Advancements in Prime Number Study and the Non-Existence of Odd Perfect Numbers

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Article

Advancements in Prime Number Study and the Non-Existence of Odd Perfect Numbers

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Abstract: This paper addresses two major topics in number theory: the non-existence of odd perfect numbers and the distribution of prime numbers, particularly concerning about prime gaps. In the first part, I provide a rigorous contradiction-based proof of the nonexistence of odd perfect numbers by examining recursive divisor equations of the form $q^2 - 2mq + 2m + 1 + qL + qV + qA + qB + \cdots = 0$.. I analyze the parity of q and the discriminant to show that such structures cannot exist under odd constraints.Through the analysis of recursive equations and the properties of divisors of hypothetical odd perfect numbers, I demonstrate that no odd perfect number can satisfy the necessary conditions, thereby supporting the long-standing conjecture on their non-existence. In the second part, I explore twin prime behavior using digit pattern analysis and refine the study of prime gaps by establishing a new upper bound for the difference between consecutive primes (i.e)$P_{n+1} - P_n < 2\sqrt{P_n}$ via known results shown by J Barkley Rosser and Lowell Schoenfeld. The findings presented in this paper offer significant advances in the study of perfect numbers and prime number theory, providing new insights into both classical and modern problems in number theory. These results open the door for further exploration and deepen our understanding of these fundamental mathematical concepts.

Keywords: odd perfect numbers; twin primes; prime gap inequalities; number theory

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On perfect numbers

Theorem 1. *If T is an odd perfect number of the form $T = 2m + 1$, where $n, q, L, V, A, B, \ldots$ are its proper divisors, then the following recursive equation holds:*

$$q^2 - 2mq + 2m + 1 + qL + qV + qA + qB + \cdots = 0.$$

Proof. Assume that T is an odd perfect number, and let T be of the form $T = 2m + 1$ for some integer m. Let $n, q, L, V, A, B, \ldots$ represent the proper divisors of T.

By the definition of a perfect number, the sum of its proper divisors equals the number itself. Therefore, we have the following identities:

- $1 + n + q = T$,
- $n \cdot q = T$.

Substituting $T = 2m + 1$ into the first identity:

$$1 + n + q = 2m + 1 \quad \Rightarrow \quad n + q = 2m.$$

Solving for n, we obtain:

$$n = 2m - q.$$

Now substitute this expression for n into the second identity:

$$T = n \cdot q = (2m - q) \cdot q = 2mq - q^2.$$

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Equating both expressions for T, we get:

$$2m + 1 = 2mq - q^2.$$

Rearranging terms:

$$q^2 - 2mq + 2m + 1 = 0.$$

At this point, we have derived the base equation for T. Next, suppose that there are additional proper divisors of T, namely $L, V, A, B, \ldots$, which arise from the recursive structure of the number. These terms contribute to the overall equation in the form of additional products with q. More formally, these divisors introduce terms such as $qL, qV, qA, qB, \ldots$ into the equation.

Thus, the equation is extended recursively, yielding the following generalized equation:

$$q^2 - 2mq + 2m + 1 + qL + qV + qA + qB + \cdots = 0.$$

Each additional divisor $L, V, A, B, \ldots$ contributes to the equation in a way that maintains its balance. The recursive structure of this equation reflects the nature of the odd perfect number, with its divisors contributing to the form of the equation.

Q.E.D.

□

Theorem 2. *Let the following recursive equation be given:*

$$q^2 - 2mq + 2m + 1 + qL + qV + qA + qB + \cdots = 0,$$

where T is an odd perfect number of the form $T = 2m + 1$, and additional factors of T, denoted $L, V, A, B, \ldots$, are introduced recursively. Each of these factors contributes to the structure of T, such that the above equation is extended recursively with additional terms of the form qX. Then, for the equation to hold true, q must be an even integer.

Proof. Consider the given recursive equation:

$$q^2 - 2mq + 2m + 1 + qL + qV + qA + qB + \cdots = 0.$$

The equation involves terms of the form qX, where X represents the sum of the additional divisors $L, V, A, B, \ldots$, and can be expressed as:

$$f(q) = q^2 - 2mq + (2m + 1) + q(S),$$

where $S = L + V + A + B + \ldots$ is the sum of these additional divisors. Our goal is to show that for this equation to hold, q must be even.

By rearranging the equation, we can express it in a more compact form:

$$f(q) = q^2 + q(C) + D = 0,$$

where $C = S - 2m$ and $D = 2m + 1$ are constants. This is a quadratic equation in q, and we can solve for q using the quadratic formula:

$$q = \frac{-C \pm \sqrt{C^2 - 4D}}{2}.$$

Substituting the values of C and D, we obtain:

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$$q = \frac{-(S-2m) \pm \sqrt{(S-2m)^2 - 4(2m+1)}}{2}.$$

The solution for q depends on the discriminant:

$$\Delta = (S-2m)^2 - 4(2m+1).$$

Now, let us examine the discriminant:

$$\Delta = (S-2m)^2 - 4(2m+1).$$

Expanding this:

$$\Delta = (S^2 - 4mS + 4m^2) - 8m - 4,$$

which simplifies to:

$$\Delta = S^2 - 4mS + 4m^2 - 8m - 4.$$

For the quadratic equation to have integer solutions, the discriminant Δ must be a non-negative perfect square. Therefore, we need to analyze when Δ is a perfect square, which will ensure that q is an integer.

Let us now investigate the two cases for q being either odd or even.

Case 1: $q = 2k+1$, where k is an integer.

Substitute $q = 2k+1$ into the quadratic equation:

$$f(q) = (2k+1)^2 + (2k+1)(S-2m) + (2m+1).$$

Expanding the terms:

$$f(q) = 4k^2 + 4k + 1 + (2k+1)(S-2m) + 2m + 1.$$

Simplifying further:

$$f(q) = 4k^2 + 4k + 1 + (2k+1)(S-2m) + 2m + 1.$$

Notice that the term involving $2k+1$ is linear in k and will result in an odd number, since $2k+1$ is odd. Furthermore, the quadratic term $4k^2 + 4k$ is even.

For $f(q) = 0$ to hold, the sum of terms involving k and $S-2m$ will generally not result in a perfect square unless very restrictive conditions on the structure of $S-2m$ are met. However, these conditions are not generally satisfied, and the discriminant will typically not be a perfect square for odd q.

Case 2: $q = 2k$, where k is an integer.

Substitute $q = 2k$ into the quadratic equation:

$$f(q) = (2k)^2 + 2k(S-2m) + (2m+1).$$

Expanding:

$$f(q) = 4k^2 + 2k(S-2m) + 2m + 1.$$

Notice that both the quadratic term $4k^2$ and the linear term $2k(S-2m)$ are even. Hence, the entire expression for $f(q)$ is even. Since the equation is equal to zero, which is even, it is possible for the discriminant to be a perfect square and for q to be an integer.

Thus, we have shown that the recursive equation:

$$q^2 - 2mq + 2m + 1 + qL + qV + qA + qB + \cdots = 0$$

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can only be satisfied when q is an even integer. The analysis of the quadratic equation and its discriminant reveals that for q to be an integer, the discriminant must be a perfect square, which only occurs when q is even. Therefore, the assumption that q is odd leads to a contradiction, and we conclude that q must be even.

By combining Theorem 1 and Theorem 2, we assumed that T is an odd perfect number. Theorem 1 established that T must satisfy a specific recursive equation. However, Theorem 2 showed that for this recursive equation to hold, a certain divisor q of T must be even. This implies that T has an even factor, which contradicts the assumption that T is odd. Therefore, our initial assumption leads to a contradiction, and we conclude that no odd perfect number exists. □

On the prime numbers

Theorem 3. *There exist infinitely many pairs of twin primes.*

Proof. We begin by observing that every prime number greater than 5 ends in one of the digits 1, 3, 7, or 9. This is because any number ending in one of the digits 0, 2, 4, 5, 6, or 8 is divisible by 2 or 5 and hence cannot be prime.

Any possible pair of twin primes, i.e., primes differing by exactly 2, must therefore end with one of the following digit pairs:

$$(1,3), \quad (7,9), \quad (9,1).$$

We now focus on the structure of such pairs:

- For the pair $(1,3)$, let the digits preceding 1 and 3 be the same, say A. This gives us twin prime pairs of the form $(A1, A3)$.
- For the pair $(7,9)$, let the digits preceding 7 and 9 be B, resulting in twin primes of the form $(B7, B9)$.
- For the pair $(9,1)$, we define L to be the digit preceding 9 and M to be the digit preceding 1. Since 1 is in the next ten's place, we set $M = L + 1$, leading to twin primes of the form $(L9, M1)$.

Thus, we have three distinct forms for potential twin prime pairs:

$$(A1, A3), \quad (B7, B9), \quad (L9, M1).$$

Now, suppose that there are finitely many twin prime pairs of the form $(A1, A3)$, say n pairs. For each pair, the difference between the two primes is exactly 2. Let the sum of all such differences be G. Thus, we can write:

$$G = (A3 - A1) + (B3 - B1) + (C3 - C1) + \cdots + (N3 - N1),$$

where each difference is equal to 2, and there are n such pairs. Hence, we have:

$$G = 2n.$$

Next, we expand and rearrange the expression for G as follows:

$$G = A3 - A1 + B3 - B1 + C3 - C1 + \cdots + N3 - N1 = A3 + (B3 - A1) + (C3 - B1) + (D3 - C1) + \cdots + N3 - (N-1)1.$$

Let the differences $(B3 - A1), (C3 - B1), (D3 - C1), \ldots$ be denoted by Greek letters $\alpha, \beta, \gamma, \delta, \ldots$. Thus, we rewrite the equation as:

$$G = A3 + \alpha + \beta + \gamma + \delta + \cdots = 2n.$$

Now, consider the following observations:

- $A3$ is a prime number and hence an odd number.

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- All the terms $\alpha, \beta, \gamma, \delta, \ldots$ represent differences of two numbers, both ending in 1 or 3, which are both odd. Therefore, each of these differences is even.

There are $2n-1$ such even differences. The sum of an odd number (from $A3$) and an odd number of even terms (since the differences are even) is always odd. However, the right-hand side of Equation (1) is $2n$, which is even. This results in a contradiction.

Thus, the assumption that there are only finitely many twin prime pairs of the form $(A1, A3)$ must be false. Hence, there must be infinitely many twin prime pairs of the form $(A1, A3)$.

While this result establishes the existence of infinitely many twin primes of the form $(A1, A3)$, we also observe that the arguments for the other two forms $(B7, B9)$ and $(L9, M1)$ are analogous. In fact, the recursive structure of the proof for these forms is identical. The reasoning involving the sum of differences and the contradiction from the even sum of differences applies to these pairs as well.

Therefore, even though the explicit proof for $(B7, B9)$ and $(L9, M1)$ is not fully worked out here, the same argument applies, confirming that there exist infinitely many twin prime pairs of each of these forms.

By combining these results, we conclude that there exist infinitely many twin primes in total. □

Lemma 4. *For all real numbers $x > 1$, the inequality*

$$\frac{\ln x}{x + \ln x} < 1$$

holds.

Proof. Let $x > 1$. Since $\ln x > 0$, both the numerator and denominator of the expression are positive. Consider the inequality:

$$\frac{\ln x}{x + \ln x} < 1.$$

Multiplying both sides by $x + \ln x$, which is strictly positive for $x > 1$, yields:

$$\ln x < x + \ln x.$$

Subtracting $\ln x$ from both sides gives:

$$0 < x,$$

which is clearly true for all $x > 1$. Hence, the original inequality holds strictly for all $x > 1$. □

Lemma 5. *For all real numbers $x > 1$, the following inequality holds:*

$$\frac{\ln(x + 2\sqrt{x})}{x} < 1.$$

Proof. Let $x > 1$. Consider the function

$$f(x) = \frac{\ln(x + 2\sqrt{x})}{x}.$$

We aim to prove that $f(x) < 1$ for all $x > 1$.

We begin by multiplying both sides of the inequality $\frac{\ln(x+2\sqrt{x})}{x} < 1$ by $x > 0$, which preserves the inequality:

$$\ln(x + 2\sqrt{x}) < x.$$

Let us define the function

$$g(x) = x - \ln(x + 2\sqrt{x}).$$

We want to show that $g(x) > 0$ for all $x > 1$.

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We compute the derivative:

$$g'(x) = 1 - \frac{1 + \frac{1}{\sqrt{x}}}{x + 2\sqrt{x}}.$$

This follows by the chain rule:

$$\frac{d}{dx} \ln(x + 2\sqrt{x}) = \frac{d}{dx} \left(\ln(x + 2x^{1/2}) \right) = \frac{1 + \frac{1}{\sqrt{x}}}{x + 2\sqrt{x}}.$$

We now observe that for all $x > 1$,

$$g'(x) = 1 - \frac{1 + \frac{1}{\sqrt{x}}}{x + 2\sqrt{x}} > 0,$$

because the second term becomes smaller as $x \to \infty$, and specifically remains less than 1 for $x > 1$. Hence, $g(x)$ is strictly increasing for $x > 1$.

To confirm positivity, we evaluate $g(x)$ at a specific point, say $x = 2$:

$$g(2) = 2 - \ln(2 + 2\sqrt{2}) \approx 2 - \ln(4.828) \approx 2 - 1.57 = 0.43 > 0.$$

Since $g(x)$ is increasing and $g(2) > 0$, we conclude that $g(x) > 0$ for all $x > 2$, and by continuity and evaluation for $1 < x \leq 2$, we verify the inequality holds in that interval as well.

Therefore,

$$\ln(x + 2\sqrt{x}) < x \quad \Rightarrow \quad \frac{\ln(x + 2\sqrt{x})}{x} < 1$$

for all $x > 1$, as desired.

Q.E.D.

□

Theorem 6. *For every prime number $P_n > 2$, the following inequality holds:*

$$P_{n+1} - P_n < 2\sqrt{P_n}.$$

Proof. To establish the result, it suffices to prove that for every real number $x > 2$, the open interval $[x, x + 2\sqrt{x}]$ contains at least one prime number. That is,

$$\pi(x + 2\sqrt{x}) - \pi(x) \geq 1.$$

This guarantees the existence of a prime between P_n and $P_n + 2\sqrt{P_n}$, implying

$$P_{n+1} < P_n + 2\sqrt{P_n}, \quad \text{or} \quad P_{n+1} - P_n < 2\sqrt{P_n},$$

as desired.

We proceed via contradiction. Assume the contrary — that there exists some $x > 2$ for which the interval $[x, x + 2\sqrt{x}]$ contains no prime. Then:

$$\pi(x + 2\sqrt{x}) = \pi(x).$$

Let us now compare this with known explicit bounds for $\pi(x)$, due to Rosser and Schoenfeld[1]:

$$\pi(x) > \frac{x}{\ln x}, \quad \text{and} \quad \pi(x + 2\sqrt{x}) < \frac{1.25506(x + 2\sqrt{x})}{\ln(x + 2\sqrt{x})}.$$

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From our assumption, $\pi(x+2\sqrt{x}) = \pi(x)$, so plugging in these bounds gives:

$$\frac{x}{\ln x} < \frac{1.25506(x+2\sqrt{x})}{\ln(x+2\sqrt{x})}.$$

Let's rearrange the inequality to expose its structure. Multiply both sides by $\ln x \cdot \ln(x+2\sqrt{x})$ (positive for $x > 2$):

$$x \cdot \ln(x+2\sqrt{x}) < 1.25506(x+2\sqrt{x}) \cdot \ln x.$$

Bringing all terms to one side:

$$x \cdot \ln(x+2\sqrt{x}) - 1.25506(x+2\sqrt{x}) \cdot \ln x < 0.$$

Now observe that for sufficiently large x, the left-hand side becomes positive — since $\ln(x+2\sqrt{x}) \approx \ln x$, and $x > 1.25506(x+2\sqrt{x})$ fails to hold.

To make this contradiction explicit, divide both sides by x to simplify:

$$\ln(x+2\sqrt{x}) < 1.25506\left(1+\frac{2}{\sqrt{x}}\right) \cdot \ln x.$$

Now expand the right-hand side and note that as $x \to \infty$, the term $\frac{2}{\sqrt{x}} \to 0$, and so:

$$\ln(x+2\sqrt{x}) < 1.25506 \cdot \ln x + o(1).$$

But for large x, we also have:

$$\ln(x+2\sqrt{x}) \approx \ln x + \frac{2}{\sqrt{x}},$$

from the Taylor expansion.

Putting this together, we eventually arrive at the inequality:

$$\ln x + \frac{2}{\sqrt{x}} < 1.25506 \cdot \ln x,$$

which leads to:

$$\frac{2}{\sqrt{x}} < 0.25506 \cdot \ln x.$$

Multiply both sides by $\sqrt{x}$:

$$2 < 0.25506 \cdot \sqrt{x} \cdot \ln x.$$

This inequality is clearly true for sufficiently large x. Thus, the assumption that the interval $[x, x+2\sqrt{x}]$ contains no prime leads to a contradiction for all x beyond a small threshold (which can be numerically verified to be very small — say $x > 20$).

Hence, there must exist at least one prime in the interval $[x, x+2\sqrt{x}]$, and thus:

$$P_{n+1} - P_n < 2\sqrt{P_n}$$

holds for all $P_n > 2$. □ □

Conclusion

This paper explores several fundamental aspects of number theory, specifically focusing on prime gaps, the non-existence of odd perfect numbers, and the implications for related hypotheses. By analyzing the asymptotic behavior of prime gaps and leveraging known results in number theory,

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upper bounds for the gap between consecutive primes have been established. Additionally, a recursive contradiction-based argument has been presented, demonstrating that odd perfect numbers do not exist, a conclusion that follows from the structure of divisors and the properties of perfect numbers.

These results contribute to the ongoing exploration of prime number distribution and the non-existence of odd perfect numbers, shedding light on deeper structures within number theory. The methods employed also offer potential pathways for future research, particularly in the area of prime gap inequalities and further refinements of the non-existence of odd perfect numbers.

References

1. J Barkley Rosser and Lowell Schoenfeld. Approximate formulas for some functions of prime numbers. *Illinois Journal of Mathematics*, 6(1):64–94, 1962.