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Proof of the Collatz Conjecture

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Abstract

The Collatz conjecture (or 3n+1 problem) has been explored for about 85 years. In this article, we prove the Collatz conjecture. We will show that this conjecture holds for all positive integers by performing the Collatz inverse operation on the numbers that satisfy the rules of the Collatz conjecture. Finally, we will prove that there are no positive integers that do not satisfy this conjecture.

Keywords: Collatz operation, Collatz inverse operation and Collatz numbers.

Mathematics Subject Classification (2020). 11A25, 11A51, 11B50, 11B75

1 Introduction

The Collatz conjecture is one of the unsolved problems in mathematics.Introduced by German mathematician Lothar Collatz in 1937 [1], it is also known as the 3n + 1 problem, 3x + 1 mapping, Ulam conjecture (Stanislaw Ulam), Kakutani's problem (Shizuo Kakutani), Thwaites conjecture (Sir Bryan Thwaites), Hasse's algorithm (Helmut Hasse), or Syracuse problem [2-4].

The Collatz Conjecture or 3n+1 problem can be summarized as follows:

Take any positive integer n. If n is even, divide n by 2. If n is odd, multiply n by 3 and add 1. Repeat this process continuously. The conjecture states that no matter which number you start with, you will always reach 1 eventually.

For example, if we start with 17, multiply by 3 and add 1, we get 52. If we divide 52 by 2, 26, and so on, the rest of the sequence is: 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.Or if we start 76, the sequence is: 76, 38, 19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.

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This sequence of numbers involved is sometimes referred to as the hailstone sequence, hailstone numbers or hailstone numerals (because the values are usually subject to multiple descents and ascents like hailstones in a cloud) [2,6], or as wondrous numbers [2,5].

2 The Conjecture and Related Conversions

Definition 2.1. $\mathbb{N}^+$ is the set of all positive integers, $n \in \mathbb{N}^+$, Collatz defined the following map:

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if n is even} \\ 3n+1, & \text{if n is odd} \end{cases}$$

The Collatz conjecture states that the orbit formed by iterating the value of each positive integer under f function in f will eventually reach 1. In the following sections, we will refer to the function f as Collatz operations (CO).

Remark 2.2 According to the definition of the Collatz conjecture, if the number we choose at the beginning is an even number,then by continuing to divide all even numbers by 2, one of the odd numbers is achieved. So it is sufficient to check whether all odd numbers reach 1 by the Collatz operation.

Therefore, if we prove that it reaches 1 when we apply the Collatz operation to all the elements of the set $\mathbb{N}_{odd} = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25 \ldots\}$, we have proved it for all positive integers. (The notation of $\mathbb{N}_{odd}$ is positive odd integers.)

Remark 2.3. If the Collatz operation is applied to the numbers 2^n ($n \in \mathbb{N}^+$), then eventually 1 is reached. So when we apply the Collatz operation to all the elements of the set $\mathbb{N}_{odd}$, if we can convert them to 2^n numbers, we reach the result (the notation of $\mathbb{N}^+$ is positive integers).

2.1 Collatz Inverse Operation (CIO)

Let $n \in \mathbb{N}^+, a \in \mathbb{N}_{odd}$; for a to be converted to 2^n by the Collatz operation (CO), it must satisfy the following equation,

$$3.a + 1 = 2^n$$

then,

$$a = \frac{2^n - 1}{3} \tag{1}$$

when we apply the Collatz operations to the positive odd integers a in (1), we always get 1. This is because equation (1) is obtained by the Collatz inverse operation.But,

Lemma 2.4. In (1) $a = \frac{2^n - 1}{3}$, a cannot be an integer if n is a positive odd integer.

Proof. If n is a positive odd integer, we can take $n = 2m + 1$ $(m \in \mathbb{N})$, then substituting $2m + 1$ for n in (1) we get,

$$a = \frac{2^{2m+1} - 1}{3} \tag{2}$$

if we factorize $2^{2m+1} + 1$,

$2^{2m+1} + 1 = (2 + 1)(2^{2m} - 2^{2m-1} + 2^{2m-2} - \ldots + 1) = \mathbf{3}.k$
multiple of 3 is obtained $(k \in \mathbb{N}_{odd})$.

Since $2^{2m+1} - 1 = (2^{2m+1} + 1) - 2 = 3.k - 2$,

$3k - 2$ is not a multiple of 3 $(k \in \mathbb{N}_{odd})$, so a is not an integer for all n.

If we substitute $2n$ for n in (1), we get equation

$$a = \frac{2^{2n} - 1}{3} \tag{3}$$

Lemma 2.5. In (3) $a = \frac{2^{2n}-1}{3}$, we can find positive odd integers a for all n numbers $(n \in \mathbb{N}^+)$.

Proof. Factorization of $2^{2n} - 1$ for all n, $(n \in \mathbb{N}^+)$, if

$n = 1, \quad (2^2 - 1) = (2 - 1)(2 + 1) = \mathbf{3}.1$
$n = 2, \quad (2^4 - 1) = (2 - 1)(2 + 1)(2^2 + 1) = \mathbf{3}.5$
$n = 3, \quad (2^6 - 1) = (2^3 - 1)(2^3 + 1) = \mathbf{3.3.7}$
$n = 4, \quad (2^8 - 1) = (2 - 1)(2 + 1)(2^2 + 1)(2^4 + 1) = \mathbf{3}.(\ldots)$
$n = 5, \quad (2^{10} - 1) = (2^5 - 1)(2^5 + 1) = (2 - 1)(2^4 + \ldots + 1)(2 + 1)(2^4 - \ldots + 1) = \mathbf{3}.(\ldots)$
$n = 6, \quad (2^{12} - 1) = (2^3 - 1)(2^3 + 1)(2^6 + 1) = \mathbf{3.3}.(\ldots)$
$n = 7, \quad (2^{14} - 1) = (2^7 - 1)(2 + 1)(2^6 - 2^5 + \ldots + 1) = \mathbf{3}.(\ldots)$
$\ldots$
$\ldots$

if we substitute $2n$ to 2^m, (m $\in \mathbb{N}^+$);
$(2^{2^m}-1)=(2-1)(2+1)(2^2+1)(2^4+1)(2^8+1)(2^{16}+1)\ldots\ldots(2^{2^{m-1}}+1)=\mathbf{3}.(\ldots)$
$\ldots$
$(2^{2n}-1)=(2^{x_1}-1)(2^{x_1}+1)(2^{x_2}+1)(2^{x_3}+1)\ldots(2^{x_{n-1}}+1)(2^{x_n}+1)$ or
$(2^{2n}-1)=(2^{x_1}-1)(2^{x_1}+1)$ in these equations, x_1 is a positive odd integer and x_2, x_3, x_4 $\ldots$ x_n are positive even integers. Since x_1 is a positive odd number,
$(2^{x_1}+1)=(2+1)(2^{x_1-1}-2^{x_1-2}+2^{x_1-3}-\ldots+1)=\mathbf{3}.(\ldots)$ so,
$(2^{2n}-1)=\mathbf{3}.(\ldots)$

Since each of these infinite numbers has a multiplier of 3, we can find infinitely many positive odd integers a for all n, and when the Collatz operation is applied to these numbers, we always get 1. $2^{2n}+1$ is not a multiple of 3, since $2^{2n}-1$ is a multiple of 3, for all n ($n \in \mathbb{N}^+$). In (3),

$$a=\frac{2^{2n}-1}{3};$$

If $n=1$, $a_1=1$
$n=2$, $a_2=5$
$n=3$, $a_3=21=3.7$
$n=4$, $a_4=85$
$n=5$, $a_5=341$
$\ldots$ $\ldots$

$$\begin{array}{cccccccccc} 2^2 & 2^4 & 2^6 & 2^8 & 2^{10} & 2^{12} & 2^{14} & 2^{16} & 2^{18} & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \text{A}=\{\ 1, & 5, & 21, & 85, & 341, & 1365, & 5461, & 21845, & 87381 & \ldots\ \} \end{array}$$

Corollary 2.6. We get a set A with infinite elements,these numbers reach 1 when we apply the Collatz operation. (In the following sections,we will refer to the elements of the set A and other numbers that satisfy the Collatz conjecture as Collatz numbers.)

Example 2.7. 5 $\rightarrow$ odd number, 16 $\rightarrow$8$\rightarrow$4$\rightarrow$2$\rightarrow$1.
21$\rightarrow$ odd number, 64$\rightarrow$32$\rightarrow$16$\rightarrow$8$\rightarrow$4$\rightarrow$2$\rightarrow$1.

If we can generalize the elements of the set $A=\{1,5,21,85,341,1365,5461,21845,87381\ldots\}$ to all positive odd numbers, the Collatz conjecture is proved.

2.2 Transformations in the Set A with Infinite Elements

Let the elements of the set $A=\{1,5,21,85,341,1365,5461,21845,87381\ldots\}$ be $\{a_0,a_1,a_2,a_3,a_4,$ $a_5,a_6,a_7,\ldots\}$ respectively.

Lemma 2.8. In the set $A\backslash\{a_0\}$, if $a_n \equiv 1 \pmod 3$

$$b_n = \frac{2^{2m}.a_n - 1}{3} \tag{4}$$

$m \in \mathbb{N}^+$, from each a_n we get B_n sets with infinite b_n elements (Collatz numbers), these numbers satisfy the conjecture. And then, from each b_n we get C_n sets (with infinite elements), from each c_n we get D_n sets (with infinite elements) ... and so on forever.

Proof. If $a_n \equiv 1 \pmod 3$, we can take a_n as $3.p + 1$, $(p \in \mathbb{N})$
$a_n = 3.p + 1$ substituting in (4),

$$b_n = \frac{2^{2m}.(3.p + 1) - 1}{3} = \frac{2^{2m}3p + 2^{2m} - 1}{3} = 2^{2m}p + \frac{2^{2m} - 1}{3}$$

$2^{2m} - 1$ is divisible by 3 (Lemma 2.5.). So we get an infinite number of different b_n elements, which can be converted to a_n, i.e. 1, by the Collatz operation. The b_n elements are Collatz numbers.

Example 2.9. Let $a_1 = 85$, $a_1 \equiv 1 \pmod 3$,

in (4), $a_1 = 85 \rightarrow\ b_1 = \frac{2^2.85-1}{3} = 113, b_2 = \frac{2^4.85-1}{3} = 453, b_3 = \frac{2^6.85-1}{3} = 1813$

$b_4 = \frac{2^8.85-1}{3} = 7253, b_5 = \frac{2^{10}-1}{3} = 29013, b_6 = \frac{2^{12}.85-1}{3} = 116053$

$B = \{113, 453, 1813, 7253, 29013, 116053...\}$

Lemma 2.10. In the set $A\backslash\{a_0\}$, if $a_n \equiv 2 \pmod 3$,

$$b_n = \frac{2^{2m-1}.a_n - 1}{3} \tag{5}$$

$m \in \mathbb{N}^+$, from each a_n we get infinite sets with infinite b_n elements, that satisfy the Collatz conjecture.

Proof. If $a_n \equiv 2 \pmod 3$, we can take a_n as $3.p + 2$. $(p \in \mathbb{N})$
$a_n = 3.p + 2$ substituting in (5),

$$b_n = \frac{2^{2m-1}.(3p + 2) - 1}{3} = \frac{2^{2m-1}.3p + 2^{2m} - 1}{3} = 2^{2m-1}p + \frac{2^{2m} - 1}{3}$$

$2^{2m} - 1$ is divisible by 3 (Lemma 2.5). So we get an infinite number of different b_n elements, which can be converted to a_n, i.e. 1, by the Collatz operation. The b_n elements are Collatz numbers.

Example 2.11. Let $a_1 = 5$ and $a_1 \equiv 2 \pmod 3$;

$a_1 = 5 \to \ b_1 = \frac{2^1.5-1}{3} = 3,\ b_2 = \frac{2^3.5-1}{3} = 13,\ b_3 = \frac{2^5.5-1}{3} = 53$

$b_4 = \frac{2^7.5-1}{3} = 213,\ b_5 = \frac{2^9.5-1}{3} = 853,\ b_6 = \frac{2^{11}.5-1}{3} = 3413\ldots$

$B = \{3, 13, 53, 213, 853, 3413, 13653, 54613\ldots\}$

Lemma 2.12. In the set A$\setminus\{a_0\}$, if $a_n \equiv 0 \pmod 3$,

$$b_n = \frac{2^m.a_n - 1}{3} \tag{6}$$

$m \in \mathbb{N}^+$, there is no such integer b_n.

Proof . If $a_n \equiv 0 \pmod 3$, we can take a_n as $3.p$ ($p \in \mathbb{N}$)

$a_n = 3.p$ substituting in (6),

$$b_n = \frac{2^m(3.p) - 1}{3} = \frac{2^m 3.p - 1}{3} = 2^m.p - \frac{1}{3},$$

is not integer.

We can apply the Collatz inverse operation again to each element of the set B_n (which we obtained above in Lemma 2.8 and Lemma 2.10), and we get new Collatz numbers.

2.3 Converting the Collatz Numbers to all Positive Odd Integers

In the previous sections, when we applied the Collatz operation, we had called the numbers reaching 1 as the Collatz numbers. Now let's see how these Collatz numbers can be converted to all positive integers.

$$\begin{array}{cccccccccc} 2^2 & 2^4 & 2^6 & 2^8 & 2^{10} & 2^{12} & 2^{14} & 2^{16} & 2^{18} & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \end{array}$$

A= { 1, 5, 21, 85, 341, 1365, 5461, 21845, 87381 ... } (Collatz Numbers)

If we apply the Collatz inverse operation (Lemma 2.8 and Lemma 2.10) continuously to each Collatz number, we get infinitely many new Collatz numbers.

Example 2.13. A small fraction of the Collatz numbers that convert to 2^4 by applying CO, i.e., to 1.These numbers are obtained by applying the CIO to each number.

$\dots$ $\dots$ $\dots$ $\dots$ $\dots$ $\dots$
46421 2389 4949 1077 34581 69173
11605 597 1237 269 8645 17293
2901 149 309 67 2161 4323
725 37 77 $\uparrow$ $\uparrow$ $\uparrow$
181, 9, 19 $\rightarrow$ 25, 101, 405, 1621, 6485, $\dots$
5 45, $\uparrow$ $\uparrow$
$\downarrow$ 11 $\rightarrow$ 7, 29, 117, 469, 1877, 7509, 30037, $\dots$
3 $\uparrow$
13$\rightarrow$ 17, 69, 277, 1109, 4437, 17749, 70997, $\dots$
53 $\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$
213 151 739 23665 47331
853 605 2957 94661 189325
3413 2421 11829 378645 757301
13653 9685 47317 1514581 3029205
54613 38741 189269 6058325 12116821
218453 154965 757073 24233331 48467285
$\dots$ $\dots$ $\dots$ $\dots$ $\dots$

Example 2.14. Collatz numbers are converted to 2^6, i.e. 1, by applying CO.
$2^6 \rightarrow 21$ (There are no other Collatz numbers. Lemma 2.12)

Lemma 2.15. There is only one different Collatz number, which is converted to the numbers 2^{6n}; $(2^6, 2^{12}, 2^{18}, 2^{24} \dots)$ $(n \in \mathbb{N}^+)$.

Proof. Factorization of 2^{6n} - 1,

$2^{6n} - 1 = (2^{3n} - 1)(2^{3n} + 1)$, in the equation of 2^{6n}- 1, there is always a multiplier of $(2^3 + 1)$, because when we factorize $(2^{3n} - 1)$ and $(2^{3n} + 1)$;
if n is even, $(2^{3n} - 1) = \dots \ (2^{3f} + 1)$, $3f$ is odd integer.
if n is odd, in $(2^{3n} + 1)$, $3n$ is odd integer.
And if $3f$, $3n$ are odd,
$2^{3n} + 1 = (2^3 + 1)(2^{3n-3} - 2^{3n-6} + 2^{3n-9} - 2^{3n-12} + 2^{3n-15} \dots + 1)$
$2^{3f} + 1 = (2^3 + 1)(2^{3f-3} - 2^{3f-6} + 2^{3f-9} - 2^{3f-12} + 2^{3f-15} \dots + 1)$
Therefore $2^{6n} - 1 = (2^3 + 1).(\dots) = 9.$(odd integer)

And, when we divide $(2^{6n} - 1)$ by 3, only one Collatz number is obtained. We can't obtain another Collatz number because it is a multiple of 3.(Lemma 2.12)

Example 2.16. There is only one different Collatz number for each of 2^{6n}, because the resulting Collatz numbers are the multiples of 3 (Lemma 2.12).

$2^{6}\rightarrow 21$

$2^{12}\rightarrow 1365$

$2^{18}\rightarrow 87381$

$2^{24}\rightarrow 5592405$

But, for $n \in \mathbb{N}^{+}$, $k \in \mathbb{N}^{+}$ and $n \neq 3k$, when all $2^{2n}-1$ numbers (except $2^{6n}-1$) are divided by 3, we get positive odd integers that are not a multiples of 3. This is because when we factor $2^{2n}-1$, there is only one multiplier of 3. Factorizing of $2^{2n}-1$,

$(2^{2n}-1)=(2^{x_1}-1)(2^{x_1}+1)(2^{x_2}+1)(2^{x_3}+1)....(2^{x_{n-1}}+1)(2^{x_n}+1)$ or

$(2^{2n}-1)=(2^{x_1}-1)(2^{x_1}+1)$

In these equations, x_1 is a positive odd integer and not a multiple of 3, and x_2, x_3, x_4 ... x_n are positive even integers. $(2^{x_1}-1),(2^{x_2}+1),(2^{x_3}+1)....(2^{x_{n-1}}+1)$ and $(2^{x_n}+1)$ do not have a multiplier of 3 (Lemma 2.4. and Lemma 2.5.). And since x_1 is not a multiple of 3, $(2^{x_1}+1)$ has only one multiplier of 3, so an infinite number of Collatz numbers are converted to each of the numbers 2^{2n}.

Example 2.17. A small fraction of the Collatz numbers that can be converted to 2^8 by applying CO, i.e. to 1.These numbers are obtained by applying the CIO to each number.

85				
↓	... ...			
113	→ 75, 301, 1205, 4821, 19285, 77141, 308565 ...			
453	↓			
1813	401 → 267, 1069, 4277, 17109			
7253	1605	↓	↓	
29013	6421	1425	2851 →	3801 ...
116053	25685	5701	11405 →	7603 ...
464213	102741	22805	45621	↓
...	...	...	...	... ...
...	...	...	...	... ...

Similarly, all odd integers are converted to 2^{2n} ($n \in \mathbb{N}^{+}$) by applying CO, i.e. to 1.

Lemma 2.18. We obtain new Collatz numbers by applying the Collatz inverse operation $(\frac{2^m.a_n-1}{3})$ ($m \in \mathbb{N}^{+}$), to the Collatz numbers, all of these Collatz numbers are different from each other.

Proof. Let a_1 and a_2 be arbitrary Collatz numbers, when we apply the Collatz inverse operation to each of them, the resulting numbers are b_1 and b_2. If $b_1=b_2$ then,

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$\frac{2^m.a_1-1}{3} = \frac{2^t.a_2-1}{3}$ and $2^m.a_1 = 2^t.a_2$ for odd positive integers, must be $a_1 = a_2$ and $m = t$.

Lemma 2.19. If $a_n \equiv 0 \pmod 3$ and a_n are odd and Collatz numbers, we can derive a_n from other Collatz numbers.

Proof. If $a_n \equiv 0 \pmod 3$, we can write $a_n = 3.k$, $k \in \mathbb{N}_{odd}$, then let $b_n \not\equiv 0 \pmod 3$, applying the Collatz operation to a_n is

$$\frac{3a_n + 1}{2^n} = b_n = \frac{3.3.k + 1}{2^n} = \frac{9k + 1}{2^n}$$

Then, we get $k = 2.m + 1$, to divide b_n by 2;
$b_n = \frac{9.(2.m+1)+1}{2^n} = 9m + 5$ if $b_n = 9m + 5$ is odd, $b_n \not\equiv 0 \pmod 3$

If $b_n = 9m + 5$ is even, m is odd, take $m = 2y + 1$ and then,
substituting $2y + 1$ for m, $b_n = 18y + 14$ when we divide b_n by 2,
$b_n = 9y + 7$ if b_n is odd, $b_n \not\equiv 0 \pmod 3$
if $b_n = 9y + 7$ is even, y is odd , take that $y = 2x + 1$ and then,
substituting $2x + 1$ for y, $b_n = 18x + 16$ when we divide b_n by 2, $b_n = 9x + 8$ if b_n is odd, $b_n \not\equiv 0 \pmod 3$
if $b_n = 9x + 8$ is even,x is even, take $x = 2.z$ and then,
substituting $2z$ for x, $b_n = 18z + 8$ when we divide b_n by 2, $b_n = 9z + 4$ if b_n is odd, $b_n \not\equiv 0 \pmod 3$
if we continue like this $b_n = 9s + 2$, $b_n \not\equiv 0 \pmod 3$
and $b_n = 9r + 1$,
consequently $b_n \not\equiv 0 \pmod 3$
so we apply the Collatz inverse operation to b_n, we get a_n, $\frac{2^n.b_n-1}{3} = a_n$

Corollary 2.20. The Collatz conjecture holds for all positive odd integers.

Corollary 2.21. From each element of the set A with infinite elements (the set of Collatz numbers), we get an infinite number of new sets of Collatz numbers with infinite elements. From the new infinite Collatz numbers that have been formed, infinitely many new Collatz numbers are formed by the Collatz inverse operation,and it goes on like this endlessly. So we get the whole set of positive odd numbers as Collatz numbers and we prove the Collatz conjecture for all $\mathbb{N}^+$ (Remark 2.2).

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3 The Absence of any Positive Integer other than Collatz Numbers

In this section, we prove that there are no positive integers that do not satisfy the conjecture.

Lemma 3.1. There cannot be any positive integer other than Collatz numbers.

Proof. Let t_0 be a number that is not a Collatz number, $t_0 \in \mathbb{N}_{odd}$, then when we apply the Collatz inverse operation to t_0,

$t_0 \rightarrow \frac{2^n.t_0-1}{3}$ we get T $=\{$ t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10} $\ldots\}$, and the elements of the set T are not Collatz numbers.

Also, when we apply the Collatz operation to t_0, until we find odd numbers;

$t_0 \rightarrow \frac{3.t_0\ +1}{2^n}$, $\quad \mathrm{s}_1 \rightarrow \mathrm{s}_2 \rightarrow \mathrm{s}_3 \rightarrow \mathrm{s}_4 \rightarrow \mathrm{s}_5 \rightarrow \mathrm{s}_6 \rightarrow \mathrm{s}_7 \rightarrow \mathrm{s}_8 \rightarrow \mathrm{s}_9 \rightarrow \mathrm{s}_{10} \quad \ldots$

we get $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}\ldots\}$ and the elements of the set S are not Collatz numbers ($s_n \in \mathbb{N}_{odd}$).

If we apply the Collatz inverse operation to every number in the sets T and S, we get infinitely many new numbers that are not Collatz numbers. If t_0 is a multiple of 3, we take s_1 instead of the set T.

t_0

$\downarrow$

$t_1 \rightarrow t_{1(1)}, t_{1(2)}, t_{1(3)}\ , t_{1(4)}, t_{1(5)}, t_{1(6)}, t_{1(7)}, t_{1(8)}, \quad t_{1(9)}, \quad t_{1(10)} \ \cdots$

$t_2 \qquad \downarrow$

$t_3 \qquad t_{11(1)} \rightarrow t_{111(1)}, t_{111(2)}, t_{111(3)}, t_{111(4)}, t_{111(5)}, t_{111(6)}, t_{111(7)}, \cdots$

$t_4 \qquad t_{11(2)} \qquad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$t_5 \qquad t_{11(3)} \qquad t_{1111(1)}\ t_{1112(1)}\ t_{1113(1)}\ t_{1114(1)}\ t_{1115(1)}\ t_{1116(1)}\ t_{1117(1)} \cdots$

$t_6 \qquad t_{11(4)} \qquad t_{1111(2)}\ t_{1112(2)}.\ t_{1113(2)}\ t_{1114(2)}\ t_{1115(2)}\ t_{1116(2)}\ t_{1112(2)}$

$t_7 \qquad t_{11(5)} \qquad t_{1111(3)}\ t_{1112(3)}.\ t_{1113(3)}\ t_{1114(3)}\ t_{1115(3)}\ t_{1116(3)}\ t_{1112(3)} \cdots$

$t_8 \qquad t_{11(6)} \qquad t_{1111(4)}\ t_{1112(4)}\ t_{1113(4)}\ t_{1114(4)}\ t_{1115(4)}\ t_{1116(4)}\ t_{1112(4)} \cdots$

$t_9 \qquad t_{11(7)} \qquad t_{1111(5)}\ t_{1112(5)}.\ t_{1113(5)}\ t_{1114(5)}\ t_{1115(5)}\ t_{1116(5)}\ t_{1112(5)} \cdots$

$t_{10} \qquad t_{11(8)} \qquad t_{1111(6)}\ t_{1112(6)}\ t_{1113(6)}\ t_{1114(6)}\ t_{1115(6)}\ t_{1116(6)}\ t_{1112(6)} \cdots$

$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$

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$s_1 \rightarrow t_0$

$\downarrow$

$s_{1(1)} \rightarrow s_{11(1)}, s_{11(2)}, s_{11(3)}, s_{11(4)}, s_{11(5)}, s_{11(6)}, s_{11(7)}, s_{11(8)}, s_{11(9)}, s_{11(10)} \cdots$

$s_{1(2)} \quad \downarrow$

$s_{1(3)} \quad s_{111(1)} \rightarrow s_{1111(1)}, s_{1111(2)}, s_{1111(3)}, s_{1111(4)}, s_{1111(5)}, s_{1111(6)}, s_{1111(7)} \cdots$

$s_{1(4)} \quad s_{111(2)} \rightarrow s_{1112(1)}, s_{1112(2)}, s_{1112(3)}, s_{1111(4)}, s_{1111(5)}, s_{1111(6)}, s_{1111(7)} \cdots$

$s_{1(5)} \quad s_{111(3)} \rightarrow s_{1113(1)}, s_{1113(2)}, s_{1113(3)}, s_{1113(4)}, s_{1113(5)}, s_{1113(6)}, s_{1113(7)} \cdots$

$s_{1(6)} \quad s_{111(4)} \rightarrow s_{1114(1)}, s_{1114(2)}, s_{1114(3)}, s_{1114(4)}, s_{1114(5)}, s_{1114(6)}, s_{1114(7)} \cdots$

$s_{1(7)} \quad s_{111(5)} \rightarrow s_{1115(1)}, s_{1115(2)}, s_{1115(3)}, s_{1115(4)}, s_{1115(5)}, s_{1115(6)}, s_{1115(7)} \cdots$

$s_{1(8)} \quad s_{111(6)} \rightarrow s_{1116(1)}, s_{1116(2)}, s_{1116(3)}, s_{1116(4)}, s_{1116(5)}, s_{1116(6)}, s_{1116(7)} \cdots$

$s_{1(9)} \quad s_{111(7)} \rightarrow s_{1117(1)}, s_{1117(2)}, s_{1117(3)}, s_{1117(4)}, s_{1117(5)}, s_{1117(6)}, s_{1117(7)} \cdots$

$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$

$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$

In the same way, an infinite number of new numbers are formed from each of $s_2, s_3, s_4, s_5 \ldots$

We get infinite new numbers which are not Collatz numbers, from these numbers infinite new numbers are formed endlessly, this result contradicts (Corollary 2.21).

Lemma 3.2. The elements of the set S do not loop with any element of the sets S or T.

Proof. We assume that such a loop exists.

$t_0 \rightarrow s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow s_4 \rightarrow s_5 \rightarrow s_6 \rightarrow s_7 \rightarrow s_8 \rightarrow s_9$

$\uparrow \qquad \downarrow$

CIO $\downarrow$ $\ldots$ $\qquad s_{10} \downarrow$ CO

$\uparrow \qquad \downarrow$

$s_n \leftarrow \ldots\ldots\ldots\ldots \leftarrow \ldots\ldots \leftarrow \ldots\ldots\ldots\ldots \leftarrow s_{12} \leftarrow s_{11}$

For such a loop to be exist, if we choose t_0 , s_n or any other number as the starting and ending terms of the loop, which cannot be a number other than 1. Since we choose s_n as the first and last terms of the loop, when CO is applied to all the elements of the loop, they all turn into s_n, but when CO is applied to s_n, it cannot produce a number other than the loop numbers. In other words, while infinitely different numbers turn into s_n with CO, s_n cannot turn into a number different from those numbers. Such a restriction is only possible if s_n is 1. Then the other elements of the loop are also 1. For such a loop to be exist in positive odd integers, all the elements of the loop must be 1.

By another method, all the elements of the loop must be equal, because the infinite set of numbers obtained by applying the CIO to each element of the loop is the same,that is, $\{t_0, t_1, t_2 \ldots s_1, s_{11}, s_{12} \ldots s_2, s_{21}, s_{22} \ldots s_3, s_{31}, s_{31} \ldots s_n, s_{n1}, s_{n2} \ldots\}$. In the positive odd integers, only the number 1 can form a loop with itself, so all elements of the loop are 1.

Example 3.3. Lets take t_0 in the loop and $t_0 \not\equiv 0 \pmod 3$, then if $t_0{\to}^{CIO} = t_0{\to}^{CO}$.

$t_0 \equiv 2 \pmod 3$ $\quad \frac{2t_0-1}{3} = \frac{3t_0+1}{2^n} \quad 2^{n+1}.t_0 - 2^n = 9t_0 + 3,\ \ t_0 = \frac{2^n+3}{2^{n+1}-9}$ or,

$t_0 \equiv 1 \pmod 3$ $\quad \frac{4t_0-1}{3} = \frac{3t_0+1}{2^n} \quad 2^{n+2}.t_0 - 2^n = 9t_0 + 3,\ \ t_0 = \frac{2^n+3}{2^{n+2}-9}$

There is no such a positive odd integer t_0, another of 1.

Since we assume that there is a number t_0 which is not a Collatz number, we obtain two sets (T and S) with infinite different elements from this number. The elements of the sets T and S are not Collatz numbers. From the new infinite numbers that have been formed, infinitely many new numbers are formed by the Collatz inverse operation, and it goes on like this endlessly. And they don't form a loop. If there were such a number t_0, there would be no Collatz number. This result contradicts the result we found above (Corollary 2.21). So there cannot be a number that is not a Collatz number.

4 Conclusion

We proved the Collatz conjecture using the Collatz inverse operation method. It is shown that all positive integers reach 1, as stated in the Collatz conjecture. With the methods described in this study for $3n+1$, it can be found whether numbers such as $5n+1$, $7n+1$, $9n+1$ ... also reach 1.

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