons, positrons, photons and gravitons in the case that the internal space is $d = (5 + 1)$, while fields have non-zero momenta in $d = (3 + 1)$.*

We follow here to some extent a similar part in the Ref. ([20], and the references therein). This toy model is to show the reader, in a simple model, what the new description of the internal spaces of fermion and boson fields offers.

In Table A2 the odd “basis vectors”, $\hat{b}_f^{m\dagger}$, appearing in four $(2^{\frac{d-6}{2}} - 1)$ families, each family having four $(2^{\frac{d-6}{2}} - 1)$ family members, are presented in the first group, as products of an odd number of nilpotents (one or three) and the remaining projectors. Their Hermitian conjugate partners are presented in the second group, again with 16 members.

The even basis vectors appear in the third and the fourth group.

Table A2 presents the eigenvalues of all Cartan subalgebra members, Eq. (3); S^{ab} for a family members, and $\tilde{S}^{ab}$ for a family. $S^{ab} = (S^{ab} + \tilde{S}^{ab})$ determine the Cartan eigenvalues of the even “basis vectors”, presenting internal spaces of boson fields.

The reader can check the relations among “basis vectors” of fermions and bosons appearing in Eqs. (6 – 20), for the case that the internal space has $d = (5 + 1)$, by taking into account Eqs. (5, A67, A68).

The corresponding creation and annihilation operators for free massless fermion fields ($\hat{\mathbf{b}}_f^{m\dagger}(\vec{p}) = \hat{b}_{\vec{p}}^{\dagger} *_{\mathcal{T}} \hat{b}_f^{m\dagger}$), and for free massless boson fields (${}^i\hat{\mathcal{A}}_{fa}^{m\dagger}, i = (I, II)$), carrying the space index a , ${}^i\hat{\mathcal{A}}_{fa}^{m\dagger}(\vec{p}) = {}^i\hat{\mathcal{C}}_{fa}^m(\vec{p}) *_{\mathcal{T}} {}^i\hat{\mathcal{A}}_f^{m\dagger}, i = (I, II)$), can be found in Eqs. (23, 25).

Let us call the first $\hat{b}_f^{m\dagger}$ of the “basis vectors” in Table A2, $\hat{b}_1^{1\dagger} = ({}^{03}_{+i})[{}^{12}_{+}] [{}^{56}_{+}]$, the “basis vector” of the “electron”, and the third “basis vector” $\hat{b}_1^{3\dagger} = [{}^{03}_{-i}] [{}^{12}_{+}] [{}^{56}_{-}]$, both belong to the first family, the “basis vector” of the “positron”, choosing the quantum numbers of the “electron” equal to ($S^{03} = \frac{i}{2}, S^{12} = \frac{1}{2}$ and $S^{56} = \frac{1}{2}$), and of the “positron” equal to ($S^{03} = -\frac{i}{2}, S^{12} = \frac{1}{2}$ and $S^{56} = \frac{1}{2}$). One can transform the “electron” to the “positron” by S^{05} .

The “basis vectors” of the “positron” and “electron” have fractional charges and both appear in four families, reachable from the first one by the application of $\tilde{S}^{ab}$.

For example, one generates the second family by applying $\tilde{S}^{05}$ on the first family.

The corresponding “photon” field, its “basis vector” indeed, describing the internal space of the “photon”, must be a product of projectors only, since the photon does not change the charge of the positron or electron.

There is only one even “basis vector”, that applying to the “basis vector” of the “electron” gives a non-zero contribution, the “basis vector” ${}^I\hat{\mathcal{A}}_3^{1\dagger} = [{}^{03}_{+i}] [{}^{12}_{+}] [{}^{56}_{+}]$. It is presented in Table A4.

There is also only one even “basis vector”, which, applying to the “basis vector” of the “positron”, gives a non-zero contribution. Both even “basis vectors” have the properties of photons. It is presented in Table A4.

$$\begin{aligned} {}^I\hat{\mathcal{A}}_{3ph}^{1\dagger} (\equiv [{}^{03}_{+i}] [{}^{12}_{+}] [{}^{56}_{+}]) *_{\mathcal{A}} \hat{b}_f^{1\dagger} (\equiv ({}^{03}_{+i}) [{}^{12}_{+}] [{}^{56}_{+}]) &\rightarrow \hat{b}_f^{1\dagger}, \\ {}^I\hat{\mathcal{A}}_{2ph}^{3\dagger} (\equiv [{}^{03}_{-i}] [{}^{12}_{+}] [{}^{56}_{-}]) *_{\mathcal{A}} \hat{b}_f^{3\dagger} (\equiv [{}^{03}_{-i}] [{}^{12}_{+}] [{}^{56}_{-}]) &\rightarrow \hat{b}_f^{3\dagger}. \end{aligned} \quad (26)$$

The same “photon” makes the same transformations on the corresponding “electron” (or “positron”) of all the families. Obviously, the Cartan subalgebra quantum numbers, Eq. (3), ($S^{ab} + \tilde{S}^{ab}$), applying on any member of the “photon” is equal to zero: ($S^{03} + \tilde{S}^{03} = 0, S^{12} + \tilde{S}^{12} = 0$ and $S^{56} + \tilde{S}^{56} = 0$) of either ${}^I\hat{\mathcal{A}}_{3ph}^{1\dagger}$ or ${}^I\hat{\mathcal{A}}_{2ph}^{3\dagger}$, are zero, since the projectors have properties that $S^{ab} = -\tilde{S}^{ab}$, Eq. (4).

Let us check the relation of Eq. (19), using Eqs. (5, A66).

$$\begin{aligned} {}^I\hat{\mathcal{A}}_3^{1\dagger} (\equiv [{}^{03}_{+i}] [{}^{12}_{+}] [{}^{56}_{+}]) &= \hat{b}_1^{1\dagger} (\equiv ({}^{03}_{+i}) [{}^{12}_{+}] [{}^{56}_{+}]) *_{\mathcal{A}} (\hat{b}_1^{1\dagger})^{\dagger} (\equiv (({}^{03}_{+i}) [{}^{12}_{+}] [{}^{56}_{+}])^{\dagger}). \\ {}^I\hat{\mathcal{A}}_2^{3\dagger} (\equiv [{}^{03}_{-i}] [{}^{12}_{+}] [{}^{56}_{-}]) &= \hat{b}_1^{3\dagger} (\equiv [{}^{03}_{-i}] [{}^{12}_{+}] [{}^{56}_{-}]) *_{\mathcal{A}} (\hat{b}_1^{3\dagger})^{\dagger} (\equiv ([{}^{03}_{-i}] [{}^{12}_{+}] [{}^{56}_{-}])^{\dagger}). \end{aligned}$$

We demonstrated on one example, that knowing the odd “basis vectors” we can reproduce all the even “basis vectors”, ${}^I\hat{\mathcal{A}}_f^{m\dagger}$.¹⁰ Tables A5 and A6 relate the odd “basis vectors” and their Hermitian conjugated partners.

We can repeat all the relations obtained for ${}^I\hat{\mathcal{A}}_f^{m\dagger}$ in this subsection also for ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$. Kipping in mind Eq. (20), we easily see the essential difference between ${}^I\hat{\mathcal{A}}_f^{m\dagger}$ and ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$. While ${}^I\hat{\mathcal{A}}_f^{m\dagger}$ transform family members of odd “basis vectors” among themselves, keeping family quantum number unchanged, transform ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$ a particular family member to the same family member of all the families, changing

¹⁰ In Ref. [20] the relations among even “basis vectors”, and the odd “basis vectors” are presented in Tables (2,3,4,5). Tables (2,3) relate ${}^I\hat{\mathcal{A}}_f^{m\dagger}$ and odd “basis vectors”, while Tables (4,5) relate ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$ and odd “basis vectors”.

the family quantum numbers.¹¹ Correspondingly, we see in the second kind of bosons, when carrying the scalar space-time index $a \geq 5$, a scalar boson - a kind of Higgs.

Let us point out that the even “basis vectors”, determining the creation and annihilation operators in a tensor product with the basis in ordinary space-time, determine spins and charges of boson fields. Having non zero momentum only in $d = (3 + 1)$, they carry space index $a = n = (0, 1, 2, 3)$. They behave in the case that internal space has $(5 + 1)$ dimensions as a “photon”, as we just discussed. Our “photon” can exchange the momentum in ordinary space-time with “electron” or “positron”, but can not influence any internal property, like there are the spins, S^{03} and S^{12} , or the charge S^{56} .

Let us find out what represents the even “basis vectors”, ${}^I\hat{\mathcal{A}}_4^{1+}$, with two nilpotents in the $SO(3, 1)$ subgroup of the group $SO(5, 1)$. The two spins, S^{03} and S^{12} , enables the creation operators, which are the tensor product of the basis in ordinary space-time and the even “basis vectors” with two nilpotents, Eq. (25), to form “gravitons”, presented in Table A3. We presents two gravitons, ${}^I\hat{\mathcal{A}}_{4n}^{1+}(\vec{p})$ and ${}^I\hat{\mathcal{A}}_{3n}^{2+}(\vec{p})$,

$${}^I\hat{\mathcal{A}}_{4n}^{1+}(\vec{p}) = {}^I\mathcal{C}_{4n}^1(\vec{p}) * {}^I\hat{\mathcal{A}}_4^{1+}(\equiv \hat{b}_1^{1+} * {}^A(\hat{b}_1^{2+})^\dagger = ({}^{03}{}^{12}{}^{56}{}^{+i}{}^{+}{}^{+})[+]),$$

$${}^I\hat{\mathcal{A}}_{3n}^{2+}(\vec{p}) = {}^I\mathcal{C}_{3n}^2(\vec{p}) * {}^I\hat{\mathcal{A}}_3^{2+}(\equiv ({}^{03}{}^{12}{}^{56}{}^{-i}{}^{-}{}^{+})[+]) = (\hat{b}_1^{2+} * {}^A(\hat{b}_1^{1+})^\dagger),$$

with the basis vectors ${}^I\hat{\mathcal{A}}_4^{1+}(\equiv ({}^{03}{}^{12}{}^{56}{}^{+i}{}^{+}{}^{+})[+])$, the first one, and with ${}^I\hat{\mathcal{A}}_3^{2+}(\equiv ({}^{03}{}^{12}{}^{56}{}^{-i}{}^{-}{}^{+})[+])$, the second one, which change the spins, S^{03} and S^{12} , of fermions. When a boson ${}^I\hat{\mathcal{A}}_{4n}^{1+}(\vec{p})$ scatters on a “electron” with the spin down, $\hat{b}_1^{2+}(\vec{p})(\equiv \hat{b}_p^\dagger * {}^T\hat{b}_1^{2+}$, Eq. (23), changes its spin from $\downarrow$ to $\uparrow$, and transfers the momentum to the “electron”. This boson ${}^I\hat{\mathcal{A}}_{4n}^{1+}(\vec{p})$, transferring the integer spin to the “electron” in addition to momentum of the space-time, is obviously “graviton” with $S^{03} = i$ and $S^{12} = 1$, changing the quantum numbers $S^{03} = -\frac{i}{2}$ and $S^{12} = -\frac{1}{2}$ of $\hat{b}_1^{2+}(\vec{p})$ to $S^{03} = \frac{i}{2}$ and $S^{12} = \frac{1}{2}$ of $\hat{b}_1^{1+}(\vec{p})$.

Let us check for two cases, how do the “basis vectors” of “gravitons” behave when “gravitons” scatter.

$$\begin{aligned} {}^I\hat{\mathcal{A}}_{3gr}^{2+}(\equiv ({}^{03}{}^{12}{}^{56}{}^{-i}{}^{-}{}^{+})[+]) * {}^I\hat{\mathcal{A}}_{4gr}^{1+}(\equiv ({}^{03}{}^{12}{}^{56}{}^{+i}{}^{+}{}^{+})[+]) &\rightarrow {}^I\hat{\mathcal{A}}_{4ph}^{2+}(\equiv ({}^{03}{}^{12}{}^{56}{}^{-i}{}^{-}{}^{+})[+]) , \\ {}^I\hat{\mathcal{A}}_{4gr}^{1+}(\equiv ({}^{03}{}^{12}{}^{56}{}^{+i}{}^{+}{}^{+})[+]) * {}^I\hat{\mathcal{A}}_{3gr}^{2+}(\equiv ({}^{03}{}^{12}{}^{56}{}^{-i}{}^{-}{}^{+})[+]) &\rightarrow {}^I\hat{\mathcal{A}}_{3ph}^{1+}(\equiv ({}^{03}{}^{12}{}^{56}{}^{+i}{}^{+}{}^{+})[+]) . \end{aligned} \quad (27)$$

There are also even “basis vectors” of the kind ${}^I\hat{\mathcal{A}}_f^{m+}$ which change spin and charges, changing for example “positrons” into “electron”¹², changing at the same time the handedness.

Looking at the even “basis vector” in this toy model, there are one fourth of ${}^I\hat{\mathcal{A}}_f^{m+}$, which are “photons” (like ${}^I\hat{\mathcal{A}}_3^{1+}$ and ${}^I\hat{\mathcal{A}}_4^{2+}$, not able to change the quantum numbers of the “electrons” and “positrons”, presented in Table A2) or “gravitons” (like ${}^I\hat{\mathcal{A}}_3^{2+}$ and ${}^I\hat{\mathcal{A}}_4^{1+}$, which change the spin of “electrons” and “positrons”).

The rest eight ${}^I\hat{\mathcal{A}}_f^{m+}$ relate “electrons” and “positrons”.

¹¹ Let us demonstrate the right application of ${}^{II}\hat{\mathcal{A}}_f^{m+}$ on $\hat{b}_{f'}^{m'+}$ for the case $d = (5 + 1)$.

The same ${}^{II}\hat{\mathcal{A}}_f^{m+}$, applying from the right-hand side on $\hat{b}_{f'}^{m'+}$, transforms any family member of one family to the same family member of another family.

$$\begin{aligned} \hat{b}_1^{1+}(\equiv ({}^{03}{}^{12}{}^{56}{}^{+i}{}^{+}{}^{+})[+]) * {}^{II}\hat{\mathcal{A}}_4^{1+}(\equiv ({}^{03}{}^{12}{}^{56}{}^{-i}{}^{-}{}^{+})[+]) &\rightarrow \hat{b}_3^{1+}(\equiv ({}^{03}{}^{12}{}^{56}{}^{+i}{}^{+}{}^{+})[+]) , \\ \hat{b}_1^{1+}(\equiv ({}^{03}{}^{12}{}^{56}{}^{+i}{}^{+}{}^{+})[+]) * {}^{II}\hat{\mathcal{A}}_1^{1+}(\equiv ({}^{03}{}^{12}{}^{56}{}^{-i}{}^{-}{}^{+})[+]) &\rightarrow \hat{b}_4^{1+}(\equiv ({}^{03}{}^{12}{}^{56}{}^{+i}{}^{+}{}^{+})[+]) , \\ \hat{b}_1^{2+}(\equiv ({}^{03}{}^{12}{}^{56}{}^{-i}{}^{-}{}^{+})[+]) * {}^{II}\hat{\mathcal{A}}_4^{1+}(\equiv ({}^{03}{}^{12}{}^{56}{}^{-i}{}^{-}{}^{+})[+]) &\rightarrow \hat{b}_3^{2+}(\equiv ({}^{03}{}^{12}{}^{56}{}^{-i}{}^{-}{}^{+})[+]) , \\ \hat{b}_1^{2+}(\equiv ({}^{03}{}^{12}{}^{56}{}^{-i}{}^{-}{}^{+})[+]) * {}^{II}\hat{\mathcal{A}}_1^{1+}(\equiv ({}^{03}{}^{12}{}^{56}{}^{-i}{}^{-}{}^{+})[+]) &\rightarrow \hat{b}_4^{2+}(\equiv ({}^{03}{}^{12}{}^{56}{}^{-i}{}^{-}{}^{+})[+]) . \end{aligned}$$

¹² The corresponding bosons transform “positrons” into “electron”, ${}^I\hat{\mathcal{A}}_1^{2+}(\equiv ({}^{03}{}^{12}{}^{56}{}^{-i}{}^{-}{}^{+})[+]) * {}^I\hat{\mathcal{A}}_1^{4+}(\equiv ({}^{03}{}^{12}{}^{56}{}^{+i}{}^{+}{}^{+})[+]) \rightarrow \hat{b}_1^{2+}(\equiv ({}^{03}{}^{12}{}^{56}{}^{-i}{}^{-}{}^{+})[+])$.

As we already said, repeating the relations for ${}^I\hat{\mathcal{A}}_f^{m+}$, Eq. (26, 27), also for ${}^{II}\hat{\mathcal{A}}_f^{m+}$, we shall not get “photons” or “gravitons”, which both transform family members of odd “basis vectors” among themselves, keeping the family quantum number unchanged. Carrying the space index equal to (5, 6), the scalar bosons of the second kind, ${}^{II}\hat{\mathcal{A}}_f^{m+}$, (“photons” and “gravitons”) cause, as a kind of “Higgs”, after breaking symmetries in this toy model, the masses of fermion fields.

b. *The realistic case, which offers the “basis vectors” for all the so far observed fermion and boson fields, requires for internal space $d = (13 + 1)$, and for the space-time, in which fermions and bosons have non zero momenta, $d = (3 + 1)$, at least at observable energies.*

In Table A1, App. A, the $2^{\frac{14}{2}-1}$ odd “basis vectors” present one irreducible representation, one family, of quarks and leptons and antiquarks and antileptons (both appearing in the same family), analysed with respect to the subgroups $SO(3, 1)$, $SU(2)_I$, $SU(2)_{II}$, $SU(3)$, $U(1)$ of the group $SO(13, 1)$. One can notice in Table A1, that the content of the subgroup $SO(7, 1)$ (including subgroups $SO(3, 1)$, $SU(2)_I$, $SU(2)_{II}$) are identical for quarks and leptons, and identical for antiquarks and antileptons; due to two $SU(2)$ subgroups $SU(2)_I$, $SU(2)_{II}$, first representing the weak charge, postulated by the *standard model*, the second $SU(2)_{II}$ group members are not (yet) observed at low energies. Quarks and leptons, and antiquarks and antileptons distinguish only in the $SU(3) \times U(1)$ part of the group $SO(13, 1)$.

From the first member, the odd “basis vector” u_R^{c1} in Table A1, follow the rest odd “basis vectors” by the application of the infinitesimal generators of the Lorentz group S^{ab} , as well as by the application of ${}^I\hat{\mathcal{A}}_f^{m+}$. All the first members of the other families follow from the one presented in Table A1 by applying on u_R^{c1} by $\tilde{S}^{ab}$, as well as by the application of ${}^{II}\hat{\mathcal{A}}_f^{m+}$.

The corresponding creation and annihilation operators of fermions are tensor products of a “basis vector” and the basis in ordinary space-time, for example, $u_R^{c1}(\vec{p}) = u_R^{c1} * \hat{b}_{\vec{p}}$.

The even “basis vectors” can be obtained, according to Eqs. (19, 20), as the algebraic products of the odd “basis vectors” and their Hermitian conjugate partners; In a tensor product with the basis in ordinary space-time, and with the space index $a = n (= 0, 1, 2, 3)$ added, ${}^I\hat{\mathcal{A}}_{fa}^{m+}(\vec{p}) = {}^I\mathcal{C}_{fa}^m(\vec{p}) * {}^I\hat{\mathcal{A}}_f^{m+}$.

${}^I\hat{\mathcal{A}}_{fa}^{m+}(\vec{p})$ manifest the properties of the tensor ($a = n$), vector ($a = n$) and scalar ($a = s \geq 5$) gauge fields, observed so far.

In a tensor product with the basis in ordinary space-time, and with the space index $a = s \geq 5$ added, ${}^I\hat{\mathcal{A}}_{fa}^{m+}(\vec{p})$ manifest the properties of the scalar fields, like the Higgs and other scalar fields, bringing masses to quarks and leptons and antiquarks and antileptons and to weak bosons, for example.

Let us look in Table A1 for e_L^{-+} , 29th line. The photon ${}^I\hat{\mathcal{A}}_{ph e_L^{-+} \rightarrow e_L^{-+}}^+$ interacts with e_L^{-+} as follows

$$\begin{aligned} & {}^I\hat{\mathcal{A}}_{ph e_L^{-+} \rightarrow e_L^{-+}}^+ (\equiv [-i][+][-][+][+][+][+]) * {}^I\mathcal{C}_{ph e_L^{-+} \rightarrow e_L^{-+}}^+ (\equiv [-i]+(+)(+)(+)(+)) \rightarrow \\ & e_L^{-+} (\equiv [-i]+(+)(+)(+)(+)) f, \quad {}^I\hat{\mathcal{A}}_{ph e_L^{-+} \rightarrow e_L^{-+}}^+ = e_L^{-+} * (e_L^{-+})^+, \end{aligned} \quad (28)$$

Let us look for the weak boson, transforming e_L^{-+} from the 29th line into ν_L^+ from the 31st line. It follows

$$\begin{aligned} & {}^I\hat{\mathcal{A}}_{w1 e_L \rightarrow \nu_L}^+ (\equiv [-i]+(-)[+][+][+]) * {}^I\mathcal{C}_{w1 e_L \rightarrow \nu_L}^+ (\equiv [-i]+(+)(+)(+)(+)) \rightarrow \\ & \nu_L^+ (\equiv [-i][+][+][-](+)(+)(+)), \quad {}^I\hat{\mathcal{A}}_{w1 e_L \rightarrow \nu_L}^+ = \nu_L^+ * (e_L^{-+})^+. \end{aligned} \quad (29)$$

Knowing “basis vectors” of fermions, we can find “basis vectors” of all bosons fields. Only few of them are among the so far observed boson fields. ¹³

¹³ We discuss in Refs. [8,20], that breaks of symmetries make that not all possibilities of the predicted vector and scalar boson fields are observable at low energies. Also among the observed fermions and antifermions the predicted right-handed neutrinos and left-handed antineutrinos wait to be observed. Our model predicts more families than observed, like the

However, studying all the boson fields might help to recognise why and how the properties of fermions and bosons change with breaking symmetries, if this theory describing the internal spaces of fermion and boson fields with odd and even “basis vectors” is what our universe obeys.

Demonstrating so many simple and elegant descriptions of the second quantized fields, explaining the assumptions of other theories, makes us hop that the theory might be what the universe obeys.

Since the graviton in this theory is understood in an equivalent way as all the gauge fields observed so far, let us at the end of this section, try to analyse the “basis vectors” of the gravitons if the internal space has $d = (13 + 1)$.

We must take into account that the “gravitons” do have the spin and handedness (non-zero S^{03} and S^{12} , which means that this part must be presented by two nilpotents, $(\pm i)(\pm i)$ in $d = (3 + 1)$, and do not have weak, colour and $U(1)$ charges (what means that all the rest must be projectors), and have, as all the vector gauge fields, the space index $n = (0, 1, 2, 3)$.

We can then easily find the “basis vector” of the graviton, ${}^I\hat{\mathcal{A}}_{gr\,u_{R\uparrow}^{c1\uparrow}\rightarrow u_{R\downarrow}^{c1\uparrow}}^\dagger$, which applying on $u_{R\downarrow}^{c1\uparrow}$ with spin up, appearing in the first line of the table A1, transforms it into $u_{R\downarrow}^{c1\uparrow}$ with spin down, appearing in the second line of the table A1.

$${}^I\hat{\mathcal{A}}_{gr\,u_{R\uparrow}^{c1\uparrow}\rightarrow u_{R\downarrow}^{c1\uparrow}}^\dagger (\equiv (-i)(-)[+][+][+][-][-]) * {}^I\hat{\mathcal{A}}_{gr\,u_{R\uparrow}^{c1\uparrow}\rightarrow u_{R\downarrow}^{c1\uparrow}}^\dagger (\equiv (+i)[+]+(+)[-][-]) \rightarrow u_{R\downarrow}^{c1\uparrow}, (\equiv [-i](-)+(+)[-][-]), \quad {}^I\hat{\mathcal{A}}_{gr\,u_{R\uparrow}^{c1\uparrow}\rightarrow u_{R\downarrow}^{c1\uparrow}}^\dagger = u_{R\downarrow}^{c1\uparrow} * {}^I\hat{\mathcal{A}}_{gr\,u_{R\uparrow}^{c1\uparrow}\rightarrow u_{R\downarrow}^{c1\uparrow}}^\dagger. \quad (30)$$

Let us look at the “scattering” (that is the algebraic application, $*_A$) of the graviton with the “basis vector” ${}^I\hat{\mathcal{A}}_{gr\,u_{R\uparrow}^{c1\uparrow}\rightarrow u_{R\downarrow}^{c1\uparrow}}^\dagger$ with the graviton with the “basis vector”

${}^I\hat{\mathcal{A}}_{gr\,u_{R\downarrow}^{c1\uparrow}\rightarrow u_{R\uparrow}^{c1\uparrow}}^\dagger (\equiv (+i)(+)[+][+][+][-][-])$, that is

$${}^I\hat{\mathcal{A}}_{gr\,u_{R\uparrow}^{c1\uparrow}\rightarrow u_{R\downarrow}^{c1\uparrow}}^\dagger (\equiv (-i)(-)[+][+][+][-][-]) *_A {}^I\hat{\mathcal{A}}_{gr\,u_{R\downarrow}^{c1\uparrow}\rightarrow u_{R\uparrow}^{c1\uparrow}}^\dagger (\equiv (+i)(+)[+][+][+][-][-]) \rightarrow (\equiv [-i](-)[+][+][+][-][-]) = u_{R\downarrow}^{c1\uparrow} *_A (u_{R\downarrow}^{c1\uparrow})^\dagger = {}^I\hat{\mathcal{A}}_{ph\,u_{R\downarrow}^{c1\uparrow}\rightarrow u_{R\downarrow}^{c1\uparrow}}^\dagger,$$

to recognize how easily one finds the internal space of bosons.

The creation operators for gravitons must carry the space index $n = (0, 1, 2, 3)$, like: ${}^I\hat{\mathcal{A}}_{gr\,u_{R\uparrow}^{c1\uparrow}\rightarrow u_{R\downarrow}^{c1\uparrow}}^\dagger(\vec{p})$.

3. General Algebraic Structure of Fermion and Boson Second Quantised Fields

We demonstrated in Sect. 2 the relations among second quantised fermion and boson fields when treating internal spaces of fermions and bosons in a unique way: fermions as a superposition of odd products of operators γ^a , bosons as a superposition of even products of operators γ^a carrying in addition the ordinary space-time index a , named odd and even “basis vectors”, respectively. Although the corresponding states of the creation and annihilation operators have, in the case that the internal space has $d = (13 + 1)$ while in the ordinary space-time fermions and bosons are active only in $d = (3 + 1)$, the desired properties as assumed by the other theories, yet there are differences, which might lead to different predictions and conclusions.

Let us point out the main differences:

- a All “basis vectors” of fermions are mutual orthogonal and so are mutually orthogonal their Hermitian conjugate partners, and correspondingly their creation and annihilation operators.
- b Fermions appear in families, bosons appear in two orthogonal groups.

fourth family to the observed three, [11,12] and the dark matter [13,15] as the (almost stable) fifth family belonging to the upper four family.

c Bosons multiply fermions either from the left-hand side (when causing transformations within one family) or from the right-hand side (when causing transformations among families).

In this section, we generalize Sect. 2, by generalising structure of massless fermion and boson second quantised fields with the non-zero momentum only in $d = (3 + 1)$ of the ordinary space-time, presented in Refs. [17–22], and explained and discussed in Sect. 2 of this paper.

The decision that the internal spaces of the second quantised fermion and boson fields are described by a superposition of odd numbers of operators γ^a 's for fermions and even numbers of γ^a 's for bosons, bosons carry the ordinary space-time index a , determines all the properties of the fermion and boson fields, as well as their mutual interactions, which are realised by the algebraic multiplication.

This decision explains Dirac's postulates for the second quantised fermion and boson fields, determines the statistics of fermions and bosons, determines families of fermions, each of which includes fermions and antifermions¹⁴, determines the Lorentz (and correspondingly gauge) symmetry of fermion and boson fields, Subsect. 3.2, determines couplings among fermion and boson fields, Subsect. 3.3, and consequently the Lagrange densities, Subsects. (3.4, 3.6), determines two orthogonal kinds of boson fields, Subsects. (3.5, 3.6), determines orthogonality of fermion states, what all influences Feynman diagrams 3.7 (which should reproduce the experimental data).

3.1. Algebraic Structure

The theory offers fermion fields, they are tensor products of "basis vectors" and basis in ordinary space-time, Eq. (23),

$$\psi(x) \in Cl^{\text{odd}}(d-1, 1), \quad (31)$$

as Clifford odd $2^{\frac{d}{2}-1}$ members in $2^{\frac{d}{2}-1}$ irreducible representations, families, together with the same number of their Clifford odd Hermitian conjugate partners. They satisfy the relations of Eq. (6)¹⁵

$$\psi *_A \psi = 0, \quad \psi^\dagger *_A \psi^\dagger = 0, \quad \psi^\dagger *_A \psi \neq 0, \quad (32)$$

where $*_A$ denotes an associative algebraic product.

The theory also contains two orthogonal bosonic fields, which are tensor products of "basis vectors" and basis in ordinary space-time, Eq. (25), each with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members,

$$^I \hat{\mathcal{A}}_a(x), \quad ^{II} \hat{\mathcal{A}}_a(x),$$

with

$$^I \hat{\mathcal{A}}_a, ^{II} \hat{\mathcal{A}}_a \in Cl^{\text{even}}(d-1, 1),$$

a is the space-time index¹⁶. They satisfy the orthogonality condition, Eq. (12),

$$^I \hat{\mathcal{A}}_a *_A ^{II} \hat{\mathcal{A}}_b = ^{II} \hat{\mathcal{A}}_a *_A ^I \hat{\mathcal{A}}_b = 0. \quad (33)$$

The chosen algebraic structure leads to the relations of Eq. (13),

$$^i \hat{\mathcal{A}}_a *_A ^i \hat{\mathcal{A}}_a \rightarrow ^i \hat{\mathcal{A}}_a, \quad i = (I, II), \quad (34)$$

and of Eq. (16),

$$^I \hat{\mathcal{A}}_a *_A \psi \rightarrow \psi, \quad (35)$$

¹⁴ Consequently, the vacuum is not the negative-energy Dirac vacuum; It is just the quantum vacuum.

¹⁵ The oddness of "basis vectors" determines the orthogonality.

¹⁶ The space-time index a is in this theory equal to $n = (0, 1, 2, 3)$ for vectors and tensors, and $a \geq 5$ for scalars.

transforming fermion fields within an irreducible representation - family, and to Eq. (17),

$$\psi *_A {}^I\hat{\mathcal{A}}_a \rightarrow \psi, \quad (36)$$

transforming fermion fields among irreducible representations - families.

3.2. Internal Lorentz Symmetry

Local internal Lorentz transformations in the internal space of fermion and boson fields are generated by, Ref. [21],

$$\mathcal{S}^{ab} = (S^{ab} + \tilde{S}^{ab}).$$

The corresponding local transformation reads

$$\Lambda(x) = e^{i\omega_{ab}(x)\mathcal{S}^{ab}}, \quad \Lambda^\dagger(x) *_A \Lambda(x) = 1,$$

provided that $\omega_{0i}^* = -\omega_{0i}$ and $\omega_{ij}^* = \omega_{ij}$.

The fermion field transforms as

$$\psi \longrightarrow \Lambda *_A \psi.$$

Translations in ordinary space-time are generated by operators p_a . Rotations in both spaces, internal and ordinary space-time, are generated by $M^{ab} = (L^{ab} + S^{ab} + \tilde{S}^{ab})$ ¹⁷.

3.3. Covariant Derivative

The covariant derivative corresponding to the algebraic properties of the fermion and boson fields acting on ψ is defined by, Eqs. (16, 35, 17, 36),

$$D_a \psi = p_a \psi - {}^I\hat{\mathcal{A}}_a *_A \psi - \psi *_A {}^I\hat{\mathcal{A}}_a. \quad (37)$$

Under a local Lorentz transformation in internal space (defined by $(S^{ab} + \tilde{S}^{ab})$, concerning the internal and space-time Lorentz transformations one needs to take $M^{ab} = (L^{ab} + S^{ab} + \tilde{S}^{ab})$.)

$$p_a \longrightarrow \Lambda^\dagger *_A p_a *_A \Lambda = p_a + (p_a \Lambda^\dagger) *_A \Lambda.$$

Lorentz covariance of $D_a \psi$ requires

$$D_a *_A (\Lambda *_A \psi) \longrightarrow (\Lambda^\dagger *_A D_a *_A \Lambda) *_A \psi, \quad (38)$$

as we shall see.

3.4. Fermion Lagrangian and Local Lorentz Transformations in Internal Space

Following the properties of the fermion and boson fields, repeated above in Sects. (2, 3 up to this subsection) we are able to consider the fermionic Lagrangian density in a Dirac way, and with our covariant derivative D_a

$$\mathcal{L}_F(\psi) = \frac{1}{2} \left[\psi^\dagger *_A \gamma^0 \gamma^a *_A D_a \psi + (D_a \psi)^\dagger *_A \gamma^0 \gamma^a *_A \psi \right]. \quad (39)$$

The covariant derivative acting on the fermion operator ψ is

$$D_a *_A \psi = (p_a - {}^I\mathcal{A}_a) *_A \psi - \psi *_A {}^I\mathcal{A}_a. \quad (40)$$

¹⁷ L^{ab} gives non-zero contributions only in $d = 3 + 1$. Only in $d = 3 + 1$ are the rotations in internal space and in ordinary space-time related.

We consider a local Lorentz transformation in internal space

$$\psi \rightarrow \psi' = \Lambda \psi, \quad (41)$$

with

$$\Lambda(x) = e^{i\omega_{ab}(x)S^{ab}}.$$

Using this in the Lagrange density it follows¹⁸

$$\mathcal{L}_F(\Lambda\psi) = \frac{1}{2} \left[(\Lambda\psi)^\dagger *_A \gamma^0 \gamma^a *_A D_a *_A (\Lambda\psi) + (D_a *_A \Lambda\psi)^\dagger *_A \gamma^0 \gamma^a *_A \Lambda\psi \right],$$

which leads to

$$\mathcal{L}_F(\Lambda\psi) = \frac{1}{2} \left[(\psi)^\dagger \Lambda^\dagger *_A \gamma^0 \gamma^a *_A \Lambda \Lambda^\dagger *_A D_a *_A \Lambda\psi + (\Lambda \Lambda^\dagger *_A D_a *_A \Lambda\psi)^\dagger *_A \gamma^0 \gamma^a *_A \Lambda\psi \right].$$

Taking into account that $\Lambda^\dagger *_A \gamma^0 \gamma^a *_A \Lambda = \gamma'^0 \gamma'^a$ is the Lorentz transformed object, and $\Lambda^\dagger *_A D_a *_A \Lambda = D'_a$ is the Lorentz transformed object, while $(\Lambda *_A D'_a \psi)^\dagger = (D'_a *_A \psi)^\dagger \Lambda^\dagger$, the new, transformed, fermion Lagrange density is

$$\mathcal{L}_F(\Lambda\psi) = \frac{1}{2} \left[\psi^\dagger *_A \gamma'^0 \gamma'^a *_A D'_a *_A \psi + (D'_a \psi)^\dagger *_A \gamma'^0 \gamma'^a *_A \psi \right]. \quad (43)$$

Expanding the derivative term yields

$$p_a(\Lambda\psi) = (p_a \Lambda)\psi + \Lambda(p_a \psi).$$

The new covariant derivative is, if we call $-\Lambda^\dagger *_A {}^I \mathcal{A}_a *_A \Lambda + p_a \Lambda = -{}^I \mathcal{A}'_a$ and $-\Lambda^\dagger *_A {}^{II} \mathcal{A}_a *_A \Lambda = -{}^{II} \mathcal{A}'_a$.

$$\Lambda^\dagger D_a (\Lambda\psi) = D'_a \psi = \{-{}^I \mathcal{A}'_a\} *_A \psi - \psi *_A {}^{II} \mathcal{A}'_a. \quad (44)$$

This determines the transformation laws of the bosonic fields

$$\begin{aligned} {}^I \mathcal{A}'_a &= \Lambda^\dagger {}^I \mathcal{A}_a \Lambda + \Lambda^\dagger (p_a \Lambda), \\ {}^{II} \mathcal{A}'_a &= \Lambda^\dagger {}^{II} \mathcal{A}_a \Lambda. \end{aligned} \quad (45)$$

With these transformation rules one obtains

$$\gamma^0 \gamma^a D_a (\Lambda\psi) = \gamma'^0 \gamma'^a *_A D'_a *_A \psi, \quad (46)$$

and the fermion Lagrange density preserves its form under local Lorentz transformations as we promised in Subsect. 3.3.¹⁹

We could as well assume the quadratic form of the Lagrange density for fermions, again with the covariant derivative ($D_a \psi$ presented in Eq. (44))

$$\mathcal{L}_F = (D_a \psi)^\dagger *_A (D^a \psi). \quad (47)$$

¹⁸ Looking at transformations in the first order in the way

$$\begin{aligned} & \frac{1}{2} \{ (\gamma^0 \gamma^a p_a (1 + i\omega_{ij} S^{ij} + i\omega_{0i} S^{0i}) \psi')^\dagger (1 + i\omega_{ij} S^{ij} + i\omega_{0i} S^{0i}) \psi' + \\ & ((1 + i\omega_{ij} S^{ij} + i\omega_{0i} S^{0i}) \psi')^\dagger \gamma^0 \gamma^a p_a (1 + i\omega_{ij} S^{ij} + i\omega_{0i} S^{0i}) \psi' \}, \\ & = \frac{1}{2} \{ (p_a \psi')^\dagger \gamma^0 \gamma^a \psi' + (\psi')^\dagger \gamma^0 \gamma^a p_a \psi' \}, \end{aligned} \quad (48)$$

after taking into account that $\omega_{ij}^* = \omega_{ij}$, while $\omega_{0i}^* = -\omega_{0i}$, and that $(S^{ij})^\dagger = S^{ij}$, $(S^{0i})^\dagger = -S^{0i}$.

¹⁹ The gauge transformations of boson fields coincide with (some of) the Lorentz transformations concerning only the internal space with $d > 3$ if the boson fields carry the space index $n = (0, 1, 2, 3)$.

This Lagrange density, obviously Hermitian, transforming under Lorentz rotations as required by Eqs. (44, 45), is waiting to be studied.

3.4.1. Equation of Motion for Fermion Fields

Varying the fermion Lagrangian density with respect to the Hermitian conjugate field $\psi^\dagger$ gives

$$\delta \mathcal{L}_F = \frac{1}{2} \left[\delta \psi^\dagger *_A \gamma^0 \gamma^a *_A D_a \psi + (D_a \delta \psi)^\dagger *_A \gamma^0 \gamma^a *_A \psi \right].$$

After integration by parts one obtains the equation of motion

$$\gamma^0 \gamma^a *_A D_a \psi = 0. \quad (48)$$

Using the definition of the covariant derivative

$$D_a \psi = p_a \psi - {}^I \hat{\mathcal{A}}_a *_A \psi - \psi *_A {}^{II} \hat{\mathcal{A}}_a,$$

the explicit form of the fermion equation becomes

$$\gamma^0 \gamma^a \left(p_a \psi - {}^I \hat{\mathcal{A}}_a *_A \psi - \psi *_A {}^{II} \hat{\mathcal{A}}_a \right) = 0. \quad (49)$$

We see that the theory, assuming that internal spaces of fermion and boson fields are described by the odd number of nilpotents (fermions) and the even number of nilpotents (bosons), the rest of projectors, fermions can interact only by exchanging bosons, either with left or right multiplication.

3.5. Algebraic Structure of Boson Fields

Boson fields in the present approach carry a space-time index

$$a = 0, 1, 2, 3, 5, 6, \dots, d,$$

20

For $a = 0, 1, 2, 3$ the fields

$$\mathcal{A}_a(x)$$

transform as vectors in space-time. For $a \geq 5$ the corresponding fields appear as scalars from the $(3+1)$ -dimensional point of view. Boson fields therefore appear either as space-time vectors or as space-time scalars.

The internal structure of bosons can be expressed in terms of fermion operators ψ and $\psi^\dagger$, Eqs. (19, 20). The algebra satisfies the nilpotent relations, Eq. (6),

$$\psi *_A \psi = 0, \quad \psi^\dagger *_A \psi^\dagger = 0,$$

while mixed products do not vanish. The internal structure of bosonic operators can be expressed through the bilinears, Eqs. (19, 20),

$$\psi *_A \psi^\dagger, \quad \psi^\dagger *_A \psi.$$

The algebra distinguishes two types of bosons, Eqs. (19, 20),

$${}^I \mathcal{A}_a \sim \psi *_A \psi^\dagger, \quad {}^{II} \mathcal{A}_a \sim \psi^\dagger *_A \psi.$$

²⁰ Here the indices $(0, 1, 2, 3)$ correspond to the usual $(3+1)$ -dimensional space-time when representing vectors and tensors (only in this space-time bosons, as well as fermions, are active), while $a \geq 5$ determine scalars. The internal space is active in all $d = 2(2n+1)$.

Both transform under Lorentz transformation as presented in Eq. (45).²¹

Because of the nilpotent relations - their mutual algebraic products are zero - these two sectors do not interact directly. They can interact only in the presence of a fermion. One finds schematically²²

$${}^I\mathcal{A}_a * {}_A\psi = (\psi * {}_A\psi^\dagger) * {}_A\psi = \psi * {}_A(\psi^\dagger * {}_A\psi) \sim \psi * {}_A{}^II\mathcal{A}_a. \quad (51)$$

Thus bosons of type I and type II are connected only through fermions, what must be recognized in Feynman diagrams, which must, of course, reproduce the experimental data.

When ${}^I\mathcal{A}_a * {}_A\psi \rightarrow \psi$, it is indeed ${}^I\mathcal{A}_a * {}_A\psi \rightarrow \psi * {}_A{}^II\mathcal{A}_a$. In this case ${}^II\mathcal{A}_a$ can either remain without energy and momentum, taken by ψ , or both - ${}^II\mathcal{A}_a$ and ψ - share energy and momentum²³.

In general a boson field can be written as an expansion over Clifford algebra elements, Eq. (25),

$$\mathcal{A}_a(x) = \sum_k A_a^{(k)}(x) \Gamma_k, \quad (52)$$

where Γ_k denote elements of the Clifford algebra built from nilpotents and projectors. A boson field is a tensor product, $*_T$, of a field in the ordinary space-time in the coordinate representation and of one of the two kinds of Clifford even "basis vectors": $\mathcal{C}^m_{fa}(x) *_T \hat{\mathcal{A}}_f^{m\dagger}$, Eq. (25). Here Γ_k replaces $\hat{\mathcal{A}}_f^{m\dagger}$ and $A_a^{(k)}(x)$ replaces $\mathcal{C}^m_{fa}(x)$.

The space-time index a determines whether the field behaves as a vector or scalar in space-time, while the Clifford algebra structure determines the physical properties such as spin and charges.

3.6. Bosonic Lagrange Density

Since the algebra distinguishes two kinds of non interacting bosons,

$${}^I\mathcal{A}_a, \quad {}^II\mathcal{A}_a,$$

the bosonic Lagrange density separates naturally into two parts,

$$\mathcal{L}_B = \mathcal{L}_B^I + \mathcal{L}_B^{II}.$$

²¹ Let us see what Eq. (45) does say about the transformation of bosons ${}^II\mathcal{A}_a \sim \psi^\dagger * {}_A\psi$ and ${}^I\mathcal{A}_a \sim \psi * {}_A\psi^\dagger$

$$\begin{aligned} {}^I\mathcal{A}_a &= \psi * {}_A\psi^\dagger, & {}^II\mathcal{A}_a &= \psi^\dagger * {}_A\psi, \\ {}^I\mathcal{A}'_a &= (\Lambda^\dagger \psi) * {}_A(\Lambda^\dagger \psi)^\dagger = \Lambda^\dagger \psi * {}_A\psi^\dagger \Lambda = {}^I\mathcal{A}'_a \\ {}^II\mathcal{A}'_a &= (\Lambda^\dagger \psi)^\dagger * {}_A(\Lambda^\dagger \psi) = \psi^\dagger \Lambda * {}_A\Lambda^\dagger \psi = {}^II\mathcal{A}'_a \\ &\text{leading to} \\ {}^I\mathcal{A}'_a &= \Lambda^\dagger \psi * {}_A\psi^\dagger \Lambda, & {}^II\mathcal{A}'_a &= \psi^\dagger * {}_A\psi. \end{aligned} \quad (50)$$

While ${}^I\mathcal{A}_a$ transform as we use to see transformation properties of vector gauge fields, the second kind does not transform at all.

²² Let us demonstrate the relation in Eq. (56) on the case that $d = (5 + 1)$, with the fermion and boson "basis vectors" presented in Table A2. Let the "basis vector" of ${}^I\mathcal{A}_a$ be equal to ${}^I\hat{\mathcal{A}}_4^{1\dagger}(\equiv (+i)(+)[+]) = \hat{b}_1^{1\dagger}(\equiv (+i)[+][+]) * {}_A(\hat{b}_1^{2\dagger})^\dagger(\equiv ([-i](-)[+])^\dagger)$, and let it applies on a fermion ψ with the "basis vector" equal to $\hat{b}_1^{2\dagger}(\equiv [-i](-)[+])$. ψ with the "basis vector" $\hat{b}_1^{2\dagger}$ scatters to $\hat{b}_1^{1\dagger}$ and ${}^II\mathcal{A}_a$ with the "basis vector" equal to ${}^II\hat{\mathcal{A}}_3^{1\dagger}(\equiv [-i][+][+])$. ψ and ${}^II\mathcal{A}_a$ either share the momentum and energy, or ${}^II\mathcal{A}_a$ remains momentumless in the vacuum, obeying Eq. (16).

²³ When applying either ${}^I\mathcal{A}_a(x)$ on ψ or ψ on ${}^II\mathcal{A}_a(x)$, we always get two fields back. In the first case, when ${}^I\mathcal{A}_a(x)$ apply on ψ we get (another) ψ and ${}^II\mathcal{A}_a(x)$. The boson field may stay in the vacuum without energy, or appear as well as a propagating field. In the second case when ψ applies on ${}^II\mathcal{A}_a(x)$ we get $\psi * {}_A(\psi^\dagger * {}_A\psi) = (\psi * {}_A\psi^\dagger) * {}_A\psi$, that is ${}^I\mathcal{A}_a(x)$ and another ψ . The boson fields may stay in the vacuum without energy, or appear as well as a propagating field. This will be very important when looking at the Feynman diagrams.

Taking into account Eqs. (13, 37), and requiring that each kind contains the kinetic term for the corresponding boson fields, and the self interacting term following Eqs. (13, 37), we end up with the field strengths

$$\begin{aligned} {}^I F_{ab} &= -i(p_a {}^I \mathcal{A}_b - p_b {}^I \mathcal{A}_a - \{ {}^I \mathcal{A}_a, {}^I \mathcal{A}_b \}_-), \\ {}^{II} F_{ab} &= -i(p_a {}^{II} \mathcal{A}_b - p_b {}^{II} \mathcal{A}_a - \{ {}^{II} \mathcal{A}_a, {}^{II} \mathcal{A}_b \}_-), \end{aligned} \quad (53)$$

and correspondingly with the bosonic Lagrange density

$$\mathcal{L}_B = \frac{1}{4} {}^I F_{ab} {}^I F^{ab} + \frac{1}{4} {}^{II} F_{ab} {}^{II} F^{ab}. \quad (54)$$

Because the two bosonic sectors correspond to different Clifford structures, orthogonal to each other, they do not interact in the bosonic part of the Lagrangian.

The interaction between the two sectors occurs only through fermions. The fermion interaction term can schematically be, taken from Eq. (39) into account, written as

$$\begin{aligned} \mathcal{L}_{int} &= \frac{1}{2} \{ (\psi^\dagger \gamma^0 \gamma^a {}^I \mathcal{A}_a \psi + \psi^\dagger \gamma^0 \gamma^a \psi {}^{II} \mathcal{A}_a + \\ &\quad [{}^I \mathcal{A}_a \psi + \psi {}^{II} \mathcal{A}_a]^\dagger \gamma^0 \gamma^a \psi \}. \end{aligned} \quad (55)$$

In this way bosons of type I and type II meet only in the presence of fermionic fields.

3.7. Feynman Rules and Interaction Structure

The algebraic structure discussed above have important consequences for the interaction vertices and the corresponding Feynman rules.

The proposed description of the internal space of bosonic operators with the even number of nilpotents, leads to the relations among fermion operators and boson operators

$$\psi *_A \psi^\dagger, \quad \psi^\dagger *_A \psi,$$

describing two kinds of bosons,

$${}^I \mathcal{A}_a, \quad {}^{II} \mathcal{A}_a.$$

Because of the nilpotent relations

$$\psi *_A \psi = 0, \quad \psi^\dagger *_A \psi^\dagger = 0,$$

the two kinds of bosons do not interact directly. They can interact only through fermionic operators. Schematically one finds relations of the form

$${}^I \mathcal{A}_a *_A \psi = (\psi *_A \psi^\dagger) *_A \psi = \psi *_A (\psi^\dagger *_A \psi) \sim \psi *_A {}^{II} \mathcal{A}_a.$$

The above implies that bosons of type I and type II can be connected only through fermion lines. Correspondingly, it is the fermionic Lagrange density which is responsible for the interaction among fermions and bosons.

$$\mathcal{L}_{int} = \psi^\dagger *_A \gamma^0 \gamma^a ({}^I \mathcal{A}_a *_A \psi + \psi *_A {}^{II} \mathcal{A}_a).$$

4. Conclusion

The proposed theory, built on the assumption that the internal spaces of fermion and boson fields are describable by odd (for fermions) and even (for bosons) products of operators γ^a , offers the unique description of spins and charges of fermion and boson second quantised fields, as we can noticed in Tables (A1 for the observed quarks and leptons, A2 for a toy model), as well as in the covariant derivatives of the Lagrange densities for fermion and boson fields in interaction, Eqs. (39, 40, 54, 55).

The theory offers the explanation for the appearance of all the so far observed fermions (quarks and leptons and antiquarks and antileptons; as well as for their families) and bosons ($SO(3,1)$ gravitons [18, 20]²⁴, two $SU(2)$ weak fields [8,18,20], $SU(3)$ gluons [8,18,20], $U(1)$ photons and new predicted fermions (right-handed neutrinos and left-handed antineutrinos ([8] and references therein), the fourth family to the observed three [11,15,31], a new group of four families of quarks and leptons, the stable one at low energies explaining the Dark matter [12]).

Both fields, fermions and bosons, are assumed to be massless and appear in a flat space-time.

Without breaking symmetries, there would also exist boson fields carrying more than one charge at the same time, like the weak and colour charge, or the spin, weak charge and colour charge, which we have not (yet) observed.

There is the break of symmetries which make fermion fields and some of the boson fields massive, as studied in Ref. [8]²⁵.

The Lorentz transformations in internal space, presented in Subsect. 3.2, manifest the gauge transformations of other theories when we treat the internal space with $d \geq 5$. Our description of the internal spaces of fermion and boson fields requires that all bosons are treated equivalently, with gravitons (with the space index $a = (0, 1, 2, 3)$) and scalars included (scalars have the space index $a \geq 5$ while they can have all the internal space properties as vectors and tensors).

The choice that the internal spaces of fermions and bosons are described by superposition of odd and even products of operators γ^a , respectively, explains the Dirac's postulates of the second quantized fermion and boson fields.

We assume the Dirac-like action for fermion fields, but with the covariant derivative required by our description of the internal spaces of fermion and boson fields, presented in Eqs. (39, 40). The fermion Lagrangian density, preserving its form under the local Lorentz transformations as shown in Subsect. 3.4, explains the gauge invariance of the usual theories if our internal space concerns $d \geq 5$ ²⁶. There is no negative energy Dirac sea for fermions. Fermions have only ordinary quantum vacuum. Fermions and anti-fermions appear in the same family. Fermions and bosons appear in the quantum vacuum.

The quadratic Lagrange density for fermion fields, preserving its form under local Lorentz transformations, presented in Eq. (47), remains to be studied.

Also the boson Lagrangian density preserves its form under the local Lorentz transformations, as shown in Eq. (50).

The proposed description of the internal spaces of fermion and boson fields requires two kinds of mutually non-interacting boson fields, Subsect. 3.5, ${}^I\mathcal{A}_a$ and ${}^{II}\mathcal{A}_a$, ${}^I\mathcal{A}_a * {}^{II}\mathcal{A}_a = 0$. Their Lorentz transformation properties are presented in Eqs. (45, 50).

The Lagrange density for bosons, generated as in usual theories, is correspondingly, the sum of two independent Lagrange densities, presented in Subsect. 3.6, Eqs. (53, 54), for ${}^I\mathcal{A}_a$ and ${}^{II}\mathcal{A}_a$, which interact only through fermions.

The boson fields either manifest the left multiplication to fermions, ${}^I\mathcal{A}_a$, or the right one, ${}^{II}\mathcal{A}_a$, Eq. 40.

Their mutual interaction term in the Lagrange densities, presented in Eq. (55), are determined by our description of internal spaces of fermion and boson fields, requiring two kinds of bosons.

We arrange in any $d = 2(2n + 1)$ dimensional internal space, the fermion and boson states to be eigenvectors of all the members of the Cartan subalgebra, Eq. (3), we call these eigenstates the "basis vectors". The "basis vectors" for fermion fields have an odd number of nilpotents, and for the boson fields, an even number of nilpotents, the rest are projectors, Eq.(4).

²⁴ Let us point out that the graviton in this theory is explained in an equivalent way as all the gauge fields observed so far.

²⁵ The breaks of symmetries were studied when the boson fields were described by $S^{ab}\omega_{ab\alpha}$ and $\tilde{S}^{ab}\tilde{\omega}_{ab\alpha}$, ([8], Subsect. 6.2 and references therein), instead of by ${}^I\hat{\mathcal{A}}_f^{m\dagger}$ and ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$.

²⁶ There are gravitons with two nilpotents in $SO(3,1)$, the rest of projectors, Eq. (30), suggesting that gravitons should be treated according to the usual gauge fields.

The fermion “basis vectors” appear in $2^{\frac{d}{2}-1}$ families, each family having $2^{\frac{d}{2}-1}$ members; and there are $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ of their Hermitian conjugate partners, appearing in a separate group.

The boson “basis vectors” appear in two orthogonal groups, each with $2^{\frac{d}{2}-1} \times 2^{\frac{d}{2}-1}$ members and have their Hermitian conjugate partners within the same group.

The “basis vectors” for bosons are expressible as the algebraic products of fermion “basis vectors” and their Hermitian conjugate partners, Eqs. (19, 20).

The second quantised fermion fields are tensor products of the “basis vectors” and basis in ordinary space time, Eq. (23).

The second quantised boson fields are tensor products of the “basis vectors” and basis in ordinary space time, and carry the space-time index, Eq. (25).

The properties of fermion and boson fields, presented in this contributions, like:
 . the algebraic product of two fermions is zero

$$\psi *_A \psi = 0, \quad \psi^\dagger *_A \psi^\dagger = 0,$$

. the algebra distinguishes two types of bosons, Eqs. (19, 20),

$$^I \mathcal{A}_a \sim \psi *_A \psi^\dagger, \quad ^{II} \mathcal{A}_a \sim \psi^\dagger *_A \psi,$$

. bosons can interact only in the presence of a fermion

$$^I \mathcal{A}_a *_A \psi = (\psi *_A \psi^\dagger) *_A \psi = \psi *_A (\psi^\dagger *_A \psi) \sim \psi *_A ^{II} \mathcal{A}_a,$$

influence the Feynman diagrams. The Feynman rules are studied in Ref. [22].

Demonstrating so many simple and elegant descriptions of the second quantised fields, explaining the assumptions of other theories, makes us hope that the theory might be what the universe obeys.

4.1. What Should We Understand

If our proposed description of the internal degrees of freedom of the second quantised fermion and boson fields is what nature uses in the case that the space-time is flat, and all the second quantised fields are massless, while all the fields are active only in $d = (3 + 1)$ in ordinary space-time, we are able not only to explain the *standard model* assumptions before the electroweak break, but also find the solution for all the open questions connected with elementary fermion and boson fields.

There are two kinds of boson fields in this theory. One kind describes the observed gauge fields and the graviton field on an equal level, unifying all the boson fields. This kind, causing transformations within members of each family of fermions and antifermions, predicts the existence of the second kind of the weak force and correspondingly, the existence of the right-handed neutrino and the left-handed anti-neutrino, as seen in Table A1.

The second kind of boson fields transforms a family member of one family to the same family member of all the families, offering the explanation for the appearance of the Higgs scalars and predicting new scalar fields [13].

The internal spaces of all the observed boson fields are describable with “basis vectors” having only two nilpotents (gravitons in $SO(3, 1)$ part of $SO(13, 1)$, two kinds of weak bosons in $SO(4)$ part of $SO(13, 1)$, gluons in $SU(3)$ part of $SO(6)$, part of $SO(13, 1)$), and the rest of projectors, photons with only projectors in all $SO(13, 1)$.

There are, however, many more boson fields. Their “basis vectors” have more than two nilpotents (they can have four or six nilpotents), carrying correspondingly, more than one kind of the so-far-observed charges.

There are many more families in this theory than the observed three. The theory predicts that the three observed families are the members of the group of four families [31]. The theory predicts also the second group of four families, contributing to the dark matter [12,16].

To be able to explain why “nature has decided” to break symmetries, we should know the properties this theory has with respect to:

- The renormalisability and anomalies in even and odd dimensional spaces.
- How does the second kind of bosons contribute to the breaking of symmetries, while the first kind of the boson “basis vectors” seems to mainly determine the properties of all the observed boson fields, with the gravity included.
- The differences in odd, $d = (2n + 1)$, and even, $d = (2(2n + 1))$, dimensional spaces. While in even dimensional spaces, $d = 2(2n + 1)$, the odd “basis vectors” anticommute and have their Hermitian conjugated “basis vectors” in a separate group, and the even “basis vectors” commute and appear in two orthogonal groups, have the “basis vectors” in $d = 2(2n + 1) + 1$ strange properties; half of the odd and even “basis vectors” behave like in $d = 2(2n + 1)$, in the second half, the anticommuting odd “basis vectors” appear in two orthogonal groups, while the commuting even “basis vectors” appear in families and have the Hermitian conjugate partners in a separate group [19].
- The differences in even dimensional internal spaces, when $d = 2(2n + 1)$ and $d = 4n$. While in $d = 2(2n + 1)$ the “basis vectors” for fermions and antifermions appear in the same family, in $d = 4n$ the “basis vectors” of a family do not include antifermions. Correspondingly, the vacuum in $d = 2(2n + 1)$ is just the quantum vacuum, while in $d = 4n$ the Dirac sea with the negative energies must be invented.
- How to present and interpret the Feynman diagrams in this theory in comparison with the Feynman diagrams so far presented and interpreted [22].
- It might be needed to extend the second quantised fermion and boson fields to strings, with the first step already done in Ref. [23].

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Appendix A. One Family Representation of Odd “Basis Vectors” in $d = (13 + 1)$

This appendix, is following similar appendices in Refs. [8,18,19].

One irreducible representation, one family, of the odd “basis vectors” describing the internal spaces of fermions in $d = (13 + 1)$, analysed with respect to the subgroups $SO(3,1) \times SU(2) \times SU(2) \times SU(3) \times U(1)$, is presented. One family contains the “basis vectors” of quarks and leptons and antiquarks and antileptons, with the quantum numbers assumed by the *standard model* before the electroweak break, with the right-handed neutrinos and left-handed antineutrinos included. There are two $SU(2)$ subgroups, $SU(2)_I$ and $SU(2)_{II}$, of $SO(4)$ with the hypercharge of the *standard model* $Y = \tau^{23} + \tau^4$, Eqs. (A70 - A72). The reader can see that the “basis vectors” of quarks distinguish from the “basis vectors” of leptons, and the “basis vectors” of antiquarks distinguish from the “basis vectors” of antileptons only in the part $SU(3) \times U(1)$ of $SO(13,1)$.

The generators S^{ab} of the Lorentz transformations in the internal space of fermions with $d = (13 + 1)$, analysed with respect to the subgroups $SO(3,1) \times SU(2) \times SU(2) \times SU(3) \times U(1)$, are presented as

$$\vec{N}_{\pm} (= \vec{N}_{(L,R)}) := \frac{1}{2}(S^{23} \pm iS^{01}, S^{31} \pm iS^{02}, S^{12} \pm iS^{03}), \quad (A56)$$

$$\vec{\tau}^1 := \frac{1}{2}(S^{58} - S^{67}, S^{57} + S^{68}, S^{56} - S^{78}), \quad \vec{\tau}^2 := \frac{1}{2}(S^{58} + S^{67}, S^{57} - S^{68}, S^{56} + S^{78}), \quad (A57)$$

$$\begin{aligned}\bar{\tau}^3 &:= \frac{1}{2} \{ S^{9\,12} - S^{10\,11}, S^{9\,11} + S^{10\,12}, S^{9\,10} - S^{11\,12}, S^{9\,14} - S^{10\,13}, \\ &\quad S^{9\,13} + S^{10\,14}, S^{11\,14} - S^{12\,13}, S^{11\,13} + S^{12\,14}, \frac{1}{\sqrt{3}}(S^{9\,10} + S^{11\,12} - 2S^{13\,14}) \}, \\ \tau^4 &:= -\frac{1}{3}(S^{9\,10} + S^{11\,12} + S^{13\,14}),\end{aligned}\tag{A58}$$

$$Y := \tau^4 + \tau^{23}, \quad Q := \tau^{13} + Y,\tag{A59}$$

The (chosen) Cartan subalgebra operators, determining the commuting operators in the above equations, is presented in Eq. (3).

The corresponding relations for $\tilde{S}^{ab}$, determining the family quantum numbers, follow if we replace in above equations S^{ab} by $\tilde{S}^{ab}$.

The hypercharge Y and the electromagnetic charge Q relate to the *standard model* quantum numbers.

For fermions, the operator of handedness Γ^d is determined as follows:

$$\Gamma^{(d)} = \prod_a (\sqrt{\eta^{aa}} \gamma^a) \cdot \begin{cases} (i)^{\frac{d}{2}}, & \text{for } d \text{ even}, \\ (i)^{\frac{d-1}{2}}, & \text{for } d \text{ odd}. \end{cases}\tag{A60}$$

All the families (all the irreducible representations) follow from this one by applying, let say, on the first member, u_R^c , all possible $\tilde{S}^{ab}$, Eq. (A68). Let us start with $\tilde{S}^{01}$ which transforms $u_{R,f=1}^c (\equiv (+i) \begin{smallmatrix} 03 & 12 \\ + & + \end{smallmatrix})$ to $u_{R,f=2}^c (\equiv [+i] \begin{smallmatrix} 03 & 12 \\ + & + \end{smallmatrix})$ or $u_{R,f=3}^c (\equiv [+i] \begin{smallmatrix} 03 & 12 \\ + & + \end{smallmatrix})$. From the first family member of the second family all the members of the second family follow by the application of S^{ab} . There are obviously, the same number of families as there is the number of the family members.

The even “basis vectors”, analysed with respect to the same subgroups, $(SO(3,1) \times SU(2) \times SU(2) \times SU(3) \times U(1))$ of the $SO(13,1)$ group, offer the description of the internal spaces of the corresponding tensor, vector and scalar gauge fields, appearing in the *standard model* before the electroweak break [24,28,30]; as explained in Sect. 2.3.1. There are breaks of symmetries which make the very limited number of families observed at observable energies.

The even “basis vectors” are expressible as products of the odd “basis vectors” and their Hermitian conjugate partners, as presented in Eqs. (19, 20).

Table A1. The left-handed ($\Gamma^{(13,1)} = -1$, Eq. (A74)) irreducible representation representing the “basis vectors” of one family of spinors — the product of the odd number of nilpotents and of projectors, the eigenvectors of all the Cartan subalgebra members of the Lorentz algebra of the group $SO(13,1)$ [5,13], manifesting the colour charged quarks and antiquarks and the colourless leptons and antileptons — is presented. It contains the left-handed ($\Gamma^{(3,1)} = -1$) weak $(SU(2)_I)$ charged ($\tau^{13} = \pm \frac{1}{2}$), and $SU(2)_{II}$ chargeless ($\tau^{23} = 0$) quarks and leptons, and the right-handed ($\Gamma^{(3,1)} = 1$) weak $(SU(2)_I)$ chargeless and $SU(2)_{II}$ charged ($\tau^{23} = \pm \frac{1}{2}$) quarks and leptons, both with the spin S^{12} up and down ($\pm \frac{1}{2}$, respectively). Quarks distinguish from leptons only in the $SU(3) \times U(1)$ part: Quarks are triplets of three colours ($c^i = (\tau^{33}, \tau^{38}) = [(\frac{1}{2}, \frac{1}{2\sqrt{3}}), (-\frac{1}{2}, \frac{1}{2\sqrt{3}}), (0, -\frac{1}{\sqrt{3}})]$), carrying the “fermion charge” ($\tau^4 = \frac{1}{6}$). The colourless leptons carry the “fermion charge” ($\tau^4 = -\frac{1}{2}$). The same multiplet - family - contains also the left-handed weak $(SU(2)_I)$ chargeless and $SU(2)_{II}$ charged antiquarks and antileptons and the right-handed weak $(SU(2)_I)$ charged and $SU(2)_{II}$ chargeless antiquarks and antileptons. Antiquarks distinguish from antileptons again only in the $SU(3) \times U(1)$ part: Antiquarks are anti-triplets carrying the “fermion charge” ($\tau^4 = -\frac{1}{6}$). The anti-colourless antileptons carry the “fermion charge” ($\tau^4 = \frac{1}{2}$). $Y = (\tau^{23} + \tau^4)$ is the hyper charge, the electromagnetic charge is $Q = (\tau^{13} + Y)$. One can calculate, taking into account Eq. (A68), also the family quantum numbers of the presented family: $\tilde{S}^{03} = \frac{i}{2}, \tilde{S}^{12} = -\frac{1}{2}, \tilde{S}^{56} = -\frac{1}{2}, \tilde{S}^{78} = \frac{1}{2}, \tilde{S}^{9\,10} = \frac{1}{2}, \tilde{S}^{11\,12} = \frac{1}{2}, \tilde{S}^{13\,14} = \frac{1}{2}$.

i		${}^a\psi_i >$	$\Gamma^{(3,1)}$	S^{12}	τ^{13}	τ^{23}	τ^{33}	τ^{38}	τ^4	Y	Q
		(Anti)octet, $\Gamma^{(7,1)} = (-1)1, \Gamma^{(6)} = (1) -1$ of (anti)quarks and (anti)leptons									
1	u_R^c	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9\,10 & 11\,12 & 13\,14 \\ (+i) & (+) & (+) & (+) & (+) & (-) & (-) \end{smallmatrix}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$

Table A1. *Cont.*

i		$ ^a\psi_i\rangle$ (Anti)octet, $\Gamma(7,1) = (-1)1, \Gamma(6) = (1) - 1$ of (anti)quarks and (anti)leptons	$\Gamma(3,1)$	s_{12}	τ^{13}	τ^{23}	τ^{33}	τ^{38}	τ^4	Y	Q
2	$u^c_1 R$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-] & (-) & & (+) & & (+) & & (-) & & (-) \end{smallmatrix}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
3	$d^c_1 R$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+) & (+) & & (-) & & (-) & & (+) & & (-) \end{smallmatrix}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
4	$d^c_1 R$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-] & (-) & & (-) & & (-) & & (+) & & (-) \end{smallmatrix}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
5	$d^c_1 L$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-] & (+) & & (+) & & (+) & & (-) & & (-) \end{smallmatrix}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
6	$d^c_1 L$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ - & (+) & (-) & & (-) & (+) & & (+) & & (-) \end{smallmatrix}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
7	$u^c_1 L$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ - & [-] & & & & (+) & & (-) & & (-) \end{smallmatrix}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$
8	$u^c_1 L$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+) & (-) & & (+) & & (-) & & (-) & & (-) \end{smallmatrix}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
9	$u^c_2 R$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+) & (+) & & (+) & & (+) & & (-) & & (-) \end{smallmatrix}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
10	$u^c_2 R$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-] & (-) & & (+) & & (+) & & (-) & & (-) \end{smallmatrix}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
11	$d^c_2 R$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+) & (+) & & (-) & & (-) & & (+) & & (-) \end{smallmatrix}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
12	$d^c_2 R$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-] & (-) & & (-) & & (-) & & (+) & & (-) \end{smallmatrix}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
13	$d^c_2 L$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-] & (+) & & (-) & & (-) & & (-) & & (-) \end{smallmatrix}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
14	$d^c_2 L$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ - & (+) & (-) & & (-) & (+) & & (-) & & (+) \end{smallmatrix}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
15	$u^c_2 L$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ - & [-] & & & & (+) & & (-) & & (+) \end{smallmatrix}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
16	$u^c_2 L$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+) & (-) & & (+) & & (-) & & (-) & & (-) \end{smallmatrix}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
17	$u^c_3 R$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+) & (+) & & (+) & & (+) & & (-) & & (-) \end{smallmatrix}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
18	$u^c_3 R$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-] & (-) & & (+) & & (+) & & (-) & & (-) \end{smallmatrix}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{2}{3}$
19	$d^c_3 R$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+) & (+) & & (-) & & (-) & & (-) & & (+) \end{smallmatrix}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
20	$d^c_3 R$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-] & (-) & & (-) & & (-) & & (-) & & (+) \end{smallmatrix}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
21	$d^c_3 L$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-] & (+) & & (-) & & (-) & & (-) & & (+) \end{smallmatrix}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
22	$d^c_3 L$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ - & (+) & (-) & & (-) & (+) & & (-) & & (-) \end{smallmatrix}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{3}$
23	$u^c_3 L$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ - & [-] & & & & (+) & & (-) & & (-) \end{smallmatrix}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
24	$u^c_3 L$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+) & (-) & & (+) & & (-) & & (-) & & (-) \end{smallmatrix}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
25	ν_R	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+) & (+) & & (+) & & (+) & & (+) & & (+) \end{smallmatrix}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
26	ν_R	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-] & (-) & & (+) & & (+) & & (+) & & (+) \end{smallmatrix}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	0
27	e_R	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+) & (+) & & (-) & & (-) & & (+) & & (+) \end{smallmatrix}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	-1	-1
28	e_R	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-] & (-) & & (-) & & (-) & & (+) & & (+) \end{smallmatrix}$	1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	-1	-1
29	e_L	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-] & (+) & & (-) & & (-) & & (+) & & (+) \end{smallmatrix}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
30	e_L	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ - & (+) & (-) & & (-) & (+) & & (+) & & (+) \end{smallmatrix}$	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
31	ν_L	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ - & [-] & & & & (+) & & (-) & & (+) \end{smallmatrix}$	-1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
32	ν_L	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+) & (-) & & (+) & & (-) & & (+) & & (+) \end{smallmatrix}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0
33	$\tilde{d}^c_1 L$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-] & (+) & & (+) & & (+) & & (-) & & (+) \end{smallmatrix}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
34	$\tilde{d}^c_1 L$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+) & (-) & & (+) & & (+) & & (-) & & (+) \end{smallmatrix}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
35	$\tilde{u}^c_1 L$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ - & [-] & & & & (-) & & (-) & & (+) \end{smallmatrix}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
36	$\tilde{u}^c_1 L$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ - & (+) & (-) & & (-) & (-) & & (-) & & (+) \end{smallmatrix}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
37	$\tilde{d}^c_1 R$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+) & (+) & & & & (-) & & (-) & & (+) \end{smallmatrix}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
38	$\tilde{d}^c_1 R$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ - & [-] & (-) & & (+) & & (-) & & (-) & (+) \end{smallmatrix}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
39	$\tilde{u}^c_1 R$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+) & (+) & & (-) & & (-) & & (+) & & (+) \end{smallmatrix}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
40	$\tilde{u}^c_1 R$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-] & (-) & & (-) & & (-) & & (-) & & (+) \end{smallmatrix}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
41	$\tilde{d}^c_2 L$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-] & (+) & & (+) & & (+) & & (-) & & (+) \end{smallmatrix}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
42	$\tilde{d}^c_2 L$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+) & (-) & & (+) & & (+) & & (-) & & (+) \end{smallmatrix}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
43	$\tilde{u}^c_2 L$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ - & [-] & & & & (-) & & (-) & & (+) \end{smallmatrix}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$

Table A1. Cont.

i		$^{id} \psi_i >$ (Anti)octet, $\Gamma^{(7,1)} = (-1)^1, \Gamma^{(6)} = (1) - 1$ of (anti)quarks and (anti)leptons	$\Gamma^{(3,1)}$	s^{12}	τ^{13}	τ^{23}	τ^{33}	τ^{38}	τ^4	Y	Q
44	$\tilde{u}_L^2$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 \\ - & (+i) & (-) & & (-) & & (-) & & (+) & & (-) & \\ & & & & 13 & 14 & & & & & & \\ & & & & (+) & & & & & & & \end{smallmatrix}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
45	$\tilde{d}_R^2$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & (+) & & (+) & & (-) & & (+) & & (-) & & \\ & & & & & & & & & & & \end{smallmatrix}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
46	$\tilde{d}_R^2$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ - & [-i] & (-) & & (+) & & (-) & & (+) & & (-) & \\ & & & & & & & & & & & \end{smallmatrix}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
47	$\tilde{u}_R^2$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & (+) & & (-) & & (+) & & (+) & & (-) & & \\ & & & & & & & & & & & \end{smallmatrix}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
48	$\tilde{u}_R^2$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-i] & (-) & & (-) & & (+) & & (+) & & (-) & & \\ & & & & & & & & & & & \end{smallmatrix}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
49	$\tilde{d}_L^3$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-i] & (+) & & (+) & & (+) & & (+) & & (-) & & \\ & & & & & & & & & & & \end{smallmatrix}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
50	$\tilde{d}_L^3$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & (+) & & (+) & & (+) & & (+) & & (+) & & \\ & & & & & & & & & & & \end{smallmatrix}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
51	$\tilde{u}_L^3$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ - & [-i] & (+) & & (-) & & (-) & & (+) & & (+) & \\ & & & & & & & & & & & \end{smallmatrix}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
52	$\tilde{u}_L^3$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ - & (+i) & (-) & & (-) & & (-) & & (+) & & (+) & \\ & & & & & & & & & & & \end{smallmatrix}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{2}{3}$	$-\frac{2}{3}$
53	$\tilde{d}_R^3$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & (+) & & (+) & & (-) & & (+) & & (-) & & \\ & & & & & & & & & & & \end{smallmatrix}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
54	$\tilde{d}_R^3$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ - & [-i] & (-) & & (+) & & (-) & & (+) & & (+) & \\ & & & & & & & & & & & \end{smallmatrix}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$
55	$\tilde{u}_R^3$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & (+) & & (-) & & (+) & & (+) & & (-) & & \\ & & & & & & & & & & & \end{smallmatrix}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
56	$\tilde{u}_R^3$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-i] & (-) & & (-) & & (+) & & (+) & & (-) & & \\ & & & & & & & & & & & \end{smallmatrix}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{\sqrt{3}}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{2}{3}$
57	$\tilde{e}_L$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-i] & (+) & & (+) & & (-) & & (-) & & (-) & & \\ & & & & & & & & & & & \end{smallmatrix}$	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
58	$\tilde{e}_L$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & (+) & & (+) & & (-) & & (-) & & (-) & & \\ & & & & & & & & & & & \end{smallmatrix}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1
59	$\tilde{\nu}_L$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ - & [-i] & (+) & & (-) & & (-) & & (-) & & (-) & \\ & & & & & & & & & & & \end{smallmatrix}$	-1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
60	$\tilde{\nu}_L$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ - & (+i) & (-) & & (-) & & (-) & & (-) & & (-) & \\ & & & & & & & & & & & \end{smallmatrix}$	-1	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	0
61	$\tilde{\nu}_R$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & (+) & & (-) & & (+) & & (-) & & (-) & & \\ & & & & & & & & & & & \end{smallmatrix}$	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
62	$\tilde{\nu}_R$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ - & [-i] & (-) & & (-) & & (+) & & (-) & & (-) & \\ & & & & & & & & & & & \end{smallmatrix}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
63	$\tilde{e}_R$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ (+i) & (+) & & (+) & & (-) & & (-) & & (-) & & \\ & & & & & & & & & & & \end{smallmatrix}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1
64	$\tilde{e}_R$	$\begin{smallmatrix} 03 & 12 & 56 & 78 & 9 & 10 & 11 & 12 & 13 & 14 \\ [-i] & (-) & & (+) & & (-) & & (-) & & (-) & & \\ & & & & & & & & & & & \end{smallmatrix}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1

Appendix B. Grassmann and Clifford Algebras

This part is taken from Ref. [20,23,24], following Refs. [1,2,8,14].
In Grassmann d -dimensional space there are d anti-commuting (operators) θ^a , and d anti-commuting operators which are derivatives with respect to θ^a , $\frac{\partial}{\partial \theta_a}$.

$$\{\theta^a, \theta^b\}_+ = 0, \quad \left\{ \frac{\partial}{\partial \theta_a}, \frac{\partial}{\partial \theta_b} \right\}_+ = 0,$$
$$\left\{ \theta_a, \frac{\partial}{\partial \theta_b} \right\}_+ = \delta_a^b, \quad (a, b) = (0, 1, 2, 3, 5, \dots, d).$$

(A61)

The choice

$$(\theta^a)^\dagger = \eta^{aa} \frac{\partial}{\partial \theta_a}, \quad \text{leads to} \quad \left(\frac{\partial}{\partial \theta_a} \right)^\dagger = \eta^{aa} \theta^a,$$

(A62)

with $\eta^{ab} = \text{diag}\{1, -1, -1, \dots, -1\}$.
 θ^a and $\frac{\partial}{\partial \theta_a}$ are, up to the sign, Hermitian conjugate to each other. The identity is a self-adjoint member of the algebra.
In d -dimensional space, there are 2^d superposition of products of θ^a , the Hermitian conjugated partners of which are the corresponding superposition of products of $\frac{\partial}{\partial \theta_a}$ [8,29].

We can make from θ^a 's and their conjugate momenta $p^{\theta^a} = i \frac{\partial}{\partial \theta_a}$ two kinds of the operators, γ^a and $\tilde{\gamma}^a$ [2],

$$\begin{aligned}\gamma^a &= (\theta^a + \frac{\partial}{\partial \theta_a}), \quad \tilde{\gamma}^a = i(\theta^a - \frac{\partial}{\partial \theta_a}), \\ \theta^a &= \frac{1}{2}(\gamma^a - i\tilde{\gamma}^a), \quad \frac{\partial}{\partial \theta_a} = \frac{1}{2}(\gamma^a + i\tilde{\gamma}^a),\end{aligned}\tag{A63}$$

each offers 2^d superposition of products of γ^a or $\tilde{\gamma}^a$ ([8] and references therein)

$$\begin{aligned}\{\gamma^a, \gamma^b\}_+ &= 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+, \\ \{\gamma^a, \tilde{\gamma}^b\}_+ &= 0, \quad (a, b) = (0, 1, 2, 3, 5, \dots, d), \\ (\gamma^a)^\dagger &= \eta^{aa} \gamma^a, \quad (\tilde{\gamma}^a)^\dagger = \eta^{aa} \tilde{\gamma}^a.\end{aligned}\tag{A64}$$

The Grassmann algebra offers the description of the internal space of *anti-commuting integer spin second quantized fields* and of the *commuting integer spin second quantized fields* [8].

Both algebras, the superposition of odd products of γ^a 's or of $\tilde{\gamma}^a$'s, offer the description of the second quantized half integer spins and charges in the fundamental representations of the group [8], Table A1 represents the internal spaces of one family of quarks and leptons and antiquarks and antileptons, the internal space of which is determined by odd products of γ^a 's.

The superposition of even products of either γ^a 's or $\tilde{\gamma}^a$'s offer the description of the internal spaces of commuting second quantized boson fields with integer spins [18,19,24]), manifesting from the point of the subgroups of the $SO(d-1, 1)$ group, spins and charges in the adjoint representations.

There is so far observed only one kind of the anti-commuting half-integer spin second quantized fields.

The *postulate*, which determines how does $\tilde{\gamma}^a$ operate on γ^a , reduces the presentations of the two Clifford subalgebras, γ^a and $\tilde{\gamma}^a$, to the one described by γ^a [2,5,14]

$$\{\tilde{\gamma}^a B = (-)^B i B \gamma^a\} |\psi_{oc} > ,\tag{A65}$$

with $(-)^B = -1$, if B is (a function of) odd products of γ^a 's, otherwise $(-)^B = 1$ [5], the vacuum state $|\psi_{oc} >$ is defined in Eq. (7) of Subsect. 2.1.

After the postulate of Eq. (A65) the vector space of γ^a 's are chosen to describe the internal space of fermions, while $\tilde{\gamma}^a$'s are used to determine the family quantum numbers of the fermion fields.

Appendix C. Some Useful Relations

In this appendix some useful relations, needed in this paper in Sects. (2.1, A) are presented, taken from Ref. [20], App. E.

One can find if taking into account Eq. (A64)

$$\begin{aligned}
 \gamma^a \begin{smallmatrix} ab \\ (k) \end{smallmatrix} &= \eta^{aa} \begin{smallmatrix} ab \\ [-k] \end{smallmatrix}, \quad \gamma^b \begin{smallmatrix} ab \\ (k) \end{smallmatrix} = -ik \begin{smallmatrix} ab \\ [-k] \end{smallmatrix}, \quad \gamma^a \begin{smallmatrix} ab \\ [k] \end{smallmatrix} = (-k), \quad \gamma^b \begin{smallmatrix} ab \\ [k] \end{smallmatrix} = -ik\eta^{aa} \begin{smallmatrix} ab \\ (-k) \end{smallmatrix}, \\
 \tilde{\gamma}^a \begin{smallmatrix} ab \\ (k) \end{smallmatrix} &= -i\eta^{aa} \begin{smallmatrix} ab \\ [k] \end{smallmatrix}, \quad \tilde{\gamma}^b \begin{smallmatrix} ab \\ (k) \end{smallmatrix} = -k \begin{smallmatrix} ab \\ [k] \end{smallmatrix}, \quad \tilde{\gamma}^a \begin{smallmatrix} ab \\ [k] \end{smallmatrix} = i \begin{smallmatrix} ab \\ (k) \end{smallmatrix}, \quad \tilde{\gamma}^b \begin{smallmatrix} ab \\ [k] \end{smallmatrix} = -k\eta^{aa} \begin{smallmatrix} ab \\ (k) \end{smallmatrix}, \\
 \begin{smallmatrix} ab \\ (k) \end{smallmatrix} \begin{smallmatrix} ab \\ (-k) \end{smallmatrix} &= \eta^{aa} \begin{smallmatrix} ab \\ [k] \end{smallmatrix}, \quad \begin{smallmatrix} ab \\ (-k) \end{smallmatrix} \begin{smallmatrix} ab \\ (k) \end{smallmatrix} = \eta^{aa} \begin{smallmatrix} ab \\ [-k] \end{smallmatrix}, \quad \begin{smallmatrix} ab \\ (k) \end{smallmatrix} \begin{smallmatrix} ab \\ [k] \end{smallmatrix} = 0, \quad \begin{smallmatrix} ab \\ (k) \end{smallmatrix} \begin{smallmatrix} ab \\ [-k] \end{smallmatrix} = \begin{smallmatrix} ab \\ (k) \end{smallmatrix}, \\
 \begin{smallmatrix} ab \\ (-k) \end{smallmatrix} \begin{smallmatrix} ab \\ [k] \end{smallmatrix} &= \begin{smallmatrix} ab \\ (-k) \end{smallmatrix}, \quad \begin{smallmatrix} ab \\ [k] \end{smallmatrix} \begin{smallmatrix} ab \\ (k) \end{smallmatrix} = \begin{smallmatrix} ab \\ (k) \end{smallmatrix}, \quad \begin{smallmatrix} ab \\ [k] \end{smallmatrix} \begin{smallmatrix} ab \\ (-k) \end{smallmatrix} = 0, \quad \begin{smallmatrix} ab \\ [k] \end{smallmatrix} \begin{smallmatrix} ab \\ [-k] \end{smallmatrix} = 0, \\
 \begin{smallmatrix} ab \\ (k) \end{smallmatrix}^{\dagger} &= \eta^{aa} \begin{smallmatrix} ab \\ (-k) \end{smallmatrix}, \quad (\begin{smallmatrix} ab \\ (k) \end{smallmatrix})^2 = 0, \quad \begin{smallmatrix} ab \\ (k) \end{smallmatrix} \begin{smallmatrix} ab \\ (-k) \end{smallmatrix} = \eta^{aa} \begin{smallmatrix} ab \\ [k] \end{smallmatrix}, \\
 \begin{smallmatrix} ab \\ [k] \end{smallmatrix}^{\dagger} &= \begin{smallmatrix} ab \\ [k] \end{smallmatrix}, \quad (\begin{smallmatrix} ab \\ [k] \end{smallmatrix})^2 = \begin{smallmatrix} ab \\ [k] \end{smallmatrix}, \quad \begin{smallmatrix} ab \\ [k] \end{smallmatrix} \begin{smallmatrix} ab \\ [-k] \end{smallmatrix} = 0.
 \end{aligned} \tag{A66}$$

$$\begin{aligned}
 \begin{smallmatrix} ab \\ (\tilde{k}) \end{smallmatrix} \begin{smallmatrix} ab \\ (k) \end{smallmatrix} &= 0, \quad \begin{smallmatrix} ab \\ (\tilde{k}) \end{smallmatrix} \begin{smallmatrix} ab \\ (-k) \end{smallmatrix} = -i\eta^{aa} \begin{smallmatrix} ab \\ [-k] \end{smallmatrix}, \quad \begin{smallmatrix} ab \\ (-\tilde{k}) \end{smallmatrix} \begin{smallmatrix} ab \\ (k) \end{smallmatrix} = -i\eta^{aa} \begin{smallmatrix} ab \\ [k] \end{smallmatrix}, \quad \begin{smallmatrix} ab \\ (\tilde{k}) \end{smallmatrix} \begin{smallmatrix} ab \\ [k] \end{smallmatrix} = i \begin{smallmatrix} ab \\ (k) \end{smallmatrix}, \\
 \begin{smallmatrix} ab \\ (\tilde{k}) \end{smallmatrix} \begin{smallmatrix} ab \\ [-k] \end{smallmatrix} &= 0, \quad \begin{smallmatrix} ab \\ (-\tilde{k}) \end{smallmatrix} \begin{smallmatrix} ab \\ [k] \end{smallmatrix} = 0, \quad \begin{smallmatrix} ab \\ (-\tilde{k}) \end{smallmatrix} \begin{smallmatrix} ab \\ [-k] \end{smallmatrix} = i \begin{smallmatrix} ab \\ (-k) \end{smallmatrix}, \quad \begin{smallmatrix} ab \\ [\tilde{k}] \end{smallmatrix} \begin{smallmatrix} ab \\ (k) \end{smallmatrix} = \begin{smallmatrix} ab \\ (k) \end{smallmatrix}, \\
 \begin{smallmatrix} ab \\ [\tilde{k}] \end{smallmatrix} \begin{smallmatrix} ab \\ (-k) \end{smallmatrix} &= 0, \quad \begin{smallmatrix} ab \\ [\tilde{k}] \end{smallmatrix} \begin{smallmatrix} ab \\ [k] \end{smallmatrix} = 0, \quad \begin{smallmatrix} ab \\ [-\tilde{k}] \end{smallmatrix} \begin{smallmatrix} ab \\ [k] \end{smallmatrix} = \begin{smallmatrix} ab \\ [k] \end{smallmatrix}, \quad \begin{smallmatrix} ab \\ [\tilde{k}] \end{smallmatrix} \begin{smallmatrix} ab \\ [-k] \end{smallmatrix} = \begin{smallmatrix} ab \\ [-k] \end{smallmatrix},
 \end{aligned} \tag{A67}$$

$$\begin{aligned}
 S^{ac} \begin{smallmatrix} ab \\ (k) \end{smallmatrix} \begin{smallmatrix} cd \\ (k) \end{smallmatrix} &= -\frac{i}{2} \eta^{aa} \eta^{cc} \begin{smallmatrix} ab \\ [-k] \end{smallmatrix} \begin{smallmatrix} cd \\ [-k] \end{smallmatrix}, \quad S^{ac} \begin{smallmatrix} ab \\ [k] \end{smallmatrix} \begin{smallmatrix} cd \\ [k] \end{smallmatrix} = \frac{i}{2} \begin{smallmatrix} ab \\ (-k) \end{smallmatrix} \begin{smallmatrix} cd \\ (-k) \end{smallmatrix}, \\
 S^{ac} \begin{smallmatrix} ab \\ (k) \end{smallmatrix} \begin{smallmatrix} cd \\ [k] \end{smallmatrix} &= -\frac{i}{2} \eta^{aa} \begin{smallmatrix} ab \\ [-k] \end{smallmatrix} \begin{smallmatrix} cd \\ (-k) \end{smallmatrix}, \quad S^{ac} \begin{smallmatrix} ab \\ [k] \end{smallmatrix} \begin{smallmatrix} cd \\ (k) \end{smallmatrix} = \frac{i}{2} \eta^{cc} \begin{smallmatrix} ab \\ (-k) \end{smallmatrix} \begin{smallmatrix} cd \\ [-k] \end{smallmatrix}, \\
 \tilde{S}^{ac} \begin{smallmatrix} ab \\ (k) \end{smallmatrix} \begin{smallmatrix} cd \\ (k) \end{smallmatrix} &= \frac{i}{2} \eta^{aa} \eta^{cc} \begin{smallmatrix} ab \\ [k] \end{smallmatrix} \begin{smallmatrix} cd \\ [k] \end{smallmatrix}, \quad \tilde{S}^{ac} \begin{smallmatrix} ab \\ [k] \end{smallmatrix} \begin{smallmatrix} cd \\ [k] \end{smallmatrix} = -\frac{i}{2} \begin{smallmatrix} ab \\ (k) \end{smallmatrix} \begin{smallmatrix} cd \\ (k) \end{smallmatrix}, \\
 \tilde{S}^{ac} \begin{smallmatrix} ab \\ (k) \end{smallmatrix} \begin{smallmatrix} cd \\ [k] \end{smallmatrix} &= -\frac{i}{2} \eta^{aa} \begin{smallmatrix} ab \\ [k] \end{smallmatrix} \begin{smallmatrix} cd \\ (k) \end{smallmatrix}, \quad \tilde{S}^{ac} \begin{smallmatrix} ab \\ [k] \end{smallmatrix} \begin{smallmatrix} cd \\ (k) \end{smallmatrix} = \frac{i}{2} \eta^{cc} \begin{smallmatrix} ab \\ (k) \end{smallmatrix} \begin{smallmatrix} cd \\ [k] \end{smallmatrix}.
 \end{aligned} \tag{A68}$$

The reader can calculate all the quantum numbers of Table A1, App. A, if taking into account the generators of the two $SU(2) \subset SO(3,1) \subset SO(7,1) \subset SO(13,1)$ groups, describing spins of fermions and the corresponding family quantum numbers

$$\begin{aligned}
 \vec{N}_{\pm} (= \vec{N}_{(L,R)}) &:= \frac{1}{2} (S^{23} \pm iS^{01}, S^{31} \pm iS^{02}, S^{12} \pm iS^{03}), \\
 \vec{\tilde{N}}_{\pm} (= \vec{\tilde{N}}_{(L,R)}) &:= \frac{1}{2} (\tilde{S}^{23} \pm i\tilde{S}^{01}, \tilde{S}^{31} \pm i\tilde{S}^{02}, \tilde{S}^{12} \pm i\tilde{S}^{03}),
 \end{aligned} \tag{A69}$$

the generators of the two $SU(2) \subset SO(4) \subset SO(7,1) \subset SO(13,1)$ groups, describing the weak charge, $\vec{\tau}^1$, and the second kind of the weak charge, $\vec{\tau}^2$, of fermions and the corresponding family quantum numbers

$$\begin{aligned}
 \vec{\tau}^1 : &= \frac{1}{2} (S^{58} - S^{67}, S^{57} + S^{68}, S^{56} - S^{78}), \quad \vec{\tau}^2 := \frac{1}{2} (S^{58} + S^{67}, S^{57} - S^{68}, S^{56} + S^{78}), \\
 \vec{\tilde{\tau}}^1 : &= \frac{1}{2} (\tilde{S}^{58} - \tilde{S}^{67}, \tilde{S}^{57} + \tilde{S}^{68}, \tilde{S}^{56} - \tilde{S}^{78}), \quad \vec{\tilde{\tau}}^2 := \frac{1}{2} (\tilde{S}^{58} + \tilde{S}^{67}, \tilde{S}^{57} - \tilde{S}^{68}, \tilde{S}^{56} + \tilde{S}^{78}),
 \end{aligned} \tag{A70}$$

and the generators of $SU(3)$ and $U(1)$ subgroups of $SO(6) \subset SO(13, 1)$, describing the colour charge and the "fermion" charge of fermions as well as the corresponding family quantum number $\tilde{\tau}^4$

$$\begin{aligned}\tilde{\tau}^3 &:= \frac{1}{2} \{ S^{9\ 12} - S^{10\ 11}, S^{9\ 11} + S^{10\ 12}, S^{9\ 10} - S^{11\ 12}, S^{9\ 14} - S^{10\ 13}, \\ &\quad S^{9\ 13} + S^{10\ 14}, S^{11\ 14} - S^{12\ 13}, S^{11\ 13} + S^{12\ 14}, \frac{1}{\sqrt{3}}(S^{9\ 10} + S^{11\ 12} - 2S^{13\ 14}) \}, \\ \tau^4 &:= -\frac{1}{3}(S^{9\ 10} + S^{11\ 12} + S^{13\ 14}), \\ \tilde{\tau}^4 &:= -\frac{1}{3}(\tilde{S}^{9\ 10} + \tilde{S}^{11\ 12} + \tilde{S}^{13\ 14}).\end{aligned}\quad (A71)$$

The (chosen) Cartan subalgebra operators, determining the commuting operators in the above equations, is presented in Eq. (3).

The hypercharge Y and the electromagnetic charge Q and the corresponding family quantum numbers then follows as

$$\begin{aligned}Y &:= \tau^4 + \tau^{23}, \quad Q := \tau^{13} + Y, \quad Y' := -\tau^4 \tan^2 \vartheta_2 + \tau^{23}, \quad Q' := -Y \tan^2 \vartheta_1 + \tau^{13}, \\ \tilde{Y} &:= \tilde{\tau}^4 + \tilde{\tau}^{23}, \quad \tilde{Q} := \tilde{Y} + \tilde{\tau}^{13}, \quad \tilde{Y}' := -\tilde{\tau}^4 \tan^2 \vartheta_2 + \tilde{\tau}^{23}, \quad \tilde{Q}' := -\tilde{Y} \tan^2 \vartheta_1 + \tilde{\tau}^{13}.\end{aligned}\quad (A72)$$

Below are some of the above expressions written in terms of nilpotents and projectors

$$\begin{aligned}N_{\pm}^{\pm} &= N_{\pm}^1 \pm i N_{\pm}^2 = -(\mp i)(\pm), \quad N_{\pm}^{\pm} = N_{\pm}^1 \pm i N_{\pm}^2 = (\pm i)(\pm), \\ \tilde{N}_{\pm}^{\pm} &= -(\mp i)(\pm), \quad \tilde{N}_{\pm}^{\pm} = (\pm i)(\pm), \\ \tau^{1\pm} &= (\mp)(\pm)(\mp), \quad \tau^{2\mp} = (\mp)(\mp)(\mp), \\ \tilde{\tau}^{1\pm} &= (\mp)(\pm)(\mp), \quad \tilde{\tau}^{2\mp} = (\mp)(\mp)(\mp).\end{aligned}\quad (A73)$$

For fermions, the operator of handedness Γ^d is determined as follows:

$$\Gamma^{(d)} = \prod_a (\sqrt{\eta^{aa}} \gamma^a) \cdot \begin{cases} (i)^{\frac{d}{2}}, & \text{for } d \text{ even}, \\ (i)^{\frac{d-1}{2}}, & \text{for } d \text{ odd}. \end{cases}\quad (A74)$$

Appendix D. Odd and Even "Basis Vectors" in $(5 + 1)$ -Dimensional Space

In this appendix, the even and odd "basis vectors" are presented for the choice $d = (5 + 1)$, needed in Subsect. (2.3). The presentation follows the paper [18].

Table A2 presents $2^{d=6}$ "eigenvectors" of the Cartan subalgebra members, Eq. (3), of the odd and even "basis vectors" which are the superposition of odd, $\hat{b}_f^{m\pm}$, appearing in $2^{\frac{d=6}{2}-1} = 4$ families with $2^{\frac{d=6}{2}-1} = 4$ members each, and their Hermitian conjugate partners, $(\hat{b}_f^{m\pm})^\dagger$, 16, and of $2^{\frac{d=6}{2}-1} \times 2^{\frac{d=6}{2}-1} = 16$ even, $^I \mathcal{A}_f^m$, and of 16 even, $^{II} \mathcal{A}_f^m$, products of γ^a 's, helpful in Sect. (2.3). Table A2 is presented in several papers ([8,18], and references therein).

Odd and even "basis vectors" are presented as products of nilpotents and projectors, Eqs. (4, A66). The odd "basis vectors" are products of odd number of nilpotents, one or three, the rest are projectors, two or zero; the even "basis vectors" are products of even number of nilpotents, zero or two, the rest are projectors, three or one.

As explained in Eqs. (19, 20), even "basis vectors" can be expressed as algebraic products of odd "basis vectors" and their Hermitian conjugated partners: $^I \hat{\mathcal{A}}_f^{m\pm} = \hat{b}_{f'}^{m'\pm} *_A (\hat{b}_{f'}^{m''\pm})^\dagger$, while $^{II} \hat{\mathcal{A}}_f^{m\pm} = (\hat{b}_{f'}^{m'\pm})^\dagger *_A \hat{b}_{f'}^{m''\pm}$.

The "basis vectors" $^I \hat{\mathcal{A}}_f^{m\pm}$ are presented in Tables (A3, A4), expressed with the odd "basis vectors" of the first group of Table A2, and the first family, and their Hermitian conjugated partners. Any of the

families generates the same ${}^I\hat{\mathcal{A}}_f^{m\dagger}$. The first, Table A3, collects the eight members with $\mathcal{S}^{12} = \pm 1$, the second, Table A4, collects the remaining eight members with $\mathcal{S}^{12} = 0$.

The “basis vectors” ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$ are presented in Tables (A5, A6), expressed with the odd “basis vectors” of the first group of Table A2, and the first members of any family, and with their Hermitian conjugated partners. Any of the family members generates the same ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$. The first Table A5 collects the eight members with $\mathcal{S}^{12} = \pm 1$, the second one, Table A4, collects the remaining eight members with $\mathcal{S}^{12} = 0$.

Table A2. This table, taken from [18], represents for the internal space $d = (5 + 1) 2^d = 64$ “eigenvectors” of the Cartan subalgebra, Eq. (3), members of the odd and even “basis vectors” which are the superposition of odd and even products of γ^a ’s in $d = (5 + 1)$ -dimensional internal space. Table is divided into four groups. The first group, *odd I*, is (chosen) to represent “basis vectors”, $\hat{b}_f^{m\dagger}$, appearing in $2^{\frac{d}{2}-1} = 4$ “families” ($f = 1, 2, 3, 4$), each “family” having $2^{\frac{d}{2}-1} = 4$ “family” members ($m = 1, 2, 3, 4$). The second group, *odd II*, contains Hermitian conjugate partners of the first group for each “family” separately, $\hat{b}_f^m = (\hat{b}_f^{m\dagger})^\dagger$. The *odd I* or *odd II* are products of an odd number of nilpotents (one or three) and projectors (two or none). The “family” quantum numbers of $\hat{b}_f^{m\dagger}$, that is the eigenvalues of $(\tilde{\mathcal{S}}^3, \tilde{\mathcal{S}}^{12}, \tilde{\mathcal{S}}^{56})$, appear for the first *odd I* group, and the two last *even I* and *even II* groups above each “family”, the quantum numbers of the “family” members $(\mathcal{S}^3, \mathcal{S}^{12}, \mathcal{S}^{56})$ are written in the last three columns. For the Hermitian conjugated partners of *odd I*, presented in the group *odd II*, the quantum numbers $(\mathcal{S}^3, \mathcal{S}^{12}, \mathcal{S}^{56})$ are presented above each group of the Hermitian conjugate partners, the last three columns tell eigenvalues of $(\tilde{\mathcal{S}}^3, \tilde{\mathcal{S}}^{12}, \tilde{\mathcal{S}}^{56})$. Each of the two groups with the even number of γ^a ’s, *even I* and *even II*, has their Hermitian conjugated partners within its group. The quantum numbers f , that is the eigenvalues of $(\tilde{\mathcal{S}}^3, \tilde{\mathcal{S}}^{12}, \tilde{\mathcal{S}}^{56})$, are written above each column of four members, the quantum numbers of the members, $(\mathcal{S}^3, \mathcal{S}^{12}, \mathcal{S}^{56})$, are written in the last three columns. The quantum numbers of $(\mathcal{S}^3, \mathcal{S}^{12}, \mathcal{S}^{56})$, for the even two groups, the *even I* and *even II*, are $\mathbf{S}^{ab} = \mathcal{S}^{ab} + \tilde{\mathcal{S}}^{ab}$, which are for all projectors, according to Eq. (4), equal zero and for nilpotents equal to $\pm i$ or ± 1 .

${}^{II}\text{basis vectors}^{II}$ $(\tilde{\mathcal{S}}^3, \tilde{\mathcal{S}}^{12}, \tilde{\mathcal{S}}^{56})$	$m \rightarrow$	$f = 1$ $(\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$	$f = 2$ $(-\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$	$f = 3$ $(-\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$	$f = 4$ $(\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$	$\mathcal{S}^3$	$\mathcal{S}^{12}$	$\mathcal{S}^{56}$
<i>odd I</i> $\hat{b}_f^{m\dagger}$	1	$\begin{smallmatrix} 03 & 12 & 56 \\ (+i)(+)(+) \end{smallmatrix}$	$\begin{smallmatrix} 03 & 12 & 56 \\ [+i](+)(+) \end{smallmatrix}$	$\begin{smallmatrix} 03 & 12 & 56 \\ [+i](+)(+) \end{smallmatrix}$	$\begin{smallmatrix} 03 & 12 & 56 \\ (+i)(+)(+) \end{smallmatrix}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	2	$\begin{smallmatrix} -i(-)(-)(+) \end{smallmatrix}$	$\begin{smallmatrix} (-i)(-)(+) \end{smallmatrix}$	$\begin{smallmatrix} (-i)(-)(+) \end{smallmatrix}$	$\begin{smallmatrix} [-i](-)(+) \end{smallmatrix}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	3	$\begin{smallmatrix} [-i](+)(-) \end{smallmatrix}$	$\begin{smallmatrix} (-i)(+)(-) \end{smallmatrix}$	$\begin{smallmatrix} (-i)(+)(-) \end{smallmatrix}$	$\begin{smallmatrix} [-i](+)(-) \end{smallmatrix}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	4	$\begin{smallmatrix} (+i)(-)(-) \end{smallmatrix}$	$\begin{smallmatrix} [+i](-)(-) \end{smallmatrix}$	$\begin{smallmatrix} [+i](-)(-) \end{smallmatrix}$	$\begin{smallmatrix} (+i)(-)(-) \end{smallmatrix}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$(\mathcal{S}^3, \mathcal{S}^{12}, \mathcal{S}^{56})$	$\rightarrow$	$(-\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ $\begin{smallmatrix} 03 & 12 & 56 \end{smallmatrix}$	$(\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ $\begin{smallmatrix} 03 & 12 & 56 \end{smallmatrix}$	$(\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ $\begin{smallmatrix} 03 & 12 & 56 \end{smallmatrix}$	$(-\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ $\begin{smallmatrix} 03 & 12 & 56 \end{smallmatrix}$	$\mathcal{S}^3$	$\mathcal{S}^{12}$	$\mathcal{S}^{56}$
<i>odd II</i> $\hat{b}_f^m$	1	$\begin{smallmatrix} (-i)(+)(+) \end{smallmatrix}$	$\begin{smallmatrix} [+i](+)(-) \end{smallmatrix}$	$\begin{smallmatrix} [+i](-)(+) \end{smallmatrix}$	$\begin{smallmatrix} (-i)(-)(-) \end{smallmatrix}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
	2	$\begin{smallmatrix} [-i](+)(+) \end{smallmatrix}$	$\begin{smallmatrix} (+i)(+)(-) \end{smallmatrix}$	$\begin{smallmatrix} (+i)(-)(+) \end{smallmatrix}$	$\begin{smallmatrix} [-i](-)(-) \end{smallmatrix}$	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	3	$\begin{smallmatrix} [-i](+)(-) \end{smallmatrix}$	$\begin{smallmatrix} (+i)(+)(-) \end{smallmatrix}$	$\begin{smallmatrix} (+i)(-)(+) \end{smallmatrix}$	$\begin{smallmatrix} [-i](-)(-) \end{smallmatrix}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	4	$\begin{smallmatrix} (-i)(+)(+) \end{smallmatrix}$	$\begin{smallmatrix} [+i](+)(-) \end{smallmatrix}$	$\begin{smallmatrix} [+i](-)(+) \end{smallmatrix}$	$\begin{smallmatrix} (-i)(-)(-) \end{smallmatrix}$	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$(\mathcal{S}^3, \mathcal{S}^{12}, \mathcal{S}^{56})$	$\rightarrow$	$(-\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ $\begin{smallmatrix} 03 & 12 & 56 \end{smallmatrix}$	$(\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ $\begin{smallmatrix} 03 & 12 & 56 \end{smallmatrix}$	$(-\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ $\begin{smallmatrix} 03 & 12 & 56 \end{smallmatrix}$	$(\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ $\begin{smallmatrix} 03 & 12 & 56 \end{smallmatrix}$	$\mathcal{S}^3$	$\mathcal{S}^{12}$	$\mathcal{S}^{56}$
<i>even I</i> ${}^I\hat{\mathcal{A}}_f^m$	1	$\begin{smallmatrix} [+i](+)(+) \end{smallmatrix}$	$\begin{smallmatrix} (+i)(+)(+) \end{smallmatrix}$	$\begin{smallmatrix} [+i](+)(+) \end{smallmatrix}$	$\begin{smallmatrix} (+i)(+)(+) \end{smallmatrix}$	$\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	2	$\begin{smallmatrix} (-i)(-)(+) \end{smallmatrix}$	$\begin{smallmatrix} [-i](-)(+) \end{smallmatrix}$	$\begin{smallmatrix} (-i)(-)(+) \end{smallmatrix}$	$\begin{smallmatrix} [-i](-)(+) \end{smallmatrix}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	3	$\begin{smallmatrix} (-i)(+)(-) \end{smallmatrix}$	$\begin{smallmatrix} [-i](+)(-) \end{smallmatrix}$	$\begin{smallmatrix} (-i)(+)(-) \end{smallmatrix}$	$\begin{smallmatrix} [-i](+)(-) \end{smallmatrix}$	$-\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	4	$\begin{smallmatrix} [+i](-)(-) \end{smallmatrix}$	$\begin{smallmatrix} (+i)(-)(-) \end{smallmatrix}$	$\begin{smallmatrix} [+i](-)(-) \end{smallmatrix}$	$\begin{smallmatrix} (+i)(-)(-) \end{smallmatrix}$	$\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$(\mathcal{S}^3, \mathcal{S}^{12}, \mathcal{S}^{56})$	$\rightarrow$	$(\frac{i}{2}, \frac{1}{2}, \frac{1}{2})$ $\begin{smallmatrix} 03 & 12 & 56 \end{smallmatrix}$	$(-\frac{i}{2}, -\frac{1}{2}, \frac{1}{2})$ $\begin{smallmatrix} 03 & 12 & 56 \end{smallmatrix}$	$(\frac{i}{2}, -\frac{1}{2}, -\frac{1}{2})$ $\begin{smallmatrix} 03 & 12 & 56 \end{smallmatrix}$	$(-\frac{i}{2}, \frac{1}{2}, -\frac{1}{2})$ $\begin{smallmatrix} 03 & 12 & 56 \end{smallmatrix}$	$\mathcal{S}^3$	$\mathcal{S}^{12}$	$\mathcal{S}^{56}$
<i>even II</i> ${}^{II}\hat{\mathcal{A}}_f^m$	1	$\begin{smallmatrix} [-i](+)(+) \end{smallmatrix}$	$\begin{smallmatrix} (-i)(+)(+) \end{smallmatrix}$	$\begin{smallmatrix} [-i](+)(+) \end{smallmatrix}$	$\begin{smallmatrix} (-i)(+)(+) \end{smallmatrix}$	$-\frac{i}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	2	$\begin{smallmatrix} (+i)(-)(+) \end{smallmatrix}$	$\begin{smallmatrix} [+i](-)(+) \end{smallmatrix}$	$\begin{smallmatrix} (+i)(-)(+) \end{smallmatrix}$	$\begin{smallmatrix} [+i](-)(+) \end{smallmatrix}$	$\frac{i}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	3	$\begin{smallmatrix} (+i)(+)(-) \end{smallmatrix}$	$\begin{smallmatrix} [+i](+)(-) \end{smallmatrix}$	$\begin{smallmatrix} (+i)(+)(-) \end{smallmatrix}$	$\begin{smallmatrix} [+i](+)(-) \end{smallmatrix}$	$\frac{i}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
	4	$\begin{smallmatrix} [-i](-)(-) \end{smallmatrix}$	$\begin{smallmatrix} (-i)(-)(-) \end{smallmatrix}$	$\begin{smallmatrix} [-i](-)(-) \end{smallmatrix}$	$\begin{smallmatrix} (-i)(-)(-) \end{smallmatrix}$	$-\frac{i}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Table A3. The even “basis vectors” ${}^I\hat{\mathcal{A}}_f^{m\dagger}$, belonging to transverse momentum in internal space, $\mathcal{S}^{12} = 1$, the first half of ${}^I\hat{\mathcal{A}}_f^{m\dagger}$, and $\mathcal{S}^{12} = -1$, the second half of ${}^I\hat{\mathcal{A}}_f^{m\dagger}$, for $d = (5 + 1)$, are presented as algebraic products of the $f = 1$ family “basis vectors” $\hat{b}_1^{m\dagger}$ and their Hermitian conjugated partners $(\hat{b}_1^{m\dagger})^\dagger: \hat{b}_1^{m\dagger} *_A (\hat{b}_1^{m\dagger})^\dagger$. Two ${}^I\hat{\mathcal{A}}_f^{m\dagger}$ which are the Hermitian conjugated partners are marked with the same symbol ($\star\star, \ddagger, \otimes, \odot\odot$). The even “basis vectors” ${}^I\hat{\mathcal{A}}_f^{m\dagger}$ are products of one projector and two nilpotents, the odd “basis vectors” and their Hermitian conjugated partners are products of one nilpotent and two projectors or of three nilpotents. The even and odd objects are eigenvectors of all the corresponding Cartan subalgebra members, Eq. (3). There are $\frac{1}{2} \times 2^{\frac{6}{2}-1} \times 2^{\frac{6}{2}-1}$ algebraic products of $\hat{b}_1^{m\dagger} *_A (\hat{b}_1^{m\dagger})^\dagger$ with $\mathcal{S}^{12}$ equal to 1 or -1 . The rest 8 of 16 members present ${}^I\hat{\mathcal{A}}_f^{m\dagger}$ with $\mathcal{S}^{12} = 0$. The members $\hat{b}_f^{m\dagger}$ together with their Hermitian conjugated partners of each of the four families, $f = (1, 2, 3, 4)$, offer the same ${}^I\hat{\mathcal{A}}_f^{m\dagger}$ with $\mathcal{S}^{12} = \pm 1$ as the ones presented in this table. (And equivalently for $\mathcal{S}^{12} = 0$.) Table is taken from Ref. [23].

$\mathcal{S}^{12}$	symbol	${}^I\hat{\mathcal{A}}_f^{m\dagger} =$	$\hat{b}_f^{m\dagger} *_A (\hat{b}_f^{m\dagger})^\dagger$
1	$\star\star$	${}^I\hat{\mathcal{A}}_1^{1\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ [+i] & (+) & (+) \end{smallmatrix}$	$\hat{b}_1^{1\dagger} *_A (\hat{b}_1^{1\dagger})^\dagger =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+i) & (+) & (+) \end{smallmatrix} *_A \begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (+) & (+) \end{smallmatrix}$
1	$\ddagger$	${}^I\hat{\mathcal{A}}_3^{1\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (+) & (-) \end{smallmatrix}$	$\hat{b}_3^{1\dagger} *_A (\hat{b}_3^{1\dagger})^\dagger =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (+) & (-) \end{smallmatrix} *_A \begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (+) \end{smallmatrix}$
1	$\odot\odot$	${}^I\hat{\mathcal{A}}_4^{1\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+i) & (+) & (+) \end{smallmatrix}$	$\hat{b}_4^{1\dagger} *_A (\hat{b}_4^{1\dagger})^\dagger =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+i) & (+) & (+) \end{smallmatrix} *_A \begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (+) & (+) \end{smallmatrix}$
1	$\otimes$	${}^I\hat{\mathcal{A}}_3^{1\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (+) & (-) \end{smallmatrix}$	$\hat{b}_3^{1\dagger} *_A (\hat{b}_3^{1\dagger})^\dagger =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (+) & (-) \end{smallmatrix} *_A \begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (+) & (+) \end{smallmatrix}$
-1	$\otimes$	${}^I\hat{\mathcal{A}}_2^{1\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (+) \end{smallmatrix}$	$\hat{b}_2^{1\dagger} *_A (\hat{b}_2^{1\dagger})^\dagger =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (+) \end{smallmatrix} *_A \begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (+) \end{smallmatrix}$
-1	$\ddagger$	${}^I\hat{\mathcal{A}}_2^{1\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+i) & (-) & (-) \end{smallmatrix}$	$\hat{b}_2^{1\dagger} *_A (\hat{b}_2^{1\dagger})^\dagger =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+i) & (-) & (-) \end{smallmatrix} *_A \begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (+) \end{smallmatrix}$
-1	$\odot\odot$	${}^I\hat{\mathcal{A}}_3^{1\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (+) \end{smallmatrix}$	$\hat{b}_3^{1\dagger} *_A (\hat{b}_3^{1\dagger})^\dagger =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (+) \end{smallmatrix} *_A \begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (+) \end{smallmatrix}$
-1	$\star\star$	${}^I\hat{\mathcal{A}}_3^{1\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+i) & (-) & (-) \end{smallmatrix}$	$\hat{b}_3^{1\dagger} *_A (\hat{b}_3^{1\dagger})^\dagger =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+i) & (-) & (-) \end{smallmatrix} *_A \begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (+) \end{smallmatrix}$

Table A4. The even “basis vectors” ${}^I\hat{\mathcal{A}}_f^{m\dagger}$, belonging to zero momentum in internal space, $\mathcal{S}^{12} = 0$, for $d = (5 + 1)$, are presented as algebraic products of the $f = 1$ family “basis vectors” $\hat{b}_1^{m\dagger}$ and their Hermitian conjugated partners $(\hat{b}_1^{m\dagger})^\dagger: \hat{b}_1^{m\dagger} *_A (\hat{b}_1^{m\dagger})^\dagger$. The two ${}^I\hat{\mathcal{A}}_f^{m\dagger}$ which are Hermitian conjugated partners, are marked with the same symbol (either $\triangle$ or $\bullet$). The symbol $\bigcirc$ presents selfadjoint members, four of eight. The even “basis vectors” ${}^I\hat{\mathcal{A}}_f^{m\dagger}$ are products of one projector and two nilpotents or three projectors (they are self-adjoint), the odd “basis vectors” and their Hermitian conjugated partners are products of one nilpotent and two projectors or of three nilpotents. The even and odd objects are eigenvectors of all the corresponding Cartan subalgebra members, Eq. (3). There are $\frac{1}{2} \times 2^{\frac{6}{2}-1} \times 2^{\frac{6}{2}-1}$ algebraic products of $\hat{b}_1^{m\dagger} *_A (\hat{b}_1^{m\dagger})^\dagger$. The rest 8 of 16 members have ${}^I\hat{\mathcal{A}}_f^{m\dagger}$ with $\mathcal{S}^{12} = +1$ (four) and with $\mathcal{S}^{12} = -1$ (four), present in Table A3. The members $\hat{b}_f^{m\dagger}$ together with their Hermitian conjugated partners of each of the four families, $f = (1, 2, 3, 4)$, offer the same ${}^I\hat{\mathcal{A}}_f^{m\dagger}$ with $\mathcal{S}^{12} = 0$ as the ones presented in this table. The table is taken from Ref. [23].

$\mathcal{S}^{12}$	symbol	${}^I\hat{\mathcal{A}}_f^{m\dagger} =$	$\hat{b}_f^{m\dagger} *_A (\hat{b}_f^{m\dagger})^\dagger$
0	$\triangle$	${}^I\hat{\mathcal{A}}_1^{2\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (+) \end{smallmatrix}$	$\hat{b}_1^{2\dagger} *_A (\hat{b}_1^{2\dagger})^\dagger =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (+) \end{smallmatrix} *_A \begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (+) \end{smallmatrix}$
0	$\bigcirc$	${}^I\hat{\mathcal{A}}_1^{4\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ [+i] & (-) & (-) \end{smallmatrix}$	$\hat{b}_1^{4\dagger} *_A (\hat{b}_1^{4\dagger})^\dagger =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+i) & (-) & (-) \end{smallmatrix} *_A \begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (+) \end{smallmatrix}$
0	$\bullet$	${}^I\hat{\mathcal{A}}_3^{1\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+i) & (+) & (+) \end{smallmatrix}$	$\hat{b}_3^{1\dagger} *_A (\hat{b}_3^{1\dagger})^\dagger =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+i) & (+) & (+) \end{smallmatrix} *_A \begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (+) & (+) \end{smallmatrix}$
0	$\bigcirc$	${}^I\hat{\mathcal{A}}_2^{3\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (+) & (-) \end{smallmatrix}$	$\hat{b}_2^{3\dagger} *_A (\hat{b}_2^{3\dagger})^\dagger =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (+) & (-) \end{smallmatrix} *_A \begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (+) & (+) \end{smallmatrix}$
0	$\bigcirc$	${}^I\hat{\mathcal{A}}_1^{1\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+i) & (+) & (+) \end{smallmatrix}$	$\hat{b}_1^{1\dagger} *_A (\hat{b}_1^{1\dagger})^\dagger =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+i) & (+) & (+) \end{smallmatrix} *_A \begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (+) & (+) \end{smallmatrix}$
0	$\bullet$	${}^I\hat{\mathcal{A}}_3^{3\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (+) & (-) \end{smallmatrix}$	$\hat{b}_3^{3\dagger} *_A (\hat{b}_3^{3\dagger})^\dagger =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (+) & (-) \end{smallmatrix} *_A \begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (+) \end{smallmatrix}$
0	$\bigcirc$	${}^I\hat{\mathcal{A}}_4^{1\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (+) \end{smallmatrix}$	$\hat{b}_4^{1\dagger} *_A (\hat{b}_4^{1\dagger})^\dagger =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (+) \end{smallmatrix} *_A \begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (+) \end{smallmatrix}$
0	$\triangle$	${}^I\hat{\mathcal{A}}_4^{4\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+i) & (-) & (-) \end{smallmatrix}$	$\hat{b}_4^{4\dagger} *_A (\hat{b}_4^{4\dagger})^\dagger =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+i) & (-) & (-) \end{smallmatrix} *_A \begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (+) \end{smallmatrix}$

Table A5. The even “basis vectors” ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$, belonging to transverse momentum in internal space, $\mathcal{S}^{12} = 1$, the first half ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$, and $\mathcal{S}^{12} = -1$, the second half ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$, for $d = (5 + 1)$, are presented as algebraic products of the first, $m = 1$, member of “basis vectors” $\hat{b}_{f'}^{m'=1\dagger}$ and the Hermitian conjugated partners $(\hat{b}_{f''}^{m'=1\dagger})^\dagger$. Two ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$ which are the Hermitian conjugated partners are marked with the same symbol. The even “basis vectors” ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$ are products of one projector and two nilpotents, the odd “basis vectors” and the Hermitian conjugated partners are products of one nilpotent and two projectors or of three nilpotents. Even and odd objects are eigenvectors of the corresponding Cartan subalgebra members, Eq. (3). There are $2^{\frac{6}{2}-1} \times 2^{\frac{6}{2}-1}$ algebraic products of $(\hat{b}_{f'}^{m'=1\dagger})^\dagger$ and $\hat{b}_{f''}^{m'=1\dagger}$, f' and f'' run over all four families. The rest of the 16 members present ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$ with $\mathcal{S}^{12} = 0$. The members $(\hat{b}_{f'}^{m'=1\dagger})^\dagger$ together with $\hat{b}_{f''}^{m'=1\dagger}$, $m' = (1, 2, 3, 4)$, offer the same ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$ with $\mathcal{S}^{12} = \pm 1$ as the ones presented in this table. (And equivalently for $\mathcal{S}^{12} = 0$.) The table is taken from Ref. [23].

$\mathcal{S}^{12}$	symbol	${}^{II}\hat{\mathcal{A}}_f^{m\dagger} =$	$(\hat{b}_{f'}^{1\dagger})^\dagger * \hat{b}_{f''}^{1\dagger}$
1	**	${}^{II}\hat{\mathcal{A}}_1^{1\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ -j & (+) & (+) \end{smallmatrix}$	$(\hat{b}_1^{1\dagger})^\dagger * \hat{b}_4^{1\dagger}$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (+) & (+) \end{smallmatrix} * \hat{A} \begin{smallmatrix} 03 & 12 & 56 \\ (+) & (+) & (+) \end{smallmatrix}$
1	$\odot\odot$	${}^{II}\hat{\mathcal{A}}_3^{1\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+) & (+) & (-) \end{smallmatrix}$	$(\hat{b}_2^{1\dagger})^\dagger * \hat{b}_1^{1\dagger}$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+) & (+) & (-) \end{smallmatrix} * \hat{A} \begin{smallmatrix} 03 & 12 & 56 \\ (+) & (+) & (+) \end{smallmatrix}$
1	$\ddagger$	${}^{II}\hat{\mathcal{A}}_1^{1\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (+) & (+) \end{smallmatrix}$	$(\hat{b}_1^{1\dagger})^\dagger * \hat{b}_3^{1\dagger}$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (+) & (+) \end{smallmatrix} * \hat{A} \begin{smallmatrix} 03 & 12 & 56 \\ (+) & (+) & (+) \end{smallmatrix}$
1	$\otimes$	${}^{II}\hat{\mathcal{A}}_4^{1\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+) & (+) & (-) \end{smallmatrix}$	$(\hat{b}_2^{1\dagger})^\dagger * \hat{b}_3^{1\dagger}$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+) & (+) & (-) \end{smallmatrix} * \hat{A} \begin{smallmatrix} 03 & 12 & 56 \\ (+) & (+) & (+) \end{smallmatrix}$
-1	$\otimes$	${}^{II}\hat{\mathcal{A}}_2^{1\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+) & (-) & (+) \end{smallmatrix}$	$(\hat{b}_1^{1\dagger})^\dagger * \hat{b}_1^{1\dagger}$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+) & (-) & (+) \end{smallmatrix} * \hat{A} \begin{smallmatrix} 03 & 12 & 56 \\ (+) & (+) & (+) \end{smallmatrix}$
-1	$\odot\odot$	${}^{II}\hat{\mathcal{A}}_4^{1\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (-) \end{smallmatrix}$	$(\hat{b}_1^{1\dagger})^\dagger * \hat{b}_2^{1\dagger}$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (-) \end{smallmatrix} * \hat{A} \begin{smallmatrix} 03 & 12 & 56 \\ (+) & (+) & (+) \end{smallmatrix}$
-1	$\ddagger$	${}^{II}\hat{\mathcal{A}}_3^{1\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+) & (-) & (+) \end{smallmatrix}$	$(\hat{b}_3^{1\dagger})^\dagger * \hat{b}_1^{1\dagger}$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+) & (-) & (+) \end{smallmatrix} * \hat{A} \begin{smallmatrix} 03 & 12 & 56 \\ (+) & (+) & (+) \end{smallmatrix}$
-1	**	${}^{II}\hat{\mathcal{A}}_3^{1\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ -j & (-) & (-) \end{smallmatrix}$	$(\hat{b}_4^{1\dagger})^\dagger * \hat{b}_1^{1\dagger}$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (-) \end{smallmatrix} * \hat{A} \begin{smallmatrix} 03 & 12 & 56 \\ (+) & (+) & (+) \end{smallmatrix}$

Table A6. The even “basis vectors” ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$, belonging to $\mathcal{S}^{12} = 0$ in internal space, for $d = (5 + 1)$, are presented as algebraic products of the first, $m = 1$, member of “basis vectors” $\hat{b}_{f'}^{m'=1\dagger}$ and the Hermitian conjugated partners $(\hat{b}_{f''}^{m'=1\dagger})^\dagger$. The Hermitian conjugated partners of two ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$ are marked with the same symbol (either $\triangle$ or $\bullet$). The symbol $\bigcirc$ presents four selfadjoint members. The even “basis vectors” ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$ are the products of one projector and two nilpotents, or of three projectors (they are self adjoint), the odd “basis vectors” and the Hermitian conjugated partners are products of one nilpotent and two projectors or of three nilpotents. Even and odd objects are eigenvectors of all the corresponding Cartan subalgebra members, Eq. (3). There are $\frac{1}{2} \times 2^{\frac{6}{2}-1} \times 2^{\frac{6}{2}-1}$ algebraic products of $(\hat{b}_{f'}^{m'=1\dagger})^\dagger * \hat{b}_{f''}^{m'=1\dagger}$, f' and f'' run over all four families. The rest of 16 members present ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$ with $\mathcal{S}^{12} = \pm 1$. The members $(\hat{b}_{f'}^{m'=1\dagger})^\dagger$ together with $\hat{b}_{f''}^{m'=1\dagger}$, $m' = (1, 2, 3, 4)$, offer the same ${}^{II}\hat{\mathcal{A}}_f^{m\dagger}$, all with $\mathcal{S}^{12} = 0$. Table is taken from [23].

$\mathcal{S}^{12}$	symbol	${}^{II}\hat{\mathcal{A}}_f^{m\dagger} =$	$(\hat{b}_{f'}^{1\dagger})^\dagger * \hat{b}_{f''}^{1\dagger}$
0	$\triangle$	${}^{II}\hat{\mathcal{A}}_1^{2\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+) & (-) & (+) \end{smallmatrix}$	$(\hat{b}_3^{1\dagger})^\dagger * \hat{b}_4^{1\dagger}$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+) & (-) & (+) \end{smallmatrix} * \hat{A} \begin{smallmatrix} 03 & 12 & 56 \\ (+) & (+) & (+) \end{smallmatrix}$
0	$\bigcirc$	${}^{II}\hat{\mathcal{A}}_1^{4\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (-) \end{smallmatrix}$	$(\hat{b}_1^{1\dagger})^\dagger * \hat{b}_4^{1\dagger}$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (-) \end{smallmatrix} * \hat{A} \begin{smallmatrix} 03 & 12 & 56 \\ (+) & (+) & (+) \end{smallmatrix}$
0	$\bullet$	${}^{II}\hat{\mathcal{A}}_3^{1\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (+) & (+) \end{smallmatrix}$	$(\hat{b}_1^{1\dagger})^\dagger * \hat{b}_2^{1\dagger}$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (+) & (+) \end{smallmatrix} * \hat{A} \begin{smallmatrix} 03 & 12 & 56 \\ (+) & (+) & (+) \end{smallmatrix}$
0	$\bigcirc$	${}^{II}\hat{\mathcal{A}}_3^{3\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+) & (+) & (-) \end{smallmatrix}$	$(\hat{b}_2^{1\dagger})^\dagger * \hat{b}_2^{1\dagger}$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+) & (+) & (-) \end{smallmatrix} * \hat{A} \begin{smallmatrix} 03 & 12 & 56 \\ (+) & (+) & (+) \end{smallmatrix}$
0	$\bigcirc$	${}^{II}\hat{\mathcal{A}}_1^{1\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (+) & (+) \end{smallmatrix}$	$(\hat{b}_1^{1\dagger})^\dagger * \hat{b}_1^{1\dagger}$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (+) & (+) \end{smallmatrix} * \hat{A} \begin{smallmatrix} 03 & 12 & 56 \\ (+) & (+) & (+) \end{smallmatrix}$
0	$\bullet$	${}^{II}\hat{\mathcal{A}}_3^{3\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+) & (+) & (-) \end{smallmatrix}$	$(\hat{b}_1^{1\dagger})^\dagger * \hat{b}_1^{1\dagger}$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+) & (+) & (-) \end{smallmatrix} * \hat{A} \begin{smallmatrix} 03 & 12 & 56 \\ (+) & (+) & (+) \end{smallmatrix}$
0	$\bigcirc$	${}^{II}\hat{\mathcal{A}}_4^{2\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+) & (-) & (+) \end{smallmatrix}$	$(\hat{b}_3^{1\dagger})^\dagger * \hat{b}_3^{1\dagger}$ $\begin{smallmatrix} 03 & 12 & 56 \\ (+) & (-) & (+) \end{smallmatrix} * \hat{A} \begin{smallmatrix} 03 & 12 & 56 \\ (+) & (+) & (+) \end{smallmatrix}$
0	$\triangle$	${}^{II}\hat{\mathcal{A}}_4^{4\dagger} =$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (-) \end{smallmatrix}$	$(\hat{b}_4^{1\dagger})^\dagger * \hat{b}_3^{1\dagger}$ $\begin{smallmatrix} 03 & 12 & 56 \\ (-i) & (-) & (-) \end{smallmatrix} * \hat{A} \begin{smallmatrix} 03 & 12 & 56 \\ (+) & (+) & (+) \end{smallmatrix}$

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