Short Note

Not peer-reviewed version

Finite Field Tarski-Maligranda Inequalities

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Posted Date: 31 March 2026

doi: 10.20944/preprints202603.2405.v1

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Finite Field Tarski-Maligranda Inequalities

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Abstract

Let $\mathbb{F}$ be a sub-modulus field such that $2 \neq 0$. Let $\mathcal{X}$ be a sub-normed linear space over $\mathbb{F}$. Then we show that $\left|\|x\|-\|y\|\right| \leq \frac{2}{|2|}\|x+y\| + \frac{2}{|2|}\max\{\|x-y\|, \|y-x\|\} - (\|x\|+\|y\|)$ and $\left|\|x\|-\|y\|\right| \leq \|x\|+\|y\| - \frac{2}{|2|}\|x+y\| + \frac{2}{|2|}\max\{\|y-x\|, \|x-y\|\}$. Above inequalities are finite field versions of important Tarski-Maligranda inequalities obained by Maligranda [*Banach J. Math. Anal., 2008*].

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MSC: 12E20; 46B20

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1. Introduction

We all know that

$$\left||r|-|s|\right| \leq \min\{|r-s|, |r+s|\}, \quad \forall r,s \in \mathbb{R}. \tag{1}$$

In 1930, Tarski observed a strengthening of Inequality (1) (Page 688, [1]):

$$\left||r|-|s|\right| = |r-s| + |r+s| - (|r|+|s|), \quad \forall r,s \in \mathbb{R}. \tag{2}$$

Note that Equality (2) fails for complex numbers [2]. In 2008, Maligranda showed that one can generalize Equality (2) by deriving inequalities that hold for any normed linear space (NLS) [2].

Theorem 1. *[2] (**Tarski-Maligranda Inequalities**) Let $\mathcal{X}$ be a NLS. Then for all $x, y \in \mathcal{X} \setminus \{0\}$,*

$$\left|\|x\|-\|y\|\right| \leq \|x+y\| + \|x-y\| - (\|x\|+\|y\|) \tag{3}$$

and

$$\left|\|x\|-\|y\|\right| \leq \|x\|+\|y\| - \left|\|x+y\|-\|x-y\|\right|. \tag{4}$$

We ask for finite field versions of Inequalities (3) and (4). We wish to remark that we have recently derived non-Archimedean versions of Tarski-Maligranda inequalities [3]. We begin by introducing the notion of sub-modulus field.

Definition 1. *Let $\mathbb{F}$ be a (finite) field. A map $|\cdot| : \mathbb{F} \to [0,\infty)$ is said to be sub-modulus or sub-valued if the following conditions hold.*

(i) *If $\lambda \in \mathbb{F}$ is such that $|\lambda| = 0$, then $\lambda = 0$.*
(ii) *$|\lambda\mu| \leq |\lambda||\mu|$ for all $\lambda, \mu \in \mathbb{F}$.*
(iii) *$|\lambda+\mu| \leq |\lambda|+|\mu|$ for all $\lambda, \mu \in \mathbb{F}$.*

In this case, we say $\mathbb{F}$ as sub-modulus or sub-valued field.

Example 1. *Let p be a prime, let $\mathbb{Z}_p := \{0, 1, \dots, p-1\}$ be the standard field of integers modulo p. Given $n \in \mathbb{Z}_p$, we define*

$$|n| := \text{unique } a \text{ such that } 0 \leq a \leq p-1 \text{ and } n \equiv a(\text{mod } p).$$

We next introduce the notion of sub-normed linear space.

Definition 2. *Let $\mathcal{X}$ be a vector space over a sub-modulus field $\mathbb{F}$. A map $\|\cdot\| : \mathcal{X} \to [0, \infty)$ is said to be sub-norm if the following conditions hold.*

(i) *If $x \in \mathcal{X}$ is such that $\|x\| = 0$, then $x = 0$.*
(ii) *$\|\lambda x\| \leq |\lambda| \|x\|$ for all $\lambda \in \mathbb{F}$, for all $x \in \mathcal{X}$.*
(ii) *$\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathcal{X}$.*

In this case, we say $\mathcal{X}$ is a sub-normed linear space.

Example 2. *Let $p \in [1, \infty)$. Let $\mathbb{F}$ be a sub-modulus field. For $d \in \mathbb{N}$, let $\mathbb{F}^d$ be the standard vector space. We define*

$$\|(a_j)_{j=1}^d\|_p := \left(\sum_{j=1}^d |a_j|^p\right)^{\frac{1}{p}}, \quad \forall (a_j)_{j=1}^d \in \mathbb{F}^d.$$

Then $\|\cdot\|_p$ is a sub-norm on $\mathbb{F}^d$.

Example 3. *Let $\mathbb{F}$ be a sub-modulus field. Define*

$$c_{00}(\mathbb{N}, \mathbb{F}) := \{\{a_n\}_{n=1}^\infty : a_n \in \mathbb{F}, \forall n \in \mathbb{N}, a_n \neq 0 \text{ only for finitely many } n\}.$$

We define

$$\|\{a_n\}_{n=1}^\infty\|_{00} := \max_{n \in \mathbb{N}} |a_n|, \quad \forall \{a_n\}_{n=1}^\infty \in c_{00}(\mathbb{N}, \mathbb{F}).$$

Then $\|\cdot\|_{00}$ is a sub-norm on $c_{00}(\mathbb{N}, \mathbb{F})$.

2. Finite Field Tarski-Maligranda Inequalities

It is interesting to note that Inequality (1) may not hold for sub-modulus field. Fundamental reason for this is $|r| \neq |-r|$ in general, in a sub-valued field $\mathbb{F}$. For example, in $\mathbb{Z}_3$, $1 = |1| \neq |-1| = |2| = 2$. Hence

$$\Big||0| - |1|\Big| = \Big|-|1|\Big| = |-1| = |2| = 2 > \min\{|0-1|, |0+1|\} = \min\{|-1|, |1|\} = \min\{|2|, |1|\} = 1.$$

We first derive finite field version of Inequality (1).

Theorem 2. *Let $\mathcal{X}$ be a sub-normed linear space. Then*

$$\Big|\|x\| - \|y\|\Big| \leq \max\{\|x-y\|, \|y-x\|\}.$$

Proof. Let $x, y \in \mathcal{X}$. Then

$$\|x\| = \|(x-y) + y\| \leq \|x-y\| + \|y\|$$

doi:10.20944/preprints202603.2405.v1

and

$$\|y\| = \|(y-x)+x\| \leq \|y-x\| + \|x\|.$$

Hence

$$\|x\| - \|y\| = \|(x-y)+y\| \leq \|x-y\|$$

and

$$\|y\| - \|x\| = \|(y-x)+x\| \leq \|y-x\|.$$

Therefore

$$\Big|\|x\| - \|y\|\Big| = \max\{\|x\| - \|y\|, \|y\| - \|x\|\} \leq \max\{\|x-y\|, \|y-x\|\}.$$

□

Following is finite field version of Theorem 1.

Theorem 3. *(**Finite Field Tarski-Maligranda Inequalities**) Let $\mathcal{X}$ be a sub-normed linear space over sub-modulus field $\mathbb{F}$. Assume $2 \neq 0$. Then for all $x, y \in \mathcal{X}$,*

$$\Big|\|x\| - \|y\|\Big| \leq \frac{2}{|2|}\|x+y\| + \frac{2}{|2|}\max\{\|x-y\|, \|y-x\|\} - (\|x\| + \|y\|)$$

and

$$\Big|\|x\| - \|y\|\Big| \leq \|x\| + \|y\| - \frac{2}{|2|}\|x+y\| + \frac{2}{|2|}\max\{\|y-x\|, \|x-y\|\}.$$

Proof. Let $x, y \in \mathcal{X}$. Then

$$\|x\| + \|y\| + \Big|\|x\| - \|y\|\Big| = 2\max\{\|x\|, \|y\|\}. \tag{5}$$

We also have

$$\|x+y\| + \|x-y\| \geq \|(x+y) + (x-y)\| = \|2x\| = |2|\|x\| \tag{6}$$

and

$$\|x+y\| + \|y-x\| \geq \|(x+y) + (y-x)\| = \|2y\| = |2|\|y\|. \tag{7}$$

Inequalities (6) and (7) give

$$\|x+y\| + \max\{\|x-y\|, \|y-x\|\} = \max\{\|x+y\| + \|x-y\|, \|x+y\| + \|y-x\|\} \geq |2|\max\{\|x\|, \|y\|\}. \tag{8}$$

Equality (5) and Inequality (8) give

$$\|x\| + \|y\| + \Big|\|x\| - \|y\|\Big| = 2\max\{\|x\|, \|y\|\} \leq \frac{2}{|2|}\|x+y\| + \frac{2}{|2|}\max\{\|x-y\|, \|y-x\|\}.$$

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Rearranging,

$$\left|\|x\| - \|y\|\right| \leq \frac{2}{|2|}\|x+y\| + \frac{2}{|2|}\max\{\|x-y\|, \|y-x\|\} - (\|x\| + \|y\|).$$

We next see that

$$\|x\| + \|y\| - \left|\|x\| - \|y\|\right| = 2\min\{\|x\|, \|y\|\}. \tag{9}$$

We note that

$$\|x+y\| \leq \|(x+y) - (y-x) + \|y-x\| = |2|\|x\| + \|y-x\| \implies \|x+y\| - \|y-x\| \leq |2|\|x\| \tag{10}$$

and

$$\|x+y\| \leq \|(x+y) - (x-y) + \|x-y\| = |2|\|y\| + \|x-y\| \implies \|x+y\| - \|x-y\| \leq |2|\|y\|. \tag{11}$$

Inequalities (10) and (11) give

$$\|x+y\| - \max\{\|y-x\|, \|x-y\|\} = \min\{\|x+y\| - \|y-x\|, \|x+y\| - \|x-y\|\} \leq |2|\min\{\|x\|, \|y\|\}. \tag{12}$$

Equality (9) and Inequality (12) give

$$\|x+y\| - \max\{\|y-x\|, \|x-y\|\} \leq \frac{|2|}{2}(\|x\| + \|y\|) - \frac{|2|}{2}\left|\|x\| - \|y\|\right|.$$

Rearranging,

$$\left|\|x\| - \|y\|\right| \leq \|x\| + \|y\| - \frac{2}{|2|}\|x+y\| + \frac{2}{|2|}\max\{\|y-x\|, \|x-y\|\}.$$

□

3. Conclusions

(1) In 1930, Tarski observed an equality in the triangle inequality for real numbers [1].

(2) In 2008, Maligranda generalized Tarski equality to inequalities which are valid in every normed linear spaces [2].

(3) In this note, we derived finite field version of Tarski-Maligranda inequalities.

References

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