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Article

Existence of Generalized Maxwell-Einstein Metrics on Completions of Certain Line Bundles

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Abstract

In this paper, we prove that for some completions of certain line bundles there is a k -generalized Maxwell-Einstein metric defined by Futaki and Ono conformally related to a metric in any given Kähler class for an integer $3 \leq k \leq 13$.

Keywords: Hermitian metrics; generalized Maxwell-Einstein metrics; complex manifolds; scalar curvature; fiber bundle; almost homogeneous manifolds

MSC: 53C10; 53C21; 53C26; 53C55; 32L05; 32M12; 32Q20

1. Introduction

In every Kähler class of a compact almost homogeneous manifold with two ends we found some Maxwell-Einstein metrics in [Gu1].

In every Kähler class of a compact almost homogeneous manifold with two hyper-surface ends we found some k -generalized Maxwell-Einstein metric defined by Futaki and Ono conformally related to a metric for any $k \geq 2$ in [CG].

In this paper, we shall prove that for almost homogeneous manifolds with two ends such that one of them is a complex hyper-surface, for an integer $3 \leq k \leq 13$ there is a Kähler metric in the given Kähler class that is related to a Futaki-Ono k generalized Maxwell-Einstein metric.

Definition 1 For any given Kähler class, there is a Maxwell-Einstein metric conformally related to the Kähler class if $h = u^{-2}g$ is an Hermitian metric with a constant scalar curvature such that u is the Hamiltonian function of a holomorphic vector field related to a Kähler metric g in the given Kähler class.

Definition 2 For any given Kähler class, there is a Futaki-Ono k generalized Maxwell-Einstein metric [FO1, 2, 3] conformally related to the Kähler class if $h = u^{-k}g$ is an Hermitian metric with a constant scalar curvature such that u is the Hamiltonian function of a holomorphic vector field related to a Kähler metric g in the given Kähler class.

Remark 1. This can be found in [FO3]. LeBrun's definition [LB1, 2] for a Einstein-Maxwell metric is following: Let (M, h) be a connected, oriented Riemannian 4-manifold. We will say that h is an Einstein-Maxwell metric if there is a 2-form F on M such that the pair (h, F) satisfies the Einstein-Maxwell equations $dF = d * F = 0$ and $[r + F \circ F]_0 = 0$. Here r is the Ricci curvature of h and the subscript $[\]_0$ indicates the trace-free part with respect to h . In [LB1, 2], LeBrun proved that Einstein-Maxwell + strongly Hermitian = Maxwell-Einstein. For example, see [LB2] Proposition 5. The condition of a strongly Hermitian is that (1) the metric is Hermitian, and (2) the curvature is some kind of Kähler like. This is far very stronger condition than to be a Riemannian metric.

Question 1. Are there Einstein-Maxwell metrics other than our Maxwell-Einstein metrics?

Proposition 1. ([Gu1]) For any Kähler class on a compact almost homogeneous manifold with two ends, there is at least one Maxwell-Einstein metrics in the given Kähler class.

In [CG], we proved:

Proposition 2. ([CG]) *For any Kähler class on a compact almost homogeneous manifold with two hyper-surface ends, there is at least one Futaki-Ono generalized Maxwell-Einstein metrics in the given Kähler class.*

In this paper, we are able to prove:

THEOREM *For any Kähler class on a compact almost homogeneous manifold with two ends and one of them being a complex hyper-surface, for an integer $3 \leq k \leq 13$, there is at least one Futaki-Ono k generalized Maxwell-Einstein metrics in the given Kähler class.*

We notice that when the complex dimension of the manifold M is 2, some Calabi extremal Kähler metrics, e. g., on $\mathbb{CP}^2$ blow up a point, are actually the same as the Maxwell-Einstein metrics. Moreover, in that case, the corresponding Hermitian metrics are actually Hermitian-Einstein as Riemannian manifolds.

Therefore, Maxwell-Einstein metrics should be as standard as Calabi extremal metrics. But Maxwell-Einstein metrics are not in general Einstein metrics just as quaternion Kähler are not in general Kähler.

Recently, after the publications of a series of LeBrun's papers, e. g., [LB1,2] and the references therein, it seems to us that Maxwell-Einstein metrics became a hot topic in the mathematical community. In fact, LeBrun has found many Maxwell-Einstein metrics on the Hirzebruch surfaces. That is the special case for our earlier results with the base manifold to be $\mathbb{CP}^1$. We finally published our earlier work in [Gu1] on this aspect.

There are also many interesting papers on this aspect, for example, [AM, AMT, La1, La2]. In [AMT], they proved that on any admissible Kähler class, there is an extremal-Maxwell-Einstein metric with a given number $a > 1$. That is, the scalar curvature is a potential function of a holomorphic vector field. This is more like a soliton version of the Maxwell-Einstein metrics. Therefore, their results do not imply our earlier results. Also, the admissible metrics are very restricted and do not include all the compact almost homogeneous Kähler manifolds with two ends.

One also notice that most people called these metrics Einstein-Maxwell. These metrics understandably came from physics. Professor LeBrun worked on these metrics for many years without our notice. Some of them are actually Hermitian-Einstein in the Riemannian sense. But our metrics are generally quite different. To make the difference, we call them Maxwell-Einstein metrics instead. They are more like some kind of pseudo-Einstein metrics.

In this paper, we basically followed the line of the proof in [Gu1]. After a transformation, the related equation becomes an ordinary differential equation (12). In [Gu1], after manipulations, the equation became a simple equation of a second derivative of certain function. One could just solve it by integrating twice. Here, as in [CG], it is a little bit more complicated. Fortunately, we can still integrate once by finding a number A and a power s , and transfer the equation into (16). Fortunately and unfortunately, s has two solutions. Even if in the classical case when $k = 2$, by (19) s can be either $2n - 1$ or $2n - 2$. In that case, the number $B = A/a$ in (18) is either 0 or -2 . 0 is classical case but -2 is not. This is interesting.

Fortunately, s and B do not depend on a . After this, we obtained (20), which is a first order linear equation and can be solved. Also, $-k - s$ in (21) and $B + 1 - k - s$ in (22) are smaller than -1 . Most arguments can be carried through. The existence Lemma (Lemma 7) still work after hard efforts. One can even see the duality of the two possible s in the proof of Lemma 7. The proof is very difficult this time and we turned out getting heavy help from computer. This could only be done in nowadays, we guess. Even so, the power of computer could only do finite work this time. We need a complete different method for the infinite possible k 's. Therefore, in this article, we only do the calculation for $3 \leq k \leq 13$. And we shall leave the rest of the proof of the existence Lemma to the next paper unfortunately. However, for the existence Lemma we can do for all of these manifolds with two ends (hyper-surfaces or not).

The next difficult part in the proof is the proof of the positive Lemma (Lemma 9). Our earlier arguments again just do not quite work. Fortunately, we could handle in the cases in which one of the end is a complex hyper-surface. That is, the manifold itself is a completion of a line bundle.

We conjecture that the similar result is true for all the compact almost homogeneous Kähler manifold with two ends.

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2. Certain Completion of Line Bundles

Our results can be regarded as a continuation of [KS1,2], [Ki2] and [Gu1,2,4,5], [CG], we state without detail proof in this section the Lemmas similar to those in these papers as following. The readers might take [Gu2,4] as the references. Most Lemmas can be actually found in [Gu2].

Let $p : L \rightarrow M$ be a holomorphic line bundle over a compact complex Kähler manifold M and h an Hermitian metric of L . Denote by L^0 the open subset $L - \{0\text{-section}\}$ of L and let $s \in C^\infty(L^0)_{\mathbf{R}}$ be defined by $s(l) = \log |l|_h$ ($l \in L^0$), where $|\cdot|_h$ is the norm defined by h . Now we consider a function $\tau = \tau(s) \in C^\infty(L^0)_{\mathbf{R}}$ which is only depending on s and is monotone-increasing with respect to s .

Let $\tilde{J}$ be the complex structure of L and J be the complex structure of M . Now we consider a Riemannian metric on L^0 of the form

$$\tilde{g} = d\tau^2 + (d\tau \circ \tilde{J})^2 + g \quad (1)$$

where $g(l) = p^*g_{\tau(s(l))}(m)$ with $m = p(l) \in M$, g_τ is a one parameter family of Riemannian metrics on M . This form of the metrics is general by using the function τ as the length of the geodesics perpendicular to the generic orbits. Define a function u on L^0 depending only on τ by $u(\tau)^2 = \tilde{g}(H, H)$, where H be the real vector field on L^0 corresponding to the $\mathbf{R}^*$ action on L^0 .

Lemma 1. (Cf [KS1,2], [Gu2 p.2257]) Suppose that the range of τ contains 0. Then $\tilde{g}$ is Kähler if and only if g_0 is Kähler and $g_\tau = g_0 - UB$, where B is the curvature of L with respect to h , $U = \int_0^\tau u(\tau) d\tau$

Throughout this paper, we assume that

- (1) $\hat{L}$ is a compactification of L^0 and $\tilde{g}$ is the restriction of a Kähler metric of $\hat{L}$ to L^0 .
 - (2) the range of τ contains 0 and
 - (3) the eigenvalues of B with respect to g_τ are constants on M .
 - (4) the traces of the Ricci curvature r of g on each eigenvector space of B are constant.
- The condition (4) here is much more general than that in [Gu1,2] in which we have:
- (4)' the eigenvalues of r are constants.

Our results cover some results which appeared in recent years. For example, if g has a constant scalar curvature and B has only one eigenvalue.

By abusing the language, we call the constants in (4) the trace eigenvalues.

Let $(z^1, \dots, z^n)$ be a system of holomorphic local coordinates on M . $n = \dim_{\mathbf{C}} M$. Using a trivialization of L^0 , we take a system of holomorphic local coordinates $(z^0, \dots, z^n)$ on L^0 such that $\partial/\partial z^0 = H - \sqrt{-1}\tilde{J}H$.

Here we notice that z^0 is corresponding to w_1 in [Gu4 p.552]. s can be regarded as $\text{Re} z^0$ near the considered point. So s is the x_1 in [Gu4 p.552]. As in [Gu2] we let $\varphi = u^2$ as a function of U and we let F be the Kähler potential as in [Gu4 p.552], then by comparing [Gu2 Lemma 2] (or the Lemma 4 below) with [Gu4 p.552] we immediately have:

Lemma 2. $4\varphi = \frac{\partial^2 F}{\partial s^2}$.

From $(\frac{d\tau}{ds})^2 = \varphi$, we obtain $\frac{d\tau}{ds} = u$. $U = \int_0^\tau u d\tau = \int_{s(0)}^s u^2 ds = \int_{s(0)}^s 4^{-1} \frac{\partial^2 F}{\partial s^2} ds$ is $\frac{\partial F}{\partial s} = y_1$ up to a constant in [Gu4 p.552], i.e.,

Lemma 3. U is the Legendre transformation of s .

Here we use the Legendre transformation in [Gu4] instead of the moment map in [Gu2] since we need the new insight in the last section.

Let $\hat{X}_i, \hat{X}_{\bar{i}}$ ($0 \leq i \leq n$) be the partial differentiations $\partial/\partial z^i, \partial/\partial \bar{z}^i$ on L^0 and $X_i, X_{\bar{i}}$ ($1 \leq i \leq n$) be the partial differentiations $\partial/\partial z^i, \partial/\partial \bar{z}^i$ on M .

Lemma 4.(Cf [KS1,2], [Gu2 Lemma 2]) *We have*

$$\tilde{g}_{00} = 2u^2, \quad \tilde{g}_{0\bar{i}} = 2u\hat{X}_{\bar{i}}\tau, \quad \tilde{g}_{i\bar{j}} = g_{i\bar{j}} + 2\hat{X}_i\tau \cdot \hat{X}_{\bar{j}}\tau \quad (2)$$

where $1 \leq i, j \leq n$. At the point $P \in L^0$ considered, we can choose a local coordinate system around $m = p(P) \in M$ such that $(\partial/\partial z^i)\tau = 0$ at m , so $\hat{X}_i\tau = \hat{X}_{\bar{j}}\tau = 0$ at the point we considering, then if f is a function on L^0 depending only on τ , we have

$$\begin{aligned} \hat{X}_0\hat{X}_{\bar{0}}f &= u\frac{d}{d\tau}\left(u\frac{df}{d\tau}\right) \quad \hat{X}_i\hat{X}_{\bar{0}}f = 0 \\ \hat{X}_i\hat{X}_{\bar{j}}f &= -\frac{1}{2}uB_{i\bar{j}}\frac{df}{d\tau} \end{aligned} \quad (3)$$

The Ricci curvature at this point is

$$\begin{aligned} \tilde{r}_{00} &= -u\frac{d}{d\tau}\left(u\frac{d}{d\tau}\log(u^2Q)\right) \quad \tilde{r}_{0\bar{i}} = 0 \\ \tilde{r}_{i\bar{j}} &= p^*r_{0\bar{i}\bar{j}} + \frac{1}{2}u\frac{d}{d\tau}\log(u^2Q) \cdot B_{i\bar{j}} \end{aligned} \quad (4)$$

where $Q = \det(g_0^{-1} \cdot g_\tau)$. In particular, we have the scalar curvature

$$\tilde{R} = \frac{\Delta}{Q} - \frac{1}{2Q}\frac{d}{dU}\left(\frac{d}{dU}Q\varphi\right) \quad (5)$$

where $\varphi = u^2$ as a function of U and $\Delta(U) = Q\sum_{i,j}r_{0\bar{i}\bar{j}}g_{\tau(U)}^{i\bar{j}}$. We also have $\varphi'(\min U) = 2, \varphi'(\max U) = -2$.

Lemma 5.(Cf. [FMS], [Mb], [Gu2 Lemma 3]) *We can also regard U as a moment map corresponding to $(\tilde{g}, \tilde{J}H)$ and g_τ just be the symplectic reduction of $\tilde{g}$ at $U(\tau)$. $\tilde{g}$ is extremal if and only if $\tilde{R} = aU + b$ for some $a, b \in \mathbf{R}$.*

Let $M_0 = U^{-1}(\min U)$ and $M_\infty = U^{-1}(\max U)$, they are complex sub-manifolds, since they are components of the fixed point set of $H - \sqrt{-1}\tilde{J}H$ which is semisimple. Let D_0 be the codimension of M_0 in $\hat{L}$, D_∞ be the codimension of M_∞ in $\hat{L}$.

Lemma 6.(Cf. [Gu2 Lemma 4]) *Suppose that there is another Kähler metrics $\tilde{g}^\vee$ on $\hat{L}$ in the same Kähler class which is of form (1) on L^0 . Let*

$$\tau^\vee, g^\vee, U^\vee, Q^\vee, \Delta^\vee, \varphi^\vee, u^\vee$$

be the corresponding metric and the corresponding functions of s . Then there is a unique corresponding $\tau^\vee$ such that $g_0^\vee = g_0$. In this case, $\min U^\vee = \min U$ (or $\max U^\vee = \max U$) and $Q^\vee = Q, \Delta^\vee = \Delta$ hold. So we may write $D = \max U$ and $-d = \min U$. Then

$$\begin{aligned} Q(U) &= (1 + \frac{U}{d})^{D_0-1}Q_{-d} \\ (or \quad &= (1 - \frac{U}{D})^{D_\infty-1}Q_D), \end{aligned} \quad (6)$$

where Q_{-d} (or Q_D) is a polynomial of U such that $Q_{-d}(-d) \neq 0$ (or $Q_D(D) \neq 0$) and

$$\Delta(U) = D_0(D_0 - 1)\frac{1}{d}(1 + \frac{U}{d})^{D_0-2}Q_{-d} \pmod{(1 + \frac{U}{d})^{D_0-1}}$$

$$(or = D_{\infty}(D_{\infty} - 1)\frac{1}{D}(1 - \frac{U}{D})^{D_{\infty}-2}Q_D \pmod{(1 - \frac{U}{D})^{D_{\infty}-1}}). \quad (7)$$

Proof: Let $\tilde{g} - \tilde{g}^{\vee} = \hat{\partial}\bar{\partial}\phi$, then

$$\tilde{g}_{i\bar{j}}^{\vee} = \tilde{g}_{i\bar{j}} + \frac{1}{2}u\frac{d\phi}{d\tau}B_{i\bar{j}} = (g_0)_{i\bar{j}} - (U - \frac{1}{2}u\frac{d\phi}{d\tau})B_{i\bar{j}} \quad (8)$$

for $1 \leq i, j \leq n$, so at $\min U$ (or $\max U$) $\tilde{g}_{i\bar{j}}^{\vee} = \tilde{g}_{i\bar{j}}$, therefore there is a τ_0 such that $g_{\tau^{\vee}(\tau_0)}^{\vee} = g_0$. By choosing $\tau^{\vee}$ such that $\tau^{\vee}(\tau_0) = 0$, one sees that $\min U^{\vee} = \min U$, $\max U^{\vee} = \max U$ as desired.

The last statement follows from the fact that the scalar curvature R is finite on both M_0 and M_{∞} . Q. E. D.

We need normalization in this paper. By rescaling we can choose $U^{\vee} = a_1^2 U + a_2$ for any $a_1 > 0$ and $a_2 \in \mathbf{R}$, allowing us to assume that $\max U - \min U = 2$ and $\min U = -1$, then $\max U = 1$.

3. Existence of the Futaki-Ono's Generalized k Maxwell-Einstein Metrics—Existence

We recall our definition of the Futaki-Ono's Generalized k Maxwell-Einstein metrics: For any given Kähler class, there is a Futaki-Ono's Generalized k Maxwell-Einstein metric conformally related to the Kähler class if $h = u^{-k}g$ is an Hermitian metric with a constant scalar curvature such that u is the Hamiltonian function of a holomorphic vector field related to a Kähler metric g in the given Kähler class.

From Lemma 5, it can be seen that if $\tilde{g}$ is a Futaki-Ono's Generalized k Maxwell-Einstein metric, then $u = aU + b$ for some $a, b \in \mathbf{R}$.

From [Dr], p.119, (6.1), or [Au], p.126, (1), we have:

$$\begin{aligned} S_{g,u,-k,2n} \\ = u^k [S + k(2n-1)\frac{\Delta u}{u} - \frac{k}{4}(2n-1)(4-2k+2kn)\frac{|\partial u|^2}{u^2}] \end{aligned} \quad (9)$$

Here $S = \tilde{R}$ for the scalar curvature of our Kähler metric in Lemma 4. Notice that here we have a different sign for the Laplacian from [FO3].

Remark 2. The notation $S_{g,u,-k,2n}$ is directly taken from [FO3] page 94, formula (2). Our n here is the m in the formula (1) there. Therefore, their n was the twice of our n .

From [CG], we then have:

$$\begin{aligned} cQ - \Delta(aU + b)^k &= -\frac{1}{2}[(aU + b)^k \frac{d}{dU}(\frac{d}{dU}Q\varphi) \\ &\quad - k(2n-1)a(aU + b)^{k-1} \frac{d}{dU}(Q\varphi) \\ &\quad + \frac{k}{2}(2n-1)(2-k+kn)a^2(aU + b)^{k-2}Q\varphi], \end{aligned}$$

$$\begin{aligned} \frac{d}{dU}(\frac{d}{dU}(\frac{Q\varphi}{(aU + b)^s})) + A\frac{d}{dU}(\frac{Q\varphi}{(aU + b)^{s+1}}) \\ = (aU + b)^{-s} \frac{d}{dU}(\frac{d}{dU}Q\varphi) + (A - 2sa)(aU + b)^{-s-1} \frac{d}{dU}(Q\varphi) \\ + [s(s+1)a^2 - (s+1)Aa](aU + b)^{-s-2}Q\varphi, \end{aligned}$$

where

$$\begin{aligned} A - 2sa &= -k(2n-1)a, \\ s(s+1)a^2 - (s+1)Aa &= \frac{k}{2}(2n-1)(2-k+kn)a^2, \end{aligned}$$

$$\begin{aligned} & \frac{cQ}{(aU+b)^{k+s}} - \frac{\Delta}{(aU+b)^s} \\ &= -\frac{1}{2} \left[\frac{d}{dU} \left(\frac{d}{dU} \frac{Q\varphi}{(aU+b)^s} \right) + A \frac{d}{dU} \left(\frac{Q\varphi}{(aU+b)^{s+1}} \right) \right], \\ s &= \frac{k(2n-1) - 1 \pm \sqrt{(2n-1)(k-1)^2 - 2(n-1)}}{2}, \end{aligned}$$

we need to find an a in $(-1, 1)$ satisfying the following identity:

$$\begin{aligned} & \frac{1}{a} \int_{-1}^1 (ax+1)^{-k-s} Q(x) dx \left[\int_{-1}^1 (ax+1)^{s-2kn+k+1} \Delta(x) dx \right. \\ & \quad \left. + (a+1)^{s-2kn+k+1} Q(1) + (1-a)^{s-2kn+k+1} Q(-1) \right] \\ &= \frac{1}{a} \int_{-1}^1 (ax+1)^{s-2kn+1} Q(x) dx \left[\int_{-1}^1 (ax+1)^{-s} \Delta(x) dx \right. \\ & \quad \left. + (a+1)^{-s} Q(1) + (1-a)^{-s} Q(-1) \right]. \end{aligned} \quad (10)$$

where

$$s = \frac{k(2n-1) - 1 + \sqrt{(2n-1)(k-1)^2 - 2(n-1)}}{2} \quad (11)$$

When $Q(-1) = 0$, a similar result holds.

Let $L_{p,l} = \int_{-1}^1 (ax+1)^{-p} (1+x)^l dx$ with $p > l+1$, Where l is an integer and p is not necessarily an integer. Then

$$\begin{aligned} L_{p,l+1} &= \int_{-1}^1 (ax+1)^{-p} (1+x)^{l+1} dx \\ &= \int_{-1}^1 (ax+1)^{-p} a^{-1} ((ax+1) - (1-a)) (1+x)^l dx \\ &= a^{-1} \left[\int_{-1}^1 (ax+1)^{-p+1} (1+x)^l dx - (1-a) \int_{-1}^1 (ax+1)^{-p} (1+x)^l dx \right] \\ &= a^{-1} [L_{p-1,l} - (1-a)L_{p,l}]. \end{aligned} \quad (12)$$

Now,

$$\begin{aligned} L_{p,0} &= \int_{-1}^1 (ax+1)^{-p} dx \\ &= \frac{1}{(-p+1)a} (ax+1)^{-p} \Big|_{-1}^1 \\ &= \frac{1}{(-p+1)a} [(a+1)^{-p+1} - (-a+1)^{-p+1}] \\ &= \frac{1}{(p-1)a} [(1-a)^{-p+1} - (1+a)^{-p+1}] \end{aligned}$$

with $p > 1$, which is equivalent to $\frac{1}{(p-1)(1-a)^{p-1}}$ when a turns to 1.

$$\begin{aligned} L_{p,1} &= a^{-1} [L_{p-1,0} - (1-a)L_{p,0}] \\ &= a^{-1} \left[\frac{1}{(p-2)a} [(1-a)^{-p+2} - (1+a)^{-p+2}] \right. \\ & \quad \left. - (1-a) \frac{1}{(p-1)a} [(1-a)^{-p+1} - (1+a)^{-p+1}] \right] \end{aligned}$$

with $p > 1$, which is equivalent to

$$\begin{aligned} & \frac{1}{(p-2)(1-a)^{p-2}} - \frac{1}{(p-1)(1-a)^{p-2}} \\ &= \frac{1}{(1-a)^{p-2}} \frac{(p-1) - (p-2)}{(p-1)(p-2)} \\ &= \frac{1}{(p-1)(p-2)(1-a)^{p-2}} \end{aligned}$$

when a turns to 1.

$$L_{p,2} = a^{-1}[L_{p-1,1} - (1-a)L_{p,1}]$$

with $p > 1$, which is equivalent to

$$\begin{aligned} & \frac{1}{(p-2)(p-3)(1-a)^{p-3}} - (1-a) \frac{1}{(p-1)(p-2)(1-a)^{p-2}} \\ &= \frac{1}{(p-2)(1-a)^{p-3}} \left(\frac{1}{p-3} - \frac{1}{p-1} \right) \\ &= \frac{1}{(p-2)(1-a)^{p-3}} \frac{(p-1) - (p-3)}{(p-1)(p-3)} \\ &= \frac{1 \times 2}{(p-1)(p-2)(p-3)(1-a)^{p-3}} \end{aligned}$$

when a turns to 1.

$$L_{p,3} = a^{-1}[L_{p-1,2} - (1-a)L_{p,2}]$$

with $p > 1$, which is equivalent to

$$\begin{aligned} & \frac{1 \times 2}{(p-2)(p-3)(p-4)(1-a)^{p-4}} - (1-a) \frac{1 \times 2}{(p-1)(p-2)(p-3)(1-a)^{p-3}} \\ &= \frac{2}{(p-2)(p-3)(1-a)^{p-4}} \left(\frac{1}{p-4} - \frac{1}{p-1} \right) \\ &= \frac{2}{(p-2)(p-3)(1-a)^{p-4}} \frac{(p-1) - (p-4)}{(p-1)(p-4)} \\ &= \frac{3!}{(p-1)(p-2)(p-3)(p-4)(1-a)^{p-4}} \end{aligned}$$

when a turns to 1.

Now, we use the mathematical induction for l , assuming that when $l-1$,

$$L_{p,l-1} = a^{-1}[L_{p-1,l-2} - (1-a)L_{p,l-2}]$$

with $p > 1$, which is equivalent to $\frac{(l-1)!}{(p-1)(p-2)(p-3)\cdots(p-l)(1-a)^{p-l}}$ when a turns to 1,

$$L_{p,l} = a^{-1}[L_{p-1,l-1} - (1-a)L_{p,l-1}]$$

with $p > 1$, which is equivalent to

$$\begin{aligned} & \frac{(l-1)!}{(p-2)(p-3)(p-4)\cdots(p-l-1)(1-a)^{p-l-1}} \\ & - (1-a) \frac{(l-1)!}{(p-1)(p-2)(p-3)\cdots(p-l)(1-a)^{p-l}} \end{aligned}$$

$$\begin{aligned}
&= \frac{(l-1)!}{(p-2)(p-3)\cdots(p-l)(1-a)^{p-l-1}} \left(\frac{1}{p-l-1} - \frac{1}{p-1} \right) \\
&= \frac{(l-1)!}{(p-2)(p-3)\cdots(p-l)(1-a)^{p-l-1}} \frac{(p-1) - (p-l-1)}{(p-1)(p-l-1)} \\
&= \frac{l!}{(p-1)(p-2)(p-3)\cdots(p-l)(p-l-1)(1-a)^{p-l-1}}
\end{aligned}$$

when a turns to 1.

Therefore, by our induction formula we can prove that $L_{p,l}$ is equivalent to

$$\frac{l!(p-l-2)!}{(p-1)!(1-a)^{p-l-1}}, \quad (13)$$

when a turns to 1.

Here, for the convenience of writing, let

$$\frac{1}{(p-1)(p-2)(p-3)\cdots(p-l)(p-l-1)} := \frac{(p-l-2)!}{(p-1)!},$$

where l is an integer and p is not necessarily an integer.

The major part of the difference of the two sides of equation (10) comes from

$$\begin{aligned}
&\frac{1}{a} \int_{-1}^1 (ax+1)^{-k-s} Q(x) dx \int_{-1}^1 (ax+1)^{s-2kn+k+1} \Delta(x) dx \\
&- \frac{1}{a} \int_{-1}^1 (ax+1)^{s-2kn+1} Q(x) dx \int_{-1}^1 (ax+1)^{-s} \Delta(x) dx.
\end{aligned}$$

By the formula of Δ near -1 in (7) of Lemma 6, we only need to check that

$$L_{k+s, D_0-1} L_{2kn-s-k-1, D_0-2} - L_{2kn-s-1, D_0-1} L_{s, D_0-2} \quad (14)$$

has a negative major part. By (13) it is proportional to:

$$\begin{aligned}
&\frac{(D_0-1)!(k+s-D_0-1)!}{(k+s-1)!(1-a)^{k+s-D_0}} \frac{(D_0-2)!(2kn-s-k-1-D_0)!}{(2kn-s-k-2)!(1-a)^{2kn-s-k-D_0}} \\
&- \frac{(D_0-1)!(2kn-s-D_0-2)!}{(2kn-s-1-1)!(1-a)^{2kn-s-D_0-1}} \frac{(D_0-2)!(s-D_0)!}{(s-1)!(1-a)^{s-D_0+1}} \\
&= (D_0-1)!(D_0-2)! \\
&\left[\frac{(k+s-D_0-1)!(2kn-s-k-D_0-1)!}{(k+s-1)!(2kn-s-k-2)!(1-a)^{2kn-2D_0}} \right. \\
&- \left. \frac{(2kn-s-D_0-2)!(s-D_0)!}{(2kn-s-2)!(s-1)!(1-a)^{2kn-2D_0}} \right] \\
&= \frac{(D_0-1)!(D_0-2)!}{(1-a)^{2kn-2D_0}} \left[\frac{(k+s-D_0-1)!(2kn-s-k-D_0-1)!}{(k+s-1)!(2kn-s-k-2)!} \right. \\
&- \left. \frac{(2kn-s-D_0-2)!(s-D_0)!}{(2kn-s-2)!(s-1)!} \right].
\end{aligned}$$

It is determined by the sign of

$$\frac{(k+s-D_0-1)!(2kn-s-k-D_0-1)!}{(k+s-1)!(2kn-s-k-2)!} - \frac{(2kn-s-D_0-2)!(s-D_0)!}{(2kn-s-2)!(s-1)!}. \quad (15)$$

Simplify (15) below

$$\begin{aligned} & \frac{(k+s-D_0-1)!(2kn-s-k-D_0-1)!}{(k+s-1)!(2kn-s-k-2)!} - \frac{(2kn-s-D_0-2)!(s-D_0)!}{(2kn-s-2)!(s-1)!} \\ &= \frac{1}{\prod_{i=1}^{D_0} (k+s-i) \prod_{i=2}^{D_0} (2kn-s-k-i)} - \frac{1}{\prod_{i=2}^{D_0+1} (2kn-s-i) \prod_{i=1}^{D_0-1} (s-i)}, \end{aligned}$$

it is determined by the sign of

$$\prod_{i=2}^{D_0+1} (2kn-s-i) \prod_{i=1}^{D_0-1} (s-i) - \prod_{i=1}^{D_0} (k+s-i) \prod_{i=2}^{D_0} (2kn-s-k-i) \quad (16)$$

Simplify (16) below if $D_0 > k$

$$\begin{aligned} (16) &= \prod_{i=k+2}^{D_0+1} (2kn-s-i) \prod_{i=1}^{D_0-k} (s-i) \\ & \left[\prod_{i=2}^{k+1} (2kn-s-i) \prod_{i=D_0-k+1}^{D_0-1} (s-i) - \prod_{i=1}^k (k+s-i) \prod_{i=D_0-k+2}^{D_0} (2kn-s-k-i) \right]. \end{aligned} \quad (17)$$

This is also true if $D_0 = k$. We just do not take out any factor. If $D_0 < k$, we multiply some linear factors instead and we get:

$$\begin{aligned} (16) &= \prod_{i=D_0+2}^{k+1} (2kn-s-i)^{-1} \prod_{i=D_0-k+1}^0 (s-i)^{-1} \\ & \left[\prod_{i=2}^{k+1} (2kn-s-i) \prod_{i=D_0-k+1}^{D_0-1} (s-i) - \prod_{i=1}^k (k+s-i) \prod_{i=D_0-k+2}^{D_0} (2kn-s-k-i) \right]. \end{aligned} \quad (18)$$

Therefore, it is determined by the sign of

$$\prod_{i=2}^{k+1} (2kn-s-i) \prod_{i=D_0-k+1}^{D_0-1} (s-i) - \prod_{i=1}^k (k+s-i) \prod_{i=D_0-k+2}^{D_0} (2kn-s-k-i). \quad (19)$$

We now divide the second term of the above equation by both sides at the same time, we get

$$\frac{\prod_{i=2}^{k+1} (2kn-s-i) \prod_{i=D_0-k+1}^{D_0-1} (s-i)}{\prod_{i=1}^k (k+s-i) \prod_{i=D_0-k+2}^{D_0} (2kn-s-k-i)} - 1.$$

Taking the log of the first term above expression, we get

$$\begin{aligned} & \log \left[\frac{\prod_{i=2}^{k+1} (2kn-s-i) \prod_{i=D_0-k+1}^{D_0-1} (s-i)}{\prod_{i=1}^k (k+s-i) \prod_{i=D_0-k+2}^{D_0} (2kn-s-k-i)} \right] \\ &= \sum_{i=2}^{k+1} \log \frac{2kn-s-i}{s+k-i+1} + \sum_{i=D_0-k+1}^{D_0-1} \log \frac{s-i}{2kn-s-k-i-1}. \end{aligned}$$

Taking the derivative of D_0 , we obtain:

$$\begin{aligned} & \frac{-1}{s-D_0+k-1} - \frac{-1}{2kn-s-D_0-2} + \cdots + \frac{-1}{s-D_0+k-2} \\ & - \frac{-1}{2kn-s-D_0-3} + \frac{-1}{s-D_0+1} - \frac{-1}{2kn-s-D_0-k} \end{aligned}$$

$$\begin{aligned} &= \frac{2s - 2kn + k + 1}{(2kn - s - D_0 - 2)(s - D_0 + k - 1)} + \cdots \\ &+ \frac{2s - 2kn + k + 1}{(2kn - s - D_0 - 3)(s - D_0 + k - 2)} + \frac{2s - 2kn + k + 1}{(2kn - s - D_0 - k)(s - D_0 + 1)}, \end{aligned}$$

from (11)

$$\begin{aligned} &2s - 2kn + k + 1 \\ &= k(2n - 1) - 1 + \sqrt{(2n - 1)(k - 1)^2 - 2(n - 1) - 2kn + k + 1} \\ &= \sqrt{(2n - 1)(k - 1)^2 - 2(n - 1)} \\ &= \sqrt{(2n - 1)(k - 1)^2 - (2n - 1) + 1} \\ &= \sqrt{k(2n - 1)(k - 2) + 1} \\ &> 0, \end{aligned}$$

so (19) is a monotonically increasing function of D_0 . Now we just need to see if (19) is positive or negative when $D_0 = n$, when $D_0 = n$, (19) becomes

$$\begin{aligned} &\prod_{i=2}^{k+1} (2kn - s - i) \prod_{i=n-k+1}^{n-1} (s - i) - \prod_{i=1}^k (k + s - i) \prod_{i=n-k+2}^n (2kn - s - k - i) \\ &= \prod_{i=1}^k (2kn - s - i - 1) \prod_{i=1}^{k-1} (s - n + k - i) - \prod_{i=1}^k (k + s - i) \prod_{i=1}^{k-1} (2kn - s - n - i - 1). \end{aligned} \tag{20}$$

let

$$t = \sqrt{(2n - 1)(k - 1)^2 - 2(n - 1)},$$

then

$$n = \frac{t^2 + (k - 1)^2 - 2}{2k(k - 2)}, \tag{21}$$

$$\begin{aligned} s &= \frac{k(2n - 1) - 1 + \sqrt{(2n - 1)(k - 1)^2 - 2(n - 1)}}{2} \\ &= \frac{2k \frac{t^2 + (k - 1)^2 - 2}{2k(k - 2)} - k - 1 + t}{2} \\ &= \frac{t^2 + (k - 2)t - (k - 1)}{2(k - 2)} \\ &= \frac{(t + (k - 1))(t - 1)}{2(k - 2)}. \end{aligned} \tag{22}$$

$$\begin{aligned} 2kn - s - k - 1 &= 2k \frac{t^2 + (k - 1)^2 - 2}{2k(k - 2)} - \frac{t^2 + (k - 2)t - (k - 1)}{2(k - 2)} - k - 1 \\ &= \frac{2t^2 + 2(k - 1)^2 - 4 - t^2 - (k - 2)t + (k - 1) - 2k(k - 2) - 2(k - 2)}{2(k - 2)} \\ &= \frac{t^2 - (k - 2)t + 2(k - 1)^2 - 4 + k - 1 - 2k^2 + 4k - 2k + 4}{2(k - 2)} \\ &= \frac{t^2 - (k - 2)t - (k - 1)}{2(k - 2)} \\ &= \frac{(t - (k - 1))(t + 1)}{2(k - 2)}, \end{aligned}$$

$$\begin{aligned} 2kn - s - k &= 2kn - s - k - 1 + 1 \\ &= \frac{t^2 - (k - 2)t - (k - 1)}{2(k - 2)} + 1 \\ &= \frac{t^2 - (k - 2)t - (k - 1) + 2(k - 2)}{2(k - 2)} \\ &= \frac{t^2 - (k - 2)t + k - 3}{2(k - 2)} \\ &= \frac{(t - (k - 3))(t - 1)}{2(k - 2)}, \end{aligned}$$

$$\begin{aligned} 2kn - s - 1 - i &= 2kn - s - k + (k - 1 - i) \\ &= \frac{t^2 - (k - 2)t + k - 3}{2(k - 2)} + (k - 1 - i) \\ &= \frac{t^2 - (k - 2)t + k - 3 + 2(k - 1)(k - 2) - 2(k - 2)i}{2(k - 2)} \\ &= \frac{t^2 - (k - 2)t + 2k^2 - 5k + 1 - 2(k - 2)i}{2(k - 2)}, \end{aligned}$$

$$\begin{aligned} s - n + 1 &= \frac{t^2 + (k - 2)t - (k - 1)}{2(k - 2)} - \frac{t^2 + (k - 1)^2 - 2}{2k(k - 2)} + 1 \\ &= \frac{kt^2 + k(k - 2)t - k(k - 1) - t^2 - (k - 1)^2 + 2 + 2k(k - 2)}{2k(k - 2)} \\ &= \frac{(k - 1)t^2 + k(k - 2)t - k + 1}{2k(k - 2)} \\ &= \frac{(k - 1)t^2 + k(k - 2)t - (k - 1)}{2k(k - 2)} \\ &= \frac{((k - 1)t - 1)(t + (k - 1))}{2k(k - 2)}, \end{aligned}$$

$$\begin{aligned} s - n + k - i &= s - n + 1 + k - 1 - i \\ &= \frac{(k - 1)t^2 + k(k - 2)t - (k - 1)}{2k(k - 2)} + k - 1 - i \\ &= \frac{(k - 1)t^2 + k(k - 2)t - (k - 1) + 2k(k - 2)(k - 1) - 2k(k - 2)i}{2k(k - 2)}, \end{aligned}$$

$$\begin{aligned} s &= \frac{t^2 + (k - 2)t - (k - 1)}{2(k - 2)} \\ &= \frac{(t + (k - 1))(t - 1)}{2(k - 2)}, \end{aligned}$$

$$\begin{aligned} s + 1 &= \frac{t^2 + (k - 2)t - (k - 1)}{2(k - 2)} + 1 \\ &= \frac{t^2 + (k - 2)t - (k - 1) + 2(k - 2)}{2(k - 2)} \\ &= \frac{t^2 + (k - 2)t + k - 3}{2(k - 2)} \end{aligned}$$

$$= \frac{(t + (k - 3))(t + 1)}{2(k - 2)},$$

$$\begin{aligned} s + k - i &= \frac{t^2 + (k - 2)t - (k - 1)}{2(k - 2)} + k - i \\ &= \frac{t^2 + (k - 2)t - k + 1 + 2k(k - 2) - 2(k - 2)i}{2(k - 2)}, \end{aligned}$$

$$\begin{aligned} 2kn - s - n - k &= 2k \frac{t^2 + (k - 1)^2 - 2}{2k(k - 2)} - \frac{t^2 + (k - 2)t - (k - 1)}{2(k - 2)} - \frac{t^2 + (k - 1)^2 - 2}{2k(k - 2)} - k \\ &= \frac{2kt^2 + 2k(k - 1)^2 - 4k - kt^2 - k(k - 2)t + k(k - 1) - t^2 - (k - 1)^2 + 2 - 2k^2(k - 2)}{2k(k - 2)} \\ &= \frac{(k - 1)t^2 - k(k - 2)t - k + 1}{2k(k - 2)} \\ &= \frac{(k - 1)t^2 - k(k - 2)t - (k - 1)}{2k(k - 2)} \\ &= \frac{((k - 1)t + 1)(t - (k - 1))}{2k(k - 2)}, \end{aligned}$$

$$\begin{aligned} 2kn - s - n - 1 - i &= 2kn - s - n - k + k - 1 - i \\ &= \frac{(k - 1)t^2 - k(k - 2)t - (k - 1)}{2k(k - 2)} + k - 1 - i \\ &= \frac{(k - 1)t^2 - k(k - 2)t - (k - 1) + 2k(k - 1)(k - 2) - 2k(k - 2)i}{2k(k - 2)}. \end{aligned}$$

Therefore, after replacing i in (20) by $k - 1 - l$, (20) becomes

$$\begin{aligned} (20) &= \frac{(t - (k - 1))(t + 1)}{2(k - 2)} \times \frac{(t - (k - 3))(t - 1)}{2(k - 2)} \times \prod_{l=1}^{k-2} \frac{t^2 - (k - 2)t + k - 3 + 2(k - 2)l}{2(k - 2)} \\ &\times \frac{((k - 1)t - 1)(t + (k - 1))}{2k(k - 2)} \times \prod_{l=1}^{k-2} \frac{(k - 1)t^2 + k(k - 2)t - (k - 1) + 2k(k - 2)l}{2k(k - 2)} \\ &- \frac{(t + (k - 1))(t - 1)}{2(k - 2)} \times \frac{(t + (k - 3))(t + 1)}{2(k - 2)} \times \prod_{l=1}^{k-2} \frac{t^2 + (k - 2)t + k - 3 + 2(k - 2)l}{2(k - 2)} \\ &\times \frac{((k - 1)t + 1)(t - (k - 1))}{2k(k - 2)} \times \prod_{l=1}^{k-2} \frac{(k - 1)t^2 - k(k - 2)t - (k - 1) + 2k(k - 2)l}{2k(k - 2)} \\ &= (2(k - 2))^{-2k+1} k^{1-k} (t + 1)(t - 1)(t + (k - 1))(t - (k - 1)) \\ &\left[\begin{aligned} &(t - (k - 3))((k - 1)t - 1) \\ &\prod_{l=1}^{k-2} (t^2 - (k - 2)t + k - 3 + 2(k - 2)l)((k - 1)t^2 + k(k - 2)t - (k - 1) + 2k(k - 2)l) \\ &- (t + (k - 3))((k - 1)t + 1) \\ &\prod_{l=1}^{k-2} (t^2 + (k - 2)t + k - 3 + 2(k - 2)l)((k - 1)t^2 - k(k - 2)t - (k - 1) + 2k(k - 2)l) \end{aligned} \right]. \end{aligned}$$

If $k > 2$, it is determined by the sign of

$$(t - (k - 3))((k - 1)t - 1)$$

$$\begin{aligned}
& \prod_{l=1}^{k-2} (t^2 - (k-2)t + k-3 + 2(k-2)l)((k-1)t^2 + k(k-2)t - (k-1) + 2k(k-2)l) \\
& - (t + (k-3))((k-1)t + 1) \\
& \prod_{l=1}^{k-2} (t^2 + (k-2)t + k-3 + 2(k-2)l)((k-1)t^2 - k(k-2)t - (k-1) + 2k(k-2)l).
\end{aligned} \tag{23}$$

because

$$t = \sqrt{(2n-1)(k-1)^2 - 2(n-1)},$$

then

$$\begin{aligned}
t^2 &= (2n-1)(k-1)^2 - 2(n-1) \\
&= (2n-1)(k-1)^2 - (2n-1) + 1 \\
&= (2n-1)((k-1)^2 - 1) + 1 \\
&= k(k-2)(2n-1) + 1,
\end{aligned}$$

$$\begin{aligned}
t^2 - (k-2)t + k-3 + 2(k-2)l \\
&= k(k-2)(2n-1) + 1 - (k-2)t + k-3 + 2(k-2)l \\
&= (k-2)[-t + k(2n-1) + 1 + 2l],
\end{aligned}$$

$$\begin{aligned}
(k-1)t^2 + k(k-2)t - (k-1) + 2k(k-2)l \\
&= (k-1)[k(k-2)(2n-1) + 1] + k(k-2)t - (k-1) + 2k(k-2)l \\
&= k(k-1)(k-2)(2n-1) + k(k-2)t + 2k(k-2)l \\
&= k(k-2)[t + (k-1)(2n-1) + 2l] \\
&= k(k-2)[t + k(2n-1) - 2n + 1 + 2l],
\end{aligned}$$

$$\begin{aligned}
t^2 + (k-2)t + k-3 + 2(k-2)l \\
&= (k-2)[t + k(2n-1) + 1 + 2l],
\end{aligned}$$

$$\begin{aligned}
(k-1)t^2 - k(k-2)t - (k-1) + 2k(k-2)l \\
&= k(k-2)[-t + k(2n-1) - 2n + 1 + 2l],
\end{aligned}$$

$$\begin{aligned}
(23) &= k^{k-2}(k-2)^{2k-4}[(t - (k-3))((k-1)t - 1) \\
& \prod_{l=1}^{k-2} (-t + k(2n-1) + 1 + 2l)(t + k(2n-1) - 2n + 1 + 2l) \\
& - (t + (k-3))((k-1)t + 1) \\
& \prod_{l=1}^{k-2} (t + k(2n-1) + 1 + 2l)(-t + k(2n-1) - 2n + 1 + 2l)],
\end{aligned}$$

it is determined by the sign of

$$\begin{aligned}
& (t - (k-3))((k-1)t - 1) \\
& \prod_{l=1}^{k-2} (-t + k(2n-1) + 1 + 2l)(t + k(2n-1) - 2n + 1 + 2l) \\
& - (t + (k-3))((k-1)t + 1)
\end{aligned} \tag{24}$$

$$\prod_{l=1}^{k-2} (t + k(2n-1) + 1 + 2l)(-t + k(2n-1) - 2n + 1 + 2l),$$

for convenience, we shall use $f_1[n, k, t]$ (or $fun1[n, k, t]$ in the Mathematica calculation) to represent expression (24) below. And our goal is to prove:

Claim: $f_1[n, k, t] < 0$.

In this paper, we only deal with the cases in which $3 \leq k \leq 13$.

3.1. $k = 3$

If $k = 3$, then

$$t^2 = 3(2n-1) + 1 = 6n - 2.$$

The expression is

$$\begin{aligned} & t(2t-1)(-t+3(2n-1)+1+2)(t+3(2n-1)-2n+1+2) \\ & - t(2t+1)(t+3(2n-1)+1+2)(-t+3(2n-1)-2n+1+2) \\ & = t(2t-1)(-t+6n)(t+4n) - t(2t+1)(t+6n)(-t+4n) \\ & = (2t^2 - t)(-t+6n)(t+4n) - (2t^2 + 1)(t+6n)(-t+4n) \\ & = (2(6n-2) - t)(-t+6n)(t+4n) - (2(6n-2) + 1)(t+6n)(-t+4n) \\ & = (-t+12n-4)(-t+6n)(t+4n) - (t+12n-4)(t+6n)(-t+4n). \end{aligned}$$

This an odd function. The coefficient of the t^3 term is 2. The constant term and the t^2 terms are zeros. We only need to calculate the coefficient of the t term. We take the derivative to t , then let $t = 0$, we get

$$2[-24n^2 - (12n-4)(4n) + (12n-4)(6n)] = 2n(-24n + 2(12n-4)) = -16n.$$

We get

$$\begin{aligned} & 2t(-8n + t^2) \\ & = 2t(-8n + 6n - 2) \\ & = 2t(-2n - 2) \\ & = -4(n+1)t, \end{aligned}$$

when $k = 3$,

$$t = \sqrt{6n-2} > 0,$$

then

$$f_1[n, 3, t] = -4(n+1)t < 0.$$

Therefore, we have that the inequality in the Claim holds for $k = 3$. That is, we can find a solution a in the formula (10).

Theorem 1. A solution a exists for $k = 3$.

3.2. $k = 4, 5, 6$

If $k = 4$, then

$$t^2 = 8(2n-1) + 1 = 16n - 7,$$

$$\begin{aligned}
fun1[n, 4, t] &= (3 + 4(-1 + 2n) - t)(5 + 4(-1 + 2n) - t)(-1 + t) \\
&\times (3 - 2n + 4(-1 + 2n) + t)(5 - 2n + 4(-1 + 2n) + t)(-1 + 3t) \\
&- (3 - 2n + 4(-1 + 2n) - t)(5 - 2n + 4(-1 + 2n) - t)(1 + t) \\
&\times (3 + 4(-1 + 2n) + t)(5 + 4(-1 + 2n) + t)(1 + 3t).
\end{aligned}$$

Using Collect[Expand[fun1[n, 4, t]], t] in mathematica, then

$$\begin{aligned}
fun1[n, 4, t] &= (-8 + 8n + 800n^2 + 384n^3 - 18432n^4)t \\
&+ (16 + 16n + 736n^2 + 1152n^3)t^3 + (-8 - 24n)t^5,
\end{aligned}$$

substituting $t^2 = 16n - 7$ into $fun1[n, 4, t]$, we obtain

$$fun1[n, 4, t] = -256(2 - 3n + 3n^2 + 8n^3)t,$$

when $k = 4$,

$$t = \sqrt{16n - 7} > 0,$$

since the n^3 and n^2 terms controls the n^1 term by $8 + 3 > 3$, then

$$fun1[n, 4, t] < 0$$

Therefore, we have that the inequality in the Claim holds for $k = 4$. That is, we can find a solution a in the formula (10).

Theorem 2. A solution a exists for $k = 4$.

If $k = 5$, then

$$t^2 = 15(2n - 1) + 1 = 30n - 14,$$

$$\begin{aligned}
fun1[n, 5, t] &= (3 + 5(-1 + 2n) - t)(5 + 5(-1 + 2n) - t) \\
&\times (7 + 5(-1 + 2n) - t)(-2 + t)(3 - 2n + 5(-1 + 2n) + t) \\
&\times (5 - 2n + 5(-1 + 2n) + t)(7 - 2n + 5(-1 + 2n) + t)(-1 + 4t) \\
&- (3 - 2n + 5(-1 + 2n) - t)(5 - 2n + 5(-1 + 2n) - t) \\
&\times (7 - 2n + 5(-1 + 2n) - t)(2 + t)(3 + 5(-1 + 2n) + t) \\
&\times (5 + 5(-1 + 2n) + t)(7 + 5(-1 + 2n) + t)(1 + 4t).
\end{aligned}$$

Using Collect[Expand[fun1[n, 5, t]], t] in mathematica, then

$$\begin{aligned}
fun1[n, 5, t] &= (128n - 23040n^2 - 128n^3 + 944640n^4 + 153600n^5 - 9216000n^6)t \\
&+ (288 + 128n - 864n^2 - 4064n^3 + 328320n^4 + 307200n^5)t^3 \\
&+ (-144 - 232n - 4104n^2 - 7616n^3)t^5 + (18 + 48n)t^7,
\end{aligned}$$

substituting $t^2 = 30n - 14$ into $fun1[n, 5, t]$, we obtain

$$fun1[n, 5, t] = -144(567 - 1863n + 4027n^2 - 3113n^3 - 1570n^4 + 8000n^5)t,$$

when $k = 5$,

$$t = \sqrt{30n - 14} > 0,$$

since the n^5 term controls the n^4 and n^3 terms by $8000 > 1570 + 3113$; 2 term controls the 1 term by $4027 > 1863$, then

$$fun1[n, 5, t] < 0$$

Therefore, we have

Theorem 3. A solution a exists for $k = 5$.

If $k = 6$, then

$$t^2 = 24(2n - 1) + 1 = 48n - 23,$$

$$\begin{aligned} fun1[n, 6, t] = & (3 + 6(-1 + 2n) - t)(5 + 6(-1 + 2n) - t) \\ & \times (7 + 6(-1 + 2n) - t)(9 + 6(-1 + 2n) - t)(-3 + t) \\ & \times (3 - 2n + 6(-1 + 2n) + t)(5 - 2n + 6(-1 + 2n) + t) \\ & \times (7 - 2n + 6(-1 + 2n) + t)(9 - 2n + 6(-1 + 2n) + t)(-1 + 5t) \\ & - (3 - 2n + 6(-1 + 2n) - t)(5 - 2n + 6(-1 + 2n) - t) \\ & \times (7 - 2n + 6(-1 + 2n) - t)(9 - 2n + 6(-1 + 2n) - t)(3 + t) \\ & \times (3 + 6(-1 + 2n) + t)(5 + 6(-1 + 2n) + t) \\ & \times (7 + 6(-1 + 2n) + t)(9 + 6(-1 + 2n) + t)(1 + 5t). \end{aligned}$$

Using Collect[Expand[fun1[n, 6, t]], t] in mathematica, then

$$\begin{aligned} fun1[n, 6, t] = & (-2592 + 2160n + 702720n^2 + 130752n^3 - 54931968n^4 - 3571200n^5 \\ & + 1124352000n^6 + 82944000n^7 - 6635520000n^8)t \\ & + (5760 + 768n + 333568n^2 + 162240n^3 - 9825280n^4 - 8002560n^5 \\ & + 210124800n^6 + 138240000n^7)t^3 \\ & + (-3776 - 4000n - 67840n^2 - 75712n^3 - 2580992n^4 - 3417600n^5)t^5 \\ & + (640 + 1152n + 14592n^2 + 28480n^3)t^7 + (-32 - 80n)t^9, \end{aligned}$$

substituting $t^2 = 48n - 23$ into $fun1[n, 6, t]$, we obtain

$$fun1[n, 6, t] = -98304(192 - 952n + 2862n^2 - 5149n^3 + 5640n^4 - 883n^5 - 6678n^6 + 9000n^7)t,$$

when $k = 6$,

$$t = \sqrt{48n - 23} > 0,$$

since the n^7 term controls the n^6 and n^5 terms by $9000 > 6678 + 883$; 4 term controls the 3 term; 2 term controls the 1 term, then

$$fun1[n, 6, t] < 0.$$

Therefore, we have

Theorem 4. A solution a exists for $k = 6$.

3.3. $k = 7$

If $k = 7$, then

$$t^2 = 35(2n - 1) + 1 = 70n - 34,$$

$$\begin{aligned}
fun1[n, 7, t] &= (3 + 7(-1 + 2n) - t)(5 + 7(-1 + 2n) - t) \\
&\times (7 + 7(-1 + 2n) - t)(9 + 7(-1 + 2n) - t) \\
&\times (11 + 7(-1 + 2n) - t)(-4 + t)(3 - 2n + 7(-1 + 2n) + t) \\
&\times (5 - 2n + 7(-1 + 2n) + t)(7 - 2n + 7(-1 + 2n) + t) \\
&\times (9 - 2n + 7(-1 + 2n) + t)(11 - 2n + 7(-1 + 2n) + t)(-1 + 6t) \\
&- (3 - 2n + 7(-1 + 2n) - t)(5 - 2n + 7(-1 + 2n) - t) \\
&\times (7 - 2n + 7(-1 + 2n) - t)(9 - 2n + 7(-1 + 2n) - t) \\
&\times (11 - 2n + 7(-1 + 2n) - t)(4 + t)(3 + 7(-1 + 2n) + t) \\
&\times (5 + 7(-1 + 2n) + t)(7 + 7(-1 + 2n) + t) \\
&\times (9 + 7(-1 + 2n) + t)(11 + 7(-1 + 2n) + t)(1 + 6t).
\end{aligned}$$

Using `Collect[Expand[fun1[n, 7, t]], t]` in mathematica, then

$$\begin{aligned}
fun1[n, 7, t] &= (65536n - 34406400n^2 - 81920n^3 + 3655680000n^4 + 252903424n^5 \\
&- 126632755200n^6 - 6177669120n^7 + 1612154880000n^8 \\
&+ 63727534080n^9 - 6691391078400n^{10})t \\
&+ (204800 + 16384n - 768000n^2 - 3066880n^3 + 771712000n^4 + 381500416n^5 \\
&- 19297152000n^6 - 10765762560n^7 + 189665280000n^8 + 95591301120n^9)t^3 \\
&+ (-128000 - 97536n - 4336000n^2 - 3784960n^3 + 10000000n^4 \\
&+ 16550656n^5 - 2202144000n^6 - 2248888320n^7)t^5 \\
&+ (26400 + 35456n + 620000n^2 + 893440n^3 + 13108000n^4 + 19999104n^5)t^7 \\
&+ (-2000 - 3760n - 40000n^2 - 79680n^3)t^9 + (50 + 120n)t^{11},
\end{aligned}$$

substituting $t^2 = 70n - 34$ into $fun1[n, 7, t]$, we obtain

$$\begin{aligned}
fun1[n, 7, t] &= -1600(3, 835, 625 - 25, 278, 075n + 96, 471, 294n^2 \\
&- 243, 090, 838n^3 + 433, 721, 305n^4 - 511, 435, 635n^5 + 281, 895, 936n^6 \\
&+ 274, 986, 628n^7 - 739, 892, 160n^8 + 580, 849, 920n^9)t \\
&= f_7(n)t,
\end{aligned}$$

Here are two ways we can prove $fun1[n, 7, t] < 0$.

Method 1: when $k = 7$.

We have

$$t = \sqrt{70n - 34} > 0.$$

Lemma 7. If $a, b, 2a - b > 0$, and x is a positive integer greater than 1, then $b - a - bx + ax^2 > 0$.

Proof Differentiating $b - a - bx + ax^2$ with respect to x yields $2ax - b$. Since $2ax - b \geq 2a - b > 0$, it follows that $b - a - bx + ax^2$ is monotonically increasing. Moreover, when $x = 1$, we have $b - a - bx + ax^2 = 0$. Therefore, when x is a positive integer greater than 1, we have $b - a - bx + ax^2 > 0$.

Now, we want to take the n^9 term and a part of the n^7 term to control the n^8 term. By $580 \times 2 > 740$ and $740 - 580 = 160 > 110$, we take out $110n^7$ from the n^7 term. We still have more than $160n^7$ left. Now, $160n^7$ and 6 term provide at least $441.8n^6 > 341.8n^6$. Again, $341.8 \times 2 > 511.5$. Taking $170n^4$ from 4 term we control the 5 term. The left 4 term controls the 3 term; 2 term controls the 1 term, then

$$f_7(n) = fun1[n, 7, t] < 0.$$

Method 2: We can also substitute $n = m + 1$ into $f_7(n)$. We notice that m is just the complex dimension of the base manifold M . we obtain:

$$\begin{aligned} f_7(n) &= f_7(m+1) \\ &= -1600(152064000 + 1540893056m + 7051945128m^2 + 18997321052m^3 \\ &\quad + 33124152870m^4 + 38707788129m^5 + 30281215132m^6 \\ &\quad + 15266446468m^7 + 4487757120m^8 + 580849920m^9) \\ &< 0, \end{aligned}$$

since

$$f_7(n) < 0 \text{ and } t = \sqrt{70n - 34} > 0,$$

then

$$fun1[n, 7, t] < 0.$$

Therefore, we have

Theorem 5. A solution a exists for $k = 7$.

3.4. $8 \leq k \leq 13$

If $k = 8$, then

$$t^2 = 48(2n - 1) + 1 = 96n - 47,$$

$$\begin{aligned} fun1[n, 8, t] &= (3 + 8(-1 + 2n) - t)(5 + 8(-1 + 2n) - t) \\ &\quad \times (7 + 8(-1 + 2n) - t)(9 + 8(-1 + 2n) - t) \\ &\quad \times (11 + 8(-1 + 2n) - t)(13 + 8(-1 + 2n) - t)(-5 + t) \\ &\quad \times (3 - 2n + 8(-1 + 2n) + t)(5 - 2n + 8(-1 + 2n) + t) \\ &\quad \times (7 - 2n + 8(-1 + 2n) + t)(9 - 2n + 8(-1 + 2n) + t) \\ &\quad \times (11 - 2n + 8(-1 + 2n) + t)(13 - 2n + 8(-1 + 2n) + t)(-1 + 7t) \\ &\quad - (3 - 2n + 8(-1 + 2n) - t)(5 - 2n + 8(-1 + 2n) - t) \\ &\quad \times (7 - 2n + 8(-1 + 2n) - t)(9 - 2n + 8(-1 + 2n) - t) \\ &\quad \times (11 - 2n + 8(-1 + 2n) - t)(13 - 2n + 8(-1 + 2n) - t)(5 + t) \\ &\quad \times (3 + 8(-1 + 2n) + t)(5 + 8(-1 + 2n) + t) \\ &\quad \times (7 + 8(-1 + 2n) + t)(9 + 8(-1 + 2n) + t) \\ &\quad \times (11 + 8(-1 + 2n) + t)(13 + 8(-1 + 2n) + t)(1 + 7t). \end{aligned}$$

Using Collect[Expand[fun1[n, 8, t]], t] in mathematica, then

$$\begin{aligned} fun1[n, 8, t] &= (-3645000 + 2331000n + 1896501600n^2 + 175165760n^3 \\ &\quad - 301282434432n^4 - 11623555200n^5 + 15196259381760n^6 \\ &\quad + 639897538560n^7 - 319321072336896n^8 - 10699931648000n^9 \\ &\quad + 2867682386903040n^{10} + 67673920634880n^{11} - 9095374933327872n^{12})t \\ &\quad + (8391600 - 49840n + 606741408n^2 + 146528064n^3 \\ &\quad - 42566169600n^4 - 20540943360n^5 + 2272539290112n^6 \\ &\quad + 961035177984n^7 - 38054441779200n^8 - 16472499814400n^9 \end{aligned}$$

$$\begin{aligned} &+ (232749948469248n^{10} + 94743488888832n^{11})t^3 \\ &+ (-5963832 - 3523736n - 164289600n^2 - 100983680n^3 \\ &- 7231230720n^4 - 5576186112n^5 + 122234918400n^6 \\ &+ 104496967680n^7 - 2525691248640n^8 - 2089633710080n^9)t^5 \\ &+ (1337760 + 1421280n + 42299712n^2 + 48386688n^3 \\ &+ 416102400n^4 + 499415040n^5 + 14891945472n^6 + 18545651712n^7)t^7 \\ &+ (-125496 - 188216n - 3175200n^2 - 5041600n^3 - 50336640n^4 - 82793088n^5)t^9 \\ &+ (5040 + 9680n + 92448n^2 + 185920n^3)t^{11} + (-72 - 168n)t^{13}, \end{aligned}$$

substituting $t^2 = 96n - 47$ into $fun1[n, 8, t]$, we obtain

$$\begin{aligned} fun1[n, 8, t] &= -4718592(571, 536 - 4, 687, 956n + 21, 635, 136n^2 - 68, 048, 739n^3 \\ &+ 158, 379, 281n^4 - 276, 334, 333n^5 + 349, 225, 527n^6 - 267, 133, 524n^7 \\ &- 14, 076, 192n^8 + 361, 688, 064n^9 - 494, 885, 888n^{10} + 275, 365, 888n^{11})t \\ &= f_8(n)t, \end{aligned}$$

substituting $n = m + 1$ into $f_8(n)$, we obtain:

$$\begin{aligned} f_8(n) &= f_8(m + 1) \\ &= -4718592(41698800 + 534448980m + 3134895984m^2 + 11079557839m^3 \\ &+ 26197486365m^4 + 43501440617m^5 + 51759958635m^6 + 44125463724m^7 \\ &+ 26406622944m^8 + 10557953024m^9 + 2534138880m^{10} + 275365888m^{11}) \\ &< 0, \end{aligned}$$

since

$$f_8(n) < 0 \text{ and } t = \sqrt{96n - 47} > 0,$$

then

$$fun1[n, 8, t] < 0.$$

Therefore, we have

Theorem 6. A solution a exists for $k = 8$.

If $k = 9$, then

$$t^2 = 63(2n - 1) + 1 = 126n - 62,$$

$$\begin{aligned} fun1[n, 9, t] &= (3 + 9(-1 + 2n) - t)(5 + 9(-1 + 2n) - t) \\ &\times (7 + 9(-1 + 2n) - t)(9 + 9(-1 + 2n) - t)(11 + 9(-1 + 2n) - t) \\ &\times (13 + 9(-1 + 2n) - t)(15 + 9(-1 + 2n) - t)(-6 + t) \\ &\times (3 - 2n + 9(-1 + 2n) + t)(5 - 2n + 9(-1 + 2n) + t)(7 - 2n + 9(-1 + 2n) + t) \\ &\times (9 - 2n + 9(-1 + 2n) + t)(11 - 2n + 9(-1 + 2n) + t)(13 - 2n + 9(-1 + 2n) + t) \\ &\times (15 - 2n + 9(-1 + 2n) + t)(-1 + 8t) \\ &- (3 - 2n + 9(-1 + 2n) - t)(5 - 2n + 9(-1 + 2n) - t)(7 - 2n + 9(-1 + 2n) - t) \\ &\times (9 - 2n + 9(-1 + 2n) - t)(11 - 2n + 9(-1 + 2n) - t)(13 - 2n + 9(-1 + 2n) - t) \\ &\times (15 - 2n + 9(-1 + 2n) - t)(6 + t)(3 + 9(-1 + 2n) + t) \end{aligned}$$

$$\begin{aligned} &\times (5 + 9(-1 + 2n) + t)(7 + 9(-1 + 2n) + t)(9 + 9(-1 + 2n) + t) \\ &\times (11 + 9(-1 + 2n) + t)(13 + 9(-1 + 2n) + t)(15 + 9(-1 + 2n) + t)(1 + 8t). \end{aligned}$$

Using Collect[Expand[fun1[n, 9, t]], t] in mathematica, then

$$\begin{aligned} fun1[n, 9, t] = & (127401984n - 149824733184n^2 - 173408256n^3 + 29569575813120n^4 \\ & + 1102331510784n^5 - 2059848696987648n^6 - 64360428208128n^7 \\ & + 62915005419356160n^8 + 1826738059345920n^9 - 921875778373681152n^{10} \\ & - 21488462532182016n^{11} + 6306734304625950720n^{12} + 95865912019648512n^{13} \\ & - 16105473219300950016n^{14})t \\ & + (520224768 - 3538944n - 2124251136n^2 - 7703076864n^3 \\ & + 4439482564608n^4 + 1477419417600n^5 - 262767538962432n^6 \\ & - 96527585353728n^7 + 7371079217381376n^8 + 2623127861329920n^9 \\ & - 86213346566602752n^{10} - 30625374060675072n^{11} \\ & + 375141954326888448n^{12} + 127821216026198016n^{13})t^3 \\ & + (-354041856 - 168376320n - 14399809536n^2 - 7712739328n^3 \\ & + 46566338560n^4 + 64649811968n^5 - 21121040582656n^6 \\ & - 14315948265472n^7 + 382511373189120n^8 + 266985965223936n^9 \\ & - 3795772159033344n^{10} - 2632120912576512n^{11})t^5 \\ & + (85524992 + 74907648n + 2893097984n^2 + 2701920256n^3 \\ & + 68397690368n^4 + 68305577984n^5 - 125687317504n^6 \\ & - 119982627328n^7 + 21644589109248n^8 + 22688637124608n^9)t^7 \\ & + (-9056768 - 11264768n - 290117632n^2 - 381751552n^3 \\ & - 4018094080n^4 - 5534462080n^5 - 75154823296n^6 - 104794322944n^7)t^9 \\ & + (460992 + 736512n + 11963840n^2 + 20034336n^3 \\ & + 158899552n^4 + 273541632n^5)t^{11} \\ & + (-10976 - 21336n - 189336n^2 - 382592n^3)t^{13} + (98 + 224n)t^{15}, \end{aligned}$$

substituting $t^2 = 126n - 62$ into $fun1[n, 9, t]$, we obtain

$$\begin{aligned} fun1[n, 9, t] = & -112896(13, 698, 355, 671 - 134, 285, 154, 003n \\ & + 726, 322, 699, 013n^2 - 2, 708, 895, 877, 195n^3 + 7, 619, 060, 616, 335n^4 \\ & - 16, 736, 362, 982, 605n^5 + 28, 905, 565, 733, 807n^6 - 38, 213, 080, 544, 569n^7 \\ & + 34, 911, 433, 711, 278n^8 - 12, 362, 350, 332, 636n^9 - 23, 108, 040, 252, 360n^{10} \\ & + 50, 254, 811, 679, 744n^{11} - 48, 163, 969, 695, 744n^{12} + 20, 804, 234, 379, 264n^{13})t \\ = & f_9(n)t, \end{aligned}$$

substituting $n = m + 1$ into $f_9(n)$, we obtain:

$$\begin{aligned} f_9(n) = & f_9(m + 1) \\ = & -112896(1808142336000 + 28168183987200m + 202595808144640m^2 \\ & + 890676868548848m^3 + 2671469113628224m^4 + 5775288985434248m^5 \\ & + 9262089456891880m^6 + 11161358996990855m^7 + 10109717043238602m^8 \\ & + 6799526137639764m^9 + 3300883920775224m^{10} + 1095017456913408m^{11} \\ & + 222291077234688m^{12} + 20804234379264m^{13}) \end{aligned}$$

$< 0,$

since

$f_9(n) < 0 \text{ and } t = \sqrt{126n - 62} > 0,$

then

$fun1[n, 9, t] < 0.$

Therefore, we have

Theorem 7. *A solution a exists for k = 9.*

If k = 10, then

$t^2 = 80(2n - 1) + 1 = 160n - 79,$

$$\begin{aligned} fun1[n, 10, t] = & (3 + 10(-1 + 2n) - t)(5 + 10(-1 + 2n) - t)(7 + 10(-1 + 2n) - t) \\ & \times (9 + 10(-1 + 2n) - t)(11 + 10(-1 + 2n) - t)(13 + 10(-1 + 2n) - t) \\ & \times (15 + 10(-1 + 2n) - t)(17 + 10(-1 + 2n) - t)(-7 + t) \\ & \times (3 - 2n + 10(-1 + 2n) + t)(5 - 2n + 10(-1 + 2n) + t) \\ & \times (7 - 2n + 10(-1 + 2n) + t)(9 - 2n + 10(-1 + 2n) + t) \\ & \times (11 - 2n + 10(-1 + 2n) + t)(13 - 2n + 10(-1 + 2n) + t) \\ & \times (15 - 2n + 10(-1 + 2n) + t)(17 - 2n + 10(-1 + 2n) + t)(-1 + 9t) \\ & - (3 - 2n + 10(-1 + 2n) - t)(5 - 2n + 10(-1 + 2n) - t) \\ & \times (7 - 2n + 10(-1 + 2n) - t)(9 - 2n + 10(-1 + 2n) - t) \\ & \times (11 - 2n + 10(-1 + 2n) - t)(13 - 2n + 10(-1 + 2n) - t) \\ & \times (15 - 2n + 10(-1 + 2n) - t)(17 - 2n + 10(-1 + 2n) - t)(7 + t) \\ & \times (3 + 10(-1 + 2n) + t)(5 + 10(-1 + 2n) + t) \\ & \times (7 + 10(-1 + 2n) + t)(9 + 10(-1 + 2n) + t) \\ & \times (11 + 10(-1 + 2n) + t)(13 + 10(-1 + 2n) + t) \\ & \times (15 + 10(-1 + 2n) + t)(17 + 10(-1 + 2n) + t)(1 + 9t). \end{aligned}$$

Using Collect[Expand[fun1[n, 10, t]], t] in mathematica, then

$$\begin{aligned} fun1[n, 10, t] = & (-15558480000 + 7974338400n + 13196390860800n^2 \\ & + 720908012160n^3 - 3505540286361600n^4 - 85155466959360n^5 \\ & + 317835041771520000n^6 + 8262050315581440n^7 - 13198191456728678400n^8 \\ & - 312620844967280640n^9 + 279098389876388659200n^{10} \\ & + 5875600478404608000n^{11} - 3090536919749099520000n^{12} \\ & - 51539240013004800000n^{13} + 16945048901910528000000n^{14} \\ & + 175535727575040000000n^{15} - 36110206815436800000000n^{16})t \\ & + (36454118400 - 1526865536n + 3187880409088n^2 + 472477625472n^3 \\ & - 393490799321088n^4 - 132388213773312n^5 + 37557677392822272n^6 \\ & + 11428084381607936n^7 - 1421799043688169472n^8 - 438371852799836160n^9 \\ & + 26525354153685811200n^{10} + 8069447927955456000n^{11} \\ & - 230883659454873600000n^{12} - 69640007476838400000n^{13} \end{aligned}$$

$$\begin{aligned} &+ 771242688774144000000n^{14} + 225688792596480000000n^{15})t^3 \\ &+ (-26924768768 - 10706238720n - 970996905984n^2 - 398276882304n^3 \\ &- 49486105202688n^4 - 26329712055808n^5 + 1889640197193728n^6 \\ &+ 1152495469867008n^7 - 81464963253338112n^8 - 48068207742320640n^9 \\ &+ 1158842140065792000n^{10} + 690194889474048000n^{11} \\ &- 7283889930240000000n^{12} - 4339633265049600000n^{13})t^5 \\ &+ (6764104704 + 5025288576n + 268815267840n^2 + 213040145024n^3 \\ &+ 4188158148608n^4 + 3462486730752n^5 + 166370838380544n^6 \\ &+ 146226205741056n^7 - 1962446366834688n^8 - 1708459425792000n^9 \\ &+ 39757033399910400n^{10} + 35894227009536000n^{11})t^7 \\ &+ (-779342592 - 826715456n - 28395776000n^2 - 31782787200n^3 \\ &- 575042506752n^4 - 673348118016n^5 - 3942540607488n^6 \\ &- 4911682807808n^7 - 137227696898048n^8 - 165555487457280n^9)t^9 \\ &+ (45755392 + 62494848n + 1498441728n^2 + 2144247168n^3 \\ &+ 22681559040n^4 + 33675013120n^5 + 306767233024n^6 + 459876354048n^7)t^{11} \\ &+ (-1408512 - 2343936n - 36857856n^2 - 63840896n^3 \\ &- 433664000n^4 - 769337856n^5)t^{13} \\ &+ (21504 + 42112n + 354304n^2 + 717696n^3)t^{15} + (-128 - 288n)t^{17}, \end{aligned}$$

substituting $t^2 = 160n - 79$ into $fun1[n, 10, t]$, we obtain

$$\begin{aligned} fun1[n, 10, t] &= -1342177280(837,591,040 - 9,545,572,608n + 59,182,879,984n^2 \\ &- 254,066,945,116n^3 + 830,279,498,864n^4 - 2,158,618,273,605n^5 \\ &+ 4,554,726,168,581n^6 - 7,813,891,508,729n^7 + 10,684,923,689,976n^8 \\ &- 10,913,113,001,707n^9 + 6,487,296,663,665n^{10} + 2,211,775,182,625n^{11} \\ &- 11,241,561,653,750n^{12} + 15,050,454,682,500n^{13} \\ &- 11,276,882,775,000n^{14} + 3,985,807,500,000n^{15})t \\ &= f_{10}(n)t, \end{aligned}$$

substituting $n = m + 1$ into $f_{10}(n)$, we obtain:

$$\begin{aligned} f_{10}(n) &= f_{10}(m + 1) \\ &= -1342177280(197604126720 + 3637553257296m + 31109278376852m^2 \\ &+ 164205485918588m^3 + 598952066679427m^4 + 1600699352651976m^5 \\ &+ 3240371121805818m^6 + 5062827823955677m^7 + 6158892608618913m^8 \\ &+ 5836186244791818m^9 + 4274524058445040m^{10} + 2377090407972625m^{11} \\ &+ 971760429193750m^{12} + 275683883332500m^{13} \\ &+ 48510229725000m^{14} + 3985807500000m^{15}) \\ &< 0, \end{aligned}$$

since

$$f_{10}(n) < 0 \text{ and } t = \sqrt{160n - 79} > 0,$$

then

$$fun1[n,10,t] < 0.$$

Therefore, we have

Theorem 8. *A solution a exists for $k = 10$.*

If $k = 11$, then

$$t^2 = 99(2n - 1) + 1 = 198n - 98,$$

$$\begin{aligned} fun1[n,11,t] = & (3 + 11(-1 + 2n) - t)(5 + 11(-1 + 2n) - t) \\ & \times (7 + 11(-1 + 2n) - t)(9 + 11(-1 + 2n) - t)(11 + 11(-1 + 2n) - t) \\ & \times (13 + 11(-1 + 2n) - t)(15 + 11(-1 + 2n) - t)(17 + 11(-1 + 2n) - t) \\ & \times (19 + 11(-1 + 2n) - t)(-8 + t)(3 - 2n + 11(-1 + 2n) + t) \\ & \times (5 - 2n + 11(-1 + 2n) + t)(7 - 2n + 11(-1 + 2n) + t) \\ & \times (9 - 2n + 11(-1 + 2n) + t)(11 - 2n + 11(-1 + 2n) + t) \\ & \times (13 - 2n + 11(-1 + 2n) + t)(15 - 2n + 11(-1 + 2n) + t) \\ & \times (17 - 2n + 11(-1 + 2n) + t)(19 - 2n + 11(-1 + 2n) + t)(-1 + 10t) \\ & - (3 - 2n + 11(-1 + 2n) - t)(5 - 2n + 11(-1 + 2n) - t) \\ & \times (7 - 2n + 11(-1 + 2n) - t)(9 - 2n + 11(-1 + 2n) - t) \\ & \times (11 - 2n + 11(-1 + 2n) - t)(13 - 2n + 11(-1 + 2n) - t) \\ & \times (15 - 2n + 11(-1 + 2n) - t)(17 - 2n + 11(-1 + 2n) - t) \\ & \times (19 - 2n + 11(-1 + 2n) - t)(8 + t)(3 + 11(-1 + 2n) + t) \\ & \times (5 + 11(-1 + 2n) + t)(7 + 11(-1 + 2n) + t) \\ & \times (9 + 11(-1 + 2n) + t)(11 + 11(-1 + 2n) + t) \\ & \times (13 + 11(-1 + 2n) + t)(15 + 11(-1 + 2n) + t) \\ & \times (17 + 11(-1 + 2n) + t)(19 + 11(-1 + 2n) + t)(1 + 10t). \end{aligned}$$

Using Collect[Expand[fun1[n, 11, t]], t] in mathematica, then

$$\begin{aligned} fun1[n,11,t] = & (695784701952n - 1549860423598080n^2 - 990526832640n^3 \\ & + 487614072854937600n^4 + 11285339565981696n^5 - 56107267939306045440n^6 \\ & - 1156736378009026560n^7 + 3020141576434640486400n^8 \\ & + 59944753082350436352n^9 - 86082343319221284372480n^{10} \\ & - 1560084441634701312000n^{11} + 1366386578033328783360000n^{12} \\ & + 21525396550739558400000n^{13} - 12048976131448268390400000n^{14} \\ & - 147903413558968320000000n^{15} + 54867763879634534400000000n^{16} \\ & + 404588840406220800000000n^{17} - 100135738000539648000000000n^{18})t \\ & + (3522410053632 - 120795955200n - 15043626270720n^2 - 51356693954560n^3 \\ & + 56619342527201280n^4 + 14165710995456000n^5 - 5855618498659614720n^6 \\ & - 1591637307914452992n^7 + 301988275707715780608n^8 \\ & + 80758706172413870080n^9 - 7838918956318379212800n^{10} \\ & - 2094397775600418816000n^{11} + 107505975067308195840000n^{12} \\ & + 28470609179443200000000n^{13} - 732854283471028224000000n^{14} \end{aligned}$$

$$\begin{aligned} &- 192157407796789248000000n^{15} + 1973749877118074880000000n^{16} \\ &+ 505736050507776000000000n^{17})t^3 \\ &+ (-2507271045120 - 850095243264n - 120313974620160n^2 \\ &- 45246953553920n^3 + 446068585267200n^4 + 555558613155840n^5 \\ &- 360584811019173888n^6 - 182560392535080960n^7 \\ &+ 14747319397971394560n^8 + 7678896417104592896n^9 \\ &- 358531238768055091200n^{10} - 185744388196663296000n^{11} \\ &+ 3867745890174566400000n^{12} + 2012282789135646720000n^{13} \\ &- 17440740864117964800000n^{14} - 9097676060098560000000n^{15})t^5 \\ &+ (654856814592 + 421865717760n + 27871134842880n^2 \\ &+ 19010320269312n^3 + 747634071969792n^4 + 544737388953600n^5 \\ &- 2080130950840320n^6 - 1684758031958016n^7 + 637155661378166784n^8 \\ &+ 493283039580487680n^9 - 9069486634826956800n^{10} - 6983839862076211200n^{11} \\ &+ 90726001758830592000n^{12} + 71816781702758400000n^{13})t^7 \\ &+ (-80004464640 - 73901629440n - 3313958510592n^2 \\ &- 3223552655360n^3 - 66516944117760n^4 - 67730179104768n^5 \\ &- 1350417537552384n^6 - 1414700352061440n^7 + 2048697462435840n^8 \\ &+ 1418683846909952n^9 - 306302240675819520n^{10} - 324942405156864000n^{11})t^9 \\ &+ (5179064832 + 6172477440n + 197262812160n^2 + 246022701056n^3 \\ &+ 3867566040576n^4 + 5006567516160n^5 + 39800129725440n^6 \\ &+ 53442261086208n^7 + 696141456081408n^8 + 921716323133440n^9)t^{11} \\ &+ (-186831360 - 271604736n - 6236165376n^2 - 9435525120n^3 \\ &- 96568899840n^4 - 150694483968n^5 - 1065059186688n^6 - 1678421514240n^7)t^{13} \\ &+ (3748032 + 6416640n + 98210880n^2 + 174120192n^3 \\ &+ 1057551552n^4 + 1916087040n^5)t^{15} \\ &+ (-38880 - 76512n - 618192n^2 - 1253760n^3)t^{17} + (162 + 360n)t^{19}, \end{aligned}$$

substituting $t^2 = 198n - 98$ into $fun1[n, 11, t]$, we obtain:

$$\begin{aligned} fun1[n, 11, t] &= -248832(4, 065, 224, 478, 654, 843 - 52, 787, 222, 642, 237, 085n \\ &+ 368, 970, 411, 257, 152, 164n^2 - 1, 787, 378, 204, 000, 874, 564n^3 \\ &+ 6, 623, 691, 055, 192, 150, 674n^4 - 19, 719, 047, 743, 827, 264, 318n^5 \\ &+ 48, 402, 567, 257, 962, 672, 852n^6 - 99, 124, 517, 278, 565, 517, 884n^7 \\ &+ 169, 334, 824, 113, 052, 450, 939n^8 - 237, 677, 174, 171, 621, 991, 677n^9 \\ &+ 262, 254, 421, 946, 901, 870, 640n^{10} - 198, 503, 307, 122, 426, 496, 344n^{11} \\ &+ 35, 823, 278, 232, 902, 875, 088n^{12} + 171, 849, 246, 273, 528, 823, 472n^{13} \\ &- 320, 304, 654, 197, 891, 545, 600n^{14} + 325, 500, 808, 553, 719, 040, 000n^{15} \\ &- 201, 657, 860, 070, 092, 800, 000n^{16} + 60, 363, 460, 889, 600, 000, 000n^{17})t \\ &= f_{11}(n)t, \end{aligned}$$

substituting $n = m + 1$ into $f_{11}(n)$, we obtain:

$$\begin{aligned} f_{11}(n) &= f_{11}(m + 1) \\ &= -248832(1698607947526963200 + 36177777457196433408m \end{aligned}$$

$$\begin{aligned} &+ 359672462989573349376m^2 + 2222293802384046581760m^3 \\ &+ 9574270081469840856672m^4 + 30562684033528394595824m^5 \\ &+ 74928658585405235919024m^6 + 144223756069606887120984m^7 \\ &+ 220803047859292560748110m^8 + 270571961581874969626443m^9 \\ &+ 265518352015567541254056m^{10} + 207569985972325872837128m^{11} \\ &+ 127736798496998714130624m^{12} + 60601804263639578385072m^{13} \\ &+ 21410417670624758054400m^{14} + 5308405728417834240000m^{15} \\ &+ 824520975053107200000m^{16} + 60363460889600000000m^{17}) \\ &< 0, \end{aligned}$$

since

$$f_{11}(n) < 0 \text{ and } t = \sqrt{198n - 98} > 0,$$

then

$$fun1[n, 11, t] < 0.$$

Therefore, we have

Theorem 9. *A solution a exists for $k = 11$.*

If $k = 12$, then

$$t^2 = 120(2n - 1) + 1 = 240n - 119,$$

$$\begin{aligned} fun1[n, 12, t] &= (3 + 12(-1 + 2n) - t)(5 + 12(-1 + 2n) - t) \\ &\times (7 + 12(-1 + 2n) - t)(9 + 12(-1 + 2n) - t)(11 + 12(-1 + 2n) - t) \\ &\times (13 + 12(-1 + 2n) - t)(15 + 12(-1 + 2n) - t)(17 + 12(-1 + 2n) - t) \\ &\times (19 + 12(-1 + 2n) - t)(21 + 12(-1 + 2n) - t)(-9 + t) \\ &\times (3 - 2n + 12(-1 + 2n) + t)(5 - 2n + 12(-1 + 2n) + t) \\ &\times (7 - 2n + 12(-1 + 2n) + t)(9 - 2n + 12(-1 + 2n) + t) \\ &\times (11 - 2n + 12(-1 + 2n) + t)(13 - 2n + 12(-1 + 2n) + t) \\ &\times (15 - 2n + 12(-1 + 2n) + t)(17 - 2n + 12(-1 + 2n) + t) \\ &\times (19 - 2n + 12(-1 + 2n) + t)(21 - 2n + 12(-1 + 2n) + t)(-1 + 11t) \\ &- (3 - 2n + 12(-1 + 2n) - t)(5 - 2n + 12(-1 + 2n) - t) \\ &\times (7 - 2n + 12(-1 + 2n) - t)(9 - 2n + 12(-1 + 2n) - t) \\ &\times (11 - 2n + 12(-1 + 2n) - t)(13 - 2n + 12(-1 + 2n) - t) \\ &\times (15 - 2n + 12(-1 + 2n) - t)(17 - 2n + 12(-1 + 2n) - t) \\ &\times (19 - 2n + 12(-1 + 2n) - t)(21 - 2n + 12(-1 + 2n) - t)(9 + t) \\ &\times (3 + 12(-1 + 2n) + t)(5 + 12(-1 + 2n) + t) \\ &\times (7 + 12(-1 + 2n) + t)(9 + 12(-1 + 2n) + t) \\ &\times (11 + 12(-1 + 2n) + t)(13 + 12(-1 + 2n) + t) \\ &\times (15 + 12(-1 + 2n) + t)(17 + 12(-1 + 2n) + t) \\ &\times (19 + 12(-1 + 2n) + t)(21 + 12(-1 + 2n) + t)(1 + 11t). \end{aligned}$$

Using Collect[Expand[fun1[n, 12, t]], t] in mathematica, then

$$\begin{aligned} fun1[n, 12, t] = & (-159498730125000 + 67976984413800n \\ & + 200154454107300000n^2 + 7201602937875456n^3 \\ & - 79790705888498668800n^4 - 1316866914026737920n^5 \\ & + 11275243176132017280000n^6 + 196683082521248581632n^7 \\ & - 761946313613997985228800n^8 - 12629426874416513095680n^9 \\ & + 27971572452964695659520000n^{10} + 436171233569375284789248n^{11} \\ & - 594505987352510463855820800n^{12} - 8389704947449797783060480n^{13} \\ & + 7470916122735349459845120000n^{14} + 90250119982351302770294784n^{15} \\ & - 54421700308230763851861196800n^{16} - 504500807763566430749982720n^{17} \\ & + 211295956125114388740833280000n^{18} + 1148174252151564980327546880n^{19} \\ & - 336797780631125727562747084800n^{20})t \\ & + (377649913410000 - 19615194170352n + 38867814375746400n^2 \\ & + 3979040301713664n^3 - 7370125006780800000n^4 - 1888683157765628928n^5 \\ & + 1089550830618361267200n^6 + 257155537198518171648n^7 \\ & - 69977886353804358144000n^8 - 16658991415687010734080n^9 \\ & + 2409555862259141247590400n^{10} + 570508887645879654776832n^{11} \\ & - 46193146985215205572608000n^{12} - 10896910417561746627624960n^{13} \\ & + 494220619745029150212096000n^{14} + 115741914913043776133922816n^{15} \\ & - 2742057146718683249246208000n^{16} - 635985561998707002609500160n^{17} \\ & + 6161288600560796927262720000n^{18} + 1403324085963023864844779520n^{19})t^3 \\ & + (-285272708668200 - 84363927570648n - 12634847452656000n^2 \\ & - 3832225552843776n^3 - 736940303686400000n^4 - 296495849097388032n^5 \\ & + 47415491875651200000n^6 + 22379027720932067328n^7 \\ & - 3361874754934798080000n^8 - 1539134642183133726720n^9 \\ & + 103103322275348105216000n^{10} + 47492932974646571630592n^{11} \\ & - 1752687584771751936000000n^{12} - 807351445609961666641920n^{13} \\ & + 14697765086296434278400000n^{14} + 6789911372558817281703936n^{15} \\ & - 51084064826464042745856000n^{16} - 23678677759017964661637120n^{17})t^5 \\ & + (76214933784000 + 43286382760896n + 3637474824931200n^2 \\ & + 2186383301103616n^3 + 74075329878144000n^4 + 46207911125781504n^5 \\ & + 3274321640390169600n^6 + 2205684223056771072n^7 \\ & - 84815043212123136000n^8 - 57722613397367480320n^9 \\ & + 3182173531515284275200n^{10} + 2190100808185809666048n^{11} \\ & - 39411495460325818368000n^{12} - 27131436318402029813760n^{13} \\ & + 252891903207936098304000n^{14} + 178018366467514369572864n^{15})t^7 \\ & + (-9743713245200 - 7965253814576n - 444127081032000n^2 \\ & - 381314399431680n^3 - 10821097657920000n^4 - 9720276535478784n^5 \\ & - 121379957922432000n^6 - 114792530839726080n^7 \\ & - 4578373716238080000n^8 - 4274674308828211200n^9 \\ & + 4139995595590144000n^{10} + 36430794620982558720n^{11}\end{aligned}$$

$$\begin{aligned} & - 828076347221409792000n^{12} - 782689494544290938880n^{13})t^9 \\ & + (676904268000 + 715176569184n + 29232266606400n^2 + 32274550725120n^3 \\ & + 630782159744000n^4 + 723885397103616n^5 + 10377088279219200n^6 \\ & + 12224945200343040n^7 + 54671514349056000n^8 + 69619141893304320n^9 \\ & + 1874354405724774400n^{10} + 2217869743655976960n^{11})t^{11} \\ & + (-27239005200 - 35138412144n - 1071131952000n^2 - 1437240724480n^3 \\ & - 20932369920000n^4 - 29005653067776n^5 - 240399401088000n^6 \\ & - 342089159239680n^7 - 2970315180288000n^8 - 4200510878136320n^9)t^{13} \\ & + (648472000 + 985749952n + 21968496000n^2 + 34587801600n^3 \\ & + 340264320000n^4 + 550318752768n^5 \\ & + 3253862784000n^6 + 5317258045440n^7)t^{15} \\ & + (-8956200 - 15655992n - 234300000n^2 - 422522880n^3 \\ & - 2357664000n^4 - 4338056448n^5)t^{17} \\ & + (66000 + 130320n + 1020000n^2 + 2069760n^3)t^{19} \\ & + (-200 - 440n)t^{21}, \end{aligned}$$

substituting $t^2 = 240n - 119$ into $fun1[n, 12, t]$, we obtain:

$$\begin{aligned} fun1[n, 12, t] &= -5798205849600(190512000000 - 2775808980000n \\ &+ 21591741371400n^2 - 116339327644260n^3 + 480775022341606n^4 \\ &- 1605016773165437n^5 + 4457641070810450n^6 - 10471263138678310n^7 \\ &+ 20958934338565527n^8 - 35707528917289404n^9 + 51171193367845430n^{10} \\ &- 59775503504618470n^{11} + 52331472403645153n^{12} - 24060622668439631n^{13} \\ &- 18949318129458510n^{14} + 59085778352886360n^{15} - 76516843948861536n^{16} \\ &+ 63922275678425472n^{17} - 33885639888059520n^{18} + 8800992597703680n^{19})t \\ &= f_{12}(n)t, \end{aligned}$$

substituting $n = m + 1$ into $f_{12}(n)$, we obtain:

$$\begin{aligned} f_{12}(n) &= f_{12}(m + 1) \\ &= -5798205849600(139992980400000 + 3393840306704400m \\ &+ 38547713078525700m^2 + 273525949933053324m^3 + 1362340323143385287m^4 \\ &+ 5068463255304985888m^5 + 14624724567669212770m^6 + 33523311577860929498m^7 \\ &+ 61997108039562645999m^8 + 93381825994658982441m^9 \\ &+ 115096299716911132510m^{10} + 116142537399367645412m^{11} \\ &+ 95600966620821423116m^{12} + 63656607753864650629m^{13} \\ &+ 33800347413266033370m^{14} + 13990210927479861336m^{15} \\ &+ 4353820766886130848m^{16} + 958950491900683392m^{17} \\ &+ 133333219468310400m^{18} + 8800992597703680m^{19}) \\ &< 0, \end{aligned}$$

since

$$f_{12}(n) < 0 \text{ and } t = \sqrt{240n - 119} > 0,$$

then

$$fun1[n,12,t] < 0.$$

Therefore, we have

Theorem 10. *A solution a exists for $k = 12$.*

If $k = 13$, then

$$t^2 = 143(2n - 1) + 1 = 286n - 142,$$

$$\begin{aligned} fun1[n,13,t] = & (3 + 13(-1 + 2n) - t)(5 + 13(-1 + 2n) - t) \\ & \times (7 + 13(-1 + 2n) - t)(9 + 13(-1 + 2n) - t)(11 + 13(-1 + 2n) - t) \\ & \times (13 + 13(-1 + 2n) - t)(15 + 13(-1 + 2n) - t)(17 + 13(-1 + 2n) - t) \\ & \times (19 + 13(-1 + 2n) - t)(21 + 13(-1 + 2n) - t) \\ & \times (23 + 13(-1 + 2n) - t)(-10 + t)(3 - 2n + 13(-1 + 2n) + t) \\ & \times (5 - 2n + 13(-1 + 2n) + t)(7 - 2n + 13(-1 + 2n) + t) \\ & \times (9 - 2n + 13(-1 + 2n) + t)(11 - 2n + 13(-1 + 2n) + t) \\ & \times (13 - 2n + 13(-1 + 2n) + t)(15 - 2n + 13(-1 + 2n) + t) \\ & \times (17 - 2n + 13(-1 + 2n) + t)(19 - 2n + 13(-1 + 2n) + t) \\ & \times (21 - 2n + 13(-1 + 2n) + t)(23 - 2n + 13(-1 + 2n) + t)(-1 + 12t) \\ & - (3 - 2n + 13(-1 + 2n) - t)(5 - 2n + 13(-1 + 2n) - t) \\ & \times (7 - 2n + 13(-1 + 2n) - t)(9 - 2n + 13(-1 + 2n) - t) \\ & \times (11 - 2n + 13(-1 + 2n) - t)(13 - 2n + 13(-1 + 2n) - t) \\ & \times (15 - 2n + 13(-1 + 2n) - t)(17 - 2n + 13(-1 + 2n) - t) \\ & \times (19 - 2n + 13(-1 + 2n) - t)(21 - 2n + 13(-1 + 2n) - t) \\ & \times (23 - 2n + 13(-1 + 2n) - t)(10 + t)(3 + 13(-1 + 2n) + t) \\ & \times (5 + 13(-1 + 2n) + t)(7 + 13(-1 + 2n) + t) \\ & \times (9 + 13(-1 + 2n) + t)(11 + 13(-1 + 2n) + t) \\ & \times (13 + 13(-1 + 2n) + t)(15 + 13(-1 + 2n) + t) \\ & \times (17 + 13(-1 + 2n) + t)(19 + 13(-1 + 2n) + t) \\ & \times (21 + 13(-1 + 2n) + t)(23 + 13(-1 + 2n) + t)(1 + 12t). \end{aligned}$$

Using Collect[Expand[fun1[n, 13, t]], t] in mathematica, then

$$\begin{aligned} fun1[n,13,t] = & (8697308774400000n - 32834080085114880000n^2 \\ & - 12729477758976000n^3 + 15041629548369779097600n^4 \\ & + 236520394486783672320n^5 - 2571022036542253053247488n^6 \\ & - 37124592602944477593600n^7 + 212562631268955569447239680n^8 \\ & + 2998666970575261610803200n^9 - 9732888877227707757785776128n^{10} \\ & - 130523412174055408454860800n^{11} \\ & + 264913717377402634610524815360n^{12} \\ & + 3303993186637994329720750080n^{13} \\ & - 4433357842952372362926555660288n^{14} \\ & - 49561443806828984673081753600n^{15} \end{aligned}$$

$$\begin{aligned} &+ 45792333787448669548362690723840n^{16} \\ &+ 432939745369767260902612008960n^{17} \\ &- 283096521710803897284407316185088n^{18} \\ &- 2026875677300174827265458176000n^{19} \\ &+ 956100802723893438111924376043520n^{20} \\ &+ 3938207304014136095515544125440n^{21} \\ &- 1351592746737651507980934743851008n^{22})t \\ &+ (52618718085120000 - 2292707229696000n - 231040021325414400n^2 \\ &- 757137446587596800n^3 + 1421887561344409927680n^4 \\ &+ 284768416242109251584n^5 - 224596918835249896488960n^6 \\ &- 48302575057617467473920n^7 + 18096378993931500902154240n^8 \\ &+ 3852949167679013220515840n^9 - 786586540469501011678986240n^{10} \\ &- 167393371682353404843130880n^{11} \\ &+ 19858676587720144162538913792n^{12} \\ &+ 4210882470112043991979524096n^{13} \\ &- 296862346988928196777554739200n^{14} \\ &- 62684013545292623044725964800n^{15} \\ &+ 2580290003781603421764018241536n^{16} \\ &+ 541342439952896382237778378752n^{17} \\ &- 11997124748246295055424500531200n^{18} \\ &- 2494756260293718178110269030400n^{19} \\ &+ 23062495401167397628578367537152n^{20} \\ &+ 4725848764816963314618652950528n^{21})t^3 \\ &+ (-38506670220902400 - 10050525336698880n \\ &- 2138765004185272320n^2 - 611569422219345920n^3 \\ &+ 8527234807994777600n^4 + 9869679680251494400n^5 \\ &- 11174070109028619386880n^6 - 4512692376544118046720n^7 \\ &+ 769900667838504055603200n^8 + 319123658704559042723840n^9 \\ &- 32287744950774169534988288n^{10} \\ &- 13333044073297848635490304n^{11} \\ &+ 736060344699204453059788800n^{12} \\ &+ 304607840794218643335413760n^{13} \\ &- 9522667329761856447609569280n^{14} \\ &- 3945517301205889752577867776n^{15} \\ &+ 64100521537888070270346854400n^{16} \\ &+ 26625068704831780932582113280n^{17} \\ &- 179947457283998689567199723520n^{18} \\ &- 75006537040210062470063063040n^{19})t^5 \\ &+ (10536776747712512 + 5347819267293184n + 535943463805911040n^2 \\ &+ 286220107452252160n^3 + 16247835905698693120n^4 \\ &+ 9232133788885843968n^5 - 54111509153841807360n^6 \\ &- 39092154191961784320n^7 + 26744908726124984795136n^8 \end{aligned}$$

$$\begin{aligned} &+ 16419029949524151369728n^9 - 835979172732409035489280n^{10} \\ &- 513401250454761605038080n^{11} + 18304224959851258005946368n^{12} \\ &+ 11335073202694099883261952n^{13} - 182937981838560229903564800n^{14} \\ &- 113717712533035416928911360n^{15} \\ &+ 847781921124244234589700096n^{16} \\ &+ 536848912585173536411222016n^{17})t^7 \\ &+ (-1394522431160320 - 1021896235909120n - 69801769093365760n^2 \\ &- 53609283676241920n^3 - 1740166429576396800n^4 \\ &- 1395740806060113920n^5 - 38827743007446564864n^6 \\ &- 32246782492185034752n^7 + 93942735279779020800n^8 \\ &+ 55961272250677657600n^9 - 22677390972932318920704n^{10} \\ &- 18909019812150185918464n^{11} + 266227454186355533414400n^{12} \\ &+ 216749253865644372787200n^{13} \\ &- 2680376585185080020828160n^{14} \\ &- 2281869728231259786706944n^{15})t^9 \\ &+ (102247088463872 + 96981136343040n + 4891039085813760n^2 \\ &+ 4840186028564480n^3 + 120197120433651712n^4 \\ &+ 123570807211900928n^5 + 1822490766823710720n^6 \\ &+ 1943481414840852480n^7 + 34004735503812919296n^8 \\ &+ 36395139178496327680n^9 - 43798142066538618880n^{10} \\ &- 24853465100843048960n^{11} + 5972435338420885880832n^{12} \\ &+ 6378735888713861038080n^{13})t^{11} \\ &+ (-4457775984640 - 5166805882880n - 199407950686208n^2 \\ &- 240184941973504n^3 - 4449603004211200n^4 - 5541275501414400n^5 \\ &- 68510678189893632n^6 - 87442264095166464n^7 \\ &- 570880570252185600n^8 - 753897562334259200n^9 \\ &- 9571210478238599168n^{10} - 12253522993446469632n^{11})t^{13} \\ &+ (119544499712 + 163361386496n + 4818109260800n^2 \\ &+ 6817154488320n^3 + 94379534430720n^4 + 137391429322752n^5 \\ &+ 1114673766835200n^6 + 1658039443399680n^7 \\ &+ 11031689926371840n^8 + 16381944158638080n^9)t^{15} \\ &+ (-1979676160 - 3112095360n - 67831678464n^2 - 110029704576n^3 \\ &- 1042089945600n^4 - 1731500770560n^5 - 8961097499904n^6 \\ &- 15050554066944n^7)t^{17} \\ &+ (19634912 + 34874752n + 512168800n^2 + 935642400n^3 \\ &+ 4884876480n^4 + 9093782016n^5)t^{19} \\ &+ (-106480 - 210760n - 1607848n^2 - 3263040n^3)t^{21} \\ &+ (242 + 528n)t^{23}, \end{aligned}$$

substituting $t^2 = 286n - 142$ into $fun1[n, 13, t]$, we obtain:

$$\begin{aligned} fun1[n, 13, t] &= -495616(2905249632176107594631 \\ &- 46928302502803364552943n + 402018522638492371206159n^2 \end{aligned}$$

$$\begin{aligned} & -2382838039393169994936957n^3 + 10845959670418978868252028n^4 \\ & -40014019759440547238072974n^5 + 123488200999983169105216142n^6 \\ & -324967727016883459291284106n^7 + 737363563917102841359781679n^8 \\ & -1449541181919249572313595611n^9 + 2465290178559546205957329779n^{10} \\ & -3593383442560082134780151225n^{11} + 4385937277986699584769831614n^{12} \\ & -4239239593460848864525121432n^{13} + 2713294427224355024817793312n^{14} \\ & + 53718291601687250864939184n^{15} - 3157044418235657174246896160n^{16} \\ & + 5270026887185699714031830016n^{17} - 5461925236805062778702856192n^{18} \\ & + 3911057430335322878801608704n^{19} - 1815788228906664210299092992n^{20} \\ & + 416639770118683017917497344n^{21})t \\ & = f_{13}(n)t, \end{aligned}$$

substituting $n = m + 1$ into $f_{13}(n)$, we obtain:

$$\begin{aligned} f_{13}(n) &= f_{13}(m + 1) \\ &= -495616(3733296365984620216320000 + 101682117530000897605632000m \\ &+ 1301415358007289983805849600m^2 + 10448258836589112860285603840m^3 \\ &+ 59178304178941628867995995648m^4 \\ &+ 251907889696319728209809732800m^5 \\ &+ 837715230242949492705917778880m^6 \\ &+ 2232156391674391772792080177552m^7 \\ &+ 4847553028528515003695476894304m^8 \\ &+ 8678053128608967382638267149780m^9 \\ &+ 12897664285717775958600104964956m^{10} \\ &+ 15972180745644306093757130820815m^{11} \\ &+ 16488999114134466122376410049462m^{12} \\ &+ 14153264507814254778006257482328m^{13} \\ &+ 10039497314193045472547921605744m^{14} \\ &+ 5825152055755159952182686293680m^{15} \\ &+ 2721282214519967135085563957728m^{16} \\ &+ 999336636418755444467711099904m^{17} \\ &+ 277979296705148285791971508224m^{18} \\ &+ 55089644577125472435494191104m^{19} \\ &+ 6933646943585679165968351232m^{20} \\ &+ 416639770118683017917497344m^{21}) \\ &< 0, \end{aligned}$$

since

$$f_{13}(n) < 0 \text{ and } t = \sqrt{286n - 142} > 0,$$

then

$$fun1[n, 13, t] < 0.$$

Therefore, we have

Theorem 11. *A solution a exists for $k = 13$.*

Actually, we already calculated with computer that $f_k(m+1)$ have all negative coefficients for $k = 14, 15$ also. Therefore, these Theorems are also true for $k = 14, 15$. However, it is too long to present them here. Obviously, one can not do these forever. And, we need a new method. We shall finish the proof for the rest in the next paper. For this paper, we just stop here.

However, we can figure out a conjecture:

Conjecture 1. *The coefficients of the m terms in $f_k(m+1)$ are all negative for any integer $k > 2$.*

We then have:

Lemma 8. *There is a solution a for equation (10) with an integer k if $3 \leq k \leq 13$.*

4. Existence of the Futaki-Ono's Generalized k Maxwell-Einstein Metrics—Positivity

Let

$$L(U) = (aU + 1)^{2s-2kn+k} \left[\int_{-1}^U 2(ax + 1)^{-s-k} [(ax + 1)^k \Delta(x) - cQ(x)] dx + 2(1-a)^{-s} Q(-1) \right], \quad (25)$$

then $(aU + 1)^{s-2kn+k} Q\varphi = \int_{-1}^U L(y) dy$.

Theorem 12. (Cf. [KS1], [Gu2 Lemma 6]) *There is a generalized Maxwell-Einstein metric in the same Kähler class of $\tilde{g}$ if $\psi_0 = \Phi/Q$ is positive on $(-1, 1)$.*

Conjecture 2. (Cf. [Gu1,2,3]) *If r has nonnegative trace eigenvalues, then for a given a , Φ as above is always positive on $(-1, 1)$.*

In this section, we shall prove this conjecture for the cases in which one of D_0 and D_∞ is 1. Therefore, it is true for a completion of a line bundle.

Combining with Lemma 8 in last section, we obtain the existence of the Futaki-Ono k -generalized Maxwell-Einstein metric with an integer $3 \leq k \leq 13$ for the completion of our line bundles.

Proof Since the derivative of $Q\varphi(aU + 1)^{s-2kn+k}$ is $L(U)$, we have that

$$\begin{aligned} & (aU + 1)^{s+k} \frac{d}{dU} ((aU + 1)^{2kn-k-2s} \frac{d}{dU} (Q\varphi(aU + 1)^{s-2kn+k})) \\ &= 2[(aU + 1)^k \Delta(U) - cQ(U)]. \end{aligned} \quad (26)$$

Diagonalizing B , we see that Q is a product of polynomials of degree 1. Let

$$-a_1^{-1} \leq \dots \leq -a_p^{-1} \leq b_1^{-1} \leq \dots \leq b_q^{-1},$$

for convenience, we let

$$a_1 = c_1, a_2 = c_2, \dots, a_p = c_p$$

$$b_1 = -c_{p+1}, b_2 = -c_{p+2}, \dots, b_q = -c_{p+q}$$

then

$$-c_1^{-1} \leq \dots \leq -c_p^{-1} \leq -c_{p+1}^{-1} \leq \dots \leq -c_{p+q}^{-1},$$

denote the distinct roots of Q for which some corresponding Ricci curvature $r_{0,ii}$ is nonzero, where a_i, b_j are positive and c_k can also be negative, $i = 1, 2, \dots, p$; $j = 1, 2, \dots, q$; $k = 1, 2, \dots, p+q$. On account of

$$\Phi(U) = Q\varphi(aU + 1)^{s-2kn+k}$$

Let

$$D = 2kn - k - 2s$$

then

$$(aU + 1)^{s+k} \frac{d}{dU} ((aU + 1)^D \frac{d}{dU} (\Phi(U))) = 2[(aU + 1)^k \Delta(U) - cQ(U)]. \quad (27)$$

Recall that $Q = \det(g_0^{-1} \cdot g_\tau)$, $\Delta(U) = Q(U) \sum_{i,j} r_{0 \ i \bar{j}} g_{\tau(U)}^{i\bar{j}}$. let $r_i = r_{0 \ i \bar{i}}$, then

$$Q(U) = \prod_{i=1}^m (1 + c_i U),$$

$$\Delta(U) = Q(U) \sum_{i,j} r_{0 \ i \bar{j}} g_{\tau(U)}^{i\bar{j}} = Q(U) \sum_{i=1}^m r_i \frac{1}{1 + c_i U} = \sum_{i=1}^m r_i \prod_{j \neq i} (1 + c_j U),$$

then

$$\begin{aligned} & (aU + 1)^{s+k} \frac{d}{dU} ((aU + 1)^D \frac{d}{dU} (\Phi(U))) \\ &= 2[(aU + 1)^k \sum_{i=1}^m r_i \prod_{j \neq i} (1 + c_j U) - c \prod_{i=1}^m (1 + c_i U)]. \end{aligned} \quad (28)$$

Let's consider the case in which only at most three of r_i are nonzero. Different from the earlier paper [CG], we need to make (7) hold. In our case, both $D = d = 1$. That is,

$$Q(U) = (1 + U)^{D_0-1} Q_{-1}$$

$$(\text{or } = (1 - U)^{D_\infty-1} Q_1),$$

one need

$$\Delta(U) = D_0(D_0 - 1)(1 + U)^{D_0-2} Q_{-1} \pmod{(1 + U)^{D_0-1}}$$

$$(\text{or } = D_\infty(D_\infty - 1)(1 - U)^{D_\infty-2} Q_1 \pmod{(1 - U)^{D_\infty-1}}).$$

Therefore, if $c_i = 1$ and only r_i is nonzero, we can only have $r_i = D_0(D_0 - 1)$. And, if $c_i = -1$ and only r_i is nonzero, we can only have $r_i = D_\infty(D_\infty - 1)$. If $c_i \neq 1, -1$ and only r_i is nonzero for those i with $c_i \neq 1, -1$, we can only require that one of $r_j = D_0(D_0 - 1)$, for some j with $c_j = 1$, and one of $r_k = D_\infty(D_\infty - 1)$, for some k with $c_k = -1$.

That is different from [CG], we need require one of $r_i \neq 0$ and equal $D_0(D_0 - 1)$ if $D_0 \neq 1$ for an i with $c_i = 1$; and one of $r_i \neq 0$ and equal $D_\infty(D_\infty - 1)$ if $D_\infty \neq 1$ for an i with $c_i = -1$.

So, we need consider several situations: (1) three $r_i \neq 0$ and two of them with $c_i = 1$ and $c_j = -1$; (2) two of them $\neq 0$ and only one with $c_i = 1$, the other $c_j \neq 1, -1$, $D_\infty = 1$; (3) two of them $\neq 0$ and only one with $c_i = -1$, the other $c_j \neq 1, -1$, $D_0 = 1$; (4) two of them $\neq 0$ and one with $c_i = 1$, the other $c_j = -1$; (5) only one $r_i \neq 0$ with $c_i = 1$, and $D_\infty = 1$; (6) only one $r_i \neq 0$ with $c_i = -1$, and $D_0 = 1$.

A. We can treat (1) and (4) as one situation. Somehow, (1) is difficult although (4) can be treated. Therefore, we might leave this situation out in this paper and shall try to treat it in the future.

B. (2) and (5) as one; (3) and (6) as one. Also, (2) and (3) are similar. Therefore, we only need to deal with (2) in which we replace $r_i > 0$ by $r_i \geq 0$. In these cases, only one of 1 and -1 might be achieved by one of c_i . Therefore, the manifold can be regarded as a completion of a line bundle. On the other hand, all the manifolds which is in a form of a completion of a line bundle are in these cases. Therefore, we only treat these situation in this section.

We want to prove the following lemma first:

Lemma 9.(Cf. [Gu1,2,3]) *If r has nonnegative trace eigenvalues and (1) there is only one nonzero r_i , say for example, $r_{m'}$ other than those with $c_i = 1$ or -1 , and (2) either $D_0 = 1$ or $D_\infty = 1$ with $r_0 = 0$ or $r_{p+q} = 0$ respectively, then for a given a , Φ as above is always positive on $(-1, 1)$. Here we let $m' = m - D_\infty + 1$.*

Proof of the Lemma 9: Let

$$\Delta_1(U) = \frac{D_0(D_0 - 1)}{1 + U} + \frac{r_{m'}}{1 + c_{m'}U} + \frac{D_\infty(D_\infty - 1)}{1 - U}$$

then

$$\begin{aligned} & (aU + 1)^{s+k} \frac{d}{dU} ((aU + 1)^D \frac{d}{dU} (\Phi(U))) \\ &= 2[(aU + 1)^k [D_0(D_0 - 1) \prod_{i=2}^m (1 + c_i U) + r_{m'} \prod_{i \neq m'} (1 + c_i U) \\ &+ D_\infty(D_\infty - 1) \prod_{i=1}^{m-1} (1 + c_i U)] - c \prod_{i=1}^m (1 + c_i U)] \\ &= 2Q(U) [(aU + 1)^k \Delta_1(U) - c] \end{aligned} \quad (29)$$

where $Q(U) > 0$.

We also have $\Delta_1(U) \rightarrow +\infty$ when $U \rightarrow -1$ if $D_0 \neq 1$; and $\Delta_1(U) \rightarrow +\infty$ when $U \rightarrow 1$ if $D_\infty \neq 1$. When $Ric(g_0) \geq 0$, $\Delta(x) \geq 0$ on $[-1, 1]$, and

$$\begin{aligned} & c \int_{-1}^1 (ax + 1)^{-k-s} Q(x) dx \\ &= (a + 1)^{-s} Q(1) + (1 - a)^{-s} Q(-1) + \int_{-1}^1 (ax + 1)^{-s} \Delta(x) dx. \end{aligned}$$

then $c > 0$.

4.1. Special Cases

Next, we shall treat our two special cases with either $D_0 = 1$ or $D_\infty = 1$. In the cases in which $D_\infty = 1$ and $D_0 \neq 1$, for convenience, we replace U with y , we need some lemmas. In this case, the second term in the right side of (29) is zero and we might just replace $D_0(D_0 - 1)$ by 1. Here, we only assume that $a > 0$. Otherwise, we just exchange the ends. Therefore, this is the case in which the manifold is a completion of a line bundle over the end at the infinity. Let

$$\begin{aligned} f_{a,r,b}(y) &= (ay + 1)^k \left(\frac{1}{1 + y} + \frac{r}{1 + ay + by} \right) - c \\ &= f(y) + \frac{(ay + 1)^k}{1 + ay + by} - c, \end{aligned}$$

here $f(y) = (ay + 1)^k \frac{1}{1+y}$.

And $f'(y) = (ay + 1)^{k-1} \left[\frac{ak}{1+y} - \frac{ay+1}{(1+y)^2} \right] = \frac{(ay+1)^{k-1}}{(1-y^2)^2} g(y)$, where

$$g(y) = (1 - y)^2 (a(k - 1)y + ak - 1).$$

$$\begin{aligned} f'_{a,r,b}(y) &= f'(y) + \left(\frac{r(ay + 1)^k}{1 + ay + by} \right)' \\ &= f'(y) + \frac{kar(ay + 1)^{k-1}}{1 + ay + by} - \frac{r(a + b)(ay + 1)^k}{(1 + ay + by)^2} \\ &= \frac{(ay + 1)^{k-1}}{(1 - y^2)^2} g(y) + \frac{kar(ay + 1)^{k-1}}{1 + ay + by} - \frac{r(a + b)(ay + 1)^k}{(1 + ay + by)^2} \\ &= \frac{(ay + 1)^{k-1}}{(1 - y^2)^2 (1 + ay + by)^2} [(1 + ay + by)^2 g(y) \end{aligned}$$

$$\begin{aligned}
& + rka(1-y^2)^2(1+ay+by) - r(a+b)(ay+1)(1-y^2)^2] \\
& = \frac{(ay+1)^{k-1}}{(1-y^2)^2(1+ay+by)^2} [(1+ay+by)^2g(y) \\
& + r(1-y^2)^2(ka(1+ay+by) - (a+b)(ay+1))].
\end{aligned}$$

Let

$$g_{r,b;a}(y) = (1+ay+by)^2g(y) + r(1-y^2)^2(a(k-1)(a+b)y + a(k-1) - b).$$

We notice that the last term is nonnegative if $b < 0$.

We also let

$$\begin{aligned}
g_{r,b;a}(y) &= (1+ay+by)^2g(y) + r(1-y^2)^2(a(k-1)(a+b)y + a(k-1) - b) \\
&= (y-1)^2[(1+(a+b)y)^2(a(k-1)y + ka - 1) \\
&+ r(1+y)^2(a(k-1)(a+b)y + a(k-1) - b)] \\
&= (y-1)^2T_{r,b;a}(y).
\end{aligned}$$

We want to prove that the degree three polynomial $T_{r,b;a}$ only has one zero in $(-1, 1)$.

We notice that $T_{r,b;a}$ has two terms:

$$T_1 = (1+(a+b)y)^2(a(k-1)y + ka - 1),$$

with a derivative:

$$\begin{aligned}
& (1+(a+b)y)[2(a+b)(a(k-1)y + ka - 1) + a(k-1)(1+(a+b)y)] \\
&= (1+(a+b)y)[3(a+b)a(k-1)y + (2k(a+b) + k-1)a - 2(a+b)],
\end{aligned}$$

which has a zero $b_1 = (2b - 2k(a+b)a - (k-3)a)/(3(a+b)a(k-1))$ other than $-1/(a+b) < -1$;

$$rT_2 = r(1+y)^2(a(k-1)(a+b)y + a(k-1) - b)$$

with a derivative

$$\begin{aligned}
& r(1+y)[2(a(k-1)(a+b)y + a(k-1) - b) + a(k-1)(a+b)(1+y)] \\
&= r(1+y)[3a(k-1)(a+b)y + a(k-1)(2+(a+b)) - 2b],
\end{aligned}$$

which has a zero $b_2 = (2b - 2(k-1)a - (k-1)a(a+b))/(3(a+b)a(k-1))$ other than -1 .

Lemma 10. T_2 has a N shape if $b_2 \neq -1$ and $b_2 > -(a+b)^{-1}$. Moreover, -1 is a locally minimal point if $b_2 < -1$ and -1 is a locally maximal point $b_2 > -1$. If $b_2 \leq -1$, T_2 is positive on $(-1, 1)$. Otherwise, $T_2(-1+\delta) < 0$ for $\delta \neq 0$ but small enough. $T_2 = a(k-1)(a+b)(1+y)^3$ has a S shape if $b_2 = -1$, i.e., it is always increasing.

Proof $b_2 > -(a+b)^{-1}$ iff $2b - 2(k-1)a - (k-1)a(a+b) > -3a(k-1)$. That is, $2b + (k-1)a(1-(a+b)) > 0$. This is always true.

Q. E. D.

Similarly, if $b_2 > -1$ then

$$2b - 2(k-1)a - (k-1)a(a+b) > -3a(k-1)(a+b).$$

That is, $b - (k-1)a(1 - (a+b)) > 0$. We have $b(1 + (k-1)a) > (k-1)a(1 - a)$. This implies that $b > ((k-1)a(1 - a))/(1 + (k-1)a)$. We notice that the right side is smaller than $1 - a$. Therefore, this could be achieved whenever $b < 1 - a$ is big enough.

We see that $b_2 < b_1$ when $b > 0$. Actually,

$$\begin{aligned} b_1 - b_2 &= \frac{2(k-1)a + (k-1)a(a+b) - 2k(a+b)a - (k-3)a}{3(a+b)a(k-1)} \\ &= \frac{(k+1)(1 - (a+b))}{3(a+b)(k-1)} \\ &= \frac{(k+1)}{3(a+b)(k-1)} - \frac{(k+1)}{3(k-1)}. \end{aligned}$$

T_1 also has a zero $a_1 = (1 - ka)/(a(k-1))$, and T_2 also has a zero $a_2 = (b - a(k-1))/(a(k-1)(b+a))$. $a_1 - a_2 = (k(1 - b - a))/((k-1)(b+a)) > 0$ when $b > 0$.

Now, we first consider the case in which $b_2 > -1$ and are ready to get some property for $T_{r,b;a}$:

$$\begin{aligned} &(T_1(-1/(a+b)), T_1(-1), T_1(b_2), T_1(b_1), T_1(a_2), T_1(a_1)) \\ &= (0, -(1 - (a+b))^2(1 - a), T_1(b_2) < 0, \\ &T_1(b_1) < 0, T_1(a_2) < 0, 0), \\ &(T'_1(-1/(a+b)), T'_1(-1), T'_1(b_2), T'_1(b_1), T'_1(a_2), T'_1(a_1)) \\ &= (0, (1 - (a+b))[(k-3)a(1 - (a+b)) - 2b] < 0, T'_1(b_2) < 0, \\ &0, T'_1(a_2) > 0, T'_1(a_1) > 0); \end{aligned}$$

$$\begin{aligned} &(T_2(-1/(a+b)), T_2(-1), T_2(b_2), T_2(b_1), T_2(a_2), T_2(a_1)) \\ &= (-b(1 - 1/(a+b))^2, 0, T_2(b_2) < 0, \\ &T_2(b_1) < 0, 0, T_2(a_1) > 0), \\ &(T'_2(-1/(a+b)), T'_2(-1), T'_2(b_2), T'_2(b_1), T'_2(a_2), T'_2(a_1)) \\ &= (-(1 - 1/(a+b))[a(k-1)(1 - (a+b)) + 2b], 0, 0, \\ &T'_2(b_1) > 0, T'_2(a_2) > 0, T'_2(a_1) > 0). \end{aligned}$$

Notice that T_1, T_2 have a N shape, we see that $T_{r,b;a}(-1/(a+b)) < 0, T_{r,b;a}(-1) < 0$, but $T'_{r,b;a}(-1/(a+b)) > 0, T'_{r,b;a}(-1) < 0$. Therefore, there is a local maximal point in $(-1/(a+b), -1)$ which is < 0 .

We also have $T_{r,b;a}(a_2) < 0, T_{r,b;a}(a_1) > 0$. Therefore, there is a zero point in (a_2, a_1) . Also, $T'_{r,b;a}(b_2) < 0, T'_{r,b;a}(b_1) > 0$. Therefore, there is a zero of the derivative in (b_2, b_1) , which must be a local minimal point.

Therefore, we obtain:

Lemma 11. $T_{r,b;a}$ has a N shape and has a negative locally maximal point in $c_1 \in (-(a+b)^{-1}, -1)$ if $b > 0$ and $b_2 > -1$. Moreover, in this case, it has a negative locally minimal point in $c_2 \in (b_2, b_1)$ and has a zero point $c_0 \in (a_2, a_1)$. In particular, $c_0 > c_2 > c_1$ is the only zero of $T_{r,b;a}$ and $T'_{r,b;a}(c_0) > 0$.

Also,

$$\begin{aligned} T_2 &= (1+y)^2(a(k-1)(a+b)y + a(k-1) - b) \\ &= (1+y)^2[(a(k-1)(a+b)y + (a+b)(ka-1)) \\ &- (a+b)(ka-1) + a(k-1) - b] \\ &= (1+y)^2[(a(k-1)(a+b)y + (a+b)(ka-1)) + (1 - (a+b))ka] \end{aligned}$$

$$= \frac{(a+b)(1+y)^2 T_1}{(1+(a+b)y)^2} + (1-(a+b))ka(1+y)^2$$

$$\begin{aligned} T_{r,b;a} &= T_1 + rT_2 \\ &= \frac{1}{(1+(a+b)y)^2} [(1+(a+b)y)^2 + r(a+b)(1+y)^2] T_1 + r(1-(a+b))ka(1+y)^2 \\ &= [(1+(a+b)y)^2 + r(a+b)(1+y)^2] \\ &\quad \left[\frac{1}{(1+(a+b)y)^2} T_1 + r(1-(a+b))ka \frac{(1+y)^2}{(1+(a+b)y)^2 + r(a+b)(1+y)^2} \right]. \end{aligned}$$

The first term in the second bracket is linear and increasing. The second term is also increasing on $(-1, +\infty)$. This gives another proof for Lemma 11.

4.2. The Cases When $D_0 = 1$ and $0 < a < 1$

If $D_0 = 1$, $\Delta_1(U) = \frac{r_{m'}}{1+c_{m'}U} + \frac{D_\infty(D_\infty-1)}{1-U}$, then we let

$$f(U) = (aU+1)^k \left(\frac{r_{m'}}{1+c_{m'}U} + \frac{1}{1-U} \right) - c.$$

4.2.1 If $r_{m'} = 0$, then $f(U) = \frac{(aU+1)^k}{1-U} - c$,

$$\begin{aligned} f'(U) &= \frac{ak(aU+1)^{k-1}}{1-U} + \frac{(aU+1)^k}{(1-U)^2} \\ &= \frac{(aU+1)^{k-1}}{(1-U)^2} (ak(1-U) + aU+1) \\ &= \frac{(aU+1)^{k-1}}{(1-U)^2} (a(1-k)U + ak+1), \end{aligned}$$

let $g(U) = a(1-k)U + ak+1$,

$$g(-1) = a(k-1) + ak+1 = 2ak+1-a > 0,$$

$$g(1) = a(1-k) + ak+1 = a+1 > 0,$$

thus, $f'(U)$ has no zero point on $(-1, 1)$.

Lemma 12. If $D_0 = 1$ and $r_{m'} = 0$, $f'(U)$ has no zero point on $(-1, 1)$.

4.2.2 If $r_{m'} > 0$, $c_{m'} = a$, then $f(U) = r_{m'}(aU+1)^{k-1} + \frac{(aU+1)^k}{1-U} - c$,

$$\begin{aligned} f'(U) &= r_{m'}a(k-1)(aU+1)^{k-2} + \frac{(aU+1)^{k-1}}{(1-U)^2} (a(1-k)U + ak+1) \\ &= \frac{(aU+1)^{k-2}}{(1-U)^2} (r_{m'}a(k-1)(1-U)^2 + (aU+1)(a(1-k)U + ak+1)), \end{aligned}$$

let

$$\begin{aligned} g_{r_{m'}}(U) &= r_{m'}a(k-1)(1-U)^2 + (aU+1)(a(1-k)U + ak+1) \\ &= r_{m'}a(k-1)(1-U)^2 + (aU+1)g(U), \end{aligned}$$

because $r_{m'} > 0, -1 < U < 1, 0 < a < 1, g(U)$ is positive on $(-1, 1)$. Also $(1 - U)^2 > 0, aU + 1 > 0, g_{r_{m'}}(U) > 0$, thus, $f'(U)$ has no zero point on $(-1, 1)$.

In general, if $r = r_{m'} > 0, b = c_{m'}$, then

$$\begin{aligned} f(U) &= (aU + 1)^k \left(\frac{r}{1 + bU} + \frac{1}{1 - U} \right) - c, \\ f'(U) &= \frac{rak(aU + 1)^{k-1}}{1 + bU} - \frac{rb(aU + 1)^k}{(1 + bU)^2} + \frac{(aU + 1)^{k-1}}{(1 - U)^2} (a(1 - k)U + ak + 1) \\ &= \frac{(aU + 1)^{k-1}}{(1 - U)^2(1 + bU)^2} (rak(1 - U)^2(1 + bU) \\ &\quad - rb(aU + 1)(1 - U)^2 + (a(1 - k)U + ak + 1)(1 + bU)^2), \end{aligned}$$

let $b = a + l$ and

$$\begin{aligned} g_{r,l}(U) &= rak(1 - U)^2(1 + (a + l)U) - r(a + l)(aU + 1)(1 - U)^2 \\ &\quad + (a(1 - k)U + ak + 1)(1 + (a + l)U)^2 \\ &= r(1 - U)^2[ak(1 + (a + l)U) - (a + l)(aU + 1)] \\ &\quad + (a(1 - k)U + ak + 1)(1 + (a + l)U)^2 \\ &= r(1 - U)^2[a(a + l)(k - 1)U + a(k - 1) - l] \\ &\quad + (a(1 - k)U + ak + 1)(1 + (a + l)U)^2, \end{aligned}$$

It is > 0 if $l < 0$. Therefore, we only need to deal with $l \in (0, 1 - a)$.

Then $g(1) = (-a + 1)(1 + a + l)^2 > 0$,

$$g(-1) = 4r[a(1 - (a + l))(k - 1) - l] + (a(2k - 1) + 1)(1 - (a + l))^2$$

the zero point a_1 of the first term of $g_{r,l}(U)$ is $\frac{l - a(k - 1)}{a(a + l)(k - 1)}$ and the double zero point $1, \frac{l - a(k - 1)}{a(a + l)(k - 1)} < -1$, if and only if

$$\begin{aligned} l - a(k - 1) &< -a(a + l)(k - 1) \\ a(k - 1) - a(k - 1)(a + l) &> l \\ a(k - 1) - a(k - 1)a - a(k - 1)l &> l \\ a(k - 1) - a(k - 1)a &> (1 + a(k - 1))l \\ (1 + a(k - 1))l &< a(k - 1)(1 - a) \\ l &< \frac{a(k - 1)(1 - a)}{1 + a(k - 1)} \end{aligned}$$

That is, when $l \leq \frac{a(k - 1)(1 - a)}{1 + a(k - 1)}, a_1 \leq -1$. The first term is also positive. We only need to deal with

$$l > \frac{a(k - 1)(1 - a)}{1 + a(k - 1)}.$$

$$\begin{aligned} &[(a(1 - k)U + ak + 1)(1 + (a + l)U)^2]' \\ &= a(1 - k)(1 + (a + l)U)^2 + 2(a(1 - k)U + ak + 1)(a + l)(1 + (a + l)U) \end{aligned}$$

$$\begin{aligned}
&= (1 + (a + l)U)[a(1 - k)(1 + (a + l)U) + 2(a + l)(a(1 - k)U + ak + 1)] \\
&= (1 + (a + l)U)[3a(1 - k)(a + l)U + a(1 - k) + 2(a + l)(ak + 1)],
\end{aligned}$$

it has zero points $-\frac{1}{a+l}$ and $b_1 = \frac{a(1-k)+2(a+l)(ak+1)}{3a(k-1)(a+l)}$, when $l \in (0, 1-a)$, $-\frac{1}{a+l} < -1$, $\frac{a(1-k)+2(a+l)(ak+1)}{3a(k-1)(a+l)} < 1$, if and only if

$$a(1 - k) + 2(a + l)(ak + 1) < 3a(k - 1)(a + l)$$

$$a(1 - k) < (a + l)[3a(k - 1) - 2(ak + 1)]$$

$$a(1 - k) < (a + l)[a(k - 3) - 2]$$

$$a(1 - k) < a[a(k - 3) - 2] + l[a(k - 3) - 2]$$

$$a[(1 - k) - a(k - 3) + 2] < l[a(k - 3) - 2]$$

$$a[(3 - k) - a(k - 3)] < l[a(k - 3) - 2]$$

$$l[a(k - 3) - 2] > a(a + 1)(3 - k)$$

If $a(k - 3) - 2 > 0$, then $l > \frac{a(a+1)(k-3)}{2-a(k-3)}$. This is always true since the right side is negative.

If $a(k - 3) - 2 < 0$, then $l < \frac{a(a+1)(k-3)}{2-a(k-3)}$. In this case, we have: $\frac{a(-a+1)(k-1)}{1+a(k-1)} < l < \frac{a(a+1)(k-3)}{2-a(k-3)}$. We have: $\frac{k+1}{2k^2-5k+1} < a < \frac{2}{k-3}$.

$$\begin{aligned}
a_1 - b_1 &= \frac{l - a(k - 1)}{a(a + l)(k - 1)} - \frac{a(1 - k) + 2(a + l)(ak + 1)}{3a(k - 1)(a + l)} \\
&= \frac{3[l - a(k - 1)] - [a(1 - k) + 2(a + l)(ak + 1)]}{3a(k - 1)(a + l)} \\
&= \frac{3l - 2a(k - 1) - 2(a + l)(ak + 1)}{3a(k - 1)(a + l)} \\
&= \frac{(1 - 2ak)l - 2ak(1 + a)}{3a(k - 1)(a + l)} \\
&= \frac{l - 2ak(1 + a + l)}{3a(k - 1)(a + l)},
\end{aligned}$$

Therefore, if $a(k - 3) - 2 > 0$, and $a_1 > b_1$ then $l - 2ak(1 + a + l) > 0$, $0 > 2ka(a + 1) + (2ak - 1)l$. This implies $1 > 2ka$. That is, $a < \frac{1}{2k}$. But $a > \frac{2}{k-3}$. So, $\frac{1}{2k} > \frac{2}{k-3}$. A contradiction. This implies that $a_1 \leq b_1$. The function only has one zero.

Therefore, we only need to deal with the case in which $a(k - 3) - 2 < 0$. In this case, we still get $a < \frac{1}{2k}$. Therefore, $\frac{k+1}{2k^2-5k+1} < a < \frac{1}{2k}$. That is, $2k^2 + 2k < 2k^2 - 5k + 1$. Again, a contradiction. We get $a_1 \leq b_1$. It only has one zero. The proof is complete.

Lemma 13. If $D_0 = 1$ and $r_{m'} > 0$, $f'(U)$ has only one zero point on $(-1, 1)$.

4.3. The Cases in Which $D_\infty = 1$ and $0 < a < 1$

If $D_\infty = 1$, $\Delta_1(U) = \frac{D_0(D_0-1)}{1+U} + \frac{r_{m'}}{1+c_{m'}U}$, then we let

$$f(U) = (aU + 1)^k \left(\frac{1}{1 + U} + \frac{r_{m'}}{1 + c_{m'}U} \right) - c.$$

4.3.1 If $r_{m'} = 0$, then $f(U) = \frac{(aU+1)^k}{1+U} - c$,

$$f'(U) = \frac{ak(aU + 1)^{k-1}}{1 + U} - \frac{(aU + 1)^k}{(1 + U)^2}$$

$$\begin{aligned}
&= \frac{(aU+1)^{k-1}}{(1+U)^2} (ak(1+U) - aU - 1) \\
&= \frac{(aU+1)^{k-1}}{(1+U)^2} (a(k-1)U + ak - 1),
\end{aligned}$$

$$\text{let } g(U) = a(k-1)U + ak - 1,$$

$$g(-1) = a(1-k) + ak - 1 = a - 1 < 0,$$

$$g(1) = a(k-1) + ak - 1 = 2ak - a - 1 = a(2k-1) - 1,$$

if $a(2k-1) < 1$, then $g(1) < 0$, $f'(U)$ has no zero point on $(-1, 1)$; if $a(2k-1) > 1$, then $g(1) > 0$, $f'(U)$ has one zero point on $(-1, 1)$; thus, $f'(U)$ has at most one zero point on $(-1, 1)$.

Lemma 14. If $D_\infty = 1$ and $r_{m'} = 0$, $f'(U)$ has only one zero point on $(-1, 1)$.

4.3.2 If $r_{m'} > 0$, $c_{m'} = a$, then $f(U) = \frac{(aU+1)^k}{1+U} + r_{m'}(aU+1)^{k-1} - c$,

$$\begin{aligned}
f'(U) &= \frac{(aU+1)^{k-1}}{(1+U)^2} (a(k-1)U + ak - 1) + r_{m'}a(k-1)(aU+1)^{k-2} \\
&= \frac{(aU+1)^{k-2}}{(1+U)^2} ((aU+1)(a(k-1)U + ak - 1) + r_{m'}a(k-1)(1+U)^2),
\end{aligned}$$

let

$$\begin{aligned}
g_{r_{m'}}(U) &= (aU+1)(a(k-1)U + ak - 1) + r_{m'}a(k-1)(1+U)^2 \\
&= (aU+1)g(U) + r_{m'}a(k-1)(1+U)^2,
\end{aligned}$$

then

$$g_{r_{m'}}(-1) = (1-a)(a(1-k) + ak - 1) = (1-a)(a-1) = -(1-a)^2 < 0,$$

$$g_{r_{m'}}(1) = (a+1)(a(k-1) + ak - 1) + 4r_{m'}a(k-1) = (a+1)(a(2k-1) - 1) + 4r_{m'}a(k-1).$$

Let's take the derivative of $g_{r_{m'}}(U)$ over U , and we'll get

$$\begin{aligned}
g'_{r_{m'}}(U) &= a(a(k-1)U + ak - 1) + a(k-1)(aU+1) + 2r_{m'}a(k-1)(1+U) \\
&= 2a(k-1)(a+r_{m'})U + a[ak-1+k-1+2r_{m'}(k-1)] \\
&= 2a(k-1)(a+r_{m'})U + a[k(a+1)+2r_{m'}(k-1)-2],
\end{aligned}$$

because of $r_{m'} > 0$, $-1 < U < 1$, $0 < a < 1$, $k \geq 3$, $g'_{r_{m'}}(U)$ is monotonically increasing on $(-1, 1)$.

$$\begin{aligned}
g'_{r_{m'}}(-1) &= 2a(k-1)(a+r_{m'})(-1) + a[k(a+1)+2r_{m'}(k-1)-2] \\
&= a[2(1-k)a-2(k-1)r_{m'}+k(a+1)+2r_{m'}(k-1)-2] \\
&= a[2(1-k)a+k(a+1)-2] \\
&= a(2a-ak+k-2) \\
&= a[k(1-a)-2(1-a)] \\
&= a(k-2)(1-a) \\
&> 0,
\end{aligned}$$

so $g_{r_{m'}}(U)$ is monotonically increasing on $(-1, 1)$ and because $g_{r_{m'}}(-1) < 0$, it implies that $f'(U)$ has at most one zero point on $(-1, 1)$.

In general, if $r = r_{m'} > 0$, $b = c_{m'}$, then

$$f(U) = (aU + 1)^k \left(\frac{1}{1+U} + \frac{r}{1+bU} \right) - c,$$

$$\begin{aligned} f'(U) &= \frac{(aU + 1)^{k-1}}{(1+U)^2} (a(k-1)U + ak - 1) + \frac{rak(aU + 1)^{k-1}}{1+bU} - \frac{rb(aU + 1)^k}{(1+bU)^2} \\ &= \frac{(aU + 1)^{k-1}}{(1+U)^2(1+bU)^2} [(a(k-1)U + ak - 1)(1+bU)^2 \\ &\quad + rak(1+U)^2(1+bU) - rb(aU + 1)(1+U)^2], \end{aligned}$$

let $b = a + l$ and

$$\begin{aligned} g_{r,l}(U) &= (a(k-1)U + ak - 1)(1 + (a+l)U)^2 \\ &\quad + rak(1+U)^2(1 + (a+l)U) - r(a+l)(aU + 1)(1+U)^2 \\ &= (a(k-1)U + ak - 1)(1 + (a+l)U)^2 \\ &\quad + r(1+U)^2[ak(1 + (a+l)U) - (a+l)(aU + 1)] \\ &= (a(k-1)U + ak - 1)(1 + (a+l)U)^2 \\ &\quad + r(1+U)^2[a(a+l)(k-1)U + a(k-1) - l] \\ &= (1 + (a+l)U)^2[(a(k-1)U + ak - 1) \\ &\quad + r\left(\frac{1+U}{1+(a+l)U}\right)^2[a(a+l)(k-1)U + a(k-1) - l]]. \end{aligned}$$

If $l < 0$, the second term is positive. By $\frac{1+U}{1+(a+l)U}$ is increasing, it can only have at most one zero. Therefore, we only need to deal with $l \in (0, 1-a)$.

Then $g(-1) = (a-1)[1 - (a+l)]^2 < 0$,

$$g(1) = (a(2k-1) - 1)(1 + a + l)^2 + 4r[a((a+l) + 1)(k-1) - l],$$

The zero point a_1 of the first term of $g_{r,l}(U)$ is $\frac{1-ak}{a(k-1)}$ and the double zero point $-\frac{1}{a+l}$.

We notice that this is exactly $T_{r,l;a}$. Applying Lemma 11, we get that it has at most one zero in $(-1, 1)$.

Lemma 15. If $D_\infty = 1$ and $r_{m'} > 0$, $f'(U)$ has only one zero point on $(-1, 1)$.

4.4. Final Proof of the Existence Theorem for the Completion of a Line Bundle

Combining the results in these last two subsections 4.2, 4.3, we shall obtain our Lemma 9.

And we shall obtain following our first positive Theorem after our final efforts to the proof of our Lemma 9:

Theorem 13. If our manifold only has one end contracted, that is, the manifold is a completion of a line bundle, the solution we got in our earlier Theorems has a positive φ on $(-1, 1)$.

Proof of the Lemma 9 (Continue):

Recall that

$$\Phi(U) = Q\varphi(aU + 1)^{s-2kn+k}.$$

And

$$D = 2kn - k - 2s,$$

then from formula (29)

$$(aU + 1)^{s+k} \frac{d}{dU} ((aU + 1)^D \frac{d}{dU} (\Phi(U))) = 2Q(U)[(aU + 1)^k \Delta_1(U) - c].$$

4.4.1 Let's first consider the case in which $D_0 = 1$, $D_\infty \neq 1$, and r_m and only another r_i are nonzero with $c_i \neq 1$.

$$\Delta_1(U) = \frac{r_{m'}}{1+c_{m'}U} + \frac{D_\infty(D_\infty-1)}{1-U}, \text{ then}$$

$$f(U) = (aU + 1)^k \left(\frac{r}{1+c_{m'}U} + \frac{1}{1-U} \right) - c.$$

Lemma 12 and 13 say that $f'(U)$ has at most one zero on $[-1, 1]$.

If $\Phi(U)$ is not all greater than zero on $(-1, 1)$, then $\Phi(U)$ has at least one zero. If it has least two zeros, we set them to be U_1 and U_2 , let $U_1 < U_2$, then $\Phi(-1) = \Phi(U_1) = \Phi(U_2) = \Phi(1) = 0$. There are at least three critical points between them. Let $\frac{d}{dU} \Phi(U_3) = \frac{d}{dU} \Phi(U_4) = \frac{d}{dU} \Phi(U_5) = 0$.

$$(aU + 1)^D \frac{d}{dU} (\Phi(U))(-1) \geq 0$$

and > 0 near -1 ;

$$(aU + 1)^D \frac{d}{dU} (\Phi(U))(U_1) \leq 0$$

and < 0 nearby;

$$(aU + 1)^D \frac{d}{dU} (\Phi(U))(U_2) \geq 0$$

and > 0 nearby;

$$(aU + 1)^D \frac{d}{dU} (\Phi(U))(1) \leq 0$$

and < 0 nearby. Then $\frac{d}{dU} ((aU + 1)^D \frac{d}{dU} (\Phi(U))) < 0$ at some point y_1 in $(-1, U_1)$, $\frac{d}{dU} ((aU + 1)^D \frac{d}{dU} (\Phi(U))) > 0$ at some point y_2 in (U_1, U_2) and $\frac{d}{dU} ((aU + 1)^D \frac{d}{dU} (\Phi(U))) < 0$ at some point y_3 in $(U_2, 1)$. That is, $f(U) < 0$ at some point y_1 in $(-1, U_1)$, $f(U) > 0$ at some point y_2 in (U_1, U_2) and $f(U) < 0$ at some point y_3 in $(U_2, 1)$. But $f(U) \rightarrow +\infty$ at 1. That is, there are at least three zeros $z_1 \in (y_1, y_2)$, $z_2 \in (y_2, y_3)$ and $z_3 \in (y_3, 1)$ in $(-1, 1)$. And $f'(U)$ has at least two zeros in $(-1, 1)$, which is a contradiction to our Lemmas 12 and 13.

Similarly, if it has only one zero, we set it to be U_1 , then $\Phi(-1) = \Phi(U_1) = \Phi(1) = 0$. Then, U_1 is a critical point and there are at least two more critical points between them. Let $\frac{d}{dU} \Phi(U_2) = \frac{d}{dU} \Phi(U_1) = \frac{d}{dU} \Phi(U_3) = 0$ with $-1 < U_2 < U_1 < U_3 < 1$.

$$(aU + 1)^D \frac{d}{dU} (\Phi(U))(-1) \geq 0$$

and > 0 near -1 ;

$$(aU + 1)^D \frac{d}{dU} (\Phi(U))(U_1 - d) < 0$$

for $d > 0$ and small; also

$$(aU + 1)^D \frac{d}{dU} (\Phi(U))(U_1 + d) > 0;$$

$$(aU + 1)^D \frac{d}{dU} (\Phi(U))(1) \leq 0$$

and < 0 nearby. Then $\frac{d}{dU} ((aU + 1)^D \frac{d}{dU} (\Phi(U))) < 0$ at some point y_1 in $(-1, U_1 - d)$, $\frac{d}{dU} ((aU + 1)^D \frac{d}{dU} (\Phi(U))) > 0$ at some point y_2 in $(U_1 - d, U_1 + d)$ and $\frac{d}{dU} ((aU + 1)^D \frac{d}{dU} (\Phi(U))) < 0$ at some point y_3 in $(U_1 + d, 1)$. But $f(U) \rightarrow +\infty$ at 1. That is, again, there are at least three zeros $z_1 \in (y_1, y_2)$, $z_2 \in (y_2, y_3)$ and $z_3 \in (y_3, 1)$ in $(-1, 1)$, which is again a contradiction.

Therefore, in the case in our Lemma 9 with $D_0 = 1$, $\Phi(U)$ as above is always positive on $(-1, 1)$.

4.4.2 Let's second consider the case in which $D_\infty = 1$, $D_0 \neq 1$, and r_1 and only another r_i are nonzero with $c_i \neq -1$.

$\Delta_1(U) = \frac{r_{m'}}{1+c_{m'}U} + \frac{D_0(D_0-1)}{1+U}$, then

$$f(U) = (aU + 1)^k \left(\frac{r}{1+c_{m'}U} + \frac{1}{1+U} \right) - c.$$

Lemma 14 and 15 say that $f'(U)$ has at most one zero on $[-1, 1]$.

If $\Phi(U)$ is not all greater than zero on $(-1, 1)$, then $\Phi(U)$ has at least one zero. If it has least two zeros, we set them to be U_1 and U_2 , let $U_1 < U_2$, then $\Phi(-1) = \Phi(U_1) = \Phi(U_2) = \Phi(1) = 0$. There are at least three critical points between them. Let $\frac{d}{dU}\Phi(U_3) = \frac{d}{dU}\Phi(U_4) = \frac{d}{dU}\Phi(U_5) = 0$.

$$(aU + 1)^D \frac{d}{dU}(\Phi(U))(-1) \geq 0$$

and > 0 near -1 ;

$$(aU + 1)^D \frac{d}{dU}(\Phi(U))(U_1) \leq 0$$

and < 0 nearby;

$$(aU + 1)^D \frac{d}{dU}(\Phi(U))(U_2) \geq 0$$

and > 0 nearby;

$$(aU + 1)^D \frac{d}{dU}(\Phi(U))(1) \leq 0$$

and < 0 nearby. Then $\frac{d}{dU}((aU + 1)^D \frac{d}{dU}(\Phi(U))) < 0$ at some point y_1 in $(-1, U_1)$, $\frac{d}{dU}((aU + 1)^D \frac{d}{dU}(\Phi(U))) > 0$ at some point y_2 in (U_1, U_2) and $\frac{d}{dU}((aU + 1)^D \frac{d}{dU}(\Phi(U))) < 0$ at some point y_3 in $(U_2, 1)$. That is, $f(U) < 0$ at some point y_1 in $(-1, U_1)$, $f(U) > 0$ at some point y_2 in (U_1, U_2) and $f(U) < 0$ at some point y_3 in $(U_2, 1)$. But $f(U) \rightarrow +\infty$ at -1 . That is, again, there are at least three zeros $z_1 \in (-1, y_1)$, $z_2 \in (y_1, y_2)$ and $z_3 \in (y_2, y_3)$ in $(-1, 1)$, which is again a contradiction to our Lemmas 14 and 15.

Similarly, if it has only one zero, we set it to be U_1 , then $\Phi(-1) = \Phi(U_1) = \Phi(1) = 0$. Then, U_1 is a critical point and there are at least two more critical points between them. Let $\frac{d}{dU}\Phi(U_2) = \frac{d}{dU}\Phi(U_1) = \frac{d}{dU}\Phi(U_3) = 0$ with $-1 < U_2 < U_1 < U_3 < 1$.

$$(aU + 1)^D \frac{d}{dU}(\Phi(U))(-1) \geq 0$$

and > 0 near -1 ;

$$(aU + 1)^D \frac{d}{dU}(\Phi(U))(U_1 - d) < 0$$

for $d > 0$ and small; also

$$(aU + 1)^D \frac{d}{dU}(\Phi(U))(U_1 + d) > 0;$$

$$(aU + 1)^D \frac{d}{dU}(\Phi(U))(1) \leq 0$$

and < 0 nearby. Then $\frac{d}{dU}((aU + 1)^D \frac{d}{dU}(\Phi(U))) < 0$ at some point y_1 in $(-1, U_1 - d)$, $\frac{d}{dU}((aU + 1)^D \frac{d}{dU}(\Phi(U))) > 0$ at some point y_2 in $(U_1 - d, U_1 + d)$ and $\frac{d}{dU}((aU + 1)^D \frac{d}{dU}(\Phi(U))) < 0$ at some point y_3 in $(U_1 + d, 1)$. But $f(U) \rightarrow +\infty$ at -1 . That is, again, there are at least three zeros $z_1 \in (-1, y_1)$, $z_2 \in (y_1, y_2)$ and $z_3 \in (y_2, y_3)$ in $(-1, 1)$, a contradiction.

We proved our Lemma 9 as desired.

Q. E. D.

4.4.3 Proof of the Theorem 13 (Continue): Now, let's consider the case of multiple nonzero r_i . First, we need to deal with the case in which all $r_i = 0$ except when $c_i = 1$ (or -1) when $D_\infty \neq 1$ (resp. $D_0 \neq 1$). In this case, we have Lemmas 12 and 14, The arguments in the proof of the Lemma 9 still go through.

Now, we consider the case in which there is at least one nonzero r_i other than the ones with $c_i = 1$ (resp. -1). Similar to the Lemma 13, let

$$\Delta_{1,i}(U) = \frac{R_i}{1 + c_i U} + \frac{D_\infty(D_\infty - 1)}{1 - U},$$

or in the $D_0 \neq 1$ case. We want to replace $\Delta_1(U)$ by $\Delta_{1,i}(U)$ and get a solution with the same numbers c and a .

First, we want to determine the number R_i . We have that

$$\begin{aligned} c \int_{-1}^1 (ax + 1)^{-k-s} Q(x) dx \\ = (1 - a)^{-s} Q(-1) + \int_{-1}^1 (ax + 1)^{-s} Q(x) \Delta_{1,i}(x) dx. \end{aligned}$$

then

$$\begin{aligned} R_i \int_{-1}^1 (ax + 1)^{-s} \prod_{j \neq i} (1 + c_j x) dx \\ = \int_{-1}^1 (ax + 1)^{-s} Q(x) [\Delta_{1,i}(x) - \frac{D_\infty(D_\infty - 1)}{1 - U}] dx \\ = \int_{-1}^1 (ax + 1)^{-s} Q(x) [\Delta_1(x) - \frac{D_\infty(D_\infty - 1)}{1 - U}] dx. \end{aligned}$$

$$R_i = \frac{\int_{-1}^1 (ax + 1)^{-s} Q(x) [\Delta_1(x) - \frac{D_\infty(D_\infty - 1)}{1 - U}] dx}{\int_{-1}^1 (ax + 1)^{-s} \prod_{j \neq i} (1 + c_j x) dx}$$

and

$$R_i > 0, i = 1, 2, \dots, n.$$

Then the corresponding solutions $\Phi_i(U)$ have the property as in our Lemma 9.

In general, we want to find a solution such that $\Phi = \sum_{i=1}^n l_i \Phi_i(U)$, where l_i are positive numbers for $i = 1, 2, \dots, m'$. Because

$$\begin{aligned} (aU + 1)^{s+k} \frac{d}{dU} ((aU + 1)^D \frac{d}{dU} (\Phi(U))) \\ = 2Q(U) [(aU + 1)^k \Delta_1(U) - c] \end{aligned} \quad (30)$$

and

$$\begin{aligned} (aU + 1)^{s+k} \frac{d}{dU} ((aU + 1)^D \frac{d}{dU} (\Phi_i(U))) \\ = 2Q(U) [(aU + 1)^k \Delta_{1,i}(U) - c] \end{aligned} \quad (31)$$

then

$$(aU + 1)^{s+k} \frac{d}{dU} ((aU + 1)^D \frac{d}{dU} (\sum_{i=1}^{m'} l_i \Phi_i(U)))$$

$$\begin{aligned}
&= 2 \sum_{i=1}^{m'} l_i Q(U) [(aU+1)^k \Delta_{1,i}(U) - c] \\
&= 2Q(U) [(aU+1)^k \sum_{i=1}^{m'} l_i \Delta_{1,i}(U) - \sum_{i=1}^{m'} l_i c].
\end{aligned}$$

then

$$l_i R_i = r_i.$$

That is,

$$l_i = \frac{r_i}{R_i} \geq 0.$$

$$\begin{aligned}
l_i &= \frac{r_i \int_{-1}^1 (ax+1)^{-s} \prod_{j \neq i} (1+c_j x) dx}{\int_{-1}^1 (ax+1)^{-s} Q(x) [\Delta_1(x) - \frac{D_\infty(D_\infty-1)}{1-U}] dx}, \\
\sum_{i=1}^n l_i &= \frac{\sum_{i=1}^n r_i \int_{-1}^1 (ax+1)^{-s} \prod_{j \neq i} (1+c_j x) dx}{\int_{-1}^1 (ax+1)^{-s} \Delta(x) dx} \\
&= \frac{\int_{-1}^1 (ax+1)^{-s} \sum_{i=1}^n r_i \prod_{j \neq i} (1+c_j x) dx}{\int_{-1}^1 (ax+1)^{-s} \Delta(x) dx} \\
&= 1.
\end{aligned}$$

therefore $\Phi(U)$ as above is always positive on $(-1, 1)$.

This, in particular, concludes our Theorem 13.

Q. E. D.

Now, combining Lemma 8 and Theorem 13, we get our THEOREM.

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