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Keywords: Castigliano Theorem; generalized Castigliano Theorem; virtual work principle; elasticity theory; numerical example

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A Formal Proof of Castigliano Theorem and a Related Generalization

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Abstract

This short communication develops a formal proof of Castilgiano Theorem in a elasticity context. The results are base on standard tools of applied functional analysis and calculus of variations. It is worth mentioning such results here presented may be easily extended to a non-linear elasticity context. Finally, in the last section we present a numerical example in order to illustrate the results applicability.

Keywords: Castigliano Theorem; generalized Castigliano Theorem; virtual work principle; elasticity theory; numerical example

MSC: 74G05

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1. Introduction

In the next section we present the mathematical formalism of a result in elasticity theory known as the Castigliano's Theorem.

We also present a generalization of such an theorem and its connection with the principle of virtual work in a elasticity theory context.

The results are obtained through an application of basic tools of functional analysis and calculus variations to a solid mechanics theory.

Our main reference in solid mechanics is [1]. Similar results have been presented in section 40 of preprint [2].

For basic topics on the applied functional analysis and calculus of variations, please see [3,4].

For the Sobolev spaces involved, we would cite [5].

2. A Formal Proof of Castigliano Theorem

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with a regular (Lipischitzian) boundary denoted by $\partial\Omega$.

In a context of linear elasticity, consider the functional $J : V \to \mathbb{R}$ where

$$J(u) = E_{in} - \langle u_i, f_i \rangle_{L^2} - \sum_{j=1}^{N} u_i(x_j) P_{ij},$$

$u = (u_1, u_2, u_3) \in W_0^{1,2}(\Omega; \mathbb{R}^3) \equiv V$, $f = (f_1, f_2, f_3) \in L^2(\Omega; \mathbb{R}^3)$, $Y = Y^* = L^2(\Omega; \mathbb{R}^3)$, and

$$P_{ij} \in \mathbb{R},\ \forall i \in \{1,2,3\},\ j \in \{1, \cdots, N\}$$

for some $N \in \mathbb{N}$.

Here we have denoted

$$E_{in} = \frac{1}{2} \int_\Omega H_{ijkl} e_{ij}(u) e_{kl}(u)\, dx,$$

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$$e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Moreover H_{ijkl} is a fourth order positive definite and constant tensor.
Observe that the variation of J in u_i give us the following Euler-Lagrange equation

$$-(H_{ijkl}e_{kl}(u))_{,j} - f_i - \sum_{j=1}^{N} P_{ij}\delta(x_j) = \mathbf{0}, \text{ in } \Omega. \tag{1}$$

Symbolically such a system stands for

$$\frac{\partial J(u)}{\partial u_i} = \mathbf{0}, \forall i \in \{1,2,3\},$$

so that

$$\frac{\partial(E_{in} - \langle u_i, f_i\rangle_{L^2} - \sum_{j=1}^{N} u_i(x_j)P_{ij})}{\partial u_i} = \mathbf{0}, \ \forall i \in \{1,2,3\}. \tag{2}$$

We denote $u \in V$ solution of (1) by $u = u(f,P)$, so that multiplying the concerning extremal equation by u_i and integrating by parts, we get

$$\begin{aligned} H_1(u(f,P),f,P) &= 2E_{in}(u(f,P)) - \langle u_i(f,P), f_i\rangle_{L^2} - \sum_{j=1}^{N} u_i(x_j,f,P)P_{ij} \\ &= 0, \ \forall f \in Y^*, \ P \in \mathbb{R}^{3N}. \end{aligned} \tag{3}$$

Therefore

$$\frac{d}{dP_{ij}}(H_1(u(f,P),f,P)) = 0,$$

so that

$$2\frac{dE_{in}}{dP_{ij}} - \frac{d}{dP_{ij}}\left(\langle u_i(f,P), f_i\rangle_{L^2} + \sum_{j=1}^{N} u_i(x_j,f,p)P_{ij}\rangle_{L^2}\right) = 0,$$

that is

$$\begin{aligned} &\frac{dE_{in}}{dP_{ij}} + \left\langle \frac{\partial(E_{in} - \langle u_i, f_i\rangle_{L^2} - \sum_{j=1}^{N} u_i(x_j)P_{ij})}{\partial u_k}, \frac{\partial u_k}{\partial P_{ij}} \right\rangle_{L^2} \\ &-\frac{\partial}{\partial P_{ij}}\left(\langle u_i, f_i\rangle_{L^2} + \sum_{j=1}^{N} u_i(x_j)P_{ij}\right) \\ =\ & 0. \end{aligned} \tag{4}$$

From this and (1) we obtain

$$\frac{dE_{in}}{dP_{ij}} - u_i(x_j) = 0,$$

so that

$$u_i(x_j) = \frac{dE_{in}}{dP_{ij}} = \frac{d}{dP_{ij}}\left(\frac{1}{2}\int_\Omega H_{ijkl}e_{ij}(u(f,P))e_{kl}(u(f,P))\,dx\right),$$

$\forall i \in \{1,2,3\}, \forall j \in \{1,\cdots,N\}$.
With such results in mind, we have proven the following theorem.

Theorem 1 (Castigliano)**.** *Considering the notations and definitions in this section, we have*

$$u_i(x_j) = \frac{dE_{in}}{dP_{ij}} = \frac{d}{dP_{ij}}\left(\frac{1}{2}\int_\Omega H_{ijkl}e_{ij}(u(f,P))e_{kl}(u(f,P))\,dx\right),$$

$\forall i \in \{1,2,3\}, \forall j \in \{1,\cdots,N\}$.

2.1. *A Generalization of Castigliano Theorem*

In this subsection, denoting by $\delta(x_k)$ a standard Dirac delta function in a distributional sense, we present a more general version of the Castigliano theorem.

Considering the context of last section, we recall that

$$\begin{aligned} H_1(u(f,P),f,P) &= 2E_{in}(u(f,P)) - \langle u_i(f,P), f_i\rangle_{L^2} - \sum_{j=1}^N u_i(x_j,f,P)P_{ij} \\ &= 0,\ \forall f \in Y^*,\ P \in \mathbb{R}^{3N}. \end{aligned} \tag{5}$$

Therefore, here denoting the Gâteaux derivative of H_1 related to f_i by

$$\frac{d}{df_i}(H_1(u(f,P),f,P)),$$

for $x_k \in \Omega$ such that

$$x_k \neq x_j,\ \forall j \in \{1,\cdots,N\},$$

we have

$$\left\langle \frac{d}{df_i}(H_1(u(f,P),f,P)), \delta(x_k)\right\rangle_{L^2} = 0,$$

so that

$$\begin{aligned} & 2\left\langle \frac{d}{df_i}(E_{in}(u(f,P))), \delta(x_k)\right\rangle_{L^2} \\ & -\left\langle \frac{d}{df_i}\left(\langle u_i(f,P), f_i\rangle_{L^2} + \sum_{j=1}^N u_i(x_j,f,p)P_{ij}\rangle_{L^2}\right), \delta(x_k)\right\rangle_{L^2} \\ = {} & 0, \end{aligned} \tag{6}$$

that is

$$\begin{aligned} & \left\langle \frac{d}{df_i}(E_{in}(u(f,P))), \delta(x_k)\right\rangle_{L^2} \\ & +\left\langle \frac{d}{du_k}\left(E_{in}(u(f,P)) - \langle u_i(f,P), f_i\rangle_{L^2} - \sum_{j=1}^N u_i(x_j,f,p)P_{ij}\right)\frac{du_k}{df_i}, \delta(x_k)\right\rangle_{L^2} \\ & -\left\langle \frac{\partial}{\partial f_i}\left(\langle u_i(f,P), f_i\rangle_{L^2} - \sum_{j=1}^N u_i(x_j,f,p)P_{ij}\right), \delta x(x_k)\right\rangle_{L^2} \\ = {} & 0. \end{aligned} \tag{7}$$

From such results, we may obtain

$$\left\langle \frac{d}{df_i}(E_{in}(u(f,P))), \delta(x_k)\right\rangle_{L^2} - \langle u_i(x), \delta(x_k)\rangle_{L^2} = 0,$$

so that

$$\left\langle \frac{d}{df_i}(E_{in}(u(f,P))), \delta(x_k)\right\rangle_{L^2} - u_i(x_k) = 0,$$

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that is

$$u_i(x_k) = \left\langle \frac{d}{df_i}(E_{in}(u(f,P))), \delta(x_k) \right\rangle_{L^2},$$

$\forall i \in \{1,2,3\}$, $\forall x_k \in \Omega$ such that $x_k \neq x_j$, $\forall j \in \{1, \cdots, N\}$.

With such results in mind, we have proven the following theorem.

Theorem 2 (The Generalized Castigliano Theorem). *Considering the notations and definitions in this section, here again denoting the Gâteaux derivative of H_1 related to f_i by*

$$\frac{d}{df_i}(H_1(u(f,P),f,P)),$$

we have

$$u_i(x_k) = \left\langle \frac{d}{df_i}(E_{in}(u(f,P))), \delta(x_k) \right\rangle_{L^2},$$

$\forall i \in \{1,2,3\}$, $\forall x_k \in \Omega$ *such that* $x_k \neq x_j$, $\forall j \in \{1, \cdots, N\}$.

2.2. *The Virtual Work Principle*

Considering the definitions, results and statements of the previous section and subsection, we may easily prove the following theorem.

Theorem 3 (The virtual work principle). *Let $x_l \in \Omega$ such that $x_l \neq x_j, \forall j \in \{1, \cdots, N\}$.*

For a virtual constant load $P_{lk} \in \mathbb{R}$ on x_l at the direction of $u_k(x_l)$, define now $J : V \to \mathbb{R}$ where

$$J(u) = E_{in} - \langle u_i, f_i \rangle_{L^2} - \sum_{j=1}^{N} u_i(x_j)P_{ij} - P_{lk}u_k(x_l).$$

Under such hypotheses,

$$u_k(x_l) = \left(\frac{d\ E_{in}(u(f,P,P_{lk}))}{dP_{lk}} \right)_{P_{lk}=0},$$

$\forall k \in \{1,2,3\}$, $\forall x_l \in \Omega$ *such that* $x_l \neq x_j$, $\forall j \in \{1, \cdots, N\}$.

Proof. The proof is exactly the same as in the Castigliano Theorem in the previous section except by setting the virtual load $P_{lk} = 0$ in the end of this calculation and will not be repeated. $\square$

2.3. *A Numerical Example Related to the Castigliano Theorem*

Let $\Omega = [0,1] \subset \mathbb{R}$ be the axis of a straight beam with a rectangular cross section of dimensions $b \times h$, where units in this subsections refer to the international system.

Let

$$I = \frac{bh^3}{12}$$

and denote by $E > 0$ the Young modulus for a steel beam.

Assume such a beam is subject to a vertical load $P > 0$ uniformly distributed on Ω.

Assume also the beam is clamped at $x = 0$ and simply supported at $x = 1$.

Denoting by $w \in V = W^{2,2}(\Omega)$ the vertical field of displacements results from the action of P, the related boundary conditions are given by

$$w(0) = w(1) = 0,\ w_{,x}(0) = 0,\ w_{,xx}(1) = 0.$$

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The total beam energy is defined by $J : V \to \mathbb{R}$, where

$$J(w) = \frac{EI}{2}\int_\Omega w_{,xx}^2\,dx - \int_\Omega Pw\,dx.$$

In order to apply the results of the previous section, we free the rotations of the beam at $x = 0$, considering a moment load M on $x = 0$ with general work

$$Mw_{,x}(0).$$

Hence, we define

$$V_1 = \{w \in W^{2,2}(\Omega)\ :\ w(0) = w(1) = w_{,xx}(1) = 0\},$$

and define $J_1 : V_1 \to \mathbb{R}$, by

$$J_1(w) = \frac{EI}{2}\int_\Omega w_{,xx}^2\,dx - \int_\Omega Pw\,dx - M\,w_{,x}(0).$$

We emphasize again, through the methods of the previous section, we intend to obtain the value of M which corresponds to

$$w_{,x}(0) = 0.$$

Let $\varphi \in W^{2,2}$. The variation

$$\delta J_1(w;\varphi),$$

stands for

$$\delta J_1(w;\varphi) = EI\int_\Omega w_{,xx}\varphi_{,xx}\,dx - \int_\Omega P\varphi\,dx - M\varphi_{,x}(0).$$

Here assuming, w is smooth enough, integrating by parts and recalling that $\varphi \in V_1$, we obtain

$$\delta J_1(w;\varphi) = EI\int_\Omega w_{,xxxx}\varphi\,dx - EIw_{,xx}(0)\varphi_{,x}(0) - \int_\Omega P\varphi\,dx - M\varphi_{,x}(0),$$

so that the extremal condition

$$\delta J_1(w;\varphi) = 0,\ \forall \varphi \in V_1,$$

provide us the following natural boundary condition

$$M = -EIw_{,xx}(0)$$

and the equation

$$EIw_{,xxxx} - P = 0,\ \text{in } \Omega.$$

Here we recall the remaining essential boundary conditions,

$$w(0) = w(1) = w_{,xx}(1) = 0.$$

A particular solution of such an equation

$$EIw_{,xxxx} - P = 0,\ \text{in } \Omega$$

stands for

$$w_p(x) = ax^4,$$

where

$$a = \frac{P}{4!\ EI}.$$

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The concerning general solution stands for

$$w(x) = w_p(x) + bx^3 + cx^2 + dx + e.$$

The boundary condition $w(0) = 0$ implies that $e = 0$.

The boundary condition

$$M = -EIw_{,xx}(0),$$

stands for

$$M = -EI(2c),$$

so that

$$c = -\frac{M}{2EI}.$$

Moreover, from

$$w_{,xx}(1) = 0,$$

we obtain

$$12a + 6b + 2c = 0.$$

From

$$w(1) = 0$$

we have

$$a + b + c + d = 0.$$

From these last two equations, we obtain

$$b = -\frac{c}{3} - 2a \equiv b(c),$$

and

$$d = a - \frac{2}{3}c.$$

From the Castigliano Theorem, $M \in \mathbb{R}$ must be such that

$$w_x(0) = \frac{dE_{in}}{dM} = 0,$$

where

$$E_{in} = \frac{EI}{2} \int_\Omega w_{,xx}^2 \, dx.$$

Observe that

$$\frac{dE_{in}}{dM} = \frac{dE_{in}}{dc} \frac{dc}{dM} = 0,$$

so that in fact, it suffices to obtain,

$$\frac{dE_{in}}{dc} = 0.$$

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Moreover, observe that

$$\begin{aligned}
E_{in} &= \frac{EI}{2}\int_\Omega w_{,xx}^2\,dx \\
&= \frac{EI}{2}\int_0^1 (12ax^2+6b(c)x+2c)^2\,dx \\
&= \frac{EI}{2}\int_0^1 (144a^2x^4+36b(c)^2x^2+4c^2 \\
&\quad +144ab(c)x^3+48acx^2+24xb(c)c)\,dx \\
&= \frac{EI}{2}\left[\frac{144a^2x^5}{5}+36\frac{b(c)^2x^3}{3}+4c^2x\right. \\
&\quad \left.+\frac{144ab(c)x^4}{4}+\frac{48acx^3}{3}+\frac{24b(c)cx^2}{2}\right]_0^1 \\
&= \frac{EI}{2}\left(\frac{144a^2}{5}+\frac{36b(c)^2}{3}\right. \\
&\quad \left.+4c^2+\frac{144ab(c)}{4}+\frac{48ac}{3}+\frac{24cb(c)}{2}\right).
\end{aligned} \tag{8}$$

Hence, we must have

$$\begin{aligned}
\frac{dE_{in}}{dc} &= \frac{\partial E_{in}}{\partial b}\frac{db}{dc}+\frac{\partial E_{in}}{\partial c} \\
&= 0
\end{aligned} \tag{9}$$

so that,

$$12(2)b(c)\frac{db(c)}{dc}+8c+36a\frac{db(c)}{dc}+16a+12b(c)+12c\frac{db(c)}{dc}=0. \tag{10}$$

Recalling that

$$\frac{db(c)}{dc}=-1/3,$$

we have got

$$-8b(c)+8c-12a+16a+12b(c)-4c=0.$$

From such a result and recalling that

$$b(c)=-\frac{c}{3}-2a,$$

we have got

$$\frac{8c}{3}+16a-12a+16a-4c-24a-4c=0,$$

so that

$$\frac{8c}{3}-4a=0,$$

which has a solution

$$c=\frac{3a}{2}.$$

Thus,

$$c=\frac{3\ P}{(2)\ 4!\ EI},$$

so that

$$M=-2cEI=-\frac{3\ P}{4!}.$$

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2.4. *Checking this Last Result For M by Solving the Concerning Ordinary Differential Equation*

In this section we check the result obtained for M by solving the following ODE,

$$EIw_{,xxxx} - P = 0, \text{ in } \Omega = [0,1],$$

with the boundary conditions,

$$w(0) = w(1) = w_{,x}(0) = w_{,xx}(1) = 0.$$

We recall the general solution stands for

$$w(x) = ax^4 + bx^3 + cx^2 + dx + e,$$

where

$$a = \frac{P}{4!EI}.$$

From $w(0) = 0$, we obtain $e = 0$.
From $w_{,x}(0) = 0$, we obtain $d = 0$.
From $w(1) = 0$, we have

$$a + b + c = 0.$$

From $w_{xx}(1) = 0$, we have

$$12a + 6b + 2c = 0.$$

From such results, we obtain

$$b = -\frac{5a}{2},$$

and

$$c = \frac{3a}{2}$$

Observe that

$$M(x) = -EIw_{,xx},$$

so that in particular

$$M = M(0) = -EIw_{,xx}(0) = -EI(2c) = -\frac{3\,P}{4!}.$$

This value for M here obtained coincide with the one obtained in the previous subsection, as expected.

The objective of this section is complete.

3. Conclusions

In this article, we have presented a formal proof Castigliano Theorem in a elasticity theory context.

We have also presented a generalization of such a result and a numerical example to exemplify its applicability.

Conflicts of Interest: The author declares no conflict of interest concerning this article.

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