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Article

On Some Specific Order Estimates for the Mertens Function and its Relation to the Non-Trivial Zeros of $\zeta(s)$

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Abstract: The primary purpose of this article is to provide a brief overview about the notion of the *Mertens Function* $M(n)$ and *Redheffer Matrices* $\mathbb{A}_n$ and, to derive specific order estimates for $M(n)$, in order to analyze the location of the non-trivial zeros of the *Riemann Zeta function* $\zeta(s)$. A priori using the above concepts, the article aims at establishing a necessary and sufficient condition for the *Riemann Hypothesis* to hold true.

Keywords: arithmetic function; möbius function; riemann zeta function; mertens function; redheffer matrix; riemann hypothesis; roesler matrix; spectral radius; mertens hypothesis

2020 MSC: Primary 11-02, 11A25, 11N37, 11N56, 15A18; Secondary 15-02; 11Y70

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1. Introduction & Motivation

Arguably one of the most intriguing problems in the field of *pure mathematics*, especially *Number Theory* to be more precise, it is by no means, a surprise that the famous *"Riemann Hypothesis"* is still considered as an open problem, and has drawn tremendous interest from mathematicians and even scholars from other areas of science, for example, Quantum Physics, Computer Science etc. since more than two centuries.

Among all the equivalent versions of the statement of the *Riemann Hypothesis*, its extremely rare that one intends to approach the problem using concepts of *Advanced Linear Algebra* and *Advanced Matrix Theory*. Although every possible headway in this daunting endeavour is extremely appreciable, its absolutely correct to say that, over the years, there have been several instances of outstanding claims made by pioneers in this field, which gave their successors enough motivation to observe this problem from a different perspective altogether.

This article is devoted to provide an exposition on one such method proposed by *Redheffer* [48] in the late 1900's, when he defined formally a special kind of *non-symmetric* matrices, which was eventually named after him as *Redheffer Matrices*, usually denoted as $\mathbb{A}_n$. These matrices were "special" in a sense that, apart from its unique spectral properties, *Redheffer* claimed and proved that,

A necessary and sufficient condition for the statement of *Riemann Hypothesis* to be true is,
$$det(\mathbb{A}_n) = O\left(n^{\frac{1}{2}+\epsilon}\right), \text{ for every } \epsilon > 0.$$

Roesler [54] also introduced another type of *non-symmetric* matrices named after him, with similar properties (5) like *Redheffer Matrices*. Furthermore, he also established a condition which is necessary and sufficient for *Riemann Hypothesis* to be true.

One can, in fact, observe that, in terms of determinant, these two matrices are very closely related to the *Mertens function* $M(n)$, which, by definition, can also be interpreted as a summatory function of the *Möbius Function* $\mu(n)$ (2.2.1) (pp. 91 [59]).

As simple as it might seem, to be honest, understanding the true essence of these results and their implications in Linear Algebra and Number Theory takes a humoungous effort from any motivated indivudual. As a prerequisite for studying these results in detail, we shall familiarize ourselves with the notion of *Mertens Function* and *Redheffer Matrices* in the earlier sections of 3 to 5, along with deriving the relation between these two.

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In addition to studying the *Spectral Properties* for $\mathbb{A}_n$ in `section 5`, our aim is to focus on trying to estimate the order of $M(n)$ and even for the *extended Mertens Function* $M(x)$ (2.2.1), a significant application of which is to analyze the location of zeros of the *Riemann Zeta Function* $\zeta(s)$. [`section 6`]

Another important use of order estimates of $M(n)$ is to find counterexamples in support of disproving the statement of the famous *Mertens Hypothesis* [ref. (2.3)]. In `section 7`, we shall discuss several numerical evidence and theoretical justifications in the form of *conjectures* proposed by experts in this field, where our objective is to learn the approach opted by researchers in order to assess the optimal value of n for which the first such violation to the statement of *Mertens Hypothesis* occurs. Although the deductions are far from acceptable, and realistically not completely verifible by rigorous computation, thus it is still considered to be an active area of research. Although almost all the results and notations pertinent to the contents of this paper has been explained properly to the best of one's ability, the paper contains several references, which should be resourceful to any motivated indivual interested in pursuing further studies or maybe research in any of these topics.

Notations and Abbreviations

- $a \mid b$ — b is divisible by a
- $a \nmid b$ — b is not divisible by a
- $\lfloor x \rfloor$ — Greatest Integer less than or equal to x
- $\lceil x \rceil$ — Least Integer greater than or equal to x
- $Re(s)$ — Real part of a complex number $s \in \mathbb{C}$
- RH — Riemann Hypothesis
- MH — Mertens Hypothesis

2. A Survey on Mertens Function

2.1. Notion of Arithmetic Functions

We start with some standard definitions in *Analytic Number Theory*.

Definition 2.1.1. (Arithmetic Function) These are real or complex valued functions defined on the set of natural numbers $\mathbb{N}$.

In this section, we shall define some specific examples of arithmetic functions pertinent to the concept of *Mertens Function*.

Definition 2.1.2. (Mӧbius Function) The Mӧbius Function $\mu : \mathbb{N} \to \{0, \pm 1\}$ is defined as follows:

$$\mu(n) := \begin{cases} (-1)^k & \text{if } n = \prod_{i=1}^{k} {p_i}^{a_i} \text{ such that, } gcd(p_i, p_j) = 1 \quad \forall \quad i \neq j \\ 1 & \text{if, } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

One can in fact use *definition* (2.1.2) to deduce the following property regarding the Mӧbius Function.

Theorem 2.1.1.

$$\sum_{d|n} \mu(d) = \left\lfloor \frac{1}{n} \right\rfloor = \begin{cases} 1 & \textit{if, } n = 1 \\ 0 & \textit{otherwise.} \end{cases} \tag{2.1}$$

With all the neccessary analytic tools already discussed in this section, we can formally introduce the notion of a specific arithemtic function proposed by *F. Mertens* in his paper [9].

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2.2. *Formal Definition of $M(n)$*

Definition 2.2.1. (Mertens Function) The Mertens Function $M : \mathbb{N} \to \mathbb{Z}$ is defined as,

$$M(n) := \sum_{d=1}^{n} \mu(d) \tag{2.2}$$

Remark 2.2.1. In general, we can define the *Extended Mertens Function* as,

$$M(x) := \sum_{1 \le n \le x} \mu(n)\,, \qquad \forall\, x \in \mathbb{R}$$

In his paper [9], *Mertens* conjectured that, for all $M(n)$ with $1 \le n \le 10^4$, we shall have,

$$|M(n)| < \sqrt{n} \tag{2.3}$$

This is also known as the *Mertens Hypothesis*. [Interested readers can refer to [2, Theorem 14.28, pp. 374]]

Extending *Mertens'* results further upto $n = 5 \times 10^6$, *Sterneck* [10] conjectured that,

$$|M(n)| < \tfrac{1}{2}\sqrt{n}\,, \qquad \forall\, n > 200\,.$$

The primary objective for *Mertens* behind introducing the function $M(x)$ (As defined in (2.2.1)) was its underlying relation to the location of the zeros of the *Riemann Zeta Function* $\zeta(s)$, the reason being largely due to it's consequences for the distribution of the primes, also hailed as one of the most important unsolved problems in Analytic Number Theory.

3. The Riemann Hypothesis Revisited

In this section, we introduce our readers to the famous *Riemann Zeta Function*, first introduced by *Leonhard Euler*, later extended by *Bernhard Riemann*, which is the integral part of the *Riemann Hypothesis*, and also has applications in the field of Modular Forms, Class Field Theory etc.

3.1. *Riemann Zeta Function $\zeta(s)$*

Definition 3.1.1. (Riemann Zeta Function) The *Riemann Zeta Function* $\zeta(s)$ is defined as,

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{where } s \in \mathbb{C},\ Re(s) > 1. \tag{3.1}$$

We can, in fact extend $\zeta(s)$ to the whole complex plane using the *Analytic Continuation Property* of this function defined for $Re(s) > 1$. Interested readers may refer to the book written by *L. Ahlfors* [6].

Using the *Euler Product Formula*, we can pursue an alternative approach to define $\zeta(s)$ as,

Definition 3.1.2. *Riemann Zeta Function* has the following *Euler Product Representation*,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p=primes} \left\{ \frac{1}{1 - p^{-s}} \right\} \tag{3.2}$$

Where, the product on the R.H.S. is taken over all primes p, and converges for $Re(s) > 1$.

3.2. *Analytic Properties of $\zeta(s)$*

We can indeed summarize some of the analytic properties of $\zeta(s)$ in a certain way as mentioned below.

Proposition 3.2.1. *The following are true for $\zeta(s)$:-*

- *$\zeta(s)$ is meromorphic on $\mathbb{C}$.*

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- $\zeta(s)$ *has a simple pole at* $s = 1$ *with residue* 1.
- $\zeta(1)$ *is a Harmonic Series that diverges to* $+\infty$.
- $\zeta(2) = \frac{\pi^2}{6}$. *(Also known as the "Basel Problem")*

Using *Definition* (2.1.2) and expanding the *R.H.S.* in *Definition* (3.1.2) of $\zeta(s)$, we can conclude,

Proposition 3.2.2.

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \tag{3.3}$$

The ***Functional Equation*** of $\zeta(s)$ further allows us to assert that, the ***trivial zeros*** of $\zeta(s)$ occurs at *negative even integers*, i.e., at $s = -2k, \forall\, k \in \mathbb{N}$.

Also, *Riemann* conjectured in his paper [5] that, all the ***non-trivial zeros*** of $\zeta(s)$ must lie on the ***critical strip***, $\{s \in \mathbb{C} \,:\, 0 < Re(s) < 1\}$, moreover, they are symmetrically located about the real axis and the *critical line*, $Re(s) = \frac{1}{2}$.

Theorem 3.2.3. *(Riemann Hypothesis) All the non-trivial zeroes of the Riemann Zeta Function* $\zeta(s)$ *lie on the critical line,* $Re(s) = \frac{1}{2}$.

RH still remains one of Number Theory's most elusive unsolved problems (also well-known as Problem 8 in *Hilbert's list* of 23 unsolved problems, and also known as one of the *Millenium Prize Problems*), and it is still open till date.

However, one of the most important consequences of RH in Number Theory is it's connection to the *prime counting function* $\pi(x)$.

Definition 3.2.1. For each $x \geq 0$, we define,

$$\pi(x) := \text{The number of primes} \leq x\ .$$

In particular, it has been established by *von Koch* [11] in 1901 that, RH is equivalent to,

$$\pi(x) = Li(x) + O(\sqrt{x}\ln x)\ .$$

To be more specific, *Schoenfield* [12] proved in 1976 that, under RH,

$$|\pi(x) - Li(x)| \leq \frac{1}{8\pi}\sqrt{x}\ln x\,, \qquad \forall\, x \geq 2657\ .$$

where, we define,

$$Li(x) := \int_2^x \frac{1}{\ln t}dt\ .$$

It is important to observe that, we already have an explicit formula for $\pi(x)$ and if we observe closely the structural similarity of this equation to the function $Li(x)$, thus approaching Riemann hypothesis using concepts of Linear Algebra, especially the notion of *Redheffer Matrices* (which we shall discuss in the next section) seems to be a promising direction.

4. Redheffer Matrices: An Introduction

4.1. Definition

Definition 4.1.1. (Redheffer Matrix) Often denoted by $\mathbb{A}_n$, *Redheffer Matrix* was first introduced by mathematician *Redheffer* (1977), and is defined to be a square $(0,1)$ matrix of the form, $\mathbb{A}_n := (a_{ij})_{n\times n}$ such that,

$$a_{ij} := \begin{cases} 1 & \text{if, } i \mid j \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

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for every $1 \le i, j \le n$.

A priori given the *invertibility* of the Redheffer Matrices being relatively complicated due to the existence of the initial column of 1's in the matrix, $\mathbb{A}_n$ is often expressed as,

$$\mathbb{A}_n := C_n + D_n$$

where, $C_n := (c_{ij})_{n\times n}$ and, $D_n := (d_{ij})_{n\times n}$ are the $(0,1)$ matrices defined as,

$$c_{ij} := \begin{cases} 1 & \text{iff, } i \neq 1, j = 1. \\ \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

$$d_{ij} := \begin{cases} 1 & \text{iff, } i \mid j \\ \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

for every $i = 1, 2, 3, ..., n$ and, $j = 1, 2, 3, ..., n$.

For example, a 4×4 *Redheffer Matrix* can be represented in the following manner:

$$\mathbb{A}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here, one can observe that,

$$C_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

And,

$$D_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4.2. Relation with Mertens Function $M(n)$

Theorem 4.2.1. *(Determinant of $\mathbb{A}_n$) We have, for the Redheffer Matrix $\mathbb{A}_n$,*

$$det(\mathbb{A}_n) = M(n). \tag{4.4}$$

Proof. Proof uses the following result.

Lemma 4.2.2. *The inverse of the matrix D_n defined in (4.3) is defined as, ${D_n}^{-1} = (\delta_{ij})_n$, where,*

$$\delta_{ij} = \begin{cases} \mu\left(\frac{j}{i}\right) & \text{if, } i \mid j \\ \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

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Using *Lemma* (4.2.2), we obtain, $det({D_n}^{-1}) = 1$. Hence, $det(\mathbb{A}_n) = det({D_n}^{-1}\mathbb{A}_n) = det({D_n}^{-1}C_n + I_n)$.
Hence, ${D_n}^{-1}C_n$ is *lower triangular*, so is $\left({D_n}^{-1}C_n + I_n\right)$, which implies that, $det\left({D_n}^{-1}C_n + I_n\right)$ is the product of its diagonal entries.

Consequently,

$$det\left({D_n}^{-1}C_n + I_n\right) = \left\{\sum_{k=2}^{n} \mu(k) + 1\right\} = M(n)\,.$$

□

4.3. Characteristic Polynomial of $\mathbb{A}_n$

We start with stating a formal definition of a *Directed Graph* .

Definition 4.3.1. (Directed Graph) A directed graph, also called a digraph, is a graph in which each of the edges has a direction.

One method to derive the *characteristic polynomial* of $\mathbb{A}_n$ is to use the concept of *directed graph* $\mathcal{D}(M_n)$ corresponding to the matrix, $M_n := xI_n - \mathbb{A}_n$.

By definition (4.3.1), $\mathcal{D}(M_n)$ has vertex set, $\Lambda = \{1, 2, ..., n\}$ and a *directed edge* from vertex i to vertex j *iff*, $m_{ij} \neq 0$. In other words, $\mathcal{D}(M_n)$ will have a directed edge from i to j *iff*, $i \mid j$ or, $j = 1$. We denote it by the ordered pair (i, j).

Definition 4.3.2. (Cycle) A set of directed edges $\{(i_1, j_1), (i_2, j_2), ..., (i_k, j_k)\}$ is called a cycle of length k if, $j_r = i_{r+1}$, $\forall\, r = 1(1)k - 1$, and $j_k = i_1$, and $i_1, i_2, ..., i_k$ are distinct.

Moreover, suppose $\mathfrak{c} = \{(i_1, i_2), (i_2, i_3), ..., (i_k, i_1)\}$ is a *cycle* in $\mathcal{D}(M_n)$.

We set, $\mathcal{P}[\mathfrak{c}] = m_{i_1 i_2} m_{i_2 i_3} .....m_{i_k i_1}$. Also, for any $\mathcal{I} \subseteq \Lambda$, we define $M_n[\mathcal{I}]$ as the *Principal Submatrix* of M_n with rows and columns in $\mathcal{I}$. Assume $\mathcal{C}_1$ to be the set of *cycles* in $\mathcal{D}(M_n)$ containg the vertex 1, and for every $\mathfrak{c} \in \mathcal{C}_1$, denote its *length* by $l(\mathfrak{c})$. Also, suppose $\mathfrak{c}'$ be the *complement* of the vertices connected by $\mathfrak{c}$ in Λ.

Using ([16] Theorem 2, pp. 503), we can assert that,

$$det(M_n) = \sum_{\mathfrak{c} \in \mathcal{C}_1} (-1)^{l(\mathfrak{c})+1}.\mathcal{P}[\mathfrak{c}].det(M_n[\mathfrak{c}']). \tag{4.6}$$

We can further claim that, the principal submatrix $M_n[\mathfrak{c}']$ is *upper-triangular* [a priori since, each $\mathfrak{c} \in \mathcal{C}_1$ contains the vertex 1] having determinant equal to $(x-1)^{n-l(\mathfrak{c})}$. Moreover, for $l(\mathfrak{c}) \geq 2$, $\mathcal{P}[\mathfrak{c}]$ doesn't contain any diagonal entry and precisely is equal to $(-1)^{l(\mathfrak{c})}$. Since, the only cycle of *length* 1 is, $\mathfrak{c} = \{(1, 1)\}$, hence, $\mathcal{P}[\mathfrak{c}] = x - 1$.

Therefore, we obtain the *characteristic equation* of $\mathbb{A}_n$ as,

$$det(M_n) = (x-1)^n - \sum_{\substack{\mathfrak{c} \in \mathcal{C}_1 \\ l(\mathfrak{c}) \geq 2}} (x-1)^{n-l(\mathfrak{c})} \tag{4.7}$$

which can be rewritten as,

$$det(M_n) = (x-1)^n - \sum_{k=2}^{n} (x-1)^{n-k} v_{n,k-1}. \tag{4.8}$$

where, $v_{n,k}$ denotes the number of *cycles* starting at vertex 1, of length $k+1$, in the graph $\mathcal{D}(M_n)$. In other words, if a *k-list* in Λ is a list $\{1, i_1, i_2, ..., i_k\}$ with $i_r \mid i_{r+1}$, $\forall\, r = 1(1)k - 1$ and $1 < i_1 < i_2 < ..... < i_k \leq n$, then, $v_{n,k}$ shall denote the number of k-lists in Λ.

Remark 4.3.1. Similar computation for a particular case $\mathbb{A}_6$ can be found in [17, pp. 676].

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***Remark* 4.3.2.** It has been deduced that,

$$v_{n,1} = n - 1\,, \quad \text{and,} \quad v_{n,2} = \sum_{k=2}^{n} \left(\lfloor \tfrac{n}{k} \rfloor - 1 \right).$$

***Remark* 4.3.3.** Also, we can conclude that, each $v_{n,k}$ is positive and bounded above, as evident from the relation,

$$v_{n,k} < n.\frac{(\log n)^{k-1}}{(k-1)!} \tag{4.9}$$

4.4. Eigenvalues of $\mathbb{A}_n$

Theorem 4.4.1. *The Redheffer Matrix $\mathbb{A}_n$ has eigenvalue 1 with multiplicity $n - \lfloor \log_2 n \rfloor - 1$.*

Proof. Here, instead of computing the eigenvalues using the characteristic polynomial of $\mathbb{A}_n$ given by,

$$p_n(x) = det(xI_n - \mathbb{A}_n)$$

We work with the following polynomial,

$$q_n(x) = det(xI_n + (\mathbb{A}_n - I_n))$$

Observation: $p_n(x)$ and $q_n(x)$ are very closely related polynomials.

Thus using properties of determinants, we obtain,

$$p_n(x) = (-1)^n det\{(1-x)I_n + (\mathbb{A}_n - I_n)\} = (-1)^n q_n(1-x)\,.$$

Implying that, each eigenvalue α_j of $\mathbb{A}_n$ can be expressed as $(1 - \alpha'_j)$, where, α'_j is a root of $q_n(x)$, $\forall$ $j = 1, 2, 3, ..., n$.

Hence it only suffices to compute roots of $q_n(x)$ in order to deduce the eigenvalues of $\mathbb{A}_n$.

Using properties of *directed graphs*, we can associate a directed graph with any given square matrix, say, $B := (b_{ij})_{n \times n}$, all of whose entries are either 0 or 1, by assigning a vertex to each row (or column equivalently) and including an edge between the i^{th} and the j^{th} vertices iff, $b_{ij} = 1, \forall\, i, j = 1, 2, 3, ..., n$.

Therefore, corresponding to the matrix $(\mathbb{A}_n - I_n)$, we can assert from (4.4),

$$q_n(x) = x^n + \sum_{k=1}^{n} (-1)^{k-1} c(n,k) x^{n-k}$$

Where, $c(n,k)$ denotes the cycles of length k in the directed graph corresponding to the matrix $(\mathbb{A}_n - I_n)$. Now, for the proof of *Theorem* (4.4.1), observe that, in the directed graph of $(\mathbb{A}_n - I_n)$, the longest cycle is,

$$1 \to 2 \to 2^2 \to .... \to 2^{\lfloor \log_2 n \rfloor} \to 1\,.$$

(Since, each entry in the cycle must be a non-trivial multiple of the previous one, and multiplying by 2 each time increases the cycle's elements as slowly as possible .)

The cycle contains $\lfloor \log_2 n \rfloor + 1$ elements, which yields, $c(n,k) = 0$ for $k > \lfloor \log_2 n \rfloor + 1$.

The lowest power of x occuring in $q_n(x)$ is, $x^{n - \lfloor \log_2 n \rfloor - 1}$. Thus, q_n has a root 0 with multiplicity $n - \lfloor \log_2 n \rfloor - 1$, implying that, p_n has a root 1 with multiplicity $n - \lfloor \log_2 n \rfloor - 1$.

□

With this information, we can make further deductions about the spectral radius of $\mathbb{A}_n$.

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4.5. Spectral Radius of $\mathbb{A}_n$

Definition 4.5.1. (Spectral Radius) Spectral Radius of a matrix is defined to be the maximum absolute value of its eigenvalues.

Remark 4.5.1. $\mathbb{A}_n$ has a positive spectral radius.
(A priori given the ***"Perron-Frobenius Theorem"***, we can deduce that, $\mathbb{A}_n$ clearly satisfies the irreducibility criterion mentioned in the theorem, hence, the statement follows.)

Remark 4.5.2. *Barrett*, *Forcade* and *Follington* [1] deduced the upper bounds for the co-efficients of the *characteristic polynomial* of $\mathbb{A}_n$. If ρ_n denotes the *spectral radius* of $\mathbb{A}_n$, then using these bounds, they proved the following asymptotic relation for ρ_n.

Theorem 4.5.3. $\lim\limits_{n\to\infty} \frac{\rho_n}{\sqrt{n}} = 1$.

Barrett proposed several interesting conjectures based on numerical evidence regarding the other eigenvalues of $\mathbb{A}_n$. Interested readers can find the details about this in [13].
One of the significant among them was,

Theorem 4.5.4. *$\mathbb{A}_n$ has a negative (real) eigenvalue of magnitude $-\sqrt{n}$, and the remaining eigenvalues are bounded and are close to the origin.*

In ([15] Theorem 2, pp. 147), the existence of such a negative real eigenvalue has been proven. Furthermore, we can conclude that it is asymptotically equal to $-\sqrt{n}$ as $n \to \infty$, and also that, the remaining eigenvalues of $\mathbb{A}_n$ have magnitude $O\left(\frac{\sqrt{n}}{\log n}\right)$.
Again, given that most of the eigenvalues of $\mathbb{A}_n$ are 1, we can comment on their corresponding eigenspaces.

4.6. Eigenspaces of $\mathbb{A}_n$ corresponding to its Eigenvalues

Theorem 4.6.1. *The eigenspace of $\mathbb{A}_n$ corresponding to the eigenvalue 1 has dimension $\lceil \frac{n}{2} \rceil - 1$. In particular, $\mathbb{A}_n$ is non-diagonalizable for $n \geq 5$.*

Proof. Follows by obtaining the *row-reduced echelon form* of the matrix $(\mathbb{A}_n - I_n)$ and then determining its nullity.
□

A priori from the fact that, the *eigenspaces* of $\mathbb{A}_n$ and $\mathbb{A}_n^T$ are *isomorphic*, we can comment further about the eigenspaces corresponding to the eigenvalues of $\mathbb{A}_n$ that are not equal to 1 .

Theorem 4.6.2. *Every eigenspace of $\mathbb{A}_n$ with eigenvalue not equal to 1 has dimension 1 .*

Proof. As for the proof, we use the following result.

Lemma 4.6.3. *$(w_1, w_2, ..., w_n)$ is an eigenvector of $\mathbb{A}_n^T$ with λ as an eigenvalue iff,*

- $\sum\limits_{d|n} w_d = \lambda w_n$
- $\sum\limits_{j=1}^{n} w_j = \lambda w_1$

Observe that, if $\lambda \neq 1$ be an *eigenvalue* of $\mathbb{A}_n^T$, then, given a value for w_1, we can *uniquely* determine the remaining w_j's using *Lemma* (4.6.3) and the following recursive relation,

$$w_j = \frac{1}{(\lambda - 1)} \sum_{\substack{d|j \\ d \neq j}} w_d$$

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Hence the result follows. □

As a result of the above concept, we can introduce another arithmetic function in the next section.

4.7. The Arithmetic Function "ν_λ"

4.7.1. Definition

Definition 4.7.1. The function $\nu_\lambda : \mathbb{N} \to \mathbb{R}$ is defined for arbitrary $\lambda \neq 1$ using the following recursive formula,

$$\nu_\lambda(n) := \begin{cases} 1 & \text{if, } n = 1 \\ \frac{1}{(\lambda-1)} \sum\limits_{\substack{d|n \\ d \neq n}} \nu_\lambda(d) & \text{otherwise .} \end{cases} \tag{4.10}$$

Remark 4.7.1. *Definition* (4.7.1) allows us to conclude that, λ is an *eigenvalue* of $\mathbb{A}_n$ iff,

$$\sum_{k=1}^{n} \nu_\lambda(k) = \lambda\,. \tag{4.11}$$

4.7.2. Dirichlet Series Representation

Given the definition of ν_λ, we can obtain the following estimate regarding it's corresponding *Dirichlet Series*.

$$V_\lambda(s) = \sum_{n=1}^{\infty} \frac{\nu_\lambda(n)}{n^s} \tag{4.12}$$

As a consequence, we obtain,

Proposition 4.7.2.

$$V_\lambda(s) = \sum_{n=1}^{\infty} \frac{\nu_\lambda(n)}{n^s} = \frac{\lambda-1}{\lambda-\zeta(s)} \tag{4.13}$$

Where, $\zeta(s)$ stands for the Riemann Zeta Function .

Proof. Using *partial summation formula*,

$$V_\lambda(s)\zeta(s) = \sum_{n=1}^{\infty}\left(\sum_{d|n} \nu_\lambda(d)\right)\frac{1}{n^s} = \left(\sum_{n=1}^{\infty} \frac{\lambda \nu_\lambda(n)}{n^s}\right) - \lambda + 1$$

$$[\text{ Since, } \sum_{d|n} \nu_\lambda(d) = \lambda \nu_\lambda(n) \text{ for every } n \in \mathbb{N} \setminus \{1\}\ ]$$

$$= \lambda V_\lambda(s) - \lambda + 1\,.$$

Solving for V_λ gives us the result.
□

Definition 4.7.2. If $a_k(n)$ denotes the number of ways of expressing a natural number n as a product of k factors in some particular order, then, we have the following recursive definition given by,

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$$a_k(n) = \begin{cases} 1 & \text{if, } k = 0, n = 1 \\ 0 & \text{if, } k = 0, n \neq 1 \\ \sum\limits_{d|n} a_{k-1}\left(\frac{n}{d}\right) & \text{if, } k > 0\,. \end{cases}$$

Using *Definition* (4.7.2), we can have another expression for ν_λ.

Theorem 4.7.3. *For any $n \in \mathbb{N}$, we have,*

$$\nu_\lambda(n) = \left(1 - \frac{1}{\lambda}\right) \sum_{k=0}^{\infty} \frac{a_k(n)}{\lambda^k} \tag{4.14}$$

Where, $\lambda > 1$.

Remark 4.7.4. The *Redheffer Matrix* does provide an alternative approach on different notions of divisibility. Being closely related to the *Riemann Zeta Functions*, it effectively bundles together the divisibility properties of many numbers into a single term, which can then be studied using techniques of *Linear Algebra* .

5. Order Estimates for $M(x)$ and its relation to the non-trivial zeros of $\zeta(s)$

As mentioned previously in the `introduction`, among the various statements of the famous *Riemann Hypothesis* proposed by mathematicians over the years which are equialent to each other, only few of them have been approached using concepts of *Linear Algebra* and *Advanced Matrix Theory*.

Mertens [9] introduced the notion of the *extended Mertens Function*, $M(x)$ and suggested a completely different method to analyze the zeros of the location of the *Riemann Zeta Function* $\zeta(s)$. A priori for $Re(s) > 1$, one can observe from *Proposition* (3.2.2),

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{n=1}^{\infty} \frac{M(n) - M(n-1)}{n^s} = \sum_{n=1}^{\infty} \left(\frac{1}{n^s} - \frac{1}{(n++1)^s}\right) = s\int_1^{\infty} \frac{M(x)}{x^{s+1}} dx \tag{5.1}$$

If we assume, $M(x) = O(x^\epsilon)$ for some $\epsilon > 0$, then the integral on the R.H.S. of (5.1) represents an analytic function in the half-plane, $Re(s) > \epsilon$. Thus, $\frac{1}{\zeta(s)}$ shall also be analytic in that region. Consequently, we can in fact conclude, $\zeta(s) \neq 0$, which, in turn will imply [18, Theorem 30, pp. 83],

$$\pi(x) = Li(x) + O(x^\epsilon \log x) \tag{5.2}$$

Clearly the above relation justifies the significance of studying the order of $M(x)$.

Remark 5.0.1. In particular, it seems obvious that, RH follows from the statement of MH [(2.3)], which can be generalized as follows:

$$M(x) = O\left(x^{\frac{1}{2}}\right) \tag{5.3}$$

A slightly more rigorous computation of the statement in *Remark* (5.0.1) yields [2, Theorem 14.25C, pp.370],

Theorem 5.0.2. *A necessary and sufficient condition for the Riemann Hypothesis to be true is,*

$$M(x) = O\left(x^{\frac{1}{2}+\epsilon}\right) \tag{5.4}$$

for every $\epsilon > 0$.

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Remark 5.0.3. Given the *definition* (4.1.1) of $\mathbb{A}_n$ and the underlying relation between the *redheffer matrices* and $M(n)$ [See (4.2.1)], we can relate to the result proved by *Redheffer* [48] that the statement of RH is true *iff*,

$$det(\mathbb{A}_n) = O\left(n^{\frac{1}{2}+\epsilon}\right) \tag{5.5}$$

for every $\epsilon > 0$.

Roesler also defined a special kind of matrix, $\mathbb{B}_n := (b_{ij})_{2\leq i,j\leq n}$ such that,

$$b_{ij} := \begin{cases} i-1 & \text{if, } i \mid j \\ \\ -1 & \text{otherwise.} \end{cases} \tag{5.6}$$

for every $2 \leq i, j \leq n$.

Known as the *Roesler Matrices*, using these matrices, he established that in a similar manner as for *Redheffer Matrices*, RH holds true iff,

$$det(\mathbb{B}_n) = O\left(n!\, n^{-\frac{1}{2}+\epsilon}\right) \tag{5.7}$$

for every $\epsilon > 0$. It is important to observe that, both $\mathbb{A}_n$ and $\mathbb{B}_n$ are extensively related to $M(n)$ via their determinants. It can be checked that, these matrices are not *symmetric*, and one has to compute many of their eigenvalues to estimate their respective determinants.

6. Conjectures based on the Order of $M(x)$

This section provides a survey on various important numerical estimates on the bounds of $M(x)$, and *conjectures* proposed by pioneers in this field of research during a timeline spanning over more than half a century. Significant to observe that, many of these theories were enough to refute the validity of *Mertens Hypothesis* (2.3).

Neubauer deduced four *isolated* value of $M(x)$ in his paper [20] for which *von Sterneck*'s conjecture (2.2) doesn't hold true. One of the smallest such values of n is, $M(7760000000) = 47465$.

Later on, *Cohen* and *Dress* further claimed and proved [21] that, the first violation of *von Sterneck*'s conjecture in the *positive direction* is, $M(7725038629) = 43947$, whereas, credit for deriving the first violation of the theorem in the *negative direction* belongs to *Dress* [22], as he showed, $M(330486258610) = -287440$.

Towards the end of the 20^{th} century, *Odlyzko* and *te Riele* [23] astonishingly came up with the following bounds which which was sufficient to conclude tha, the statemnt of MH is indeed *false* along both the directions.

$$\liminf_{x\to\infty} \frac{|M(x)|}{\sqrt{x}} < -1.009$$

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x}} > 1.06$$

Although, *Pintz* established in his paper [24], that the violation occurs for some, $x \lesssim e^{3.21\times10^{64}} \simeq 10^{1.4\times10^{64}}$. As for further research in this area, interested readers can refer to the works of *Jurkat* [25], [26], *Spira* [27], *Jurkat* and *Peyerimhoff* [28], *te Riele* [29], *Möller* [30] and *Anderson* [31].

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Many claim that, a priori assuming RH to be true, (5.4) is satisfied, although the statement of the *generalized Mertens Hypothesis* (5.3) is indeed false. One of the conjectures which supports this claim is as follows.

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x \log\log x}} = C \tag{6.1}$$

where one can obtain estimates for C in the form of, $C = \frac{6\sqrt{2}}{\pi^2}$ (*Lévy* in a comment to *Saffari* [49]), and $C = \frac{\sqrt{12}}{\pi}$ (*Good* and *Churchhouse* [32]).

El Marraki [33] did provide one of the strongest unconditional order estimate for $M(x)$. He claimed,

$$M(x) = O\left(\frac{x}{\log^{\frac{236}{75}} x}\right) \tag{6.2}$$

A similar result can be found in *Walfisz*'s article [34],

$$M(x) = O\left(x \exp\left(-A\frac{(\log x)^{\frac{3}{5}}}{(\log\log x)^{\frac{1}{5}}}\right)\right) \tag{6.3}$$

for some costant $A > 0$. One valid estimate for A can be obtained as, $A = 0.2098$. (*Ford* [35])

There have been significant progress in the direction of obtaing estimates for $M(x)$ of the form, $|M(x)| < \frac{x}{K}$, for some constant $K > 0$ and $\forall\, x > x_0$ for some $x_0 \in \mathbb{R}$. Interested readers can read from the papers of *Hackel* [36], *MacLeod* [37], *dress* [38], *Diamond* and *McCurley* [39], *Costa Pereira* [40] and *Dress* and *El Marraki* [41].

Some more article pertinent to this topic can be cited as *Landau* [42,43] , *te Riele* [44,45] and, *Odlyzko* and *te Riele* [46].

Some of the authors even opted for rigorous computation of the function $M(n)$ and the ratio $\frac{M(n)}{\sqrt{n}}$ with high hopes of finding possible counterexample to MH. For refernce, one can study articles by *Mertens* [9], *von Sterneck* [10,50,51], *Neubauer* [20], *Yorinaga* [52], *Cohen* and *Dress* [21], *dress* [22], *Lioen* and *van de Lune* [53].

7. Research Prospects in Mertens Hypothesis

In `section` 7, we have encountered enough theoretical justification and numerical data which clearly suggests that anyhow *Mertens Hypothesis* isn't true. Furthermore, looking at the already derived estimates for $n \in \mathbb{N}$ at which the first violation occurs for MH, one should spontaneously come up with this obvious question,

Is it justifiable to study the Mertens Hypothesis, and if so, why ?

As ludacris as it may sound, the appropriate answer to this question is, **YES**. Even though MH have been disproved convincingly, the everlasting desire among many mathematicians in this era to pursue research in this area using different techniques is truly commendable. One of the most important among all the problems in this field of study is to estimate the least possible value of n for which first violation of MH occurs.

Recall that, *Pintz* provided a proper bound for n satisfying such condition as, $n < 10^{1.4\times10^{64}}$. An extensive study of the conjectures of *Good* and *Churchhouse*, and also that of *Lévy*, enables us to conclude, $n \simeq 10$ and, $n \simeq 48$ respectively. Naturally, these bounds are not preferable, as we can obtain by hands on computation, $n > 10^{14}$.

In their article [46], *Odlyzko* and *te Riele* claimed that, the first violation for MH won't occur for $n < 10^{20}$, and probably not even for $n < 10^{30}$. We recall the following theorem from *Titchmarsh* ([2] Theorem 14.27, pp. 372).

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Theorem 7.0.1. *A priori assuming RH, suppose we denote the zeros of $\zeta(s)$ on the critical line, $Re(s) = \frac{1}{2}$ by, $\rho = \frac{1}{2} + it$. Further consider all the zeros to be simple. Then, $\exists$ a sequence, T_k, satisfying, $k \le T_k \le k+1$, such that,*

$$M(x) = 2 \lim_{k\to\infty} \sum_{0<t<T_k} Re\left(\frac{x^{\rho}}{\rho\zeta'(\rho)}\right) + O(1) \tag{7.1}$$

Kotnik and *van de Lune* [47] illustrated an experimentdepending on the evaluation of *partial sums* of the series on the R.H.S. of (7.1). Thus, eventually, we can obtain,

$$\limsup_{x\to\infty} \frac{|M(x)|}{\sqrt{x \log\log\log x}} = C$$

where, a valid estimate for C can be derived as, $C \approx 0.5$. As a consequence, we deduce the following estimate for n as, $n \simeq 10^{2.3\times10^{23}}$ for the occurance of the first violation of MH.

Remark 7.0.2. This approximation for n is significantly better than the bound derived by *Pintz*, although being too large makes it difficult for mathematicians to verify all the details by direct computation. It is the very reason why people are still motivated as ever to investigate such a violation of MH by computing $M(n)$ consistently.

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