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[Bartłomiej Gawel](#)*, Bogdan Rębiasz, [Andrzej Paliński](#)

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Article

Integrating Probability and Possibility Theory: A Novel Approach to Valuing Real Options in Uncertain Environments

Bartłomiej Gawęł, Bogdan Rębiasz and Andrzej Paliński *

AGH University

* Correspondence: palinski@agh.edu.pl

Featured Application: This paper introduces a novel hybrid approach to real option valuation that can be particularly useful in industries where expert judgment plays a significant role and historical data is limited or unreliable—such as the metallurgical, energy, and infrastructure sectors. The method enables decision-makers to rigorously incorporate both objective data and subjective beliefs into investment analysis, providing more realistic valuation under deep uncertainty. It is especially applicable to long-term capital-intensive projects where strategic flexibility and uncertainty about future market conditions are crucial.

Abstract: The article presents a new method for evaluating investment projects in uncertain conditions, assuming that uncertainty may have been two origins: aleatory (related to randomness) and epistemic (due to incomplete knowledge). Epistemic uncertainty is rarely considered in investment analysis which can result in undervaluing the future opportunities and risks. Our approach utilizes the Datar-Mathews Method (DM Method) to gather relevant information regarding the potential value of real option. By combining probabilistic and possibilistic approaches, proposed valuation model incorporates hybrid Monte Carlo simulation and a random-fuzzy Geometric Brownian Motion, considering the interdependence between parameters. The result of the hybrid simulation is a pair of upper and lower cumulative probability distributions, known as a p-box, which represents the uncertainty range of the Net Present Value (NPV). We propose three transformations of the p-box into a subjective probability distribution, which allow decision-makers to incorporate their subjective beliefs and risk preferences when performing real option valuation. Thus, our approach allows the combination of objective available information about valuation of investment with the decision-maker's attitude in front of partial ignorance. To demonstrate the effectiveness of our approach in practical scenarios, we provide a numerical illustration that clearly showcases how our approach delivers a more precise valuation of real options.

Keywords: Real options, Datar-Mathews Method, possibility theory, investment decision, hybrid Monte Carlo simulation.

1. Introduction

The Datar-Mathews Method (DM Method) is a particularly interesting but still relatively underexplored area in simulation-based real option valuation (ROV). The method proposed by Datar and Mathews in 2004 provides an easy way to determine the real option value of a project by using the average of positive outcomes [1,2]. It uses Monte Carlo simulation to propagate uncertainty and to create payoff distribution of an investment alternative. The resulting payoff distribution is flexible, allowing for any compounding interval and free selection of discount rates [3]. Additionally, the results of the DM Method converge with the Black & Scholes result when the Monte Carlo simulation is run enough times. In this article, we propose a method for performing DM Method in a hybrid environment, which refers to a scenario where the parameters utilized for evaluating the investment cannot be precisely determined.

In real option valuation the accuracy of input parameters greatly influences the financial terms, such as cash flow and profit. Analysts frequently estimate parameters using past data and simultaneously make subjective guesses about future economic changes [4–6]. It results in the additional type of uncertainty in describing parameters, which, apart from random uncertainty, arises from a lack of knowledge or inherent vagueness [7]. The exclusive reliance on probability theory in these situations can result in a false sense of precision and an incorrect assumption that outcomes are inherently predictable and repeating in nature [8]. There is therefore a natural need to distinguish between different types of uncertainty in ROV.

Entrepreneurial activities encompass different levels of informational reliability, giving rise to situations where risks are linked to the probabilities of expected outcomes, while uncertainty arises when these probabilities cannot be precisely determined. It follows from the fact that valuing investment projects usually involves both types of uncertainty - aleatoric and epistemic. Aleatoric uncertainty is observed variability while epistemic uncertainty arises from incomplete knowledge. To effectively capture non-probabilistic uncertainty, it becomes crucial to incorporate to the model both probabilistic and possibilistic approaches [9]. Experts' opinions or imprecise estimates may be introduced to the model in the form of possibilistic distribution as they better express vagueness, imprecision, or ambiguity [10]. In the field of ROV, the probabilistic Black-Scholes (B-S) model provides robust framework to model aleatoric uncertainty of many variables such as market size or product prices [5,6,11–13]. Therefore, a combination of both probabilistic and possibilistic approaches is necessary for accurately modelling and evaluating the uncertainties that exist in ROV.

Several factors led us to writing of this article. The first is that pure probabilistic approach is difficult to use in valuing investment projects using DM Method due to inherent vagueness that investors encounter. The second factor is the significant reliance on expert estimates in the metallurgical industry, where investment lifecycles typically span 10-12 years. Moreover, the relevance of historical data diminishes rapidly after 2-3 years, making it impractical to assert any predictive value of time series beyond 5 years [14]. The final reason is the interdependency among different products' demands. This connection creates a challenge when trying to forecast the demand for a single product in isolation, as it depends on the demand for other products within the same product line [15].

This study introduces a novel method for valuing investment projects that accounts for both imprecise and random uncertainties, while explicitly modeling the interdependencies among parameters within the Brownian Motion framework. Conventional ROV models tend to overlook the relationships between parameters, which can result in inaccurate forecasts and less precise valuation [4,7,15,16]. While some research has examined the correlation between parameters when using a probabilistic approach [5], current methods have limitations in fully addressing the complexities of aleatory and epistemic uncertainty. By incorporating these correlations in the DM, the proposed approach can provide a more realistic and precise evaluation of investment projects, while also accounting for the vagueness and randomness of the decision-making context.

The structure of the article is as follows: Firstly, we conducted a comprehensive review of real option in a hybrid environment. Next, we introduced the method of propagating uncertainty in such an environment using Monte Carlo simulation. Subsequently, we explored strategies for selecting subjective probability distributions that consider the decision-maker's perspective when faced with incomplete knowledge. To illustrate the effectiveness of our approach, we provided a clear example demonstrating how it enhances the precision of real options valuation.

2. Fuzzy Sets in Real Options Pricing

Fuzzy set theory has emerged as a prominent tool for addressing uncertainty in real option valuation, particularly when precise probabilistic data is unavailable. Many papers explore the possibility of using fuzzy sets or possibility theory to real options valuations. Carlsson *et al.* [17] developed a methodology for valuing options on R&D projects, in which future cash flows were estimated by trapezoidal fuzzy numbers. They presented a fuzzy mixed integer programming model

for the R&D optimal portfolio selection problem. Carlsson and Fuller [8] use possibility theory to study fuzzy real option valuation. The present values of expected cash flows and expected costs are estimated by authors also by trapezoidal fuzzy numbers. They determined the optimal exercise time by the help of possibilistic mean value and variance of fuzzy numbers. Garcia [18] used the fuzzy real options valuation model in a real investment project from the energy sector. In Garcia's study, the theoretical framework proposed by Carlsson and Fullér [8] is applied to analyze the timing and selection of power plants from among various investment alternatives. Further, the study conducted by Allenator and Thulasiram [19] employs a fuzzy trinomial real options model for the valuation of grid resources, substantiating the model's applicability through empirical analysis. Additionally, Tao [20] formulated an extensive methodology that integrates fuzzy risk assessment with a real options strategy, specifically for the evaluation of information technology investments in nuclear power stations. Zeng *et al.* [21] compared the application of traditional net present value method with real options in investment evaluations, analysed the uncertainty of power grid investment project, and discussed how to make the investment decision when investment cost and cash flow are both fuzzy numbers. Kahraman and Ucal [22] used the certainty equivalent approach for real options valuation in an oil investment with fuzzified data. Ho and Liao [23] propose a fuzzy approach for investment project valuation in uncertain environments from the aspect of real options. Lee *et al.* [12] utilized the principles of fuzzy decision theory combined with Bayes' theorem to assess fuzziness in option analysis practices.

However, it should be emphasized that none of the above-mentioned studies consider the problem defined in the first part of this article. In systematic review of the interactions between fuzzy set theory and option pricing conducted in [9], the fuzzy-random approach to option pricing was represented in approximately 35% of 240 reviewed papers. However, fuzzy real option was only represented by 13 papers. To the knowledge of the authors, there are few approaches to the analysis of real options in a hybrid environment. In [24] valuation of portfolio of real option under both type of uncertainty is presented. In [25] the generalised fuzzy-stochastic multi-mode real options model is developed. The [26] introduces a fuzzy-random extension of the Ho–Lee term structure model by incorporating fuzzy volatility into a binomial lattice framework, enhancing the pricing of interest rate derivatives like caplets under uncertainty.

The equivalent of the DM Method in a fuzzy environment is the "fuzzy pay-off" method proposed by Collan (a good summary can be found in [27]) (FROV). Collan in [28] compares the use of the Datar-Mathews method and the fuzzy pay-off method, in the analysis of investment cases with different levels of complexity. The results of the two methods are compared through numerical illustrations, showing that FROV is sufficient for problems with low complexity, but the DM Method is better suited for more complex ones. In [29] Borges states that there are situations in which the original "fuzzy pay-off" calculates the real option value to be negative, what is theoretically incorrect. He proposed the centre of gravity approach to real option valuation (PROBROV). Stoklasa in [30,31] presents possibilistic method for real ROV and compares aforementioned methods and fully possibilistic fuzzy ROV (POSROV).

Many prevailing theories in judgment and decision-making treat uncertainty as a singular concept. However, in our approach, we differentiate between two distinct types of uncertainty: epistemic (events that are potentially knowable) and aleatory (inherently random events). It's crucial to emphasize that, within our framework, this differentiation leans more towards a philosophical perspective rather than a purely pragmatic one. This approach gaining again popularity [32,33].

3. Materials And Methods

3.1. Modelling Uncertainty by Correlated Random-Fuzzy Geometric Brownian Motion (GBM)

The standard method for evaluating real options assumes that the value of the parameter will fluctuate according to a Geometric Brownian Motion (GBM) over a defined period. The parameter q follows GBM, if it satisfies the diffusion equation [4,34–36]:

$$dq_t = \mu q_t dt + \sigma q_t dW_t, \quad (1)$$

where $W_t \sim N(0, t)$, and μ and σ are drift and volatility. Applying Ito's Lemma and Euler discretization over interval $\Delta t = [t, t - 1]$, we can write the formulas for the prediction q_t [4,34–36]

$$q_t = q_{t-1}(\mu, \sigma, Z_t) = q_{t-1} \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma Z\sqrt{\Delta t}\right), \quad (2)$$

where Z is independent, normal random variate. Frequently, the available statistical data are inadequate or unsuitable for estimating parameters such as μ and σ . In scenarios where statistical data are insufficient, these parameters are estimated or refined by subject matter experts. Consequently, μ and σ become imprecise variables, aptly represented using fuzzy numbers. In our approach, when confronted with insufficient statistical data, we address the uncertainty in drift and volatility by representing them as possibilistic parameters: $\tilde{\mu}, \tilde{\sigma}$, described by fuzzy numbers. Consequently, Equation (2) can be rewritten as

$$\tilde{q}_t = \tilde{q}_{t-1}(\tilde{\mu}, \tilde{\sigma}, Z_t) = \tilde{q}_{t-1} \exp\left[\left(\tilde{\mu} - \frac{\tilde{\sigma}^2}{2}\right)\Delta t + Z_t \tilde{\sigma} \sqrt{\Delta t}\right], \quad (3)$$

where \tilde{q}_t is random-fuzzy variable.

This method allows for the differentiation between uncertainties stemming from aleatory and epistemic sources to provide more realistic representation of GBM. It is noteworthy that when predicting the fluctuation of q , the literature typically assumes that all values in triple $(\tilde{\mu}, \tilde{\sigma}, Z_t)$ are represented by fuzzy numbers, resulting in a fuzzy variable as the outcome of the process. However, in our scenario, we encounter a combination of possibilistic $\tilde{\mu}, \tilde{\sigma}$ and probabilistic Z_t .

To better understand what is meant by a possibilistic distribution, it is helpful to consider its roots in possibility theory. As described by Dubois and Prade [37], a possibility distribution—also known as a fuzzy number or a nested interval—represents an alternative to probability for encoding expert opinions. It maps intervals of parameter values to degrees of possibility (μ), which indicate how plausible each value is. These intervals, called α -cuts, provide bounds that reflect varying levels of confidence. For instance, the support of the distribution includes all possible values ($\mu = 0$), while the core identifies the most plausible value ($\mu = 1$). This allows a flexible and intuitive way to express epistemic uncertainty when precise probabilities are difficult to assign.

The versatility of Equation (3) causes that it can effectively represent various parameters such as unit prices, raw materials prices, or market volumes. These parameters are often correlated that deeply impact the final valuation outcome. Therefore, incorporating these interdependencies in the model is crucial for achieving precise and reliable results. To incorporate this correlation, we use the method describe in [38]. Let's suppose that we analyse I parameters. Additionally, it is assumed that there may be defined subsets I_k of correlated parameters $k \in K$. Each I_k includes subset of correlated parameters. Therefore, Eq. (3) can be expressed as the following forecasting model for parameter q_i

$$\tilde{q}_{i,t} = \tilde{q}_{i,t-1} \exp\left[\left(\tilde{\mu}_i - \frac{\tilde{\sigma}_i^2}{2}\right)\Delta t + \hat{Z}_{i,t} \tilde{\sigma}_i \sqrt{\Delta t}\right], i \in I_k, \quad (4)$$

Here, it is assumed that $\tilde{\mu}_i$ and $\tilde{\sigma}_i$ are possibilistic parameters, $\hat{Z}_{i,t}$ is correlated random normal variate and \tilde{q}_t is random-fuzzy variable. To define $\hat{Z}_{i,t}$, the Cholesky decomposition may be utilized to decompose the correlation matrix (as described in [39]).

Possibilistic distribution may be viewed as a nested set of intervals (called α -cut)

$$\tilde{\mu}_\alpha = \{x \in R | \tilde{\mu}_\alpha \geq \alpha\}, 0 \leq \alpha \leq 1, \quad (5)$$

Interdependence between $\tilde{\mu}_i$ and $\tilde{\mu}_j, i, j \in I_k$ can be modelled by interval regression [40]. Below, we present the equations that define the relationships between these variables, using the notation of α -levels [38]

$$\tilde{\mu}_{i,\alpha} \geq \inf(a_1^{ij}) \times \tilde{\mu}_{j,\alpha} + \inf(a_2^{ij}), i, j \in I_k, \quad (6)$$

$$\tilde{\mu}_{i,\alpha} \leq \sup(a_1^{ij}) \times \tilde{\mu}_{j,\alpha} + \sup(a_2^{ij}), i, j \in I_k \quad (7)$$

where a_1^{ij} , a_2^{ij} represent the coefficients of the interval regression equations, which are pivotal in establishing the dependencies between parameters $\tilde{\mu}_i$ and $\tilde{\mu}_j$. These coefficients a_1^{ij} and a_2^{ij} can be ascertained using the method proposed by Hladik and Černý (*crisp input-crisp output* variant) [40,41].

3.2. Uncertainty Propagation

The uncertainty propagation is achieved through hybrid simulation, which combines the Monte Carlo technique with the extension principle of fuzzy set theory. The method has been applied across various fields, such as LCA [42], human health risk assessment [43], construction [44] or environmental science [45]. It's based on assumptions of IRS (Independent Random Set) simulation proposed by Baudrit [46].

Let $ICF_t(q_i)$, represents incremental cashflow of investment in year $t \in T$ (T being the economic life-cycle of project). To perform hybrid simulation, all fuzzy variables are partitioned into α -cuts with step $\Delta\alpha$.

Using the α -levels notation, the constraints on the possibilistic parameters $\tilde{\mu}_i$ and $\tilde{\sigma}_i$ can be defined as follows:

$$\inf(\tilde{\mu}_{i,\alpha}) \leq \tilde{\mu}_i \leq \sup(\tilde{\mu}_{i,\alpha}), 0 \leq \alpha \leq 1, i \in I \quad (8)$$

$$\inf(\tilde{\sigma}_{i,\alpha}) \leq \tilde{\sigma}_i \leq \sup(\tilde{\sigma}_{i,\alpha}), 0 \leq \alpha \leq 1, i \in I \quad (9)$$

Consequently, Eq. (8) and (9) constitute additional constraints in ICF calculation. The realization of $\hat{Z}_{i,t}$ is sampled from the probability distributions with Cholesky decomposition [47]. Formally, the hybrid simulation is presented in Algorithm 1. The prediction of future parameter values in the simulation loop is based on Equation (4), which captures the hybrid uncertainty propagation through random-fuzzy Geometric Brownian Motion with parameter interdependencies.

Algorithm 1 Calculation of payoff distribution in hybrid environment

Require: $J, \Delta\alpha, T$

begin

Set $j=0$

while $j \leq J$

$\alpha = 0$

while $\alpha \leq 1$

calculate $\sup()$ and $\inf()$ of $\tilde{q}_{i,0,\alpha}(\tilde{\sigma}_{i,\alpha}, \tilde{\mu}_{i,\alpha}, 0)$ (Eq.4)

subject to

α -level constraints $\tilde{\sigma}_{i,\alpha}, \tilde{\mu}_{i,\alpha}$ (Eq 8,9)

interdependence between $\tilde{\mu}_{i,\alpha}$ (Eq.6,7)

for $t = 0$ to $T-1$

Generate vector of $\hat{Z}_{i,t+1}$ (MC Sampling with Cholesky decomposition)

Forecast vector of $\tilde{q}_{i,t+1,\alpha}(\tilde{\sigma}_{i,\alpha}, \tilde{\mu}_{i,\alpha}, \hat{Z}_{i+1,t})$ (Eq. 4)

Solve $\overline{ICF}_{t,\alpha,j}[\tilde{q}_{i,t,\alpha}(\tilde{\sigma}_{i,\alpha}, \tilde{\mu}_{i,\alpha}, \hat{Z}_{i,t})] \rightarrow \max$

subject to

financial constraints

Solve $\underline{ICF}_{t,\alpha,j}[\tilde{q}_{i,t,\alpha}(\tilde{\sigma}_{i,\alpha}, \tilde{\mu}_{i,\alpha}, \hat{Z}_{i,t})] \rightarrow \min$

subject to

financial constraints

Compute $f_{\alpha,j} = [\underline{npv}_{\alpha,j}, \overline{npv}_{\alpha,j}]$

$\alpha = \alpha + \Delta\alpha$

$j = j + 1$

end

The output of hybrid simulation is set of random intervals $Result_{symhyb}$. To obtain a payoff distribution from these results, the Dempster-Shafer theory is applied [48]. Let θ be a non-empty set

called a Frame of Discernment, that describes all possible interval values of the NPV (Net Present Value) of the evaluated investment. The simulation results yield a collection of premises (called *focal elements*)

$$Result_{symhyb} = \{f_{\alpha,j} \subseteq \theta \mid f_{\alpha,j} = [\underline{npv}_{\alpha,j}, \overline{npv}_{\alpha,j}], 0 \leq \alpha \leq 1, j \in J, npv \in R\}. \quad (10)$$

Let m be a function assigning a mass probability to each $f_{\alpha,j}$, such as $m(f_{\alpha,j}) = (A * J)^{-1}$, where A is the number of α -levels used in the simulation. The belief regarding the occurrence of a given NPV value can be described by a pair of probabilities (belief):

$$\underline{P}(npv = npv_0) = \sum_{npv \subseteq f_{\alpha,j}} m(f_{\alpha,j}) \quad (11)$$

and (plausibility)

$$\overline{P}(npv = npv_0) = \sum_{npv \cap f_{\alpha,j} \neq \emptyset} m(f_{\alpha,j}). \quad (12)$$

This pair defines an interval commonly interpreted as the upper and lower probability bounds of $P(npv = npv_0)$.

Let $\underline{\Phi}, \overline{\Phi}: R \rightarrow [0,1]$ be non-decreasing functions such as, $\forall_x \overline{\Phi}(x) \geq \underline{\Phi}(x)$. An interval

$$[\underline{\Phi}, \overline{\Phi}] = \{\Phi: R \rightarrow [0,1], \text{non-decreasing} \mid \forall_{x \in R} \underline{\Phi}(npv) \leq \Phi(npv) \leq \overline{\Phi}(npv)\} \quad (13)$$

is called probability box (*p-box*), and functions $\overline{\Phi}$ i $\underline{\Phi}$ are called upper and lower cumulative payoff distribution. The exact form of the cumulative distribution function $\Phi^*(npv)$ is unknown, but it is known to lie between these bounds. It can be shown [46] that if the $npv \subseteq R$, then $\underline{\Phi}(x) = \underline{P}(npv \in (-\infty, x])$, $\overline{\Phi}(x) = \overline{P}(npv \cap (-\infty, x] \neq \emptyset)$. Thus, the set of premises $Result_{symhyb}$ can be transformed into a *p-box* representing the upper and lower bounds of the cumulative distribution for NPV outcomes.

The p-box can be interpreted in two ways: as bounds on the cumulative probability associated with a specific npv value, or as the tightest possible bounds on the distribution function given the available information. It is important to note that the p-boxes obtained from hybrid simulation are **non-parametric**—they are constructed without any assumptions about the shape of the underlying distribution of the variable of interest.

Ellsberg's experiments [49] show that choices in risky situations may differ from those in situations where information about chances are imprecise. Ellsberg introduced the concept of ambiguity, which is a state between complete ignorance and full knowledge of probability. Ambiguity arises from low information reliability or information inconsistencies [50]. Mathematically, ambiguity is reflected in situations where a decision maker deals with risk described using upper and lower distribution functions [51]. The decision maker aims to reduce ambiguity by finding a transformation that allows to convert the p-box into a subjective cumulative payoff distribution function $\Phi(npv)$. The construction of a subjective probability distribution is, in a sense, imposed on the decision-maker, who, upon encountering the ambiguity of simulation outcomes, is compelled to express their belief in the form of a probability distribution, even if his or her actual knowledge may not sufficiently justify the selection of one distribution over another.

3.3. Transformation of p-Box into Subjective Payoff Distribution

Different strategies can be applied to choose the subjective cumulative distribution function $\Phi(npv)$ depending on the decision-maker's preferences and the decision context.

The *pignistic transformation* approach was proposed by Smets [52,53], is grounded in Laplace's principle of insufficient reason. It is identical to the so-called Shapley value used in cooperative game theory as a fairness principle for sharing benefits across members of coalitions. It treats each focal element $f_{\alpha,j}$ from p-box result set as a set of payoffs with a unit mass probability $m(f_{\alpha,j})$. The

expected value of each interval is derived by assuming a uniform distribution across the interval $[npv_{\alpha,j}, \overline{npv}_{\alpha,j}]$. The value of $\Phi_{bet}(npv)$ is calculated as [52]

$$\Phi_{bet}(npv) = \sum_{f_{\alpha,j} \in \{npv \leq \overline{npv}_{\alpha,j}\}} m(f_{\alpha,j}) \quad (14)$$

This distribution reflects a neutral attitude, where the decision-maker "bets" equally on all possible outcomes within the specified intervals. It serves as the least biased representation of the decision maker's state of knowledge, aligning with the observed outcome of simulation [52]. In this transformation, the decision maker does not introduce any additional information into the model.

The Hurwicz criterion provides a way to interpolate between the best-case (upper bound) and worst-case (lower bound) scenarios represented by the p-box [54]. By using this criterion, Jeffrey introduced the concept of ambiguity, which represents a state between complete ignorance and perfect knowledge of probability. The ambiguity distribution may be defined as follows:

$$\Phi_{amb}(\kappa, \lambda) = \lambda \cdot \overline{\Phi}^{-1}(\kappa) + (1 - \lambda) \cdot \underline{\Phi}^{-1}(\kappa), \kappa \in [0,1] \quad (15)$$

where λ takes values from 0 to 1 and is called the *indicator of the pessimism (optimism)* of a decision maker. Optimism index may be treated as a simple measure of belief in expert's opinion.

The third approach is a generalization of the credibility theory proposed by Liu [55]. Dubois [54] extends the measure proposed by Liu with the coefficient of risk aversion β

$$\Phi_{cred}(npv, \beta) = \beta \cdot \overline{\Phi}((-\infty, npv]) + (1 - \beta) \cdot \underline{\Phi}((-\infty, npv]) \quad (16)$$

For $\beta=0.5$, $\Phi_{cred}(npv, 0.5)$ is equal to the Liu's credibility distribution. Credibility distribution may be treated as decision maker attitude in the face of uncertainty. Polarization and vagueness represent systematic biases in decision-making when confronted with uncertainty. Polarization occurs when probabilities significantly deviate from the average, reflecting overconfidence, while vagueness involves adjusting probabilities closer to the average, often as an overcorrection to polarization or due to manipulation for favourable average outcomes. The Hurwitz criterion and Liu's Credibility distribution are complementary to each other, but they should be used in different scenarios. This approach is similar to those propose in [27,29,56].

The choice of strategy to transform the p-box into a cumulative distribution function depends on the analyst and scenario. Each of the strategies is valuable in different situations (which will be discussed in the case study). It is usually necessary to compare all strategies simultaneously.

After choosing a strategy, the cumulative distribution function $\Phi_{\{pig, amb, cred\}}$ is transformed into a probability density function j or payoff distribution. This allows for the specification of a payoff distribution that enables the determination of the ROV.

3.4. Datar-Mathews Method in Random-Fuzzy Environment

In its classical form, the DM method relies on Monte Carlo simulation to model the uncertainty associated with investment project. The analysis begins by identifying the parameters that drive profitability and describing their uncertainty in terms of costs and revenues. A Monte Carlo run then generates the probability distribution of the project's net present value (NPV). In the case of complex computational models, it may be necessary to implement parallel computing to speed up the Monte Carlo simulation [57]. The project is subsequently interpreted as a call option: negative NPVs are set to zero, reflecting the real-world practice of abandoning unprofitable ventures.

The ROV is calculated as

$$ROV_{hyb} = U_r * E(npv > 0) = \frac{\int_0^\infty \varphi(npv) dnpv}{\int_{-\infty}^\infty \varphi(npv) dnpv} \int_0^\infty npv \varphi(npv) dnpv \quad (17)$$

where U_r is probability that the investment will be profitable (*success ratio*) and $E(npv > 0)$ denotes the expected value of the positive part of the NPV distribution. Figure 1 illustrates how the Datar-Mathews method can be applied in a hybrid (probabilistic-possibilistic) environment.

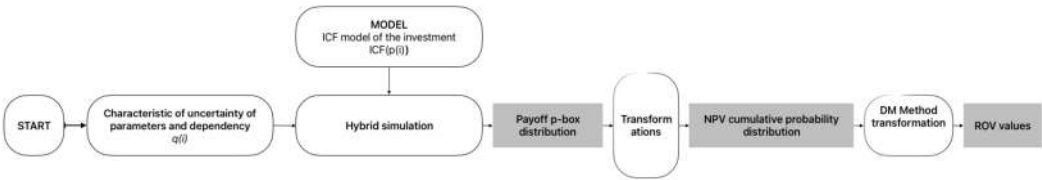


Figure 1. Calculation of Datar-Mathews method in hybrid environment.

3.5. Comparison with Existing Approaches

Before we proceed to the analysis of the numerical case, we wanted to describe the similarities and differences between our approach and the fuzzy-pay off method for real option valuation (FROV) [27]. Both approaches are based on the Datar-Mathews method but differ in their assumptions. Although, both approaches have their roots in possibility theory [54], we employ a fuzzy-random approach, whereas FPOV utilizes pure fuzzy.

In both approaches, the value of real option is calculated by multiplying the “centroid” of the positive side of the NPV distribution by the proportion of the area of the NPV distribution that represents positive outcomes. The distinction lies in the fact that in our approach, we determine this value based on the subjective probability distribution of NPV derived from the p-box, while FROV utilizes fuzzy NPV for this purpose. Additionally, we use the expected value as the centroid, while variants of FPOV utilize possibilistic mean, center of gravity, or other interpretations. In essence, we have control over the shape of the NPV distribution, whereas FPOV interprets its expected value based on the distribution.

It should be emphasized that using the method proposed by us, ROV is possible with both methods - ours and Collan's. This is made possible by relying on hybrid simulation, which yields a p-box. It can be demonstrated that this structure can be transformed into both a probability distribution and a fuzzy random variable, thereby resulting in a fuzzy NPV [58,59]. Further research is needed to determine the relationship between our approach and the approach proposed in Collan's work.

4. Results

This example considers a steel manufacturer that is weighing the construction of a new production line for organic-coated sheets. The firm currently produces conventional sheets but now has an opportunity to adopt organic-coating technology. Management must determine whether building the new facility is economically justified. Before construction can begin, the company must prepare project documentation and complete environmental assessments to obtain the required permits; we assume these preparatory activities will cost USD 20,000.

To assist in the decision-making process, the company conducts net present value (NPV) and real option analyses. These analyses involve assessing the future cash flows with the help of experts. The key question is whether the company should invest the funds to obtain a return on investment.

Model for Estimating the Value of Project in Hybrid Environment

In this section we introduce the NPV model that underpins the real-option valuation of a hypothetical production system (see Fig. 2). The model extends the framework described in [58].

The project under consideration is the construction of a new plant for manufacturing organic-coated (OC) steel sheet. OC sheet is produced from hot-dip-galvanised (HDG) sheet and enjoys both a broader range of applications and a higher unit value. The main input to the process is cold-rolled (CR) sheet, which is first converted to HDG sheet; part of that output is sold directly, while the rest is further processed into OC sheet and sold. Both the HDG and OC stages generate steel-scrap by-products that are likewise sold.

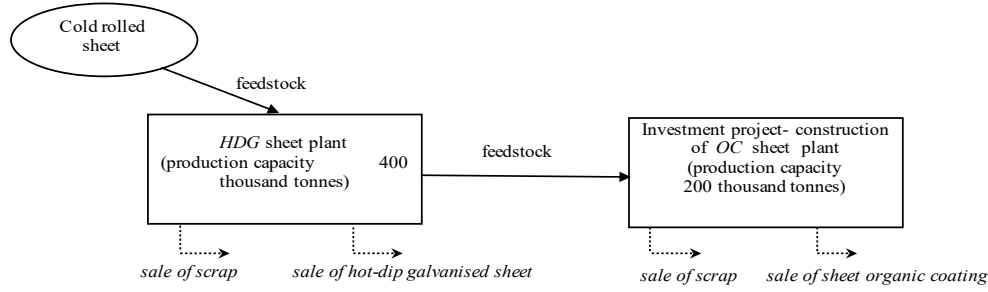


Figure 2. Description of the analysed production setup.

To apply the Datar–Mathews (DM) method we must derive φ (npv), the NPV density obtained from a Monte-Carlo simulation of the new OC plant. This calls for incremental cash-flow modelling: cash flows in the investment scenario (which includes the OC facility) are compared with those in the baseline scenario (status quo without the new plant). The project NPV can then be computed with Equations (18)–(32).

$$NPV = \frac{\sum_{t=1}^T ICF_t}{(1 + WACC)^t} - CF_0, t = 1, \dots, T \quad (18)$$

$$ICF_t = \left[(PAT_t + DA_{OC,t} + DA_{HDG,t} + \Delta WC_t)_{inv} - (PAT_t + DA_{HDG,t} + \Delta WC_t)_{base} \right] + 1_{t=T} (RV_{T,inv} - RV_{T,base}) \quad (19)$$

$$PAT_t^{scen} = Rev_t^{scen} - TC_t^{scen} - \max(Rev_t^{scen} - TC_t^{scen}, 0) * TA, \quad scen \in \{base, inv\} \quad (20)$$

$$Rev_t^{inv} = P_t^{OC} SV_t^{OC} + P_t^{HDG} SV_t^{HDG,inv} \quad (21)$$

$$Rev_t^{base} = SV_t^{HDG,base} P_t^{HDG} \quad (22)$$

$$SV_t^{OC,inv} = \min(SF_t^{OC}, CAP^{OC}) \quad (23)$$

$$SV_t^{HDG,base} = \min(SF_t^{HDG}, CAP^{HDG}) \quad (24)$$

$$SF_t^{prod} = ACF_t^{prod} MS^{prod}, prod \in \{OC, HDG\} \quad (25)$$

$$TC_t^{base} = SV_t^{HDG,base} (VC_RM_t^{HDG} + VC_O^{HDG}) + DA^{HDG} + FC^{HDG} \quad (26)$$

$$TC_t^{inv} = SV_t^{OC,inv} (VC_RM_t^{OC} + VC_O^{OC}) + SV_t^{HDG,inv} (VC_RM_t^{HDG} + VC_O^{HDG}) + DA^{HDG} + FC^{HDG} + DA^{OC} + FC^{OC} \quad (27)$$

$$VC_RM_t^{HDG} = PUC^{HDG} P_t^{CR} + (1 - PUC^{HDG}) P_t^{scrap} \quad (28)$$

$$VC_RM_t^{OC} = PUC^{OC} P_t^{HDG} - (1 - PUC^{OC}) P_t^{scrap} \quad (29)$$

$$\Delta NWC_t^{scen} = NWC_t^{scen} - NWC_{t-1}^{scen}, t = 1, \dots, T \quad (30)$$

$$NWC_t^{scen} = \frac{Rev_t^{scen}}{CTR} + \frac{Rev_t^{scen}}{DTR} + \frac{TC_t^{scen} - \sum_{prod} DA_t^{prod}}{ITO} - \frac{TC_t^{scen} - \sum_{prod} DA_t^{prod}}{DTP}, scen \in \{base, inv\} \quad (31)$$

$$RV_T^{base} = \frac{Rev_T^{scen}}{CTR} + 0.7 \frac{Rev_T^{scen}}{DTR} + 0.7 \frac{TC_T^{scen} - \sum_{prod} DA_T^{prod}}{ITO} - \frac{TC_T^{scen} - \sum_{prod} DA_T^{prod}}{DTP}, scen \in \{base, inv\}, \quad (32)$$

The notation used in model are presented in Table 1.

The net present value of the proposed OC-sheet line is obtained with Eq. (18), in which the incremental cash-flows are defined by Eq. (19) and decomposed by Eqs. (20)–(29) into operating profit after tax, depreciation, variable input costs and fixed overheads. The annual variation in net working capital is captured by Eq. (30), while the stock of working capital itself is evaluated with the revised Eq. (31), which relates cash, inventories, receivables and payables to four turnover ratios (CTR, ITO, DTR and DTP). At the end of the planning horizon the model releases tied-up capital according to the Wilcox convention [47] implemented in the revised Eq. (32), i.e. it recovers 100% of cash, 70% of the book value of inventories and receivables, and the full 100% of outstanding payables

Table 1. Notation for Parameters and Variables Used in NPV Calculation

Indices

t – investment period: $\{1, \dots, T\}$,

$scen$ – scenario: $base$ – baseline, inv – investment,

$prod$ – product: OC – oc sheet, HDG – hdg sheet, CR – cr sheet, $scrap$ – scrap,

Input parameters

TA – tax,

CAP^{prod} – capacity of $prod$ plan,

MS^{prod} – market share of $prod$ product,

DTR – receivable turnover ratio,

CTR – cash turnover ratio,

ITO – inventory turnover ratio,

DTP – payable turnover ratio,

PUC^{prod} – per-unit consumption of sheet required to produce $prod$ sheet,

FC^{prod} – fixed cost of $prod$ plan,

VC_O^{prod} – other annual variable production costs per ton for $prod$ sheet,

Auxiliary variables

PAT_t^{scen} – profit after tax in $scen$ scenario in t ,

DA_t^{prod} – annual depreciation of $prod$ plan in t

ΔWC_t^{scen} – the change in net working capital in $scen$ scenario in t ,

RV_T^{scen} – residual value in $scen$ scenario in T ,

Rev_t^{scen} – revenue in $scen$ scenario in t ,

TC_t^{scen} – total cost in $scen$ scenario in t ,

$SV_t^{prod,scen}$ – sales volume of $prod$ product in $scen$ scenario in t ,

NWC_t^{scen} – net working capital in year t in scenario $scen$,

$VC_{RM}_t^{prod}$ – cost of CR sheet per ton of $prod$ sheet in year t ,

SF_t^{prod} – sales forecast for $prod$ product,

Uncertain variables

P_t^{prod} – price of $prod$ per ton in year t ,

ACF_t^{prod} – apparent consumption forecast for $prod$ product in t .

Prices and apparent consumption for the analyzed products are inter-related rather than independent, displaying complex patterns of correlation and mutual dependence. Two relationship types can be distinguished: correlations and reciprocal dependencies. The pair-wise price correlations are summarized in Table 2, and the correlation coefficient between the apparent consumption of HDG sheet and OC sheet is 0.532.

Interval-regression coefficients that characterize the interdependencies among the drift parameters (μ_i) of the individual price series are provided in Table 3; the corresponding coefficients for apparent consumption appear in Table 4.

For a steel plant operating with the leverage typical of the industry, we assume a real weighted-average cost of capital (WACC) of 10% p.a. and a real risk-free rate of 5% p.a. [4].

Historical data for 1996–2022 show that, in real terms, OC-sheet prices increased at an average annual rate of 0.89%. Over the same period, average annual growth rates were 1.46% for HDG sheet, 1.73% for CR sheet, and 1.63% for scrap. To generalize these trends we represent the common drift term across all products by the trapezoidal fuzzy number (0.85%, 1.10%, 1.30%, 1.50%), reflecting the structural shifts observed in the market—particularly during the 2006–2007 economic boom.

Table 2. Correlation matrix for products prices

	Scrap	CR sheet	HDG sheet	OC sheet
Scrap	1.000	0.955	0.892	0.853
CR sheet	0.955	1.000	0.961	0.921
HDG sheet	0.892	0.961	1.000	0.911
OC sheet	0.853	0.921	0.911	1.000

Table 3. Interval-regression coefficient capturing the interdependence between drift parameters of the individual product-price series

Independent variable		Dependent variable			
		Scrap	CR sheet	HDG sheet	OC sheet
Scrap	a_1		[-0.060; 1.943]	[0.369; 1.378]	[0.589; 1.554]
	a_2		[-0.018; 0.001]	[-0.006; -0.002]	[0.003; 0.0098]
CR sheet	a_1	[-0.897; 2.914]		[-1.611; 3.287]	[-1.689; 3.951]
	a_2	[-0.013; 0.041]		[-0.015; 0.039]	[-0.031; 0.069]
HDG sheet	a_1	[0.721; 1.489]	[0.399; 1.598]		[0.531; 1.811]
	a_2	[0.003; 0.006]	[-0.011; -0.003]		[0.004; 0.016]
OC sheet	a_1	[0.063; 1.689]	[-2.160; 3.866]	[-0.599; 2.112]	
	a_2	[-0.012; 0.003]	[-0.062; 0.040]	[-0.022; 0.007]	

Table 4. Interval-regression coefficient capturing the interdependence between drift parameters of apparent consumption for HDG sheet and OC sheet

Independent variable		Dependent variable	
		HDG sheet	OC sheet
HDG sheet	a_1		[-0.239; 0.800]
	a_2		[-0.050; 0.167]
OC sheet	a_1	[-1.589; 3.461]	
	a_2	[-0.040; 0.088]	

A review of apparent-consumption data for 1996 – 2022 shows that demand for HDG sheet and OC sheet grew on average by 7.55% and 8.65% per year, respectively. To generalise these dynamics, we adopt a common drift term for all products, expressed as the trapezoidal fuzzy number (6.0%,

6.5%, 7.0%, 7.5%), which captures the structural changes the sector underwent—particularly in the boom years of 2006 – 2007.

Volatility is estimated as the standard deviation of log-returns over 2000 – 2022. For prices the resulting annual volatilities are 15.02% (scrap), 19.04% (CR sheet), 18.67% (HDG sheet) and 12.86% (OC sheet); for apparent consumption they are 10.43% for HDG sheet and 15.03% for OC sheet. Consistent with the treatment of drift, we represent each volatility by a trapezoidal fuzzy number:

- Prices
- Scrap: (13.0%, 14.0%, 15.0%, 16.0%)
- CR sheet and HDG sheet: (15.0%, 17.0%, 18.0%, 20.0%)
- OC sheet: (10.0%, 11.0%, 12.0%, 13.0%)
- Apparent consumption
- HDG sheet: (8.0%, 9.0%, 10.0%, 11.0%)
- OC sheet: (12.0%, 13.0%, 14.0%, 15.0%)

Other cost assumptions are as follows. Variable production costs (energy, labour, maintenance) amount to roughly 116 USD per ton for HDG sheet and 182 USD per ton for OC sheet. Fixed operating costs are estimated at 31.28 million USD per year for the existing HDG line and an incremental 13.44 million USD per year for the proposed OC line. The OC-line capital outlay is about 42 million USD. Market-share targets are 30% for HDG sheet and 25% for OC sheet, reflecting their expected competitive positions.

Under these assumptions, the NPV of building OC-sheet plant was built. Figure 3 reports the resulting probability box, $[\underline{\Phi}, \overline{\Phi}]$, for two cases: (a) when interdependencies among input parameters are included and (b) when they are ignored. P-boxes let us disentangle the effects of the two main kinds of uncertainty. Relaxing the dependency assumptions (case b) widens the gap between the lower and upper cumulative curves, signalling greater epistemic uncertainty. The vertical distance between these curves therefore measures epistemic uncertainty, whereas the shape of each curve captures the aleatory—or purely random—component. Recognising the difference helps decision-makers see whether risk stems mainly from limited knowledge and expert-judgement bias (epistemic) or from intrinsic variability in historical data (aleatory).

5. Discussion

Figure 3 shows the simulation results in the form of a p-box, accompanied by three cumulative distributions payoffs: pignistic, ambiguity ($\lambda=0.5$), and credibility ($\beta=0.5$). The corresponding results represented as a density function can be seen in Figure 4 and cumulative distribution function in Figure 5. Utilizing these transformations, the option's value was calculated using Eq (16). The NPV and ROV values can be found in Table 5.

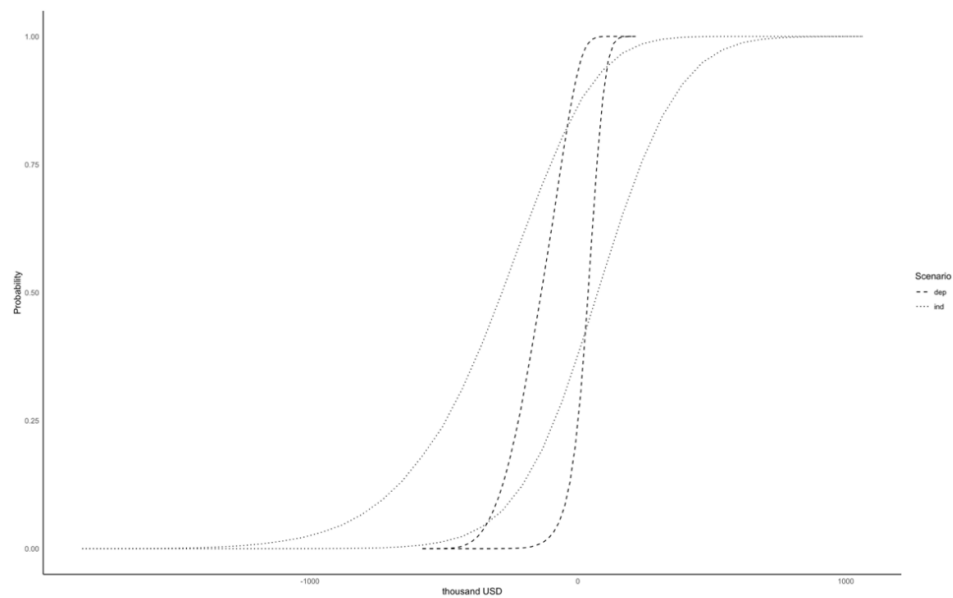


Figure 3. Net Present Value (NPV) p-box of the investment project for the construction of a new organic-coated sheet plant: (a) considering interdependencies between parameters; (b) assuming independence between parameters.

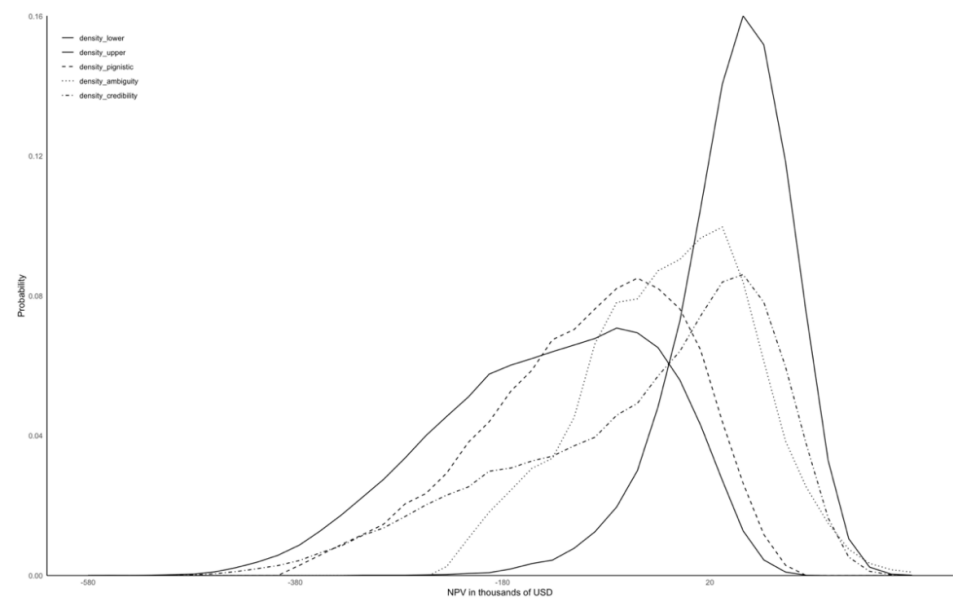


Figure 4. Comparison of NPV pay-off functions for credibility, ambiguity and pignistic transformations.

The results presented in Table 5 provide a comprehensive comparison of the various approaches considered. Utilizing the data from Table 5 allows for conducting valuation analyses using both the Net Present Value (NPV) and Real Option Valuation (ROV) methods. The classical NPV analysis, irrespective of the transformation employed, indicates a negative decision regarding the initiation of the investment. Essentially, initiating a project that does not add value to the company would not be rational. Although the standard deviation remains consistent across scenarios, the substantial percentage difference between the optimistic and pessimistic probability bounds highlights the critical importance of considering the option value in investment decisions.

Table 5. Comparison of simulation results and transformations (thousand USD).

1.		2. $\underline{\Phi}$	3. $\overline{\Phi}$	4. $\Phi_{bet}(npv)$	5. $\Phi_{amb}(\kappa, 0.5)$	6. $\Phi_{cred}(npv, 0.5)$
7. $E(npv)$		8. -3.42 USD	9. 1.05 USD	10. -2.48 USD	11. -492 USD	12. -1.18 USD
13. $Std\ dev$		14. 4.13 USD	15. 3.16 USD	16. 3.45 USD	17. 2603 USD	18. 3.06 USD
19. $Success\ ratio$		20. 8%	21. 80%	22. 15%	23. 54%	24. 55%
25. ROV		26. 2.41 USD	27. 50.91 USD	28. 4.60 USD	29. 23.72 USD	30. 26.66 USD

Conversely, adopting a real option analysis perspective provides an alternative viewpoint. Real option valuation offers insights into the project's potential upside value. If the potential benefits from the project exceed the cost of obtaining the real option, which enables capturing upside potential in response to changes, then acquiring the option becomes a sensible investment. In this analysis, three distinct outcomes are derived due to the application of three different ROV methodologies.

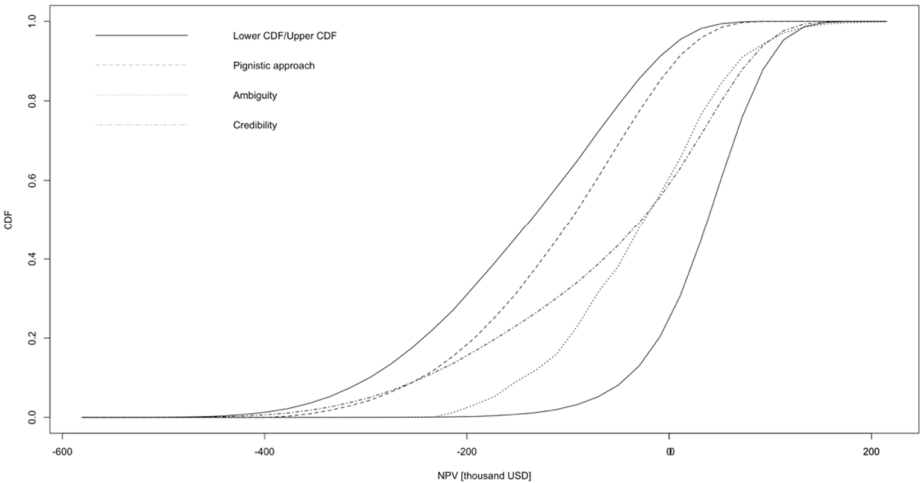


Figure 5. Transformation of the NPV p-box into subjective cumulative payoff distributions: comparison of pignistic, ambiguity ($\lambda = 0.5$), and credibility ($\beta = 0.5$) approaches.

Using pure results from the hybrid simulation, the probability of success exhibits a broad range from 8% to 80%, and the corresponding ROV spans from 2 to over 26 thousands USD. The considerable disparity between the optimistic and pessimistic bounds emphasizes the importance of distinguishing between data accuracy and expert estimations in the valuation process. Decision-makers must evaluate whether to rely on this information for investment decisions. If they find the range excessively broad, they may opt for additional research to reduce subjective uncertainty represented by fuzzy numbers for GBM parameters. However, further analysis may not always be feasible due to various constraints. In such instances, decision-makers might opt for one of the available transformations.

The pignistic probability transformation can serve as a baseline reference. The unit mass probability represents the subjective probability for each interval $f_{\alpha,j}$. Here, "subjective" implies that $m(f_{\alpha,j})$ is proportional to its appearance in simulation outcomes. This interpretation suggests the probability of epistemic states represented by $f_{\alpha,j}$. When each state within intervals is equally probable, the pignistic transformation aligns closely with pure Monte Carlo simulation. In our

example, $\Phi_{bet}(npv)$ closely mirrors Φ (with a success probability of 15%). Thus, the probability of success is influenced by the decision-maker's stance toward uncertainty and expert assessments.

The selection of other transformations depends on the perceived nature of epistemic uncertainty. The ambiguity transformation, as illustrated in this study using a weighting of 0.5, is particularly suitable for sensitivity analyses. This measure reflects the decision-maker's confidence level in expert opinions, influencing the weighting of optimistic versus pessimistic probability bounds.

Finally, the third transformation reflects the decision-maker's attitude towards uncertainty, indicating their preference for optimism or pessimism. This approach resembles the possibilistic fuzzy pay-off method for real option valuation described in [56]. It should be emphasized, however, that this transformation does not take into account investors' risk aversion. This transformation represents the investor's mental attitude towards the success of the investment or his epistemic knowledge about the effects of the investment.

Based on the pignistic transformation, the real option's value is lower than the acquisition cost, implying a negative investment decision. However, when the ROV_{amb} and ROV_{cred} transformations are considered, the real option value exceeds the acquisition cost. Therefore, these transformations support the initiation of preparatory steps for building the organic coating plant, as this would create an opportunity to capitalize on future investments in the production of the replacement product.

6. Conclusions

This study introduces a novel hybrid approach to real option valuation that integrates probabilistic and possibilistic frameworks within the Datar-Mathews Method (DMM). To the best of our knowledge, this is the first adaptation of the DMM to explicitly accommodate both aleatory and epistemic uncertainties throughout the entire valuation process. By combining random and fuzzy representations of uncertainty, the proposed method provides a more comprehensive and realistic assessment of investment projects operating under deep uncertainty and limited data availability.

One of the key strengths of the presented hybrid DMM approach lies in its ability to differentiate between uncertainty arising from inherent randomness (aleatory) and that caused by incomplete knowledge (epistemic). This distinction enables decision-makers not only to model variability in market conditions and input parameters more accurately but also to explicitly recognize the role of subjective expert judgment in scenarios where historical data are scarce or unreliable. By avoiding the premature aggregation of these two types of uncertainty during simulation, the method allows for a transparent and traceable propagation of uncertainty throughout the valuation process.

In contrast to traditional real option valuation models, which often rely solely on probabilistic assumptions, the hybrid DMM introduces an additional analytical layer that reflects the decision-maker's attitude toward uncertainty. Through the application of three different strategies for transforming p-box results into subjective probability distributions—pignistic, ambiguity-based, and credibility-based transformations—the method provides flexibility in capturing diverse decision-maker preferences, including optimism, pessimism, and varying degrees of confidence in expert assessments. This feature empowers practitioners to adjust the valuation according to their individual risk tolerance, epistemic beliefs, and strategic posture.

One of the key advantages of this hybrid DMM is its ability to effectively separate aleatoric and epistemic uncertainties in the decision-making process. As research shows [32], understanding the nature of uncertainty can help in designing interventions to improve judgment accuracy. Investors who see market uncertainty as stemming from a lack of knowledge or skill tend to seek expert guidance and react swiftly to new information, while those who view market uncertainty as driven by random processes are more inclined to diversify their portfolios to mitigate risk [33]. Unlike other methods, such as Fuzzy Real Options Valuation (FROV), this approach avoids aggregating these uncertainties during propagation through simulation. Additionally, it introduces an additional layer that explicitly requires decision-makers to address simulation results by considering potential issues like experts' over-optimism, reliance on heuristics, or other cognitive biases. Consequently, the

proposed approach allows decision-makers to assess the influence of such factors on the final decision, while also providing quantitative descriptions of their approach to uncertainty.

Overall, this paper introduces a comprehensive and robust hybrid DMM that accounts for various sources of uncertainty and provides valuable insights for decision-makers grappling with real options under high uncertainty. By incorporating both aleatoric and epistemic uncertainties, the method enables a thorough analysis of investment projects and facilitates the integration of expert knowledge, ultimately bridging the gap between real-world decision-making and real option valuation.

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