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Finite Termination and Transcendence Obstructions for Exponential Orbit Sums

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Abstract

We study finite orbit-sum termination for the two-sided exponential iteration generated by $f(x) = 2^x - 1$ and its inverse $f^{-1}(x) = \log_2(1+x)$ on the unit interval. For $w \in (0,1]$, put $u_k(w) = f^k(w)$ for $k \in \mathbb{Z}$. A binary digit sequence $a = (a_k)_{k\in\mathbb{Z}}$ with $a_0 = 1$ is normalized by $\sum\limits_{k\in\mathbb{Z}} a_k u_k(w) = 1$. Thus the expansion scale is generated by the point being expanded, rather than by an external base or partition, and finite termination is governed by finite orbit-sum equations. We prove existence and uniqueness of normalization roots for admissible digit sequences, construct the associated greedy code, and characterize finite termination by finite orbit-sum hitting equations compatible with the greedy order. The finite terminal set is countable and has Lebesgue measure zero. On the arithmetic side, the first positive and first negative boundary roots are transcendental, and the first positive second-order boundary root is irrational. Assuming Schanuel's conjecture, we exclude all non-trivial rational finite terminal points and prove transcendence for all purely positive finite roots, for all two-term boundary roots paired by the mirror identity, and for all roots with one first negative layer and arbitrary finite positive support.

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1. Introduction

Iteration theory and functional equations naturally lead to orbit identities in which finite algebraic closure is difficult to detect directly. This paper studies such identities for a two-sided exponential iteration on the unit interval. The main finite equations are orbit-sum equations: a finite set of integer iterates is selected, and the corresponding orbit values are required to sum to one. The iteration-theoretic background for equations generated by iterates is represented by the monograph of Kuczma, Choczewski, and Ger [1]. The comparison with classical expansion theory is supplied by beta-expansions, originating with Rényi and Parry; see [2–4]. The transcendence input used below is classical: the Gelfond–Schneider theorem, Baker's theory, and Schanuel-type conditional arguments; see [5–10].

The canonical map in this paper is

$$f(x) = 2^x - 1.$$

It fixes the endpoints,

$$f(0) = 0, \qquad f(1) = 1,$$

and moves every interior point to the left:

$$0 < f(x) < x, \qquad 0 < x < 1.$$

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Its inverse is logarithmic,

$$f^{-1}(x) = \log_2(1+x),$$

so the full integer orbit contains both exponential and logarithmic layers. For $w \in (0,1]$, define

$$u_k(w) = f^k(w), \qquad k \in \mathbb{Z},$$

where negative iterates are understood through f^{-1}. In particular,

$$u_0(w) = w, \qquad u_1(w) = 2^w - 1, \qquad u_{-1}(w) = \log_2(1+w).$$

An expansion is a binary choice of orbit layers. The central layer is always present, and the remaining layers are selected by a sequence

$$a = (a_k)_{k\in\mathbb{Z}}, \qquad a_k \in \{0,1\}, \qquad a_0 = 1.$$

The normalization equation is

$$G_a(w) = 1, \qquad G_a(w) = \sum_{k\in\mathbb{Z}} a_k u_k(w).$$

The point w therefore determines the scale on which it is expanded. The central arithmetic question is whether a non-trivial finite endogenous greedy expansion can terminate at an algebraic point. The conjectural answer is negative:

$$W_{\text{fin}} \cap \overline{\mathbb{Q}} = \{1\}.$$

The paper develops the analytic and arithmetic structure needed for this question. Section 2 gives the orbit estimates, admissible digit sequences, and normalization roots. Section 3 constructs the greedy code, proves exhaustion of the remainder, and describes finite termination by prefix cylinders. Section 4 reformulates finite termination as an arithmetic hitting problem, proves uniqueness and simplicity for finite-support roots, records shift covariance under integer support translations, and derives the countability and zero-measure property of the finite terminal set. Section 5 proves the unconditional first-order transcendence theorems, specializes shift covariance to the two-term mirror and ordering relations, and isolates the first second-order obstruction. Section 6 proves Schanuel-conditional rational non-termination and finite-root transcendence theorems for the positive and one-first-negative families. Section 7 summarizes the resulting theorem-level picture and identifies the remaining mixed-support cases.

2. The Canonical Orbit and Normalization Roots

Let

$$f(x) = 2^x - 1, \qquad g(x) = \log_2(1+x).$$

Then g is the inverse of f on $[0,1]$. For $k \in \mathbb{Z}$, define

$$u_k(w) = \begin{cases} f^k(w), & k \ge 0, \\ g^{-k}(w), & k < 0. \end{cases}$$

The functions u_k satisfy the recurrence laws

$$u_{k+1}(w) = 2^{u_k(w)} - 1, \tag{2.1}$$

$$u_{k-1}(w) = \log_2\left(1 + u_k(w)\right). \tag{2.2}$$

They also satisfy the inverse relation

$$u_{k+j}(w) = u_k(u_j(w))$$

whenever both sides are read as iterates of the same map f.

Lemma 2.1 (Monotonicity and endpoint behavior)**.** *For each fixed $k \in \mathbb{Z}$, the function u_k is strictly increasing on $(0,1)$. Moreover,*

$$0 < u_k(w) < 1 \tag{2.3}$$

for $0 < w < 1$, the positive tail tends to 0 as k increases without bound, and the negative tail tends to 1 as k decreases without bound.

Proof. For $0 < x < 1$, define

$$H(x) = x - f(x) = x + 1 - 2^x.$$

Then

$$H''(x) = -(\log 2)^2 2^x < 0,$$

so H is strictly concave on $[0,1]$. Since

$$H(0) = 0, \qquad H(1) = 0,$$

strict concavity gives

$$H(x) > 0, \qquad 0 < x < 1.$$

Hence

$$0 < 2^x - 1 < x, \qquad 0 < x < 1.$$

Let $x_n = f^n(w)$ with $0 < w < 1$. The preceding inequality implies

$$0 < x_{n+1} < x_n < 1.$$

Therefore (x_n) is decreasing and bounded below by 0, and hence has a limit $L \in [0,1)$. By continuity of f,

$$L = f(L) = 2^L - 1.$$

Equivalently, $H(L) = 0$. Since $H(x) > 0$ for $0 < x < 1$ and $L < 1$, it follows that $L = 0$. Thus the positive orbit tends to 0.

For the inverse map $g = f^{-1}$, the same inequality gives $f(x) < x$ for $0 < x < 1$. Since g is strictly increasing, applying g to both sides yields

$$x < g(x) = \log_2(1 + x).$$

Moreover, for $0 < x < 1$,

$$\log_2(1 + x) < 1.$$

Thus

$$x < \log_2(1 + x) < 1, \qquad 0 < x < 1.$$

Let $y_n = g^n(w)$ with $0 < w < 1$. Then

$$0 < y_n < y_{n+1} < 1.$$

Hence (y_n) is increasing and bounded above by 1, so it has a limit $M \in (0,1]$. By continuity of g, $M = g(M)$, or equivalently $f(M) = M$. The only fixed points of f in $[0,1]$ are 0 and 1, because

$H(x) > 0$ for $0 < x < 1$, while $H(0) = H(1) = 0$. Since $M \geq w > 0$, one has $M = 1$. Thus the negative orbit tends to 1.

The functions f and g are real analytic and strictly increasing on $(0, 1)$. Therefore every finite iterate u_k is real analytic and strictly increasing on $(0, 1)$. $\square$

We also record the derivative recurrences used below. Differentiating (2.1) and (2.2) gives

$$u'_{k+1}(w) = (\log 2)2^{u_k(w)}u'_k(w), \qquad k \in \mathbb{Z}, \tag{2.4}$$

$$u'_{k-1}(w) = \frac{u'_k(w)}{(1 + u_k(w)) \log 2}, \qquad k \in \mathbb{Z}. \tag{2.5}$$

Since $u'_0(w) = 1$ and all multiplicative factors in (2.4) and (2.5) are positive for $0 < w < 1$, one obtains

$$u'_k(w) > 0, \qquad k \in \mathbb{Z}, \quad 0 < w < 1.$$

We now pass from individual orbit functions to digit sequences and normalization roots. A binary sequence is written

$$a = (a_k)_{k\in\mathbb{Z}}, \qquad a_k \in \{0, 1\}.$$

The central digit is always fixed:

$$a_0 = 1.$$

Definition 2.2 (Admissible sequence)**.** *A sequence a is admissible when the following three conditions hold:*

(1) $a_0 = 1$;
(2) *the negative support is finite;*
(3) *the positive tail produces a locally uniformly convergent series*

$$\sum_{k\geq 1} a_k u_k(w) \tag{2.6}$$

on compact subintervals of $(0, 1)$.

For such an admissible sequence define

$$G_a(w) = \sum_{k\in\mathbb{Z}} a_k u_k(w). \tag{2.7}$$

For the canonical binary system, condition (3) is satisfied by every binary positive tail. The proof is included for completeness.

Lemma 2.3 (Local C^1 summability of the positive tail)**.** *For every compact subinterval $I \subset (0, 1)$, the two series*

$$\sum_{k\geq 1} u_k(w) \tag{2.8}$$

and

$$\sum_{k\geq 1} u'_k(w) \tag{2.9}$$

converge uniformly on I. Consequently, for every binary positive tail $(a_k)_{k\geq 1}$, the series

$$\sum_{k\geq 1} a_k u_k(w) \tag{2.10}$$

defines a C^1 function on $(0, 1)$ and may be differentiated term by term on compact subintervals of $(0, 1)$.

Proof. Since

$$f(0) = 0, \qquad f'(0) = \log 2 < 1,$$

there exist $\eta > 0$ and $q \in (0,1)$ such that

$$0 \le f(x) \le qx, \qquad 0 \le f'(x) \le q, \qquad 0 \le x \le \eta.$$

Fix a compact subinterval $I \subset (0,1)$, and let $s = \max I$. Since $u_n(s) \to 0$, there exists N such that $u_N(s) \le \eta$. Because u_N is increasing, for every $w \in I$ one has

$$0 < u_N(w) \le u_N(s) \le \eta.$$

Therefore, by induction,

$$u_{N+j}(w) \le q^j u_N(w) \le q^j \eta, \qquad j \ge 0, \quad w \in I.$$

This proves uniform convergence of (2.8) on I.

For the derivatives, write

$$u'_{N+j}(w) = u'_N(w) \prod_{m=0}^{j-1} f'(u_{N+m}(w)).$$

Since u'_N is continuous on the compact interval I, there exists $M_I > 0$ such that

$$u'_N(w) \le M_I, \qquad w \in I.$$

Using $f'(x) \le q$ for $0 \le x \le \eta$, we obtain

$$u'_{N+j}(w) \le M_I q^j, \qquad j \ge 0, \quad w \in I.$$

The Weierstrass test gives uniform convergence of (2.9) on I.

Since $0 \le a_k \le 1$, the same estimates apply to every subseries (2.10) and to its derivative series. Hence the positive tail defines a C^1 function on $(0,1)$ and may be differentiated term by term on compact subintervals. □

Proposition 2.4 (Strict monotonicity of G_a)**.** *For each admissible sequence a, the function G_a is C^1 and strictly increasing on $(0,1)$. Moreover,*

$$G'_a(w) \ge 1, \qquad 0 < w < 1. \tag{2.11}$$

Proof. The negative support of a is finite, so the negative part of G_a is a finite sum of analytic functions. The central term is $u_0(w) = w$. By Lemma 2.3, the positive tail is C^1 on $(0,1)$ and may be differentiated term by term.

Therefore

$$G'_a(w) = \sum_{k \in \mathbb{Z}} a_k u'_k(w).$$

Every u_k is strictly increasing, hence $u'_k(w) > 0$ for all $k \in \mathbb{Z}$ and $0 < w < 1$. Since $a_0 = 1$ and $u'_0(w) = 1$, we obtain (2.11). Consequently, if $0 < w_1 < w_2 < 1$, then

$$G_a(w_2) - G_a(w_1) = \int_{w_1}^{w_2} G'_a(t)\, dt \ge w_2 - w_1 > 0.$$

Thus G_a is strictly increasing on $(0,1)$. □

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Theorem 2.5 (Existence and uniqueness of the normalization root)**.** *For each admissible sequence a, the equation*

$$G_a(w) = 1 \tag{2.12}$$

has exactly one solution in $(0, 1]$. If a is the trivial sequence, namely $a_0 = 1$ and $a_k = 0$ for all $k \neq 0$, then the unique solution is $w = 1$. If a is non-trivial, namely $a_k = 1$ for at least one $k \neq 0$, then the unique solution lies in $(0, 1)$.

Proof. We first record the endpoint behavior. As $w \to 0^+$, each fixed orbit term $u_k(w)$ tends to 0. Since the negative support of a is finite, the negative part of G_a tends to 0.

It remains to control the positive tail. Since $f(0) = 0$ and $f'(0) = \log 2 < 1$, there exist $\eta > 0$ and $q \in (0, 1)$ such that

$$0 \le f(x) \le qx, \qquad 0 \le x \le \eta.$$

For $0 < w \le \eta$, induction gives

$$0 \le u_k(w) \le q^k w, \qquad k \ge 0.$$

Therefore

$$0 \le \sum_{k\ge1} a_k u_k(w) \le \sum_{k\ge1} u_k(w) \le \sum_{k\ge1} q^k w = \frac{qw}{1-q}.$$

Hence the positive tail tends to 0 as $w \to 0^+$. Together with the central term $u_0(w) = w$ and the finite negative part, this proves

$$\lim_{w\to0^+} G_a(w) = 0.$$

Consequently there exists $\varepsilon > 0$ such that $G_a(w) < 1$ for $0 < w < \varepsilon$.

Next consider the right endpoint. For every fixed $k \in \mathbb{Z}$,

$$\lim_{w\to1^-} u_k(w) = 1,$$

because 1 is a fixed point of both f and $g = f^{-1}$.

If a is the trivial sequence, then $G_a(w) = w$, so (2.12) has the unique solution $w = 1$.

Now assume that a is non-trivial. Then there exists $k_1 \neq 0$ such that $a_{k_1} = 1$. Since $a_0 = 1$, we have

$$G_a(w) \ge u_0(w) + u_{k_1}(w).$$

Taking $w \to 1^-$ gives

$$\lim_{w\to1^-} \left(u_0(w) + u_{k_1}(w)\right) = 2.$$

Therefore there exists $\delta > 0$ such that $G_a(w) > 1$ for $1 - \delta < w < 1$. Since G_a is continuous on $(0, 1)$, the intermediate value theorem gives at least one solution of (2.12) in $(0, 1)$.

It remains to prove uniqueness. By Proposition 2.4, G_a is strictly increasing on $(0, 1)$. Hence the equation has at most one solution in $(0, 1)$. For a non-trivial sequence, the preceding right-endpoint estimate also shows that $w = 1$ cannot be a solution: if the support is finite, the extended endpoint value is the number of non-zero digits and is at least 2; if the positive support is infinite, then $G_a(w)$ diverges to $+\infty$ as $w \to 1^-$. Thus the unique solution lies in $(0, 1)$. □

3. Greedy Coding, Exhaustion, and Finite Termination

The normalization equation always contains the central term $u_0(w) = w$. Thus, for a fixed $w \in [0, 1]$, the non-central part of the expansion must represent the remainder

$$R_0(w) = 1 - w.$$

The endpoint $w = 1$ gives the trivial code. In this case, $R_0(1) = 0$, and the greedy code is the sequence $\chi(1)$ satisfying

$$a_0 = 1, \qquad a_k = 0 \quad (k \neq 0).$$

For the rest of this section, assume $0 < w < 1$. For fixed $w \in (0,1)$, the orbit weights satisfy

$$\cdots > u_{-3}(w) > u_{-2}(w) > u_{-1}(w) > u_0(w),$$
$$u_0(w) > u_1(w) > u_2(w) > u_3(w) > \cdots .$$

and

$$\lim_{k \to -\infty} u_k(w) = 1, \qquad \lim_{k \to +\infty} u_k(w) = 0.$$

Since the central digit $a_0 = 1$ has already been fixed, the greedy search is performed only over the non-central index set

$$\mathcal{K} = \mathbb{Z} \setminus \{0\}.$$

The set $\mathcal{K}$ is ordered by the usual integer order with 0 removed:

$$\cdots < -3 < -2 < -1 < 1 < 2 < 3 < \cdots . \tag{3.1}$$

This order is not globally well ordered. However, for every remainder $R \in (0,1)$, the admissible candidate set

$$\mathcal{K}(R;w) = \{k \in \mathcal{K} : u_k(w) \le R\}$$

has a least element. Indeed, since $u_k(w) \to 1$ as $k \to -\infty$ and $R < 1$, all sufficiently negative indices are excluded. Since $u_k(w) \to 0$ as $k \to +\infty$, all sufficiently large positive indices are included. Therefore $\mathcal{K}(R;w)$ is a nonempty subset of $\mathbb{Z}$ bounded below.

Let $\operatorname{pred}(k)$ denote the immediate predecessor of k in the order on $\mathcal{K}$. Thus

$$\operatorname{pred}(k) = k-1 \ (k \le -1), \quad \operatorname{pred}(1) = -1, \quad \operatorname{pred}(k) = k-1 \ (k \ge 2).$$

Definition 3.1 (Greedy code)**.** *Let $w \in (0,1)$. Set $R_0(w) = 1 - w$. Initialize*

$$a_0(w) = 1, \qquad a_k(w) = 0 \quad (k \neq 0). \tag{3.2}$$

The first selected index is defined by

$$k_1(w) = \min\{k \in \mathcal{K} : u_k(w) \le R_0(w)\}. \tag{3.3}$$

Set $a_{k_1(w)}(w) = 1$. If $u_{k_1(w)}(w) = R_0(w)$, the algorithm terminates. Otherwise define

$$R_1(w) = R_0(w) - u_{k_1(w)}(w) \tag{3.4}$$

and continue.

Suppose that $k_1(w), \dots, k_{j-1}(w)$ have already been selected and that $R_{j-1}(w) > 0$. The next selected index is

$$k_j(w) = \min\{k \in \mathcal{K} : k > k_{j-1}(w),\ u_k(w) \le R_{j-1}(w)\}. \tag{3.5}$$

Set $a_{k_j(w)}(w) = 1$. If $u_{k_j(w)}(w) = R_{j-1}(w)$, the algorithm terminates. Otherwise set

$$R_j(w) = R_{j-1}(w) - u_{k_j(w)}(w) \tag{3.6}$$

and continue.

The resulting sequence

$$\chi(w) = \big(a_k(w)\big)_{k \in \mathbb{Z}} \tag{3.7}$$

is called the greedy code of w.

Lemma 3.2 (Well-definedness of the greedy step)**.** *At every stage with $R_{j-1}(w) > 0$, the index $k_j(w)$ exists and is unique.*

Proof. For $j = 1$, the candidate set is $\{k \in \mathcal{K} : u_k(w) \leq R_0(w)\}$. Since $0 < R_0(w) < 1$ and $\lim_{k \to -\infty} u_k(w) = 1$, all sufficiently negative indices are excluded. Since $\lim_{k \to +\infty} u_k(w) = 0$, all sufficiently large positive indices are included. Therefore the candidate set is a nonempty subset of $\mathbb{Z}$ bounded below, hence has a unique least element.

The same argument applies at step j. The set

$$\{k \in \mathcal{K} : k > k_{j-1}(w),\ u_k(w) \leq R_{j-1}(w)\}$$

is nonempty because $u_k(w) \to 0$ as $k \to +\infty$. It is bounded below by the condition $k > k_{j-1}(w)$. Hence it has a unique least element. Therefore the greedy step is well defined and deterministic. □

Lemma 3.3 (Finite negative support)**.** *For every $w \in (0,1)$, the greedy code $\chi(w)$ has finite negative support.*

Proof. Since $R_0(w) = 1 - w < 1$ and $\lim_{k \to -\infty} u_k(w) = 1$, there exists an integer $K_-(w) < 0$ such that

$$u_k(w) > R_0(w)$$

for all $k \leq K_-(w)$. The remainders satisfy $0 \leq R_j(w) \leq R_0(w)$ for every stage at which they are defined. Hence no index $k \leq K_-(w)$ can ever be selected, because $u_k(w) > R_0(w) \geq R_j(w)$. Therefore every selected negative index belongs to the finite set

$$\{K_-(w) + 1, \ldots, -1\}.$$

Thus the negative support of $\chi(w)$ is finite. □

Lemma 3.4 (Forward tail domination)**.** *For every $x \in (0,1)$,*

$$x < \sum_{n \geq 1} f^n(x), \qquad f(x) = 2^x - 1. \tag{3.8}$$

Consequently, for every $k \in \mathcal{K}$,

$$u_k(w) < \sum_{\substack{m \in \mathcal{K} \\ m > k}} u_m(w). \tag{3.9}$$

Proof. For $0 < x < 1$, convexity of the exponential function gives

$$2^x \geq 1 + (\log 2)x.$$

Therefore

$$f(x) = 2^x - 1 \geq (\log 2)x.$$

Iterating gives

$$f^n(x) \geq (\log 2)^n x, \qquad n \geq 1.$$

Hence

$$\sum_{n \geq 1} f^n(x) \geq \sum_{n \geq 1} (\log 2)^n x = \frac{\log 2}{1 - \log 2} x.$$

Since $\log 2/(1 - \log 2) > 1$, one obtains (3.8).

Now fix $k \in \mathcal{K}$ and put $x = u_k(w)$. The future orbit terms are $u_{k+n}(w) = f^n(x)$. If the future index set does not cross 0, the preceding inequality directly gives (3.9).

If the future index set crosses 0, the term $u_0(w)$ is omitted from the greedy index set $\mathcal{K}$. In that case, for some $q \geq 1$, the omitted term is $f^q(x)$. The remaining forward tail is bounded below by

$$\sum_{\substack{n\geq 1\\ n\neq q}} (\log 2)^n x = \left(\frac{\log 2}{1-\log 2} - (\log 2)^q\right) x.$$

The smallest value of the coefficient occurs at $q = 1$, and it equals

$$\frac{\log 2}{1-\log 2} - \log 2 = \frac{(\log 2)^2}{1-\log 2} > 1.$$

The same strict tail domination holds after the central index 0 is removed. □

Lemma 3.5 (Exhaustion of the remainder). *For every $w \in (0,1)$, either the greedy algorithm terminates after finitely many steps, or*

$$\lim_{j\to\infty} R_j(w) = 0. \tag{3.10}$$

Consequently, in the infinite case,

$$1 = w + \sum_{j\geq 1} u_{k_j(w)}(w), \tag{3.11}$$

and in the finite case,

$$1 = w + \sum_{j=1}^{n} u_{k_j(w)}(w) \tag{3.12}$$

for the terminal step n.

Proof. If the algorithm terminates, the conclusion follows immediately from the definition of termination. Assume that the algorithm does not terminate. Then

$$R_j(w) = R_{j-1}(w) - u_{k_j(w)}(w), \qquad 0 < R_j(w) < R_{j-1}(w),$$

so $(R_j(w))$ is decreasing and has a nonnegative limit.

For each selected index put

$$T_j(w) = \sum_{\substack{m\in\mathcal{K}\\ m>k_j(w)}} u_m(w).$$

This is the full admissible tail after the selected index. We prove the invariant

$$R_j(w) < T_j(w) \tag{3.13}$$

for every non-terminal step j.

For $j = 1$, the predecessor $\mathrm{pred}(k_1(w))$ exists in the order on $\mathcal{K}$ and is rejected by minimality of $k_1(w)$. Therefore

$$R_0(w) < u_{\mathrm{pred}(k_1(w))}(w).$$

The terms strictly after $\mathrm{pred}(k_1(w))$ are exactly $u_{k_1(w)}(w)$ together with the tail $T_1(w)$. Lemma 3.4 gives

$$u_{\mathrm{pred}(k_1(w))}(w) < u_{k_1(w)}(w) + T_1(w).$$

Consequently

$$R_1(w) = R_0(w) - u_{k_1(w)}(w) < T_1(w).$$

Assume that (3.13) has been proved through step $j-1$. If $k_j(w)$ is the immediate successor of $k_{j-1}(w)$ in $\mathcal{K}$, then

$$T_{j-1}(w) = u_{k_j(w)}(w) + T_j(w),$$

and the induction hypothesis gives

$$R_j(w) = R_{j-1}(w) - u_{k_j(w)}(w) < T_j(w).$$

If $k_j(w)$ is not the immediate successor of $k_{j-1}(w)$, then $\mathrm{pred}(k_j(w))$ is still available at step j and was rejected. Hence

$$R_{j-1}(w) < u_{\mathrm{pred}(k_j(w))}(w).$$

Again Lemma 3.4 gives

$$u_{\mathrm{pred}(k_j(w))}(w) < u_{k_j(w)}(w) + T_j(w),$$

so

$$R_j(w) = R_{j-1}(w) - u_{k_j(w)}(w) < T_j(w).$$

The invariant is proved.

The selected indices are strictly increasing in the order on $\mathcal{K}$. By Lemma 3.3, only finitely many selected indices can be negative. Therefore

$$k_j(w) \to +\infty.$$

The positive orbit tail is convergent, and hence $T_j(w) \to 0$. From (3.13) we obtain $R_j(w) \to 0$.

Finally, for every n,

$$R_n(w) = 1 - w - \sum_{j=1}^{n} u_{k_j(w)}(w).$$

Letting $n \to \infty$ gives (3.11). The finite case is the same identity with the finite terminal sum. $\square$

Definition 3.6 (Finite and infinite greedy expansion)**.** *A point $w \in (0,1]$ is called finite terminal if its greedy algorithm terminates after finitely many selections. Equivalently, either $w = 1$, or there exists $n \geq 1$ such that $R_n(w) = 0$. The set of finite terminal points is denoted by*

$$W_{\mathrm{fin}} = \{w \in (0,1] : \chi(w) \text{ terminates after finitely many selections}\}. \tag{3.14}$$

Its complement is denoted by

$$W_{\mathrm{inf}} = (0,1] \setminus W_{\mathrm{fin}}. \tag{3.15}$$

If $w \in W_{\mathrm{fin}} \setminus \{1\}$ and the selected indices are $k_1(w), \dots, k_n(w)$, then

$$w + \sum_{j=1}^{n} u_{k_j(w)}(w) = 1. \tag{3.16}$$

Conversely, a finite orbit-sum equation gives a finite terminal point only when its support is compatible with the greedy order.

Definition 3.7 (Prefix cylinder)**.** *Let $\mathbf{k} = (k_1, \dots, k_n)$ be a finite list of indices in $\mathcal{K}$ satisfying*

$$k_1 < k_2 < \cdots < k_n \tag{3.17}$$

in the order (3.1). The prefix cylinder $C(\mathbf{k})$ is the set of all $w \in (0,1)$ for which the first n greedy selections are exactly $k_1, \dots, k_n$.

For a fixed prefix $\mathbf{k} = (k_1, \ldots, k_n)$, define the associated remainders by

$$\begin{aligned} &R_0(w) = 1 - w, \\ &R_j(w) = R_{j-1}(w) - u_{k_j}(w), \qquad 1 \le j \le n. \end{aligned} \tag{3.18}$$

The condition $w \in C(\mathbf{k})$ is equivalent to the finite system of greedy comparisons described below. At the first step, k_1 is selected precisely when k_1 is admissible and its immediate predecessor is inadmissible:

$$u_{\mathrm{pred}(k_1)}(w) > R_0(w), \qquad u_{k_1}(w) \le R_0(w). \tag{3.19}$$

For each $2 \le j \le n$, the selected index k_j must be admissible at stage j:

$$u_{k_j}(w) \le R_{j-1}(w). \tag{3.20}$$

If k_j is not the immediate successor of k_{j-1} in the order on $\mathcal{K}$, then the immediate predecessor of k_j is still available at stage j and must be inadmissible:

$$u_{\mathrm{pred}(k_j)}(w) > R_{j-1}(w), \qquad \mathrm{pred}(k_j) > k_{j-1}. \tag{3.21}$$

The positivity conditions

$$R_j(w) > 0, \qquad 1 \le j \le n-1, \tag{3.22}$$

ensure that the greedy procedure reaches the n-th selection.

The immediate predecessor test suffices because the orbit weights are strictly decreasing in the order on $\mathcal{K}$. Hence, once

$$u_{\mathrm{pred}(k_j)}(w) > R_{j-1}(w)$$

holds, every earlier available candidate is also larger than the current remainder and is therefore rejected. Thus each prefix cylinder is described by finitely many analytic inequalities in the single variable w.

Boundary relations arise when one of the strict rejection comparisons degenerates into equality. In that case, for some available candidate index m, one obtains an equation of the form

$$R_{j-1}(w) - u_m(w) = 0. \tag{3.23}$$

If the prefix terminates at step n, then

$$R_n(w) = 0. \tag{3.24}$$

By (3.18), this is equivalent to

$$1 - w - \sum_{j=1}^{n} u_{k_j}(w) = 0, \tag{3.25}$$

or, equivalently,

$$w + \sum_{j=1}^{n} u_{k_j}(w) = 1. \tag{3.26}$$

Consequently, every non-trivial finite terminal point satisfies a finite orbit-sum equation. Conversely, a root of a finite orbit-sum equation gives a finite terminal point only when its support is compatible with the greedy order, namely when the corresponding prefix satisfies (3.19), (3.20), (3.21), and (3.22).

4. Finite-Support Roots and the Arithmetic Hitting Problem

By (3.16), every non-trivial finite terminal point satisfies a finite orbit-sum equation compatible with the greedy order:

$$w + \sum_{k \in K} u_k(w) = 1. \tag{4.1}$$

For a finite set $K \subset \mathcal{K}$, define the finite support sum and its residual by

$$S_K(w) = u_0(w) + \sum_{k\in K} u_k(w), \qquad E_K(w) = S_K(w) - 1. \tag{4.2}$$

Thus (4.1) is equivalent to

$$E_K(w) = 0. \tag{4.3}$$

Proposition 4.1 (Uniqueness and simplicity of finite-support roots)**.** *Let $K \subset \mathcal{K}$ be finite. Then S_K is real analytic and strictly increasing on $(0,1)$, with*

$$S_K'(w) = 1 + \sum_{k\in K} u_k'(w) > 0. \tag{4.4}$$

Consequently E_K has at most one zero in $(0,1)$, and every such zero is simple. If K is non-empty, then E_K has exactly one zero in $(0,1)$.

Proof. Analyticity follows from the analyticity of the finitely many orbit functions. Since $u_0'(w) = 1$ and $u_k'(w) > 0$ for every $k \in \mathcal{K}$, one obtains (4.4). Hence S_K is strictly increasing, and E_K has at most one zero in $(0,1)$.

Assume now that K is non-empty. Every fixed orbit term tends to 0 as $w \to 0^+$, while every fixed orbit term tends to 1 as $w \to 1^-$. Therefore

$$\lim_{w\to 0^+} S_K(w) = 0, \qquad \lim_{w\to 1^-} S_K(w) = 1 + |K| > 1. \tag{4.5}$$

The intermediate value theorem gives one zero of E_K in $(0,1)$, and uniqueness has already been proved. The zero is simple because $E_K' = S_K' > 0$. $\square$

For every non-empty finite $K \subset \mathcal{K}$, denote the unique zero of E_K by ρ_K. The symbol K denotes the noncentral support in (4.1). For a full finite support $S \subset \mathbb{Z}$ containing 0, let r_S denote the unique root of

$$\sum_{k\in S} u_k(w) = 1.$$

Thus $r_{\{0\}\cup K} = \rho_K$ whenever $K \cap \{0\} = \varnothing$.

The roots are order-reversing with respect to support inclusion: if $K, V \subset \mathcal{K}$ are finite, non-empty, and $K \subsetneq V$, then

$$\rho_V < \rho_K. \tag{4.6}$$

Indeed, at $w = \rho_K$ one has $S_K(\rho_K) = 1$, and hence

$$S_V(\rho_K) = S_K(\rho_K) + \sum_{k\in V\setminus K} u_k(\rho_K) > 1. \tag{4.7}$$

Since S_V is strictly increasing and crosses the level 1 exactly once, its root must lie to the left of ρ_K.

More generally, finite-sum roots are covariant under integer translations of the support. For a finite set $S \subset \mathbb{Z}$ with $|S| \geq 2$, the function

$$H_S(w) = \sum_{k\in S} u_k(w) - 1 \tag{4.8}$$

is strictly increasing on $(0,1)$ and satisfies

$$\lim_{w\to 0^+} \sum_{k\in S} u_k(w) = 0, \qquad \lim_{w\to 1^-} \sum_{k\in S} u_k(w) = |S| > 1.$$

Hence it has a unique zero in $(0,1)$, denoted by r_S.

Proposition 4.2 (Shift covariance of finite-sum roots). *Let $S \subset \mathbb{Z}$ be finite with $|S| \geq 2$, and let $h \in \mathbb{Z}$. Write*

$$S + h = \{k + h : k \in S\}.$$

Then

$$r_{S+h} = u_{-h}(r_S). \tag{4.9}$$

Equivalently, for every $s \in \mathbb{Z}$,

$$r_{S-s} = u_s(r_S). \tag{4.10}$$

Proof. Let $x = r_S$ and set

$$y = u_{-h}(x).$$

For every $k \in S$, the composition law of the orbit gives

$$u_{k+h}(y) = u_{k+h}\big(u_{-h}(x)\big) = u_k(x).$$

Therefore

$$\sum_{m \in S+h} u_m(y) = \sum_{k \in S} u_{k+h}(y) = \sum_{k \in S} u_k(x) = 1.$$

Thus y is a root of the finite-sum equation with support $S + h$. By uniqueness of the root for this support,

$$y = r_{S+h}.$$

This proves (4.9). Taking $h = -s$ gives (4.10). □

The two-term mirror identity is a special case of this shift covariance. If $S = \{0, N\}$, then (4.10) with $s = N$ gives

$$r_{\{-N,0\}} = u_N\Big(r_{\{0,N\}}\Big).$$

Since

$$r_{\{0,N\}} + u_N\Big(r_{\{0,N\}}\Big) = 1,$$

one obtains

$$r_{\{-N,0\}} = 1 - r_{\{0,N\}}. \tag{4.11}$$

For a shifted support $J \subset \{0, \ldots, d\}$ containing both 0 and d, the endpoint shift gives

$$r_{J-d} = u_d(r_J). \tag{4.12}$$

In the two-term case $J = \{0, d\}$, this identity becomes $r_{\{-d,0\}} = 1 - r_{\{0,d\}}$.

Proposition 4.3 (Finite terminal points have zero measure). *The set W_{fin} is countable. In particular,*

$$\lambda(W_{\text{fin}}) = 0, \qquad \lambda(W_{\text{inf}}) = 1, \tag{4.13}$$

where λ denotes Lebesgue measure on $(0, 1]$.

Proof. There are only countably many finite subsets of the countable set $\mathcal{K}$. By Proposition 4.1, each finite support gives at most one root in $(0, 1)$. Every non-trivial finite terminal point is among these roots, and the trivial terminal point $w = 1$ contributes one additional point. Hence W_{fin} is countable. A countable subset of the real line has Lebesgue measure zero, so $W_{\text{inf}} = (0, 1] \setminus W_{\text{fin}}$ has full measure in $(0, 1]$. □

The preceding reduction turns finite termination into an arithmetic hitting problem for the roots ρ_K.

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Conjecture 4.4 (Algebraic non-termination)**.** *Every algebraic number $w \in (0,1)$ belongs to W_{inf}. In set form,*

$$W_{\mathrm{fin}} \cap \overline{\mathbb{Q}} = \{1\}. \tag{4.14}$$

Equivalently, every non-trivial finite-support root ρ_K is transcendental.

The Schanuel arguments separate two input regimes. For algebraic irrational input, the forward orbit produces algebraically independent variables. For rational input, the first positive image is algebraic irrational, and the remaining positive and negative layers become algebraically independent under Schanuel. These two independence mechanisms give the global rational exclusion and the finite-root transcendence results below.

5. Low-Depth Arithmetic: First-Order Transcendence and the Second-Order Obstruction

We first settle the two depth-one boundary roots. The same orbit structure then gives an exact mirror relation for the entire two-term boundary ladder.

Theorem 5.1 (The first positive boundary root is transcendental)**.** *The unique root of*

$$w + u_1(w) = 1 \tag{5.1}$$

is transcendental. Equivalently, the unique root of

$$w + 2^w = 2 \tag{5.2}$$

in $(0,1)$ is transcendental.

Proof. Suppose $w \in \overline{\mathbb{Q}} \cap (0,1)$ satisfies (5.2). If w is algebraic irrational, then 2^w is transcendental by the Gelfond–Schneider theorem. But the equation gives

$$2^w = 2 - w,$$

which is algebraic. This is impossible.

It remains to exclude rational w. Write

$$w = \frac{p}{q}, \qquad 0 < p < q, \qquad (p,q) = 1.$$

Then

$$2^{p/q} = \frac{2q-p}{q}.$$

Raising to the q-th power gives

$$2^p q^q = (2q-p)^q.$$

Since $(q, 2q-p) = (q,p) = 1$, the right side is coprime to q, while the left side is divisible by q^q. Hence $q = 1$, contradicting $0 < p < q$. This excludes rational w. Thus no algebraic $w \in (0,1)$ satisfies (5.2). ☐

Theorem 5.2 (The first negative boundary root is transcendental)**.** *The unique root of*

$$w + u_{-1}(w) = 1 \tag{5.3}$$

is transcendental. Equivalently, the unique root of

$$w + \log_2(1+w) = 1 \tag{5.4}$$

in $(0,1)$ *is transcendental.*

Proof. The equation is equivalent to

$$1 + w = 2^{1-w}. \tag{5.5}$$

Suppose that $w \in \overline{\mathbb{Q}} \cap (0,1)$. If w is algebraic irrational, then $1 - w$ is also algebraic irrational. By the Gelfond–Schneider theorem, 2^{1-w} is transcendental. This contradicts (5.5), whose left side is algebraic.

It remains to exclude rational w. Write

$$w = \frac{p}{q}, \qquad 0 < p < q, \qquad (p,q) = 1.$$

Then (5.5) gives

$$2^{(q-p)/q} = \frac{p+q}{q}.$$

Raising both sides to the q-th power yields the integer identity

$$2^{q-p} q^q = (p+q)^q. \tag{5.6}$$

Since

$$(q, p+q) = (q,p) = 1,$$

no odd prime can divide q. Indeed, if an odd prime ℓ divided q, then ℓ would divide the left side of (5.6). Hence ℓ would also divide the right side, forcing

$$\ell \mid p+q,$$

contrary to $(q, p+q) = 1$. Thus q has no odd prime divisor, so q is a power of 2.

Since $0 < p < q$, this power of 2 is non-trivial, and hence q is even. From $(p,q) = 1$, it follows that p is odd. Therefore $p + q$ is odd, so the right side of (5.6) is odd. On the other hand, the left side is even because $q - p > 0$. This contradiction excludes rational w.

Therefore no algebraic $w \in (0,1)$ can satisfy (5.4). The unique root is transcendental. ☐

Corollary 5.3 (Two-term mirror and ordering). *For each integer* $N \geq 1$, *let* w_N^+ *be the unique root of*

$$w + u_N(w) = 1, \tag{5.7}$$

and let w_N^- *be the unique root of*

$$w + u_{-N}(w) = 1. \tag{5.8}$$

Then

$$w_N^- = 1 - w_N^+. \tag{5.9}$$

Moreover,

$$w_1^+ < w_2^+ < w_3^+ < \cdots < 1 \tag{5.10}$$

and

$$1 > w_1^- > w_2^- > w_3^- > \cdots > 0. \tag{5.11}$$

In particular, the transcendence of the first negative boundary root also follows from Theorem 5.1 and (5.9) with $N = 1$.

Proof. The mirror identity follows from Proposition 4.2 applied to the support $\{0, N\}$. Indeed,

$$r_{\{-N,0\}} = u_N(r_{\{0,N\}}),$$

and the equation defining $r_{\{0,N\}}$ gives

$$u_N(r_{\{0,N\}}) = 1 - r_{\{0,N\}}.$$

Thus

$$w_N^- = 1 - w_N^+.$$

It remains to prove the ordering. Put

$$A_N(w) = w + u_N(w).$$

For every $w \in (0,1)$ one has

$$u_{N+1}(w) = f\big(u_N(w)\big) < u_N(w).$$

Hence

$$A_{N+1}(w) < A_N(w)$$

for all $w \in (0,1)$. Evaluating at $w = w_N^+$ gives

$$A_{N+1}(w_N^+) < A_N(w_N^+) = 1.$$

Since A_{N+1} is strictly increasing, it cannot attain the level 1 at any point $w \le w_N^+$. Its unique root therefore satisfies

$$w_{N+1}^+ > w_N^+.$$

This proves (5.10). The ordering of the negative roots follows from (5.9). □

The identity (5.9) is the two-term specialization of the general shift covariance (4.9). For multi-term supports, the corresponding shifted root is the orbit iterate described in (4.12).

The next positive boundary root, corresponding to $N = 2$, already leaves the direct first-order setting. It satisfies

$$w + u_2(w) = 1. \tag{5.12}$$

Since

$$u_2(w) = 2^{2^w - 1} - 1,$$

the equation is

$$2^{2^w} = 2(2 - w).$$

Equivalently,

$$2^w \log 2 = \log\big(2(2 - w)\big).$$

This is the first obstruction beyond the direct Gelfond–Schneider regime.

Proposition 5.4 (The second positive boundary root is irrational)**.** *The unique root of* (5.12) *is irrational.*

Proof. Let $w = p/q \in (0,1)$, with $(p,q) = 1$. Then 2^w is algebraic irrational. Hence $2^w - 1$ is algebraic irrational. By the Gelfond–Schneider theorem,

$$2^{2^w - 1}$$

is transcendental. But the equation $w + u_2(w) = 1$ gives

$$2^{2^w - 1} = 2 - w,$$

which is rational. This is impossible. □

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Problem 5.5 (Minimal second-order separation)**.** *Prove that no algebraic* $\alpha \in (0,1)$ *satisfies*

$$2^{2^{\alpha}} = 2(2-\alpha). \tag{5.13}$$

Equivalently, prove

$$2^{\alpha}\log 2 \neq \log\left(2(2-\alpha)\right) \tag{5.14}$$

for algebraic $\alpha \in (0,1)$*. This problem is called the minimal linked Gelfond–Baker separation problem.*

6. Schanuel-Conditional Rational Exclusion and Finite-Root Transcendence

We use Schanuel's conjecture in the following standard form. For a field extension $F \subset E$, the transcendence degree $\operatorname{trdeg}_F E$ is the maximal cardinality of a subset of E that is algebraically independent over F. Equivalently, when E is finitely generated over F, it is the number of independent transcendental parameters needed before the remaining generators become algebraic. Thus, if $E = F(y_1, \ldots, y_r)$, then $\operatorname{trdeg}_F E \leq r$, and equality holds precisely when $y_1, \ldots, y_r$ are algebraically independent over F.

Conjecture 6.1 (Schanuel)**.** *If* $z_1, \ldots, z_n$ *are linearly independent over* $\mathbb{Q}$*, then*

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}) \geq n. \tag{6.1}$$

Let

$$t = \log 2.$$

For $j \geq 0$ and $i \geq 0$, write

$$p_j = u_j(w), \qquad b_i = u_{-i}(w).$$

Thus

$$p_0 = b_0 = w, \qquad p_{j+1} = e^{tp_j} - 1,$$

and

$$e^{tb_i} = 1 + b_{i-1}, \qquad i \geq 1.$$

In the arguments below this interpretation is used repeatedly. Schanuel's conjecture supplies a lower bound for the transcendence degree of a field generated by exponential inputs and their exponentials. In the present orbit, the exponentials are identified with orbit generators through

$$e^t = 2, \qquad e^{tp_j} = 1 + p_{j+1} \quad (j \geq 0), \qquad e^{tb_i} = 1 + b_{i-1} \quad (i \geq 1).$$

Since $\overline{\mathbb{Q}}$ is algebraic over $\mathbb{Q}$, adjoining $\overline{\mathbb{Q}}$ does not change transcendence degree:

$$\operatorname{trdeg}_{\mathbb{Q}} \overline{\mathbb{Q}}(Y) = \operatorname{trdeg}_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}(Y)$$

for every finite set $Y \subset \mathbb{C}$. Hence, whenever Schanuel's conjecture gives a lower bound equal to the number of displayed field generators, those generators are algebraically independent over $\overline{\mathbb{Q}}$.

Lemma 6.2 (Forward independence for algebraic irrational input)**.** *Assume Schanuel's conjecture. Let* $w \in \overline{\mathbb{Q}} \cap (0,1)$ *be irrational. Then, for each* $m \geq 1$*, the numbers*

$$t, p_1, \ldots, p_m \tag{6.2}$$

are algebraically independent over $\overline{\mathbb{Q}}$*.*

Proof. We first prove the case $m = 1$. The numbers t and tw are linearly independent over $\mathbb{Q}$. Indeed, if

$$a_0 t + a_1 tw = 0, \qquad a_0, a_1 \in \mathbb{Q},$$

then, since $t \neq 0$,

$$a_0 + a_1 w = 0.$$

The irrationality of w forces $a_0 = a_1 = 0$.

By Schanuel's conjecture,

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(t, tw, e^t, e^{tw}) \geq 2.$$

Since

$$e^t = 2, \qquad e^{tw} = p_1 + 1, \qquad tw \in \overline{\mathbb{Q}}(t),$$

one has

$$\mathbb{Q}(t, tw, e^t, e^{tw}) \subset \overline{\mathbb{Q}}(t, p_1).$$

Hence

$$\operatorname{trdeg}_{\mathbb{Q}} \overline{\mathbb{Q}}(t, p_1) \geq 2.$$

As $\overline{\mathbb{Q}}$ is algebraic over $\mathbb{Q}$, this is the same as

$$\operatorname{trdeg}_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}(t, p_1) \geq 2.$$

The field $\overline{\mathbb{Q}}(t, p_1)$ is generated over $\overline{\mathbb{Q}}$ by the two elements t and p_1. Therefore equality holds, and t, p_1 are algebraically independent over $\overline{\mathbb{Q}}$.

Now assume the statement has been proved for some $m \geq 1$. We show it for $m + 1$. Consider the family

$$t, \quad tw, \quad tp_1, \ldots, \quad tp_m. \tag{6.3}$$

It is linearly independent over $\mathbb{Q}$. To see this, suppose that

$$a_0 t + a_1 tw + a_2 tp_1 + \cdots + a_{m+1} tp_m = 0$$

with $a_0, \ldots, a_{m+1} \in \mathbb{Q}$. Since $t \neq 0$, division by t gives

$$a_0 + a_1 w + a_2 p_1 + \cdots + a_{m+1} p_m = 0.$$

Here $a_0 + a_1 w \in \overline{\mathbb{Q}}$. By the induction hypothesis, $p_1, \ldots, p_m$ are algebraically independent over $\overline{\mathbb{Q}}$. Thus the displayed linear relation forces

$$a_2 = \cdots = a_{m+1} = 0.$$

The remaining relation $a_0 + a_1 w = 0$ then gives $a_0 = a_1 = 0$, since w is irrational. Hence the family in (6.3) is $\mathbb{Q}$-linearly independent.

Schanuel's conjecture applied to (6.3) gives

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(t, tw, tp_1, \ldots, tp_m, e^t, e^{tw}, e^{tp_1}, \ldots, e^{tp_m}) \geq m + 2.$$

Moreover,

$$e^t = 2, \qquad e^{tw} = p_1 + 1, \qquad e^{tp_j} = p_{j+1} + 1 \quad (1 \leq j \leq m),$$

and the inputs $tw, tp_1, \ldots, tp_m$ all lie in $\overline{\mathbb{Q}}(t, p_1, \ldots, p_m)$. Therefore

$$\mathbb{Q}(t, tw, tp_1, \ldots, tp_m, e^t, e^{tw}, e^{tp_1}, \ldots, e^{tp_m}) \subset \overline{\mathbb{Q}}(t, p_1, \ldots, p_{m+1}).$$

It follows that

$$\operatorname{trdeg}_{\mathbb{Q}} \overline{\mathbb{Q}}(t, p_1, \ldots, p_{m+1}) \geq m + 2.$$

Equivalently,

$$\operatorname{trdeg}_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}(t, p_1, \dots, p_{m+1}) \geq m + 2.$$

Since the latter field is generated over $\overline{\mathbb{Q}}$ by the $m+2$ elements $t, p_1, \dots, p_{m+1}$, equality holds. Hence $t, p_1, \dots, p_{m+1}$ are algebraically independent over $\overline{\mathbb{Q}}$. ☐

Lemma 6.3 (Bidirectional independence for rational input)**.** *Assume Schanuel's conjecture. Let* $w \in \mathbb{Q} \cap (0,1)$*. Then* $p_1 = 2^w - 1$ *is algebraic irrational. Moreover, for every* $A \geq 0$ *and every* $B \geq 1$*, the numbers*

$$t, b_1, \dots, b_A, p_2, \dots, p_B \tag{6.4}$$

are algebraically independent over $\overline{\mathbb{Q}}$*, where the positive list is omitted when* $B = 1$.

Proof. Write

$$w = \frac{p}{q}, \qquad 0 < p < q, \qquad (p,q) = 1.$$

Then $2^{p/q}$ is algebraic. We claim that it is not rational. If $2^{p/q} = a/b$ in lowest terms, then

$$2^p b^q = a^q.$$

Since $(a,b) = 1$, this forces $b = 1$. Hence $a^q = 2^p$, which is impossible for $0 < p < q$ and $(p,q) = 1$. Thus $2^{p/q}$ is irrational, and therefore $p_1 = 2^{p/q} - 1$ is algebraic irrational.

We first prove the backward assertion: for every $A \geq 0$, the numbers $t, b_1, \dots, b_A$ are algebraically independent over $\overline{\mathbb{Q}}$. For $A = 0$, Schanuel's conjecture applied to the single non-zero number t gives

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(t, e^t) \geq 1.$$

Since $e^t = 2$, it follows that t is transcendental.

Assume that $t, b_1, \dots, b_A$ are algebraically independent over $\overline{\mathbb{Q}}$. We prove the assertion with $A+1$ in place of A. The numbers

$$t, \quad tb_1, \dots, \quad tb_{A+1}$$

are linearly independent over $\mathbb{Q}$. Indeed, suppose that

$$c_0 t + c_1 t b_1 + \cdots + c_{A+1} t b_{A+1} = 0$$

with $c_i \in \mathbb{Q}$. Since $t \neq 0$, and after clearing denominators, we may assume that

$$c_0 + c_1 b_1 + \cdots + c_{A+1} b_{A+1} = 0, \qquad c_i \in \mathbb{Z}.$$

If $c_{A+1} = 0$, the induction hypothesis forces $c_0 = \cdots = c_A = 0$.

Assume now that $c_{A+1} \neq 0$. Exponentiating base 2 gives

$$2^{c_0} \prod_{i=1}^{A+1} (1 + b_{i-1})^{c_i} = 1, \qquad b_0 = w.$$

If $A = 0$, this would imply a rational linear relation between 1 and b_1, and hence $b_1 \in \mathbb{Q}$. Since $1 + w = 2^{b_1}$ is rational, the standard lowest-terms argument for rational powers of 2 shows that b_1 must be an integer. This is impossible because $1 + w \in (1,2)$.

If $A \geq 1$, clearing the negative powers gives a polynomial relation

$$P(b_1, \dots, b_A) = 0, \qquad P \in \overline{\mathbb{Q}}[X_1, \dots, X_A].$$

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Since $c_{A+1} \neq 0$, after clearing negative powers one side contains a non-zero power of $1 + X_A$ while the other does not. Thus P has positive degree in X_A and is not the zero polynomial. This contradicts the induction hypothesis. Hence $c_{A+1} = 0$, and consequently all c_i vanish. Thus the displayed family is linearly independent over $\mathbb{Q}$.

Schanuel's conjecture gives

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(t, tb_1, \ldots, tb_{A+1}, e^t, e^{tb_1}, \ldots, e^{tb_{A+1}}) \geq A + 2.$$

Moreover,

$$e^t = 2, \qquad e^{tb_i} = 1 + b_{i-1}, \qquad 1 \leq i \leq A + 1.$$

Hence the Schanuel field above is contained in

$$\overline{\mathbb{Q}}(t, b_1, \ldots, b_{A+1}).$$

It follows that

$$\operatorname{trdeg}_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}(t, b_1, \ldots, b_{A+1}) \geq A + 2.$$

The latter field is generated over $\overline{\mathbb{Q}}$ by the $A + 2$ elements $t, b_1, \ldots, b_{A+1}$. Therefore equality holds, and $t, b_1, \ldots, b_{A+1}$ are algebraically independent over $\overline{\mathbb{Q}}$. This completes the backward induction.

Now fix $A \geq 0$ and extend in the positive direction. The case $B = 1$ is exactly the backward assertion just proved. Assume that, for some $B \geq 2$, the numbers

$$t, b_1, \ldots, b_A, p_2, \ldots, p_{B-1}$$

are algebraically independent over $\overline{\mathbb{Q}}$. Consider the family

$$t, \quad tb_1, \ldots, \quad tb_A, \quad tp_1, \quad tp_2, \ldots, \quad tp_{B-1}. \tag{6.5}$$

It is linearly independent over $\mathbb{Q}$. Indeed, after division by $t \neq 0$, a rational linear relation has the form

$$a_0 + a_1 p_1 + \beta_1 b_1 + \cdots + \beta_A b_A + \nu_2 p_2 + \cdots + \nu_{B-1} p_{B-1} = 0.$$

Since $a_0 + a_1 p_1 \in \overline{\mathbb{Q}}$ and $b_1, \ldots, b_A, p_2, \ldots, p_{B-1}$ are algebraically independent over $\overline{\mathbb{Q}}$ by the induction hypothesis, all coefficients β_i and ν_j vanish. The remaining relation $a_0 + a_1 p_1 = 0$ forces $a_0 = a_1 = 0$, because p_1 is irrational. Hence the family in (6.5) is $\mathbb{Q}$-linearly independent.

Schanuel's conjecture applied to (6.5) gives transcendence degree at least $A + B$. Its exponentials are

$$e^t = 2, \qquad e^{tb_i} = 1 + b_{i-1} \quad (1 \leq i \leq A), \qquad e^{tp_j} = p_{j+1} + 1 \quad (1 \leq j \leq B - 1).$$

Thus the corresponding Schanuel field is contained in

$$\overline{\mathbb{Q}}(t, b_1, \ldots, b_A, p_2, \ldots, p_B).$$

Consequently,

$$\operatorname{trdeg}_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}}(t, b_1, \ldots, b_A, p_2, \ldots, p_B) \geq A + B.$$

The latter field is generated over $\overline{\mathbb{Q}}$ by the $A + B$ displayed elements. Hence equality holds. Therefore $t, b_1, \ldots, b_A, p_2, \ldots, p_B$ are algebraically independent over $\overline{\mathbb{Q}}$. The induction on B is complete. □

Theorem 6.4 (Rational non-termination under Schanuel)**.** *Assume Schanuel's conjecture. Let $S \subset \mathbb{Z}$ be finite with $0 \in S$ and $|S| \geq 2$. Then the unique root in $(0, 1)$ of*

$$\sum_{k \in S} u_k(w) = 1 \tag{6.6}$$

is irrational. Consequently,

$$W_{\mathrm{fin}} \cap \mathbb{Q} = \{1\}. \tag{6.7}$$

Proof. Suppose that $w \in \mathbb{Q} \cap (0,1)$ satisfies (6.6). Put

$$S_- = \{i \geq 1 : -i \in S\}, \qquad S_+ = \{j \geq 1 : j \in S\}.$$

Let $\varepsilon = 1$ if $1 \in S_+$, and let $\varepsilon = 0$ otherwise. Since $u_{-i}(w) = b_i$, $u_0(w) = w$, and $u_j(w) = p_j$, the equation becomes

$$\sum_{i \in S_-} b_i + \sum_{\substack{j \in S_+ \\ j \geq 2}} p_j = 1 - w - \varepsilon p_1. \tag{6.8}$$

By Lemma 6.3, the number p_1 is algebraic. Hence the right-hand side of (6.8) belongs to $\overline{\mathbb{Q}}$.

Assume that S_- is non-empty or that S_+ contains an index $j \geq 2$. Choose

$$A_0 = \begin{cases} \max S_-, & S_- \neq \varnothing, \\ 0, & S_- = \varnothing, \end{cases}$$

and

$$B_0 = \begin{cases} \max S_+, & \text{if } S_+ \text{ contains an index } j \geq 2, \\ 1, & \text{otherwise.} \end{cases}$$

Then the variables occurring on the left-hand side of (6.8) form a non-empty subset of

$$b_1, \ldots, b_{A_0}, p_2, \ldots, p_{B_0}.$$

By Lemma 6.3, the larger family

$$t, b_1, \ldots, b_{A_0}, p_2, \ldots, p_{B_0}$$

is algebraically independent over $\overline{\mathbb{Q}}$. Hence no non-empty subset of the displayed variables can satisfy a non-trivial linear relation over $\overline{\mathbb{Q}}$. This contradicts (6.8).

It remains to consider the case where $S_- = \varnothing$ and $S_+ \subset \{1\}$. Since $0 \in S$ and $|S| \geq 2$, one has $S = \{0,1\}$. Then (6.6) gives

$$w + p_1 = 1.$$

Thus $p_1 = 1 - w \in \mathbb{Q}$, contradicting the fact that p_1 is irrational by Lemma 6.3. This proves that the root in $(0,1)$ is irrational.

By the finite-support representation above, every non-trivial finite terminal point has a full finite support S with $0 \in S$ and $|S| \geq 2$. The only finite terminal point with support $\{0\}$ is $w = 1$. Therefore (6.7) follows. □

Theorem 6.5 (Positive roots under Schanuel)**.** *Assume Schanuel's conjecture. Let $P \subset \mathbb{Z}_{\geq 1}$ be non-empty and finite. The unique root of*

$$w + \sum_{j \in P} u_j(w) = 1 \tag{6.9}$$

is transcendental.

Proof. Let $N = \max P$, and suppose that the root w is algebraic. The rational case is excluded by Theorem 6.4. Hence w is algebraic irrational. The equation gives

$$p_N = 1 - w - \sum_{\substack{j \in P \\ j < N}} p_j.$$

Thus

$$p_N \in \overline{\mathbb{Q}}(p_1, \ldots, p_{N-1}),$$

where the field on the right is interpreted as $\overline{\mathbb{Q}}$ when $N = 1$. This contradicts Lemma 6.2. Therefore the root is transcendental. □

Corollary 6.6 (Two-term roots under Schanuel). *Assume Schanuel's conjecture. For each $N \geq 1$, the roots of*

$$w + u_N(w) = 1 \tag{6.10}$$

and

$$w + u_{-N}(w) = 1 \tag{6.11}$$

are transcendental.

Proof. The positive case is Theorem 6.5 with $P = \{N\}$. By Corollary 5.3, the negative root is 1 minus the corresponding positive root. Hence the negative root is also transcendental. □

Theorem 6.7 (One first negative layer under Schanuel). *Assume Schanuel's conjecture. Let $P \subset \mathbb{Z}_{\geq 1}$ be finite and non-empty. The unique root of*

$$u_{-1}(w) + w + \sum_{j \in P} u_j(w) = 1 \tag{6.12}$$

is transcendental.

Proof. Let $N = \max P$, and suppose that $w \in \overline{\mathbb{Q}} \cap (0, 1)$ is a root. The rational case is excluded by Theorem 6.4. Hence w is algebraic irrational. Put $b_1 = u_{-1}(w)$. The equation gives

$$p_N = 1 - b_1 - w - \sum_{\substack{j \in P \\ j < N}} p_j. \tag{6.13}$$

By Lemma 6.2, the numbers $t, p_1, \ldots, p_N$ are algebraically independent over $\overline{\mathbb{Q}}$.

Consider the family

$$t, \quad tw, \quad tp_1, \ldots, \quad tp_{N-1}, \quad tb_1, \tag{6.14}$$

where the list $tp_1, \ldots, tp_{N-1}$ is omitted when $N = 1$. We claim that this family is linearly independent over $\mathbb{Q}$. Suppose that

$$a_0 t + a_1 tw + a_2 tp_1 + \cdots + a_N tp_{N-1} + ctb_1 = 0$$

with $a_0, a_1, a_2, \ldots, a_N, c \in \mathbb{Q}$. Since $t \neq 0$, division by t gives

$$a_0 + a_1 w + a_2 p_1 + \cdots + a_N p_{N-1} + cb_1 = 0.$$

If $c \neq 0$, then

$$b_1 \in \overline{\mathbb{Q}}(p_1, \ldots, p_{N-1}),$$

because $w \in \overline{\mathbb{Q}}$. Then (6.13) gives

$$p_N \in \overline{\mathbb{Q}}(p_1, \ldots, p_{N-1}),$$

contradicting the algebraic independence of $t, p_1, \ldots, p_N$.

Thus $c = 0$. The relation becomes

$$a_0 + a_1 w + a_2 p_1 + \cdots + a_N p_{N-1} = 0.$$

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Since $a_0 + a_1 w \in \overline{\mathbb{Q}}$ and $p_1, \dots, p_{N-1}$ are algebraically independent over $\overline{\mathbb{Q}}$, we obtain

$$a_2 = \cdots = a_N = 0.$$

The remaining relation $a_0 + a_1 w = 0$ forces $a_0 = a_1 = 0$, because w is irrational. Hence the family in (6.14) is $\mathbb{Q}$-linearly independent.

Schanuel's conjecture applied to (6.14) gives transcendence degree at least $N+2$ for the family and its exponentials. These exponentials are

$$\begin{aligned} e^{t} &= 2, & e^{tw} &= p_1 + 1, \\ e^{tp_j} &= p_{j+1} + 1, & 1 \le j &\le N-1, \\ e^{tb_1} &= 1 + w. \end{aligned}$$

Using (6.13) to eliminate p_N, all these quantities lie in

$$\overline{\mathbb{Q}}(t, p_1, \dots, p_{N-1}, b_1),$$

which is generated over $\overline{\mathbb{Q}}$ by $N+1$ elements and therefore has transcendence degree at most $N+1$. This contradicts the Schanuel lower bound $N+2$. Hence the root is transcendental. ☐

7. Conclusions

This paper gives an arithmetic formulation of finite termination for the endogenous greedy expansion generated by $f(x) = 2^x - 1$. The analytic part establishes unique normalization roots for admissible digit sequences and a deterministic greedy code with exhaustive remainder. Finite termination is then reduced to a finite orbit-sum equation compatible with the greedy order. These finite-support equations are rigid in one dimension: every finite support has a unique simple root, roots reverse the inclusion order of supports, and finite-sum roots are covariant under integer shifts of the support. In particular,

$$W_{\text{fin}}$$

is countable and has Lebesgue measure zero. The arithmetic conjecture isolated by this structure is

$$W_{\text{fin}} \cap \overline{\mathbb{Q}} = \{1\}. \tag{7.1}$$

The unconditional transcendence results settle the first-order boundary layer. The two roots of

$$w + u_1(w) = 1, \qquad w + u_{-1}(w) = 1$$

are transcendental. For every $N \ge 1$, the two-term boundary roots are paired by the mirror identity

$$w_N^- = 1 - w_N^+.$$

The positive roots increase strictly with N, and the negative roots decrease strictly with N. The first positive second-order boundary root, defined by

$$w + u_2(w) = 1,$$

is irrational. Transcendence of this root is equivalent to the assertion that the linked exponential-logarithmic equation

$$2^{\alpha} \log 2 = \log(2(2-\alpha)), \qquad \alpha \in \overline{\mathbb{Q}} \cap (0,1), \tag{7.2}$$

has no algebraic solution. This equation is the first obstruction in the system that is not covered by the direct first-order Gelfond–Schneider argument.

Under Schanuel's conjecture, the rational part of the conjecture is completely settled:

$$W_{\mathrm{fin}} \cap \mathbb{Q} = \{1\}.$$

The same hypothesis gives transcendence for every purely positive finite root

$$w + \sum_{j \in P} u_j(w) = 1, \qquad \varnothing \neq P \subset \mathbb{Z}_{\geq 1},$$

for all two-term boundary roots

$$w + u_N(w) = 1, \qquad w + u_{-N}(w) = 1, \qquad N \geq 1,$$

and for every finite root with one first negative layer and arbitrary finite positive support,

$$u_{-1}(w) + w + \sum_{j \in P} u_j(w) = 1, \qquad \varnothing \neq P \subset \mathbb{Z}_{\geq 1}.$$

Thus the conjecture is proved here for rational points, for all positive finite supports, for all two-term boundary supports, and for the one-first-negative family under Schanuel's conjecture. The remaining algebraic cases are genuinely mixed: they involve deeper negative layers interacting with positive orbit layers, where the available independence mechanisms no longer reduce immediately to a single Schanuel family.

Supplementary Materials: The following supporting information can be downloaded at the website of this paper posted on Preprints.org.

Data Availability Statement: No datasets were generated or analysed during the present theoretical study. The numerical values in Appendix A are reproducible from the displayed finite orbit-sum equations and the bisection method described there. Accompanying Python and Mathematica scripts are included with the submission files for independent verification of the appendix table.

Appendix A. Initial Finite-Support Roots

This appendix records initial finite-support roots for orientation in the arithmetic hierarchy. For a finite set $K \subset \mathbb{Z} \setminus \{0\}$, the displayed value solves

$$u_0(w) + \sum_{k \in K} u_k(w) = 1.$$

The roots were computed by bisection using the strict monotonicity proved in Proposition 4.1. For the full support $S = \{0\} \cup K$, the columns N and m record $\max S - \min S$ and $-\min S$, respectively. The table records finite-support roots; greedy compatibility is governed by the prefix inequalities in Section 3.

K	N	m	Root w_K	Current status
$\{1\}$	1	0	0.543000440865408174	unconditional transcendental
$\{-1\}$	1	1	0.456999559134591826	unconditional transcendental
$\{2\}$	2	0	0.585370493533997433	irrational; transcendental under Schanuel
$\{-2\}$	2	2	0.414629506466002567	irrational by mirror; transcendental under Schanuel
$\{1,2\}$	2	0	0.411998734043774446	transcendental under Schanuel
$\{-1,1\}$	2	1	0.330527873549963373	transcendental under Schanuel
$\{-1,2\}$	3	1	0.352639118320622143	transcendental under Theorem 6.7
$\{2,3\}$	3	0	0.465321884988051543	transcendental under Schanuel
$\{-2,1\}$	3	2	0.301906061717133723	rationally excluded under Schanuel; algebraic case open
$\{-2,2\}$	4	2	0.322180580023365803	rationally excluded under Schanuel; algebraic case open

The table illustrates the transition from settled first-order roots to higher-depth finite-support patterns. Purely positive supports and the one-first-negative cases fall under the Schanuel-conditional theory, while examples with deeper negative layers remain open in the algebraic case.

References

1. Kuczma, M.; Choczewski, B.; Ger, R. *Iterative Functional Equations*; Vol. 32, *Encyclopedia of Mathematics and its Applications*, Cambridge University Press: Cambridge, 1990.
2. Hofbauer, F. Beta-shifts have unique maximal measure. *Monatsh. Math.* **1978**, *85*, 189–198.
3. Parry, W. On the beta-expansions of real numbers. *Acta Math. Acad. Sci. Hungar.* **1960**, *11*, 401–416.
4. Rényi, A. Representations for real numbers and their ergodic properties. *Acta Math. Acad. Sci. Hungar.* **1957**, *8*, 477–493.
5. Ax, J. On Schanuel's conjectures. *Ann. of Math. (2)* **1971**, *93*, 252–268.
6. Baker, A. *Transcendental Number Theory*; Cambridge University Press: Cambridge, 1975.
7. Gelfond, A.O. *Transcendental and Algebraic Numbers*; Dover Publications: New York, 1960.
8. Lang, S. *Introduction to Transcendental Numbers*; Addison-Wesley: Reading, MA, 1966.
9. Schneider, T. *Einführung in die transzendenten Zahlen*; Springer: Berlin, 1957.
10. Waldschmidt, M. *Diophantine Approximation on Linear Algebraic Groups*; Vol. 326, *Grundlehren der mathematischen Wissenschaften*, Springer: Berlin, 2000.