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Article

New Formulas for the Ramanujan Tau Function

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Abstract: The Ramanujan tau function is the Fourier coefficient of the discriminant modular form. We obtain some new formulas for the Ramanujan tau function.

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1. Introduction

The Ramanujan tau function $\tau(n)$, first studied by Ramanujan [8], is the Fourier coefficient of the discriminant modular form $\Delta(z)$:

$$\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24} = \sum_{n\geq 1}\tau(n)q^n$$

where $q := e^{2\pi iz}$ with $\mathrm{Im}(z) > 0$. In his famous paper, Ramanujan proposed following three properties of $\tau(n)$.

- $\tau(ab) = \tau(a)\tau(b)$ where a and b are coprime.
- $\tau(p^{n+1}) = \tau(p)\tau(p^n) - p^{11}\tau(p^{n-1})$ where p is prime and $n > 0$ is an integer.
- $|\tau(p)| \leq 2p^{\frac{11}{2}}$ for all primes.

The first two properties were proved by Mordell [6], who used the operator which is now called Mordell operator, a special kind of Hecke operator. The third was proved by Deligne [3], who used the methods of algebraic geometry, especially the étale theory. For the coefficients $\tau(n)$, Ramanujan showed the following interesting congruence

$$\tau(p) \equiv 1 + p^{11} (\mathrm{mod}\, 691).$$

Besides this there are many other congruences on $\tau(n)$, for instance see ([4], [2]).

The exact values of $\tau(n)$ is also an interesting problem. It is well known that $\tau(n) \in \mathbb{Z}$ for all positive integers n. Lehmer [5] conjectured that $\tau(n) \neq 0$ for all positive integers n. Although this conjecture is still mysterious today, there are some formulas for the Ramanujan tau function. One of the most beautiful formula is the following:

$$\tau(n) = \frac{65}{756}\sigma_{11}(n) + \frac{691}{756}\sigma_5(n) - \frac{691}{3}\sum_{m=1}^{n-1}\sigma_5(m)\sigma_5(n-m)$$

where $\sigma_5(n)$ and $\sigma_{11}(n)$ are divisor functions. For more formulas see for example [1,7].

Our goal of this paper is to obtain some new formulas for the Ramanujan tau function. The first one (Theorem 2.2) is by combinatorial method, the second one (Theorem 3.5) is an integral representation and the third one (Theorem 4.5) is related to the Chebyshev polynomials.

Our approaches are completely elementary and we expect applications of the methods used in the paper to Fourier coefficients of other newforms as well.

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2. Combinatorial method

For a series expansion $f(q) = \sum_{n=0}^{\infty} a_n q^n$ we denote $\text{Coef}(f(q), q^n) := a_n$. We begin with following observation.

Lemma 2.1. *Let $f_n(q) := q \prod_{k=1}^{n} (1-q^k)^{24}$. Then*

$$\tau(n) = \text{Coef}(f_{n-1}(q), q^n).$$

Proof. This is just because the expansion of $(1-q^n)^{24}$ starts with

$$(1-q^n)^{24} = 1 - 24q^n + \cdots$$

The lemma then clearly follows from above expansion and the definition of $\tau(n)$:

$$q \prod_{n\geq 1} (1-q^n)^{24} = \sum_{n\geq 1} \tau(n) q^n.$$

□

We now give our first formula.

Theorem 2.2. *We have*

$$\tau(n) = \sum_{\mathbf{x}} (-1)^{x_1+\cdots+x_{n-1}} \binom{24}{x_1}\binom{24}{x_2}\cdots\binom{24}{x_{n-1}} \tag{2.1}$$

where the sum is over all nonnegative integer solutions $\mathbf{x} = (x_1, x_2, \cdots, x_{n-1})$ of the equation

$$x_1 + 2x_2 + \cdots + (n-1)x_{n-1} = n-1. \tag{2.2}$$

Remark 2.3. It is clear by definition that the number of nonnegative integer solutions of equation (2.2) $\#\{\mathbf{x}\}$ is equal to the partition function $p(n)$.

Proof. By Lemma 2.1, $\tau(n) = \text{Coef}(q\prod_{k=1}^{n-1}(1-q^k)^{24}, q^n)$. We have

$$(1-q)^{24} = \sum_{k=0}^{24} (-1)^k \binom{24}{k} q^k,$$

$$(1-q^2)^{24} = \sum_{k=0}^{24} (-1)^k \binom{24}{k} q^{2k},$$

$$\vdots$$

$$(1-q^{n-1})^{24} = \sum_{k=0}^{24} (-1)^k \binom{24}{k} q^{(n-1)k}.$$

Thus

$$q \prod_{k=1}^{n-1} (1-q^k)^{24}$$

$$= q\left(\sum_{k=0}^{24} (-1)^k \binom{24}{k} q^k\right)\left(\sum_{k=0}^{24} (-1)^k \binom{24}{k} q^{2k}\right)\cdots\left(\sum_{k=0}^{24} (-1)^k \binom{24}{k} q^{(n-1)k}\right).$$

Therefore if $\mathbf{x} = (x_1, x_2, \cdots, x_{n-1})$ is a nonnegative integer solution of the equation $x_1 + 2x_2 + \cdots + (n-1)x_{n-1} = n-1$, then

$$(-1)^{x_1}\binom{24}{x_1}(-1)^{x_2}\binom{24}{x_2}\cdots(-1)^{x_{n-1}}\binom{24}{x_{n-1}}$$

is a part of the coefficient of q^n in the expansion of $q\prod_{k=1}^{n-1}(1-q^k)^{24}$. Summing over all such solutions we obtain (2.1). □

Example 2.4.

As an example we compute $\tau(4)$. The nonnegative integer solutions of the equation

$$x_1 + 2x_2 + 3x_3 = 3$$

are $\mathbf{x} = (3,0,0)$, $\mathbf{x} = (0,0,1)$ and $\mathbf{x} = (1,1,0)$. Notice that $\binom{24}{0} = 1$, we then have

$$\tau(4) = (-1)^3\binom{24}{3} + (-1)^1\binom{24}{1} + (-1)^1\binom{24}{1}(-1)^1\binom{24}{1} = -1472.$$

One sees that the summand $(-1)^3\binom{24}{3}$ corresponds to the term $(-1)^3\binom{24}{3}q^3$ in the expansion of $(1-q)^{24}$, $(-1)^1\binom{24}{1}$ corresponds to the term $(-1)^1\binom{24}{1}q^3$ in the expansion of $(1-q^3)^{24}$ and $(-1)^1\binom{24}{1}(-1)^1\binom{24}{1}$ corresponds to the term $(-1)^1\binom{24}{1}q^1$ in the expansion of $(1-q)^{24}$ and the term $(-1)^1\binom{24}{1}q^2$ in the expansion of $(1-q^2)^{24}$.

3. An integral formula

In this section we give an integral formula for the Ramanujan tau function.

The first tau value is $\tau(1) = 1$. In the following we let $m \geq 2$. It follows from Lemma 2.1 that

$$\tau(m) = \mathrm{Coef}\left(\prod_{k=1}^{m-1}(1-q^k)^{24}, q^{m-1}\right). \tag{3.1}$$

We have following polynomial expansion:

$$\prod_{k=1}^{m-1}(1-q^k)^{24} = 1 - 24q + \cdots + a_{m-1}q^{m-1} + \cdots - 24q^{12m(m-1)-1} + q^{12m(m-1)}. \tag{3.2}$$

It is not hard to see the symmetry of the coefficients of the expansion above.

Lemma 3.1. *Let $i \geq 1$. We have*

$$\mathrm{Coef}\left(\prod_{k=1}^{m-1}(1-q^k)^{24}, q^i\right) = \mathrm{Coef}\left(\prod_{k=1}^{m-1}(1-q^k)^{24}, q^{12m(m-1)-i}\right).$$

Proof. We leave to the readers as an exercise. □

Before we proceed further we recall that the Chebyshev polynomials of the first kind $T_n(x)$ is defined as

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x). \end{aligned}$$

$T_n(x)$ has the property that for $x \neq 0$,

$$T_n\left(\frac{x+x^{-1}}{2}\right) = \frac{x^n + x^{-n}}{2}, \tag{3.3}$$

for more about Chebyshev polynomials see [9].

Set $t := \frac{q+q^{-1}}{2}$. Combining (3.2), (3.3) and Lemma 3.1 we have

$$\frac{(1-q)^{24}}{q^{12}} = \left(q+q^{-1}-2\right)^{12} = \sum_{n=0}^{12} c_{1,n} T_n(t) \tag{3.4}$$

where

$$c_{1,n} = \begin{cases} \mathrm{Coef}\big((1-q)^{24}, q^{12}\big), & n = 0; \\ 2\mathrm{Coef}\big((1-q)^{24}, q^{12-n}\big), & 1 \leq n \leq 12. \end{cases}$$

That is,

$$2^{12}(T_1(t)-1)^{12} = \sum_{n=0}^{12} c_{1,n} T_n(t). \tag{3.5}$$

Similarly we have

$$2^{24}(T_1(t)-1)^{12}(T_2(t)-1)^{12} = \sum_{n=0}^{36} c_{2,n} T_n(t), \tag{3.6}$$

where

$$c_{2,n} = \begin{cases} \mathrm{Coef}\big((1-q)^{24}(1-q^2)^{24}, q^{36}\big), & n = 0; \\ 2\mathrm{Coef}\big((1-q)^{24}(1-q^2)^{24}, q^{36-n}\big), & 1 \leq n \leq 36. \end{cases}$$

In general we conclude that

Proposition 3.2.

$$2^{12(m-1)} \prod_{k=1}^{m-1} (T_k(t)-1)^{12} = \sum_{n=0}^{6m(m-1)} c_{m-1,n} T_n(t),$$

where

$$c_{m-1,n} = \begin{cases} \mathrm{Coef}\left(\prod_{k=1}^{m-1}(1-q^k)^{24}, q^{6m(m-1)}\right), & n = 0; \\ 2\mathrm{Coef}\left(\prod_{k=1}^{m-1}(1-q^k)^{24}, q^{6m(m-1)-n}\right), & 1 \leq n \leq 6m(m-1). \end{cases}$$

By the proposition above and (3.1) we obtain

$$\tau(m) = \frac{1}{2} c_{m-1,(6m-1)(m-1)}. \tag{3.7}$$

Numerical experiments suggest that all the coefficients in the expansion (3.2) are nonzero:

Conjecture 3.3. *For all $m \geq 2$ and $0 \leq n \leq 6m(m-1)$, $c_{m-1,n} \neq 0$. In particular the Ramanujan tau function $\frac{1}{2}c_{m-1,(6m-1)(m-1)} \neq 0$.*

Next we recall the following result of Chebyshev expansion [9, §3].

Lemma 3.4. *Let $f(x)$ be a function integrable on the interval $[-1,1]$. There is a Chebyshev expansion*

$$f(x) \sim {\sum_{n\geq 0}}' a_n T_n(x) = \frac{1}{2}a_0 T_0(x) + a_1 T_1(x) + a_2 T_2(x) + \cdots$$

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where

$$a_n = \frac{2}{\pi}\int_{-1}^{1} \frac{T_n(x)f(x)}{\sqrt{1-x^2}}dx, \qquad n = 0, 1, \cdots$$

and the dash means that the first term in the series is halved.

By Proposition 3.2,

$$2^{12(m-1)}\prod_{k=1}^{m-1}(T_k(t)-1)^{12} = \sum_{n=0}^{6m(m-1)} c_{m-1,n}T_n(t).$$

We apply Lemma 3.4 by taking $f(t) = 2^{12(m-1)}\prod_{k=1}^{m-1}(T_k(t)-1)^{12}$ to obtain

$$c_{m-1,n} = \frac{2^{12(m-1)+1}}{\pi}\int_{-1}^{1}\frac{T_n(t)\prod_{k=1}^{m-1}(T_k(t)-1)^{12}}{\sqrt{1-t^2}}dt \tag{3.8}$$

for $1 \le n \le 6m(m-1)$. In particular let $n = (6m-1)(m-1)$ and combine (3.7) we obtain our main result of this section.

Theorem 3.5. *Let $m \ge 2$. The Ramanujan tau function is*

$$\tau(m) = \frac{2^{12(m-1)}}{\pi}\int_{-1}^{1}\frac{T_{(6m-1)(m-1)}(t)\prod_{k=1}^{m-1}(T_k(t)-1)^{12}}{\sqrt{1-t^2}}dt.$$

Thus we can interpret the Ramanujan tau function $\tau(m)$ as inner products of functions $T_{(6m-1)(m-1)}(t)$ and $\prod_{k=1}^{m-1}(T_k(t)-1)^{12}$ in certain inner-product space.

4. Another formula

From the second and third properties of $\tau(n)$ proposed by Ramanujan, it can be deduced the following well known formula.

Lemma 4.1 ([10], p.30.). *One has*

$$\tau(p^n) = p^{\frac{11n}{2}}\frac{\sin((n+1)\theta)}{\sin(\theta)} \tag{4.1}$$

where

$$\cos\theta = \frac{\tau(p)}{2p^{\frac{11}{2}}},$$

and p is prime, $n \ge 0$.

We now relate the Ramanujan τ function to the Chebyshev polynomials of the second kind $U_n(x)$. Recall from [9] that $U_n(x)$ is defined as

$$U_0(x) = 1$$
$$U_1(x) = 2x$$
$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$$

and it has the following well known property.

Lemma 4.2. *One has*

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin(\theta)} \tag{4.2}$$

for $n \geq 0$.

There is an explicit expression for $U_n(x)$.

Lemma 4.3. *One has for integers* $n \geq 0$.

$$U_n(x) = \frac{\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}}{2\sqrt{x^2-1}} \tag{4.3}$$

Now by Lemma 4.1 and Lemma 4.2, the following result can be deduced.

Proposition 4.4. *One has*

$$\tau(p^n) = p^{\frac{11n}{2}} U_n\left(\frac{\tau(p)}{2p^{\frac{11}{2}}}\right) \tag{4.4}$$

for $n \geq 0$.

Together with Lemma 4.3 and after a short calculation one obtains the following formula for $\tau(p^n)$ and hence for $\tau(n)$.

Theorem 4.5. *For a prime* p *and* $n \geq 0$,

$$\tau(p^n) = \frac{\left(\tau(p)+\sqrt{\tau(p)^2-4p^{11}}\right)^{n+1} - \left(\tau(p)-\sqrt{\tau(p)^2-4p^{11}}\right)^{n+1}}{2^{n+1}\sqrt{\tau(p)^2-4p^{11}}} \tag{4.5}$$

Proof. In view of Lemma 4.3 and Proposition 4.4, it is just a direct computation. □

The formula above for $\tau(p^n)$ extends multiplicatively to $\tau(n)$, and the formula of $\tau(n)$ is obtained.

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