Case Report

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Expansions for the Conditional Density and Distribution of a Standard Estimate

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Abstract

Conditioning is a very useful way of using correlated information to reduce the variability of an estimate. Inference based on a conditioned estimate, can be much more precise than on an unconditioned estimate. Here we give expansions in powers of $n^{-1/2}$ for the conditional density and distribution of a multivariate standard estimate based on a sample of size n. Standard estimates include most estimates of interest, including smooth functions of sample means and other empirical estimates. We also show that a conditional estimate is not a standard estimate, so that Edgeworth-Cornish-Fisher expansions cannot be applied directly.

Keywords: conditional distributions; multivariate Edgeworth expansions; Edgeworth coefficients; standard estimates

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1. Introduction and Summary

Suppose that $\hat{w}$ is *a standard estimate* of an unknown parameter $w \in R^q$ of a statistical model, based on a sample of size n. That is, $\hat{w}$ is a consistent estimate, and for $r \geq 1$, its rth order cumulants have magnitude $n^{-(r-1)/2}$ and can be expanded in powers of n^{-1}. This is a very large class of estimates, with potential application to a range of practical problems. For example, $\hat{w}$ may be a smooth function of one or more sample means, or a smooth functional of one or more empirical distributions. A smooth function of a standard estimate is also a standard estimate: see [29]. [32] gave the multivariate Edgeworth expansions for the distribution and density of

$$X_n = n^{1/2}(\hat{w} - w),$$

in powers of $n^{-1/2}$ about the multivariate normal in terms of the *Edgeworth coefficients* of (2.3). (For typos, see p25 of [29]. Also replace $\hat{\theta}$ by $\hat{\theta}/\theta$ on 4th to last line p1121 and in (23). To line 3 p1138, add $P_{12} = B_{23}/2$.) [30] gave the Edgeworth coefficients explicitly for the Edgeworth expansions to $O(n^{-2})$. [15].

We now turn to conditioning. This is a very useful way of using correlated information to reduce the variability of estimates, and to make inference on unknown parameters more precise. This is the motivation for this paper. In Section 3 we take $q \geq 2$, and write w, $\hat{w}$ and X_n as $\binom{\mathbf{w}_1}{\mathbf{w}_2}$, $\binom{\hat{\mathbf{w}}_1}{\hat{\mathbf{w}}_2}$ and $\binom{\mathbf{X}_{n1}}{\mathbf{X}_{n2}}$ of dimensions $\binom{q_1}{q_2}$. Just as the distribution of X_n allows inference on w, the conditional distribution of $\mathbf{X}_{n1}$ given $\mathbf{X}_{n2}$, allows inference on $\mathbf{w}_1$ for a given $\mathbf{w}_2$. The covariance of $\hat{\mathbf{w}}_1|\hat{\mathbf{w}}_2$ can be substantially less than that of $\hat{\mathbf{w}}_1$. Only when $\hat{\mathbf{w}}_1$ and $\hat{\mathbf{w}}_2$ are uncorrelated, is there no advantage in conditioning.

Theorems 3.1 and 3.2 give our main results: explicit expansions to $O(n^{-2})$ for the conditional density and distribution of $\mathbf{X}_{n1}$ given $\mathbf{X}_{n2}$, that is, for the conditional density and distribution of $\hat{\mathbf{w}}_1 - \mathbf{w}_1$ given $\hat{\mathbf{w}}_2 - \mathbf{w}_2$. In other words *it gives the likely position of* $\mathbf{w}_1$ *for any given* $\mathbf{w}_2$. The main difficulty is integrating the density. Theorem 3.2 does this in terms of $\bar{I}^{1-k}$ of (3.28), the integral of the multivariate Hermite polynomial, with respect to the conditional normal density. Note 3.1 gives $\bar{I}^{1-k}$

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in terms of derivatives of the multivariate normal distribution. Theorem 3.3 gives $\bar{I}^{1-k}$ in terms of the partial moments of the conditional distribution. If $q_1 = 1$, then Theorem 3.4 gives $\bar{I}^{1-k}$ in terms of the unit normal distribution and density.

Section 4 specialises to the case $q_1 = q_2 = 1$. Examples are the condtional distribution and density of a bivariate sample mean, of entangled gamma random variables, and of a sample mean given the sample variance. Section 5 and Section 6 give conclusions, discussion, and suggestions for future research. Appendix A gives expansions for the conditional moments of $\mathbf{X}_{n1}|(\mathbf{X}_{n2} = \mathbf{x}_2)$. It shows that $\hat{\mathbf{w}}_1$ given $\mathbf{X}_{n2}$, is neither a standard estimate, nor a Type B estimate, so that Edgeworth-Cornish-Fisher expansions do not apply to it.

Conditional expansions for the sample mean were given in Chapter 12 of [4], and used in Sections 2.3 and 2.5 of [13] to show bootstrap consistency.

2. Multivariate Edgeworth Expansions

Suppose that $\hat{w}$ is a *standard estimate* of $w \in R^q$ with respect to n. (n is typically the sample size.) That is, $E\,\hat{w} \to w$ as $n \to \infty$, where we use E for expected value, and for $r \geq 1$ and $1 \leq i_1, \ldots, i_r \leq q$, the rth order cumulants of $\hat{w} = (\hat{w}_1, \ldots, \hat{w}_q)'$ can be expanded as

$$\bar{k}^{1-r} = k^{i_1\ldots i_r} = \kappa(\hat{w}_{i_1}, \ldots, \hat{w}_{i_r}) \approx \sum_{d=r-1}^{\infty} n^{-d}\, \bar{k}_d^{1-r}, \text{ where } \bar{k}_d^{1-r} = k_d^{i_1\ldots i_r}, \tag{2.1}$$

where $\approx$ indicates an asymptotic expansion, and the *cumulant coefficients* $\bar{k}_d^{1-r}$ may depend on n but are bounded as $n \to \infty$. So *the bar replaces each i_k by k.* For example $\bar{k}_0^1 = w^{i_1}$ and $\bar{k}_1^{12} = k_1^{i_1 i_2}$. *We reserve i_k for this bar notation* to avoid double subscripts.

$$\text{As } n \to \infty,\ X_n = n^{1/2}(\hat{w} - w) \xrightarrow{\mathcal{L}} X = \mathcal{N}_q(0, V),\ \text{for } V = (\bar{k}_1^{12}),\ q \times q, \tag{2.2}$$

the multivariate normal on R^q, with density and distribution

$$\phi_V(x) = (2\pi)^{-q/2}(det\ V)^{-1/2}\exp(-x'V^{-1}x/2),\ \Phi_V(x) = \int_{-\infty}^{x} \phi_V(x)dx.$$

V may depend on n, but we assume that $det\ V$ is bounded away from 0.

$$\text{Let } \bar{P}_r^{1-k} = P_r^{i_1\ldots i_k} \text{ be } \textit{the rth Edgeworth coefficient} \text{ of } \hat{w}, \tag{2.3}$$

for $q \geq 1 \leq r \leq 3$. These are Bell polynomials in the cumulant coefficients of (2.1), as defined and given in [30]. Their importance lies in their central role in the Edgeworth expansions of X_n of (2.2).

(When $q = 1$ and $\hat{w}$ is a sample mean, the Edgeworth coefficients were given for *all* r in [31]. For typos, see p24–25 of [29].) Set $P(A)$ =Probability A is true. By [32], or [29], for $\hat{w}$ non-lattice, the distribution and density of X_n can be expanded as

$$P(X_n \leq x) \approx \sum_{r=0}^{\infty} n^{-r/2} P_r(x),\ p_{X_n}(x) \approx \sum_{r=0}^{\infty} n^{-r/2} p_r(x),\ x \in R^q, \tag{2.4}$$

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$$\text{where } P_0(x) = \Phi_V(x),\ p_0(x) = \phi_V(x),$$

$$\text{and for } r \geq 1,\ P_r(x) = \sum_{k=1}^{3r} [P_{rk}(x) :\ k - r \text{ even}],$$

$$p_r(x)/\phi_V(x) = \sum_{k=1}^{3r} [\tilde{p}_{rk} :\ k - r \text{ even}] = \tilde{p}_r(x) \text{ say}, \tag{2.5}$$

$$P_{rk}(x) = \bar{P}_r^{1-k}\, \bar{H}_*^{1-k},\ \tilde{p}_{rk} = \bar{P}_r^{1-k} \bar{H}^{1-k}, \tag{2.6}$$

$$\bar{H}_*^{1-k} = \bar{H}_*^{1-k}(x, V) = \bar{O}^{1-k}\, \Phi_V(x) = \int_{-\infty}^{x} \bar{H}^{1-k} \phi_V(x) dx, \tag{2.7}$$

$$\bar{O}^{1-k} = (-\bar{\partial}_1) \ldots (-\bar{\partial}_k),\ \bar{\partial}_k = \partial_{i_k},\ \partial_i = \partial/\partial x_i,$$

$$\bar{H}^{1-k} = H^{i_1 \ldots i_k} = \phi_V(x)^{-1} \bar{O}^{1-k}\, \phi_V(x) = E\,(\bar{y}_1 + I\bar{Y}_1) \ldots (\bar{y}_k + I\bar{Y}_k) \tag{2.8}$$

$$\text{and } I = \sqrt{-1},\ y = V^{-1}x,\ Y = V^{-1}X \sim \mathcal{N}_q(0, V^{-1}). \tag{2.9}$$

$\bar{H}^{1-k}(x, V) = \bar{H}^{1-k}$ is *the multivariate Hermite polynomial*. We use the tensor summation convention, repetition of $i_1, \ldots, i_k$ in (2.6) implies their implicit summation over their range, $1, \ldots, q$. [30] gave $\bar{H}^{1-k}$ explicitly for $k \leq 6$ and for $k \leq 9$ when $q = 2$.

$$\text{Set } \bar{\mu}^{1-2k} = E\, \bar{Y}_1 \ldots \bar{Y}_{2k} = \sum^{1.3\ldots(2k-1)} \bar{V}^{12} \ldots \bar{V}^{2k-1,2k}, \tag{2.10}$$

where $\sum^N \bar{f}^{1-2k}$ sums $\bar{f}^{1-2k}$ over all N permutations of $i_1, \ldots, i_{2k}$ giving distinct values. For example,

$$\bar{H}^1 = \bar{y}_1,\ \bar{H}^{12} = \bar{y}_1\bar{y}_2 - \bar{V}^{12},$$

$$\bar{H}^{1-3} = \bar{y}_1\bar{y}_2\bar{y}_3 - \sum^3 \bar{y}_1 \bar{V}^{23} = \bar{y}_1\bar{y}_2\bar{y}_3 - \bar{y}_1\bar{V}^{23} - \bar{y}_2\bar{V}^{13} - \bar{y}_3\bar{V}^{12},$$

$$\bar{H}_*^1 = \bar{J}^1,\ \bar{H}_*^{12} = \bar{J}^{12} - \bar{V}^{12}\Phi_V(x),\ \bar{H}_*^{1-3} = \bar{J}^{123} - \sum^3 \bar{J}^1 \bar{V}^{23}, \text{ where}$$

$$\bar{J}^{1-k} = \bar{J}^{1-k}(x, V) = E\, \bar{Y}_1 \ldots \bar{Y}_k\, I(X \leq x) = \bar{V}^{1,k+1} \ldots \bar{V}^{k,2k} \bar{M}_V^{k+1-2k}, \tag{2.11}$$

$$\text{and } \bar{M}_V^{a-b} = \bar{M}^{a-b}(x, V) = \int_{-\infty}^{x} \bar{x}_a \ldots \bar{x}_b\, \phi_V(x) dx, \text{ for } \bar{x}_a = x_{i_a}. \tag{2.12}$$

$$\text{So, } P_1(x) = \sum_{k=1,3} P_{1k}(x),\ P_{11}(x) = \bar{k}_1^1\, \bar{H}_*^1,\ P_{13}(x) = \bar{k}_2^{1-3}\, \bar{H}_*^{1-3}/6,$$

$$\tilde{p}_1(x) = p_1(x)/\phi_V(x) = \sum_{k=1,3} \tilde{p}_{1k},\ \tilde{p}_{11} = \bar{k}_1^1\, \bar{H}^1,\ \tilde{p}_{13} = \bar{k}_2^{1-3}\, \bar{H}^{1-3}/6. \tag{2.13}$$

(So the repeated $i_{k+1}, \ldots, i_{2k}$ in (2.11) implies their repeated summatioin over $1, \ldots, q$.) $P_2(x)$, $P_3(x)$ are given explicitly in [30]. So (2.4) with the $\bar{P}_r^{1-k}$ in [30] give the Edgeworth expansions for the distribution and density of X_n of (2.2) to $O(n^{-2})$. $\tilde{p}_{rk}$ and P_{rk} each have q^k terms, but many are duplicates as $\bar{P}_r^{1-k}$ is symmetric in $i_1, \ldots, i_k$. This is exploited by the notation of Section 4 of [30] to greatly reduce the number of terms in (2.6).

By (2.5), the density of X_n relative to its asymptotic value is

$$p_{X_n}(x)/\phi_V(x) \approx 1 + \sum_{r=1}^{\infty} n^{-r/2} \tilde{p}_r(x) = 1 + n^{-1/2} \tilde{p}_1(x) + O(n^{-1}), \text{ for } x \in R^q,$$

and for measurable $C \subset R^q$,

$$P(X_n \in C) \approx \Phi_V(C) + \sum_{r=1}^{\infty} n^{-r/2} p_{rC}, \text{ where for },$$
$$p_{rC} = E\, p_r(X)\; I(X \in C) = \int_C p_r(x)\phi_V(x)dx = \sum_{k=1}^{3r} [\tilde{p}_{rk}(C) : k - r \text{ even}],$$
$$\tilde{p}_{rk}(C) = E\, \tilde{p}_{rk}(X) I(X \in C) = \int_C \tilde{p}_{rk}(x)\phi_V(x)dx = \bar{P}_r^{1-k} \bar{H}^{1-k}(C),$$
$$\text{and } \bar{H}^{1-k}(C) = E\, \bar{H}^{1-k}(X, V)\; I(X \in C) = \int_C \bar{H}^{1-k}\phi_V(x)dx.$$

If $-C = C$, then for r odd, $\bar{Q}^{1-r} = \tilde{p}_{rk}(C) = p_{rC} = 0$, so that

$$P(X_n \in C) \approx \Phi_V(C) + \sum_{r=1}^{\infty} n^{-r} p_{2rC} = \Phi_V(C) + n^{-1} p_{2C} + O(n^{-2}). \tag{2.14}$$

Examples 3 and 4 of [30] gave $p_{2C}$for
$C = \{x : x'V^{-1}x \leq u\}$, and $C = \{x : |(V^{-1/2}x)_j| \leq u_j,\ j = 1, \ldots, q\}$.

3. The Conditional Density and Distribution

For $q = q_1 + q_2$, $q_1 \geq 1$, and $q_2 \geq 1$, partition w, $\hat{w}$, $X \sim \mathcal{N}_q(0, V)$,
$X_n = n^{1/2}(\hat{w} - w)$, x and $y = V^{-1}x$ as $\binom{\mathbf{w}_1}{\mathbf{w}_2}$, $\binom{\hat{\mathbf{w}}_1}{\hat{\mathbf{w}}_2}$, $\binom{\mathbf{X}_1}{\mathbf{X}_2}$, $\binom{\mathbf{X}_{n1}}{\mathbf{X}_{n2}}$, $\binom{\mathbf{x}_1}{\mathbf{x}_2}$ and $\binom{\mathbf{y}_1}{\mathbf{y}_2}$, where $\mathbf{w}_i$, $\hat{\mathbf{w}}_i$, $\mathbf{X}_i$, $\mathbf{X}_{ni}$, $\mathbf{x}_i$, $\mathbf{y}_i$ are vectors of length q_i. Partition V, V^{-1} as $(\mathbf{V}_{ij}), (\mathbf{V}^{ij})$, 2×2, where $\mathbf{V}_{ij}$, $\mathbf{V}^{ij}$ are $q_i \times q_j$.

$$\text{Set } \mathbf{X}_{1.2} = \mathbf{X}_1 | (\mathbf{X}_2 = \mathbf{x}_2), \quad \mathbf{X}_{n1.2} = \mathbf{X}_{n1} | (\mathbf{X}_{n2} = \mathbf{x}_2), \tag{3.1}$$
$$\hat{\mathbf{w}}_{1.2} = \hat{\mathbf{w}}_1 | (\mathbf{X}_{n2} = \mathbf{x}_2) = \mathbf{w}_1 + n^{-1/2}\mathbf{X}_{n1.2}. \tag{3.2}$$

Now we come to the main purpose of this paper. Theorem 3.1 expands the conditional density of $\mathbf{X}_{n1.2}$ about the conditional density of $\mathbf{X}_{1.2}$. Its derivation is straightforward, the only novel feature being the use of Lemma 3.2 to find the reciprocal of a series, using Bell polynomials. Theorem 3.2 integrates the conditional density to obtain the expansion for the conditional distribution of $\mathbf{X}_{n1.2}$ about the conditional distribution of $\mathbf{X}_{1.2}$ in terms of $\bar{I}^{1-k}$ of (3.28) below, the integral of the Hermite polynomial $\bar{H}^{1-k}$ of (2.8), with respect to the conditional normal density. Note 3.1 gives $\bar{I}^{1-k}$ in terms of derivatives of the multivariate normal distribution. Theorem 3.3 gives $\bar{I}^{1-k}$ in terms of the partial moments of the conditional normal distribution. For $\mathbf{X}_{1.2}$ of (3.1), set

$$\mu_{1.2} = E\, \mathbf{X}_{1.2} = \mathbf{V}_{12}\mathbf{V}_{22}^{-1}\mathbf{x}_2 \in R^{q_1}, \tag{3.3}$$
$$V_{1.2} = covar\ \mathbf{X}_{1.2} = \mathbf{V}_{11} - \mathbf{V}_{12}\mathbf{V}_{22}^{-1}\mathbf{V}_{21} = V_0 \text{ say}, \in R^{q_1 \times q_1}. \tag{3.4}$$
$$\text{So, } \mathbf{X}_{1.2} \sim \mathcal{N}_{q_1}(\mu_{1.2}, V_{1.2}). \tag{3.5}$$

Lemma 3.1. *The elements of* $(\mathbf{V}^{ij}) = V^{-1}$ *are*

$$\mathbf{V}^{11} = V_{1.2}^{-1},\ V^{12} = -\mathbf{V}^{11}\mathbf{V}_{12}\mathbf{V}_{22}^{-1} = -\mathbf{V}_{11}^{-1}\mathbf{V}_{12}\mathbf{V}^{22},$$
$$V^{21} = -\mathbf{V}^{22}\mathbf{V}_{21}\mathbf{V}_{11}^{-1} = -\mathbf{V}_{22}^{-1}\mathbf{V}_{21}\mathbf{V}^{11},$$
$$\mathbf{V}^{22} = V_{2.1}^{-1},\ \textit{where } V_{2.1} = \mathbf{V}_{22} - \mathbf{V}_{21}\mathbf{V}_{11}^{-1}\mathbf{V}_{12} \in R^{q_2 \times q_2}.$$
$$\textit{For } i = 1, 2, \textit{ set } A_i = \mathbf{V}^{i1}\mathbf{V}_{12}\mathbf{V}_{22}^{-1} + \mathbf{V}^{i2}.\ \textit{Then } A_1 = 0_{q_1 \times q_2},\ A_2 = \mathbf{V}_{22}^{-1}. \tag{3.6}$$

PROOF $VV^{-1} = V^{-1}V = I_q$ gives 8 equations relating $\{\mathbf{V}^{ij}\}$ and $\{\mathbf{V}_{ij}\}$. Now solve for $\{\mathbf{V}^{ij}\}$.
So $A_1 = 0_{q_1 \times q_2}$, $A_2 = \mathbf{V}^{22} B \mathbf{V}_{22}^{-1}$ for $B = \mathbf{V}_{22} - \mathbf{V}_{21}\mathbf{V}_{11}^{-1}\mathbf{V}_{12} = (\mathbf{V}^{22})^{-1}$. □
Since $Q = \mathbf{V}_{11} - V_{1.2} \geq 0_{q_1 \times q_1}$ in the sense that $x'Qx \geq 0$ for $x \in R^{q_1}$, $\mathbf{X}_{1.2}$ is less variable than $\mathbf{X}_1$, and $\mathbf{X}_{n1.2}$ is less variable than $\mathbf{X}_{n1}$, unless $\mathbf{X}_1$ and $\mathbf{X}_2$ are uncorrelated, that is, $\mathbf{V}_{12}$ is a matrix of zeros.

The *conditional density* of $\mathbf{X}_{n1.2}$ is

$$p_{n1.2}(\mathbf{x}_1) = p_{X_n}(x)/p_{X_{n2}}(\mathbf{x}_2) = \phi_{1.2}(\mathbf{x}_1)\ (1+S)/(1+S_2), \tag{3.7}$$

$$\text{where } S = p_{X_n}(x)/\phi_V(x) - 1 \approx \sum_{r=1}^{\infty} n^{-r/2}\tilde{p}_r(x) \text{ of (2.6)},$$
$$S_2 = p_{X_{n2}}(\mathbf{x}_2)/\phi_{\mathbf{V}_{22}}(\mathbf{x}_2) - 1 \approx \sum_{r=1}^{\infty} n^{-r/2} f_r, \text{ for } f_r = p_r^*(\mathbf{x}_2), \tag{3.8}$$

where $p_r^*(\mathbf{x}_2)$ is $\tilde{p}_r(x)$ of (2.6) for $\mathbf{X}_{n2}$, and $\phi_{1.2}(\mathbf{x}_1)$ is the density of $\mathbf{X}_{1.2}$ of (3.1). By (4)–(6), Section 2.5 of [1], for V_0 of (3.4),

$$\phi_{1.2}(\mathbf{x}_1) = \phi_V(x)/\phi_{\mathbf{V}_{22}}(\mathbf{x}_2) = \phi_{V_0}(u), \text{ where } u = \mathbf{x}_1 - \mu_{1.2} \in R^{q_1}. \tag{3.9}$$

So the distribution of $\mathbf{X}_1|(\mathbf{X}_2 = \mathbf{x}_2)$ is

$$\Phi_{1.2}(\mathbf{x}_1) = \Phi_{V_0}(u), \text{ for } V_0 \text{ of (3.4)}. \tag{3.10}$$

For $\mu_{1.2}$ of (3.3), $V_{1.2}$ of (3.4), and $v \in R^{q_1}$, set

$$x_1(x_2, v) = \mu_{1.2} + V_{1.2}^{1/2} v = \mathbf{V}_{12}\mathbf{V}_{22}^{-1}\mathbf{x}_2 + V_{1.2}^{1/2} v. \tag{3.11}$$

Corollary 3.1. *Suppose that $q_1 = 1$. Then for $v = V_0^{-1/2} u$ of (3.9),*

$$\begin{aligned}
&\phi_{1.2}(\mathbf{x}_1) = V_0^{-1/2}\phi(v),\ P(X_{1.2} < x_1(x_2, v)) = \Phi(v),\\
&P(\hat{w}_{1.2} < w_1 + n^{-1/2} x_1(x_2, v)) = P(X_{n1.2} < x_1(x_2, v)) = \Phi(v) + O(n^{-1/2}),\\
&P(\ |w_1 - \hat{w}_{1.2}| < n^{-1/2} x_1(x_2, v)) =\ P(\ |X_{n1.2}| < x_1(x_2, v))\\
&= 2\Phi(v) - 1 + O(n^{-1}) \textit{ if } v > 0.
\end{aligned} \tag{3.12}$$

If $q_1 = 1$, this gives an asymptotic conditional confidence limit for $\hat{w}_1 - w_1$ given $\hat{\mathbf{w}}_2 - \mathbf{w}_2$. So if $q_1 = 1$, by (2.14), for $x_1(x_2, v)$ of (3.11), 2-sided limits are

$$P(x_1(x_2, -v) < X_{n1.2} < x_1(x_2, v)) = 2\Phi(v) - 1 + O(n^{-1}), \text{ if } v > 0.$$
$$\text{Set } \bar{H}_q^{1-k} = \bar{H}^{1-k} = \bar{H}^{1-k}(x, V), \text{ and } \bar{H}_{q_2}^{1-k} = \bar{H}^{1-k}(\mathbf{x}_2, \mathbf{V}_{22}).$$

So $\bar{H}_{q_2}^{1-k}$ is given by replacing $y = V^{-1}x$ and $(V^{ij}) = V^{-1}$ in $\bar{H}_q^{1-k}$ by

$$z = \mathbf{V}_{22}^{-1}\mathbf{x}_2 \text{ and } (U^{ij}) = \mathbf{V}_{22}^{-1}. \text{ For example } \bar{H}_{q_2}^{12} = \bar{z}_1\bar{z}_2 - \bar{U}^{12}. \tag{3.13}$$

By (2.5) and (2.6), for $r \geq 1$, $p_r^*(\mathbf{x}_2)$ of (3.8) is given by

$$p_r^*(\mathbf{x}_2) = \sum_{k=1}^{3r} [p_{rk}^* :\ k - r \text{ even}], \text{ where } p_{rk}^* = \bar{P}_r^{1-k}\bar{H}_{q_2}^{1-k}, \tag{3.14}$$

and *implicit summation in* (3.14) *for* $i_1, \ldots, i_k$ *is now over* $q_1+1, \ldots, q$. So,

$$p_1^*(\mathbf{x}_2) = \sum_{k=1,3} p_{1k}^*,\ p_{11}^* = \sum_{i_1=q_1+1}^{q} \bar{k}_1^1\, \bar{H}_{q_2}^1,\ p_{13}^* = \sum_{i_1,i_2,i_3=q_1+1}^{q} \bar{k}_2^{1-3}\, \bar{H}_{q_2}^{1-3}/6,$$

$$\text{where } \bar{H}_{q_2}^1 = \bar{z}_1,\ \bar{H}_{q_2}^{1-3} = \bar{z}_1\bar{z}_2\bar{z}_3 - \sum^3 \bar{U}^{12}\bar{z}_3,$$

$$p_2^*(\mathbf{x}_2) = \sum_{k=2,4,6} p_{2k}^*,\ p_3^*(\mathbf{x}_2) = \sum_{k=1,3,5,7,9} p_{3k}^*,\ \text{for } p_{rk}^* \text{ of (3.14).}$$

Ordinary Bell polynomials. For a sequence $e = (e_1, e_2, \ldots)$ from R, the *partial ordinary Bell polynomial* $\tilde{B}_{rs} = \tilde{B}_{rs}(e)$, is defined by the identity

$$\text{for } s = 0,1,2,\ldots \text{ and } z \in R,\ S^s = \sum_{r=s}^{\infty} z^r \tilde{B}_{rs}(e),\ \text{where } S = \sum_{r=1}^{\infty} z^r e_r. \tag{3.15}$$

$$\text{So, } \tilde{B}_{r0} = \delta_{r0},\ \tilde{B}_{r1} = e_r,\ \tilde{B}_{rr} = e_1^r,\ \tilde{B}_{32} = 2e_1e_2,$$

where $\delta_{00} = 1,\ \delta_{r0} = 0$ for $r \neq 0$. They are tabled on p309 of [7]. To obtain (3.7), we use

Lemma 3.2. *Take* $\tilde{B}_{rs}(e)$ *of* (3.15). *Set* $S_2 = \sum_{r=1}^{\infty} z^r f_r$ *for* $f_r \in R$. *Then*

$$(1+S_2)^{-1} = \sum_{r=0}^{\infty} z^r C_r,\ \text{where } C_r = B_r^*(-f),\ B_r^*(e) = \sum_{s=0}^{r} \tilde{B}_{rs}(e). \tag{3.16}$$

$$\text{So, } B_0^*(e) = 1,\ B_1^*(e) = e_1,\ B_2^*(e) = e_2 + e_1^2,\ B_3^*(e) = e_3 + 2e_1e_2 + e_1^3,$$

$$C_0 = 1,\ C_1 = -f_1,\ C_2 = f_1^2 - f_2,\ C_3 = -f_1^3 + 2f_1f_2 - f_3. \tag{3.17}$$

PROOF

$$(1+S_2)^{-1} = \sum_{s=0}^{\infty} (-S_2)^s,\ \text{and } (-S_2)^s = \sum_{r=s}^{\infty} z^r \tilde{B}_{rs}(-f).$$

Now swap summations. □

Theorem 3.1. *Take* $\tilde{p}_r(x)$ *of* (2.6) *and* $C_r = B_r^*(-f)$ *of* (3.16) *with*

$$f_r = p_r^*(\mathbf{x}_2) \text{ of (3.14).} \tag{3.18}$$

The conditional density $p_{n1.2}(\mathbf{x}_1)$ *of* (3.7), *relative to* $\phi_{1.2}(\mathbf{x}_1)$ *of* (3.9), *is*

$$p_{n1.2}(\mathbf{x}_1)/\phi_{1.2}(\mathbf{x}_1) \approx \sum_{r=0}^{\infty} n^{-r/2} D_r,\ \text{where } D_r = C_r \otimes \tilde{p}_r(x), \tag{3.19}$$

and for sequences $(a_0, a_1, \ldots)$ *and* $(b_0, b_1, \ldots)$, $a_r \otimes b_r = \sum_{i=0}^{r} a_i b_{r-i}$. *So,*

$$D_0 = \tilde{p}_0(x) = 1,\ D_1 = C_1 + \tilde{p}_1(x),\ D_2 = C_2 + C_1\tilde{p}_1(x) + \tilde{p}_2(x), \tag{3.20}$$

$$D_3 = C_3 + C_2\tilde{p}_1(x) + C_1\tilde{p}_2(x) + \tilde{p}_3(x). \tag{3.21}$$

PROOF This follows from (3.7) and Lemma 3.2. □

So $D_0, \ldots, D_3$ of (3.20) and (3.21) give the conditional density to $O(n^{-2})$. We call (3.19) *the relative conditional density.* We now give our main result, an expansion for the conditional distribution of $\mathbf{X}_{n1}|(\mathbf{X}_{n2} = \mathbf{x}_2)$. As noted, Theorem 3.2 gives this in terms of $\bar{I}^{1-k}$ of (3.28) below, an integral of the Hermite polynomial $\bar{H}^{1-k}$ of (2.8), and Note 3.1 gives $\bar{I}^{1-k}$ in terms of derivatives of the multivariate

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normal distribution. Theorem 3.3 gives $\bar{I}^{1-k}$ in terms of the partial moments of the conditional distribution $\Phi_{1.2}(\mathbf{x}_1)$ of (3.10). When $q_1 = 1$, Theorem 3.4 gives $\bar{I}^{1-k}$ in terms of $\Phi(v)$ and $\phi(v)$ for

$$v = V_{1.2}^{-1/2} u = V_{1.2}^{-1/2}(x_1 - \mu_{1.2}) \in R^{q_1}. \tag{3.22}$$

Theorem 3.2. *Take C_r, D_r of Theorem 3.1. Set $\tilde{p}_0(x) = 1$. The conditional distribution of $\mathbf{X}_{n1}$ given $\mathbf{X}_{n2}$, about $\Phi_{1.2}(\mathbf{x}_1)$ of (3.10), has the expansion*

$$P_{n1.2}(\mathbf{x}_1) = P(\mathbf{X}_{n1.2} \leq \mathbf{x}_1 \approx \sum_{r=0}^{\infty} n^{-r/2} G_r, \tag{3.23}$$

$$\text{where } G_r = \int_{-\infty}^{\mathbf{x}_1} D_r \, d\Phi_{1.2}(\mathbf{x}_1) \ = C_r \otimes g_r, \text{ and} \tag{3.24}$$

$$g_r = \int_{-\infty}^{\mathbf{x}_1} \tilde{p}_r(x) \, d\Phi_{1.2}(\mathbf{x}_1). \text{ So, } G_0 = g_0 = \Phi_{1.2}(\mathbf{x}_1) = \Phi_{V_0}(u) \text{ of (3.10)}, \tag{3.25}$$

$$\text{for } r \geq 1,\ g_r = \sum_{k=1}^{3r} [g_{rk} :\ k - r \text{ even}], \text{ where for } \tilde{p}_{rk} \text{ of (2.6)}, \tag{3.26}$$

$$g_{rk} = \int_{-\infty}^{\mathbf{x}_1} \tilde{p}_{rk} \, d\Phi_{1.2}(\mathbf{x}_1) = \bar{P}_r^{1-k} \bar{I}^{1-k}, \text{ and for } 1 \leq i_1, \ldots, i_k \leq q, \tag{3.27}$$

$$\bar{I}^{1-k} = I^{i_1 \ldots i_k} = \int_{-\infty}^{\mathbf{x}_1} \bar{H}_q^{1-k} \phi_{1.2}(\mathbf{x}_1) d\mathbf{x}_1, \text{ for } \phi_{1.2}(\mathbf{x}_1) \text{ of (3.9)}. \tag{3.28}$$

PROOF (3.26) holds by (2.6). (3.27) holds by (2.6). Now use (3.9). □

$$\text{So } P_{n1.2}(\mathbf{x}_1) = \Phi_{V_0}(u) + \sum_{r=1}^{3} n^{-r/2} G_r + O(n^{-2}), \text{ where } G_1 = g_1 - f_1 g_0,$$
$$G_2 = g_2 - f_1 g_1 + C_2 g_0,\ G_3 = g_3 - f_1 g_2 + C_2 g_1 + C_3 g_0, \tag{3.29}$$

for C_r of (3.17). $g_1 = g_{11} + g_{13}$ is given by $\bar{I}^1, \bar{I}^{1-3}$, $g_2 = g_{22} + g_{24} + g_{26}$ is given by $\bar{I}^{12}, \bar{I}^{1-4}, \bar{I}^{1-6}$, and $g_3 = \sum(g_{3k} :\ k = 1, 3, 5, 7, 9)$ is given by $\bar{I}^1, \bar{I}^{1-3}, \ldots, \bar{I}^{1-9}$.

Note 3.1. *Set $\partial^\dagger = \Pi_{i=q_1+1}^{q} \partial_i$. By (3.9),*

$$\bar{I}^{1-k} = \phi_{\mathbf{V}_{22}}(\mathbf{x}_2)^{-1} \, \bar{L}^{1-k}, \text{ where}$$
$$\bar{L}^{1-k} = \int_{-\infty}^{\mathbf{x}_1} \bar{H}_q^{1-k} \phi_V(x) d\mathbf{x}_1 = (-\bar{\partial}_1) \ldots (-\bar{\partial}_k) \partial^\dagger \, \Phi_V(x). \tag{3.30}$$

Comparing $\bar{L}^{1-k}$ with the Hermite function $\bar{H}_*^{1-k}$ *of (2.7), we can call $\bar{L}^{1-k}$* the partial Hermite function. *When $q = 2$, see (4.1).*

By (3.25), G_r in (3.23) is given by C_r of (3.18) and g_r of (3.26). Viewing $\bar{H}_q^{1-k}$ as a polynomial in $\mathbf{x}_1 = \mu_{1.2} + u$ for u of (3.9), $\bar{I}^{1-k}$ is linear in

$$\int_{-\infty}^{\mathbf{x}_1} x_{i_1} \ldots x_{i_s} \, \phi_{1.2}(\mathbf{x}_1) d\mathbf{x}_1 = \int_{-\infty}^{u} (\mu_{1.2} + u)_{i_1} \ldots (\mu_{1.2} + u)_{i_s} \, \phi_{V_0}(u) du$$

for $0 \leq s \leq k$, $1 \leq i_1, \ldots, i_s \leq q_1$. So $\bar{I}^{1-k}$ can be expanded in terms of *the partial moments* of

$$M = \Phi_{1.2}(\mathbf{x}_1) = \Phi_{V_0}(u), \tag{3.31}$$

$$\bar{M}^{a-b} = \bar{M}^{a-b}(u, V_0) = M_{i_a \ldots i_b}(u, V_0) = \int_{-\infty}^{u} u_{i_a} \ldots u_{i_b} \, \phi_{V_0}(u) du. \tag{3.32}$$

This has only q_1 integrals, while (2.12) has q integrals.

Lemma 3.3. *For* $u = \mathbf{x}_1 - \mu_{1.2}$, $y = V^{-1}x = \alpha + \Lambda u$, *where*

$$\Lambda = \begin{pmatrix} \mathbf{V}^{11} \\ \mathbf{V}^{21} \end{pmatrix} \in R^{q\times q_1},\ \alpha = \begin{pmatrix} \mathbf{ff}_1 \\ \mathbf{ff}_2 \end{pmatrix},\ \mathbf{ff}_1 = 0_{q_1},\ \mathbf{ff}_2 = \mathbf{V}_{22}^{-1}\mathbf{x}_2. \tag{3.33}$$

PROOF $y = \binom{\mathbf{y}_1}{\mathbf{y}_2}$, where $\mathbf{y}_i = \mathbf{V}^{i1}\mathbf{x}_1 + \mathbf{V}^{i2}\mathbf{x}_2 = \mathbf{ff}_i + \mathbf{V}^{i1}u$, and $\mathbf{ff}_i = A_i\mathbf{x}_2$ for A_i of (3.6). □
Our main result, Theorem 3.2, gave the conditional distribution expansion in terms of $\bar{I}^{1-k}$ of (3.28). Note 4.1 gave these in terms of the derivatives of $\Phi_V(x)$. We now give $\bar{I}^{1-k}$ in terms of $\bar{J}^{1-k}$, the partial moments of the conditional distribution $\Phi_{1.2}(\mathbf{x}_1)$ of (3.10). As in (2.10), for any $\pi = (m,\ldots,n)$, set $\sum^N c_\pi = \sum c_\pi$ summed over all, N say, permutations of π giving distinct c_π. For example, $\sum^2 c_{23} = c_{23} + c_{32}$.

Theorem 3.3. *Take* $\bar{J}^{1-k}(x,V)$ *of* (2.11), u *of* (3.10), M *of* (3.31), $\bar{M}^{a-b}$ *of* (3.32), Λ, α *of* (3.33), *and* $1 \le i_1,\ldots,i_k \le q$. *Set*

$$\bar{K}^{1-k} = K^{i_1\ldots i_k} = \int_{-\infty}^{u} (\Lambda u)_{i_1}\ldots(\Lambda u)_{i_k}\,\phi_{V_0}(u)du = \Lambda_{i_1j_1}\ldots\Lambda_{i_kj_k}M^{j_1,\ldots j_k},$$

where $j_1,\ldots j_k$ *sum over their range* $1,\ldots,q_1$. *So* ,

$$\bar{K}^{1-k} = \bar{\Lambda}_{1,k+1}\ldots\bar{\Lambda}_{k,2k}\,\bar{M}^{k+1-2k}. \tag{3.34}$$

For example, $\bar{K}^1 = \bar{\Lambda}_{12}\bar{M}^2$, $\bar{K}^{12} = \bar{\Lambda}_{13}\bar{\Lambda}_{24}\bar{M}^{34}$, $\bar{K}^{123} = \bar{\Lambda}_{14}\bar{\Lambda}_{25}\bar{\Lambda}_{36}\bar{M}^{456}$.

$$\text{Set } \bar{J}_0^{1-k} = \bar{J}^{1-k}(u,V_0) = \int_{-\infty}^{u} \bar{y}_1\ldots\bar{y}_k\,\phi_{V_0}(u)du: \tag{3.35}$$

$$\bar{J}_0^{1-k} = \bar{\alpha}_1\ldots\bar{\alpha}_k M + \sum^{\binom{k}{1}}\bar{\alpha}_1\ldots\bar{\alpha}_{k-1}\bar{K}^k + \sum^{\binom{k}{2}}\bar{\alpha}_1\ldots\bar{\alpha}_{k-2}\bar{K}^{k-1,k} + \cdots + \bar{K}^{1-k}.$$

For example, $\bar{J}_0^1 = \bar{\alpha}_1 M + \bar{K}^1$, $\bar{J}_0^{12} = \bar{\alpha}_1\bar{\alpha}_2 M + \sum^2 \bar{\alpha}_1\bar{K}^2 + \bar{K}^{12}$,

$$\bar{J}_0^{1-3} = \bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3 M + \sum^3(\bar{\alpha}_1\bar{\alpha}_2\bar{K}^3 + \bar{\alpha}_1\bar{K}^{23}) + \bar{K}^{1-3},$$

$$\bar{J}_0^{1-4} = \bar{\alpha}_1\ldots\bar{\alpha}_4 M + \sum^4(\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3\bar{K}^4 + \bar{\alpha}_1\bar{K}^{234}) + \sum^6\bar{\alpha}_1\bar{\alpha}_2\bar{K}^{34} + \bar{K}^{1-4},$$

$$\bar{J}_0^{1-5} = \bar{\alpha}_1\ldots\bar{\alpha}_5 M + \sum^5(\bar{\alpha}_1\ldots\bar{\alpha}_4\bar{K}^5 + \bar{\alpha}_1\bar{K}^{2-5}) + \sum^{10}(\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3\bar{K}^{45} + \bar{\alpha}_1\bar{\alpha}_2\bar{K}^{3-5}) + \bar{K}^{1-5},$$

$$\bar{J}_0^{1-6} = \bar{\alpha}_1\ldots\bar{\alpha}_6 M + \sum^6(\bar{\alpha}_1\ldots\bar{\alpha}_5\bar{K}^6 + \bar{\alpha}_1\bar{K}^{2-6}) + \sum^{15}(\bar{\alpha}_1\ldots\bar{\alpha}_4\bar{K}^{56} + \bar{\alpha}_1\bar{\alpha}_2\bar{K}^{3-6}) + \sum^{20}\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3\bar{K}^{4-6} + \bar{K}^{1-6},$$

For $\bar{\mu}^{1-2k}$ *of* (2.10), $\bar{I}^1 = \bar{J}_0^1$, $\bar{I}^{12} = \bar{J}_0^{12} - M\bar{V}^{12}$, $\bar{I}^{1-3} = \bar{J}_0^{1-3} - \sum^3 \bar{J}_0^1\bar{V}^{23}$,

$$\bar{I}^{1-4} = \bar{J}_0^{1-4} - \sum^6 \bar{J}_0^{12}\bar{V}^{34} + M\bar{\mu}^{1-4},\ \bar{I}^{1-5} = \bar{J}_0^{1-5} - \sum^{10}\bar{J}_0^{1-3}\bar{V}^{45} + \sum^5 \bar{J}_0^1\bar{\mu}^{2-5},$$

$$\bar{I}^{1-6} = \bar{J}_0^{1-6} - \sum^{15}\bar{J}_0^{1-4}\bar{V}^{56} + \sum^{15}\bar{J}_0^{12}\bar{\mu}^{3-6} - M\bar{\mu}^{1-6}.$$

PROOF Since $\mathbf{x}_1 = \mu + u$, $y = \Lambda\mathbf{x}_1 + \binom{\mathbf{V}^{12}}{\mathbf{V}_{22}}\mathbf{x}_2 = \alpha + \Lambda u \in R^q$. Substitute $y = \alpha + \Lambda u$ into the expressions for $\bar{H}^{1-k}$. Now multiply by $\phi_{V_0}(u)$ and integrate from $-\infty$ to u. □
This gives the $\bar{I}^{1-k}$ needed for g_1, g_2, G_1, G_2. The $\bar{I}^{1-k}$, $k = 7,9$ needed for g_3, G_3 can be written down similarly in terms of the partial moments using $\bar{H}_q^{1-k}$ for $k = 7,9$. We now show that if $q_1 = 1$, we only need the partial moments of $\Phi(v)$ at v of (3.22), and that these are easily written in terms of $\Phi(v)$ and $\phi(v)\times$ a polynomial in v of (3.22).

The case $q_1 = 1$. So $\mathbf{w}_1 = w_1$, $\hat{\mathbf{w}}_1 = \hat{w}_1$, $\mathbf{X}_1 = X_1$, $\mathbf{X}_{n1} = X_{n1}$, $\mathbf{V}_{11} = V_{11}$.

Theorem 3.4. *For $q_1 = 1, 1 \leq k \leq 6,$ $\bar{I}^{1-k}$ is given by Theorem 3.3 with*

$$\alpha_1 = 0,\ \bar{K}^{1-k} = \bar{\Lambda}_1 \ldots \bar{\Lambda}_k \sigma^k \gamma_k,\ where\ \Lambda = \begin{pmatrix} V^{11} \\ \mathbf{V}^{21} \end{pmatrix} \in R^q,$$

$$\sigma = V_{1.2}^{1/2}\ of\ (3.4),\ \gamma_k = \int_{-\infty}^{v} v^k\ \phi(v)dv,\ for\ v\ of\ (3.22). \tag{3.36}$$

$$So\ \gamma_0 = \Phi(v),\ \gamma_1 = -\phi(v),\ \gamma_k = v^{k-1}\gamma_1 + (k-1)\gamma_{k-2},\ for\ k \geq 2: \tag{3.37}$$

$$\gamma_2 = \gamma_0 + v\gamma_1,\ \gamma_3 = (v^2+2)\gamma_1,\ \gamma_4 = 3\gamma_0 + (v^3+3v)\gamma_1,$$
$$\gamma_5 = (v^4+4v^2+4.2)\gamma_1,\ \gamma_6 = 5.3\gamma_0 + (v^5+5v^3+5.3v)\gamma_1,$$
$$\gamma_7 = (v^6+6v^4+6.4v^2+6.4.2)\gamma_1,$$
$$\gamma_8 = 7.5.3\gamma_0 + (v^7+7v^5+7.5v^3+7.5.3v)\gamma_1,$$
$$\gamma_9 = (v^8+8v^6+8.6v^4+8.6.4v^2+8.6.4.2)\gamma_1,$$

where dot denotes multiplication. Also, $G_0 = \Phi(v)$.

PROOF For v of (3.22), by (3.9), $\phi_{1.2}(x_1) = \sigma^{-1}\phi(v)$. (3.37) follows from integration by parts. By (3.34), $K^{1-k} = \bar{\Lambda}_1 \ldots \bar{\Lambda}_k M^{1^k}$ where
$M^{1^k} = \int_{-\infty}^{u} u^k d\Phi(u/\sigma) = \sigma^k \gamma_k$. That $G_0 = \Phi(v)$, follows from (3.25). □
By (3.23), for C_r of (3.18) and v of (3.22), the conditional distribution of $X_{n1.2}$ is

$$P(X_{n1.2}/\sigma \leq v) \approx \Phi(v) + \sum_{r=1}^{\infty} n^{-r/2} G_r,\ \text{where } G_r = C_r \otimes g_r, \tag{3.38}$$

as in (3.29), and g_r is given by (3.26) in terms of the integrated Hermite polynomial, $\bar{I}^{1-k}$ of (3.28) given by Theorems 3.3, 3.4.

4. The Case $q_1 = q_2 = 1.$

Theorem 3.2 gave the conditional Edgeworth expansion in terms of $\bar{I}^{1-k}$ of (3.28). Theorem 3.3 gave $\bar{I}^{1-k}$ needed for g_{rk} of (3.27) and G_1, G_2 of (3.23), in terms of the partial moments $\bar{M}^{a-b}$ of (3.32). When $q_1 = 1$, Theorem 3.4 gave $\bar{I}^{1-k}$ in terms of $\Phi(v)$ and its partial moments for v of (3.22). But now $q = 2$ so that $i_1, \ldots, i_k = 1$ or 2. So for (I, y, Y) of (2.9), we switch notation to

$$H_{ab} = (-\partial_1)^a(-\partial_2)^b\ \phi_V(x) = E\,(y_1 + IY_1)^a(y_2 + IY_2)^b,$$
$$H^*_{ab} = (-\partial_1)^a(-\partial_2)^b\ \Phi_V(x) = \int_{-\infty}^{x} H_{ab}\ \phi_V(x)dx.$$
$$\text{So } H^*_{ab} = H_{a-1,b-1}\ \phi_V(x) \text{ if } a \geq 2, b \geq 1,$$
$$H^*_{10} = \int_{-\infty}^{x_2} \phi_V(x)dx_2 = \partial_1\Phi_V(x),\ H^*_{a0} = (-\partial_1)^{a-1}H^*_{10} \text{ if } a \geq 2,$$
$$H^*_{01} = \int_{-\infty}^{x_1} \phi_V(x)dx_1 = \partial_2\Phi_V(x),\ H^*_{0b} = (-\partial_2)^{b-1}H^*_{01} \text{ if } b \geq 1,$$
$$L^{1^a 2^b} = (-\partial_1)^a(-\partial_2)^{b+1}\Phi_V(x) = H^*_{a,b+1}, \tag{4.1}$$

for $\bar{L}^{1-k}$ of (3.30). Similarly, write (2.1) as

$$\kappa_{ab}(\hat{w}_1, \hat{w}_2) \approx \sum_{d=a+b-1}^{\infty} n^{-d} k_{abd},\ \text{for } a+b \geq 1,\ \text{where } k_{abd} = k_d^{1^a 2^b},\ \text{and}$$
$$\text{set } K_{ab} = K^{1^a 2^b},\ J_{ab} = J_0^{1^a 2^b},\ I_{ab} = I^{1^a 2^b} = \int_{-\infty}^{x_1} H_{ab}(x)\ \phi_{1.2}(x_1)dx_1. \tag{4.2}$$

Also, we switch from $\bar{P}_r^{1-k}$ to

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$$P_r(ab) = \binom{a+b}{a} P_r^{1^a 2^b}.$$

given for $r \leq 3$ in Section 4 of [30]. So,

$$\tilde{p}_{rk} = \sum_{b=0}^{k} P_r(k-b,b)\ H_{k-b,b},\ P_{rk}(x) = \sum_{b=0}^{k} P_r(k-b,b)\ H^*_{k-b,b}. \tag{4.3}$$
$$\text{So, } \tilde{p}_{r1} = P_r(10)H_{10} + P_r(01)H_{01},\ \tilde{p}_{11} = k_{101}y_1 + k_{011}y_2,$$
$$\tilde{p}_{13} = P_1(30)H_{30} + P_1(21)H_{21} + P_1(12)H_{12} + P_1(03)H_{03}.$$

$P_r(ba)$ is just $P_r(ab)$ with 1 and 2 reversed. For the other $\tilde{p}_{rk}$ and $P_{rk}(x)$ needed for $r \leq 3$, see Section 4 of [30]. Our main result for this section, Theorem 4.3, gives simple formulas for I_{ab} and for g_r of (3.26), the main ingredient needed in Theorem 3.2 for the expansion of the conditional distribution.

Theorem 4.1. *The conditional density of $X_{n1.2}$ of (3.1), is given by Theorem 3.1 where $f_r = p^*_r(\mathbf{x}_2)$ is given by (3.14) in terms of*

$$p^*_{rk} = P_r(0k)H^*_k,\ \text{where } H^*_k = H_k(z, V_{22}) \text{ and } z = V_{22}^{-1}x_2. \tag{4.4}$$
$$\text{For example, } H^*_1 = z,\ H^*_2 = z^2 - V_{22}^{-1},\ H^*_3 = z^3 - 3zV_{22}^{-1}, \tag{4.5}$$
$$H^*_4 = z^4 - 15z^2V_{22}^{-1} + 3V_{22}^{-2},\ H^*_5 = z^5 - 10z^3V_{22}^{-1} + 15zV_{22}^{-2}, \tag{4.6}$$
$$H^*_6 = z^6 - 15z^4V_{22}^{-1} + 45z^2V_{22}^{-2} - 15V_{22}^{-3}. \tag{4.7}$$

PROOF This follows from Theorem 3.1. □

Theorem 4.2 gives a laborious expression for the conditional distribution.

However Theorem 4.3 gives a huge simplification.

Theorem 4.2. *The conditional distribution of $X_{n1.2}$ of (3.1), is given by Theorem 3.2 with $\Lambda, \sigma, \gamma_s$ of Theorem 3.4 as follows. For $k-r$ even, g_{rk} of (3.27) is given by*

$$g_{rk} = \sum_{b=0}^{k} P_r(k-b,b)\ I_{k-b,b}, \tag{4.8}$$

where I_{ab} of (4.2) is given for $a+b=k$, as follows in terms of $\Lambda_i = V^{i1}$.

$$K_{ab} = \Lambda_1^a\Lambda_2^b\sigma^{a+b}\gamma_{a+b},\ \text{and } J_{k0} = \sum_{s=0}^{k} \alpha_1^{k-s}\ K_{s0},\ \text{for } K_{s0} = (\Lambda_1\sigma)^s\gamma_s. \tag{4.9}$$
$$\text{For } k=1:\ I_{10} = J_{10} = \alpha_1\gamma_0 + \Lambda_1\sigma\gamma_1.$$
$$\text{For } k=2:\ I_{20} = J_{20} - \gamma_0V^{11},\ I_{11} = J_{11} - \gamma_0V^{12},$$
$$J_{11} = \alpha_1\alpha_2\gamma_0 + \sigma\gamma_1\sum^{2}\alpha_1\Lambda_2 + \Lambda_1\Lambda_2\sigma^2\gamma_2.$$
$$\text{For } k=3:\ I_{30} = J_{30} - 3J_{10}V^{11},\ I_{21} = J_{21} - (2J_{10}V^{12} + J_{01}V^{11}),$$
$$J_{21} = \alpha_1^2\alpha_2\gamma_0 + X_{21} + X_{12} + K_{21},\ \text{where}$$
$$X_{21} = \alpha_1^2K_{01} + 2\alpha_1\alpha_2K_{10},\ X_{12} = 2\alpha_1K_{11} + \alpha_2K_{20}.$$
$$\text{For } k=4:\ I_{40} = J_{40} - 6J_{20}V^{11} + 3\gamma_0(V^{11})^2,$$
$$I_{31} = J_{31} - S_6 + \gamma_0S_3,\ \text{where } S_6 = 3J_{20}V^{22} + 3J_{11}V^{12},\ \ S_3 = 3V^{11}V^{12},$$
$$J_{31} = \alpha_1^3\alpha_2\gamma_0 + X_{31} + X_{22} + X_{13} + K_{31},\ \text{where } X_{31} = \alpha_1^3K_{01} + 3\alpha_1^2\alpha_2K_{10},$$
$$X_{22} = 4\alpha_1^2K_{11} + 2\alpha_1\alpha_2K_{20},\ X_{13} = 3\alpha_1K_{21} + 6\alpha_2K_{30},$$
$$I_{22} = J_{22} - S_6 + \gamma_0S_3,\ \text{where } S_6 = J_{20}V^{22} + 4J_{11}V^{12} + J_{02}V^{11},$$

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$$
\begin{aligned}
& S_3 = \mu^{1122} = V^{11}V^{22} + 2(V^{12})^2, \\
& J_{22} = \alpha_1^2\alpha_2^2\gamma_0 + X_{31} + X_{22} + X_{13} + K_{22},\ where\ X_{31} = 2\alpha_1^2\alpha_2 K_{01} + 2\alpha_1\alpha_2^2 K_{02}, \\
& X_{22} = \alpha_1^2 K_{02} + 4\alpha_1\alpha_2 K_{11} + \alpha_2^2 K_{20},\ X_{13} = 2\alpha_1 K_{12} + 2\alpha_2 K_{21}. \\
& For\ k = 5:\ \ I_{50} = J_{50} - 10J_{30}V^{11} + 15J_{10}(V^{11})^2, \\
& I_{41} = J_{41} - S_{10} + S_{15},\ where\ S_{10} = 6J_{21}V^{11} + 4J_{30}V^{12}, \\
& S_{15} = 12J_{10}V^{11}V^{12} + 3J_{01}(V^{11})^2, \\
& J_{41} = \alpha_1^4\alpha_2\gamma_0 + X_{41} + X_{32} + X_{23} + X_{14} + K_{41},\ where, \\
& X_{41} = 4\alpha_1^3\alpha_2 K_{10} + \alpha_1^4 K_{01},\ X_{32} = 5\alpha_1^2\alpha_2 K_{20} + 5\alpha_1^3 K_{11}, \\
& X_{23} = 6\alpha_1^2 K_{21} + 4\alpha_1\alpha_2 K_{30},\ X_{14} = 4\alpha_1 K_{31} + \alpha_2 K_{40}, \\
& I_{32} = J_{32} - S_{10} + S_{15},\ where\ S_{10} = 3J_{12}V^{11} + 6J_{21}V^{12} + J_{30}V^{22}, \\
& S_{15} = 3J_{10}\mu^{1122} + 6J_{01}V^{11}V^{12}, \\
& J_{32} = \alpha_1^3\alpha_2^2\gamma_0 + X_{41} + X_{32} + X_{23} + X_{14} + K_{32},\ where, \\
& X_{41} = 3\alpha_1^2\alpha_2^2 K_{10} + 2\alpha_1^3\alpha_2 K_{01},\ X_{32} = 3\alpha_1\alpha_2^2 K_{20} + 6\alpha_1^2\alpha_2 K_{11} + \alpha_1^3 K_{02}, \\
& X_{23} = 3\alpha_1^2 K_{12} + 6\alpha_1\alpha_2 K_{21} + \alpha_2^2 K_{30},\ X_{14} = 3\alpha_1 K_{22} + 2\alpha_2 K_{31}. \\
& For\ k = 6:\ \ I_{60} = J_{60} - 15J_{40}V^{11} + 45J_{20}(V^{11})^2 - 15\gamma_0(V^{11})^3, \\
& I_{51} = J_{51} - S_{15} + S_{45} - \gamma_0 S'_{15},\ where\ S_{15} = 5V^{12}J_{40} + 10V^{11}J_{31}, \\
& S_{45} = 30V^{11}V^{12}J_{20} + 15(V^{11})^2 J_{11},\ S'_{15} = 15\gamma_0(V^{11})^2 V^{12}, \\
& J_{51} = \alpha_1^5\alpha 2\gamma_0 + X_{51} + X_{42} + X_{33} + X_{24} + X_{15} + K_{51},\ where \\
& X_{51} = \alpha_1^5 K_{01} + 5\alpha_1^4\alpha_2 K_{10},\ X_{42} = 5\alpha_1^4 K_{11} + 10\alpha_1^3\alpha_2 K_{20}, \\
& X_{33} = 10\alpha_1^3 K_{21} + 10\alpha_1^2\alpha_2 K_{30},\ X_{24} = 10\alpha_1^2 K_{31} + 5\alpha_1\alpha_2 K_{40}, \\
& X_{15} = \alpha_2 K_{60} + 5\alpha_1 K_{51}, \\
& I_{42} = J_{42} - S_{15} + S_{45} - \gamma_0 S'_{15},\ where\ \ S_{15} = V^{22}J_{40} + 6V^{11}J_{22} + 8V^{12}J_{31}, \\
& S_{45} = 3(V^{11})^2 J_{02} + 6\mu^{1122}J_{20} + 24V^{11}V^{12}J_{11}, \\
& S'_{15} = 3(V^{11})^2 V^{22} + 24V^{11}V^{12})^2 + 6V^{11}\mu^{1122}, \\
& J_{42} = \alpha_1^4\alpha_2^2\gamma_0 + X_{51} + X_{42} + X_{33} + X_{24} + X_{15} + K_{42},\ where \\
& X_{51} = 2\alpha_1^5 K_{01} + 4\alpha_1^4\alpha_2 K_{10},\ \ X_{42} = \alpha_1^4 K_{02} + 8\alpha_1^3\alpha_2 K_{11} + 6\alpha_1^2\alpha_2^2 K_{20}, \\
& X_{33} = 10\alpha_1^2\alpha_2 K_{12} + 10\alpha_1\alpha_2^2 K_{21}, \\
& X_{24} = \alpha_2^2 K_{40} + 8\alpha_1\alpha_2 K_{31} + 6\alpha_1^2 K_{22},\ X_{15} = 2\alpha_2 K_{50} + 4\alpha_1 K_{41},
\end{aligned}
$$

$$
\begin{aligned}
& I_{33} = J_{33} - S_{15} + S_{45} - \gamma_0 S'_{15},\ where\ \ S_{15} = 6V^{11}J_{04} + 9V^{12}J_{22} + 6V^{22}J_{40}, \\
& S_{45} = 9V^{12}V^{22}J_{20} + 9\mu^{1122}J_{11} + 9V^{12}V^{11}J_{02},\ \ S'_{15} = 6V^{11}V^{12}V^{22} + 3V^{12}\mu^{1122}, \\
& J_{33} = \alpha_1^3\alpha_2^3\gamma_0 + X_{51} + X_{42} + X_{33} + X_{24} + X_{15} + K_{33},\ where \\
& X_{51} = 2\alpha_1^2\alpha_2^3 K_{10} + 3\alpha_1^3\alpha_2^2 K_{01},\ X_{42} = 6\alpha_1\alpha_2^3 K_{20} + 3\alpha_1^3\alpha_2^3 K_{11} + 6\alpha_1^3\alpha_2 K_{02}, \\
& X_{33} = \alpha_1^3 K_{03} + 9\alpha_1^2\alpha_2 K_{12} + 9\alpha_1\alpha_2^2 K_{21} + \alpha_2^3 K_{30}, \\
& X_{24} = 6\alpha_1^2 K_{13} + 3\alpha_1\alpha_2 K_{22} + 8\alpha_2^2 K_{31},\ \ X_{15} = 3\alpha_1 K_{23} + 3\alpha_2 K_{32}.
\end{aligned}
$$

Also $J_{ba},\ I_{ba}$ are $J_{ab},\ I_{ab}$ with $\alpha_1, \Lambda_1 = V^{11}$ and $\alpha_2, \Lambda_2 = V^{21}$ of (3.33) reversed, before setting $\alpha_1 = 0$ and $\alpha_2 = z = V_{22}^{-1}x_2$ by (3.13). For example, by (4.9), for $\Lambda, \sigma, \gamma_s$ of Theorem 3.4,

$$I_{10} = \alpha_1\gamma_0 + \Lambda_1\sigma\gamma_1 = V^{11}\sigma\gamma_1,\ I_{01} = \alpha_2\gamma_0 + \Lambda_2\sigma\gamma_1 = z\gamma_0 + V^{11}\sigma\gamma_1, \tag{4.10}$$

$$J_{k0} = K_{k0} = (V^{11}\sigma)^k\gamma_k,\ J_{0k} = \sum_{s=0}^{k} z^{k-s}\ K_{0s} = zJ_{0,k-1} + K_{0k} \tag{4.11}$$

where $K_{0s} = (V^{21}\sigma)^s\gamma_s$.

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PROOF This follows from Theorems 3.3 and 3.4. □

This gives the $\bar{I}_{ab}$ needed for g_1, g_2, G_1, G_2 for the conditional distribution of (3.23)–(3.25) to $O(n^{-3/2})$. The $\bar{I}_{ab}$ needed for g_3, G_3 can be written down similarly. We now give *a much simpler method* for obtaining g_{rk} of (3.27), and so g_r by (3.26), and G_r needed for (3.23) by (3.24). Theorem 4.3 gives g_{rk} and g_r in terms of I_{0k} of (4.2). Theorem 4.4 gives I_{0k} in terms of J_{0k} of (4.11), a function of $(\Lambda, \sigma, \gamma_s)$ of Theorem 3.4.

Theorem 4.3. *For v of (3.22), I_{ab} of (4.2) is given by*

$$I_{ab} = -H_{a-1,b}\, \sigma^{-1}\phi(v), \text{ for } a \geq 1. \tag{4.12}$$

For $k \geq r \geq 1$ and $k - r$ even, g_{rk} of (3.27) is given by

$$g_{rk} = P_r(0k)\, I_{0k} - b_{rk}\, \sigma^{-1}\phi(v), \text{ for } b_{rk} = \sum_{a=1}^{k} P_r(a, k-a)\, H_{a-1,k-a}. \tag{4.13}$$

So by (3.26), for $r \geq 1$, g_r of (3.25) is given by

$$g_r = \sum_{k=1}^{3r} [P_r(0k)\, I_{0k} - b_{rk}\sigma^{-1}\phi(v) :\ k - r \text{ even}]. \tag{4.14}$$

PROOF By (4.8), $g_{rk} = P_r(0k) I_{0k} + \sum_{a=1}^{k} P_r(a, k-a)\, I_{a,k-a}$.

By (3.9), $\phi_{1.2}(x_1)/\phi_V(x) = \theta^{-1}$ where $\theta = \phi_{V_{22}}(x_2)$ and $\phi_{1.2}(x_1) = \sigma^{-1}\phi(v)$. So,

$$\text{for } a \geq 1,\ H_{ab}\, \phi_V(x) = (-\partial_1)^a(-\partial_2)^b = -\partial_1[H_{a-1,b}\, \phi_V(x)],$$
$$\text{so } I_{ab} = \theta^{-1} \int_{-\infty}^{x_1} H_{ab}\, \phi_V(x) dx_1 = -\theta^{-1} H_{a-1,b}\, \phi_V(x) = -H_{a-1,b}\, \sigma^{-1}\phi(v).$$

This proves (4.12). So,

$$g_{rk} = P_r(0k) I_{0k} - \theta^{-1}\phi_V(x) \sum [P_r(ab) H_{a-1,b} :\ a + b = k, a \geq 1].$$

(4.13) follows. (4.14) now follows from (3.14). □

Note 4.1. *b_{rk} is just $\tilde{p}_{rk}$ of (4.3) with (H_{0b}, H_{ab}) replaced by $(0, H_{a-1,b})$ for $a \geq 1$.*

So for $r = 1, 2, 3,\ b_{rk}$ is given in terms of $P_r(.)$ of Section 3, by

$$b_{r1} = P_r(10) = k_r^1 = k_{10r},\ b_{r3} = P_r(30) H_{20} + P_r(21) H_{11} + P_r(12) H_{02}, \tag{4.15}$$

$$b_{22} = P_2(20)\, H_{10} + P_2(11)\, H_{01}, \tag{4.16}$$

$$b_{24} = P_2(40)\, H_{30} + P_2(31)\, H_{21} + P_2(22)\, H_{12} + P_2(13)\, H_{03}. \tag{4.17}$$

$$b_{26} = P_2(60) H_{50} + P_2(51) H_{41} + P_2(42)\, H_{32} + P_2(33)\, H_{23} + P(24) H_{14} + P_2(15)\, H_{05},$$
$$b_{35} = P_3(50) H_{40} + P_3(41) H_{31} + P_3(32) H_{22} + P_3(23) H_{13} + P_3(14) H_{04},$$
$$b_{37} = P_3(70) H_{60} + P_3(61) H_{51} + P_3(52) H_{42} + P_3(43) H_{33} + P_3(34) H_{24} + P_3(25) H_{15} + P_3(16) H_{06},$$
$$b_{39} = P_3(90) H_{80} + P_3(81) H_{71} + P_3(72) H_{62} + P_3(63) H_{53} + P_3(54) H_{44} + P_3(45) H_{35} + P_3(36) H_{26} + P_3(27) H_{17} + P_3(18) H_{08}. \tag{4.18}$$

This gives g_{rk} and g_r of (3.26) for $r \leq 3$, and so the conditional distribution $P_{1.2}(x_1)$ of (3.23), to $O(n^{-2})$, in terms of I_{0k} of (4.2) and the coefficients $P_r(ab)$.

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Theorem 4.4. *The I_{0k} needed for g_1, g_2, g_3 of (4.14) and (3.24) are given in terms of $\gamma_0 = \Phi(v)$, v of (3.22), and J_{0k} of (4.11), by*

$$\begin{aligned}
&I_{01} = J_{01},\ I_{02} = J_{02} - \gamma_0 V^{22},\ I_{03} = J_{03} - 3J_{01}V^{22},\\
&I_{04} = J_{04} - 6J_{02}V^{22} + 3\gamma_0(V^{22})^2,\\
&I_{05} = J_{05} - 10J_{03}V^{22} + 15J_{01}(V^{22})^2,\\
&I_{06} = J_{06} - 15J_{04}V^{22} + 45J_{02}(V^{22})^2 - 15\gamma_0(V^{22})^3,\\
&I_{07} = J_{07} - 21J_{05}V^{22} + 105J_{03}(V^{22})^2 - 105J_{01}(V^{22})^3,\\
&I_{08} = J_{08} - 28J_{06}V^{22} + 210J_{04}(V^{22})^2 - 420J_{02}(V^{22})^3 + 105\gamma_0(V^{22})^4,\\
&I_{09} = J_{09} - 36J_{07}V^{22} + 378J_{05}(V^{22})^2 - 1260J_{03}(V^{22})^3 + 945J_{01}(V^{22})^4.
\end{aligned}$$

PROOF For $k \leq 6$, I_{0k} follow from Theorem 3.2. By the proof of Theorem 3.3, I_{0k} can be read off [30] and the univariate Hermite polynomials $H_k(u)$ given in terms of $I = \sqrt{-1}$ by expanding

$$H_k = H_k(u) = \phi(u)^{-1}(-d/du)^k\phi(u) = E\ (u + IN)^k,\ \text{for}\ k \geq 0. \qquad \square$$

To summarise, the conditional density of $X_{n1.2}$ of (3.1), is given by Theorem 4.1, and the conditional distribution is given by (3.23), (3.27) in terms of g_r of (4.14) and I_{0k} of Theorem 4.4.

Example 4.1. *Conditioning when $\hat{w} \in R^2$ is the mean of a sample with cumulants κ_{ab}. The non-zero $P_r(ab)$ were given in Example 6 of [30]. So $b_{rk} = 0$ for $(rk) = (11), (22), (31), (33)$, and for $r = 1, 2, 3$, other b_{rk} are given by (4.15)–(4.18) starting*

$$6b_{13} = \kappa_{30}H_{20} + 3\kappa_{21}H_{11} + \kappa_{12}H_{02}, \tag{4.19}$$

$$24b_{24} = \kappa_{40}H_{30} + 4\kappa_{31}H_{21} + 6\kappa_{22}H_{12} + 4\kappa_{13}H_{03}, \tag{4.20}$$

$$\begin{aligned}72b_{26} = {}&\kappa_{30}^2H_{50} + 6\kappa_{30}\kappa_{21}H_{41} + 3(2\kappa_{30}\kappa_{12} + 3\kappa_{21}^2)H_{32} + 12(\kappa_{30}\kappa_{03}\\
&+ 9\kappa_{21}\kappa_{12})H_{23} + 3(2\kappa_{03}\kappa_{21} + 3\kappa_{12}^2)H_{14} + 6\kappa_{03}\kappa_{12}H_{05}.\end{aligned} \tag{4.21}$$

The relative conditional density is given to $O(n^{-2})$ by (3.19) in terms of $\tilde{p}_r$ of (2.6), $\tilde{p}_{rk}$ of (4.3), $f_r = p_r^*(x_2)$ of (3.14) for $r \leq 3$, and H_k^* of (4.4) for $k \leq 9$.

$$\begin{aligned}
&\text{So, } f_1 = p_{13}^* = P_1(03)H_3^*,\ P_1(03) = \kappa_{03}/6,\\
&f_2 = p_{24}^* + p_{26}^*,\ p_{24}^* = P_2(04)H_4^*,\ P_2(04) = \kappa_{04}/24,\\
&p_{26}^* = P_2(06)H_6^*,\ P_2(06) = \kappa_{03}^2/72,
\end{aligned}$$

$$\begin{aligned}
&f_3 = \sum_{k=5,7,9} p_{3k}^*,\ p_{3k}^* = P_3(0k)H_k^*,\ P_3(05) = \kappa_{05}/120,\\
&P_3(07) = \kappa_{04}\kappa_{03}/144,\ P_3(09) = (\kappa_{03}/6)^3.
\end{aligned}$$

The conditional distribution is given by (3.38) with g_r of (4.14), starting

$$\begin{aligned}
&G_0 = g_0 = \Phi(v),\ g_1 = \kappa_{03}I_{03}/6 - b_{13}\sigma^{-1}\phi(v),\\
&\text{for } v \text{ of (3.22), with } \sigma^2 = V_0 = \kappa_{20} - \kappa_{11}^2/\kappa_{02},\ \mu_{1.2} = \kappa_{11}\kappa_{02}^{-1}x_2,
\end{aligned} \tag{4.22}$$

I_{03} of Theorem 4.4, and b_{13} of (4.19). As noted this is a far simpler result than using Theorem 4.2.

$$\begin{aligned}
&\text{Similarly, } g_2 = \kappa_{04}I_{04}/24 + \kappa_{03}^2I_{06}/72 - (b_{24} + b_{26})\sigma^{-1}\phi(v),\\
&g_3 = \sum_{k=5,7,9}[P_3(0k)I_{0k} - b_{3k}\sigma^{-1}\phi(v)],
\end{aligned}$$

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for b_{24}, b_{26} of (4.20), (4.21) and b_{3k} above.

Example 4.2. *We now build on* **the entangled gamma** *model of Example 7 of [30], which gave the $P_r(ab)$ needed. Let G_0, G_1, G_2 be independent gamma random variables with means $\gamma = \gamma_0, \gamma_1, \gamma_2$. For $i = 1, 2$, set $X_i = G_0 + G_i$, $w_i = E\ X_i = \gamma + \gamma_i$, and let $\hat{w}$ be the mean of a random sample of size n distributed as (X_1, X_2). So, $E\hat{w} = w$, and $n\hat{w} \stackrel{\mathcal{L}}{=} (G_{n0} + G_{n1}, G_{n0} + G_{n2})'$ where G_{n0}, G_{n1}, G_{n2} are independent gamma random variables with means $n\gamma, n\gamma_1, n\gamma_2$. The rth order cumulants of $X = (X_1, X_2)'$ are $\kappa^{i^r} = (r-1)!w_i$, and otherwise $(r-1)!\gamma$. Now suppose that $\gamma_i \equiv 1$,* **the entangled exponential** *model. So $q = 2$, X_{n1} and X_{n2} have correlation $1/2$,*

$$V = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},\ V_{12}V_{22}^{-1} = 1/2,\ V_{1.2} = 3/2,\ V^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}/3,$$
$$P(X_{n1}|(\mathbf{X}_{n2} = \mathbf{x}_2) < x_1(x_2, u)) = \Phi(u) + O(n^{-1/2}),$$

for $x_1(x_2, v)$ of (3.11), that is, $x_1(x_2, u) = x_2/2 + (3/2)^{1/2}u$. Figure 4.1 plots the conditional asymptotic quantiles of $X_{n1.2}$, that is, $x_1(x_2, u)$, for $\Phi(u) = .01, .025, .1, 0, .9, .975, .99$. To $O(n^{-1/2})$, given n and $\hat{\mathbf{w}}$, this figure is equivalent to a figure of w_1 versus w_2. That is, Figure 4.1 shows to $O(n^{-1/2})$, the likely value of $w_1 = \hat{w}_1 - n^{-1/2}x_1$ for a given value of $w_2 = \hat{w}_2 - n^{-1/2}x_2$. In fact by (3.12), $-X_{n1.2} = n^{1/2}(w_1 - \hat{w}_{1.2})$ lies between the outer limits with probability .98+$O(n^{-1})$. So although labelled as x_1 versus x_2, the figure can be viewed as showing the likely value of $w_1 = \hat{w}_1 - n^{-1/2}x_1$ for a given value of $w_2 = \hat{w}_2 - n^{-1/2}x_2$.

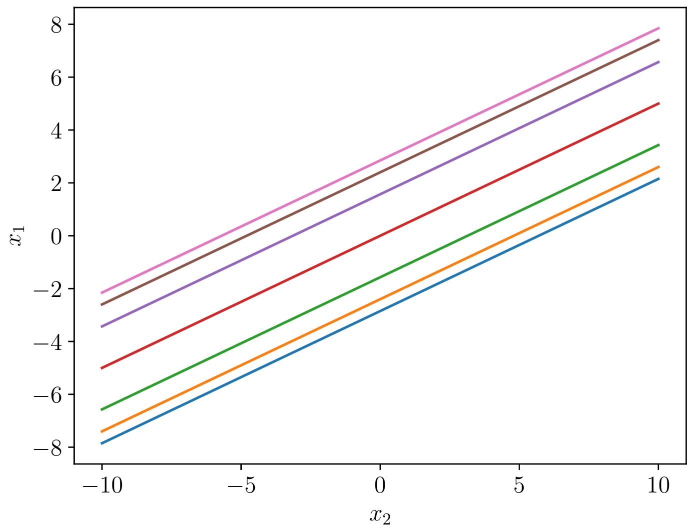

Figure 4.1. $x_1(x_2, v) = x_2/2 + (3/2)^{1/2}v$ of (3.11) for $\Phi(v) = .01, .1, 0, .9, .99$ - courtesy of Dr Paul Teal:

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We now give C_r of (3.17), D_r of (3.19), H_k^* and p_{rk}^* of (4.4), and g_r for G_r, the coefficients of the expansion for the conditional distribution of (3.23).

$$\begin{aligned}
&\textit{So, } P_1(03) = 2/3,\ P_1(21) = 1,\ P_2(04) = 1/2,\ P_2(31) = 1,\ P_2(22) = 3/2,\\
&P_2(06) = 2/9,\ P_2(51) = 2/3,\ P_2(42) = 7/6,\ P_2(33) = 26/3\\
&P_3(05) = 2/5,\ P_3(41) = 1,\ P_3(32) = 2,\ P_3(07) = 1/3,\ P_3(61) = P_3(52) = 5/2,\\
&P_3(43) = 3,\ P_3(09) = 8/27,\ P_3(81) = 4/27,\ P_3(72) = 10/27,\\
&P_3(63) = 47/756,\ P_3(54) = 59/945.\\
&\textit{By Theorem 4.4, to 3 decimal places,} I_{03} = J_{03} - 2J_{01} = -.586,\\
&I_{04} = J_{04} - 4J_{02} + 4\gamma_0/3 = .871,\ I_{05} = J_{05} - 20J_{03}/3 + 20J_{01}/9 = .709,\\
&I_{06} = J_{06} - 10J_{04} + 20J_{02} - 40\gamma_0/9 = -3.187,\\
&I_{07} = J_{07} - 14J_{05} + 140J_{03}/3 - 280J_{01}/9 = -12.857,\\
&I_{08} = J_{08} - 56J_{06}/3 + 280J_{04}/3 - 1120J_{02}/3 + 560\gamma_0/27 = 12.077,\\
&I_{09} = J_{09} - 24J_{07} + 168J_{05} - 1120J_{03} + 560J_{01}/3 = 28.278.
\end{aligned}$$

By Note 4.1, $\tilde{p}_{rk}$ of Example 7 of [30], symmetry, and (4.14),

$$\begin{aligned}
&b_{13} = 5H_{20}/3 + H_{11},\ b_{24} = 3H_{40}/2 + 2H_{31},\ b_{26} = [7H_{50} + 12H_{41} + 19H_{32}]/9,\\
&b_{35} = [3H_{40} + 2H_{31} + H_{22}]/5,\ b_{37} = [9H_{60} + 19H_{51} + 30H_{42} + 18H_{33}]/6,\\
&b_{39} = [44H_{80} + 83H_{71} + 206H_{62} + 159H_{53}]/27,\\
&g_1 = 2I_{03}/3 + b_{13}\sigma^{-1}\gamma_1,\ g_2 = I_{04}/2 + 2I_{06}/9 - b_{24} - b_{26},\\
&g_3 = 2I_{05}/5 + I_{07}/3 + 8I_{09}/27 - b_{35} - b_{37} - b_{39}.
\end{aligned}$$

Let us work through 2 numerical examples to get the conditional distribution to $O(n^{-2})$. We build on Example 7 of [30]. By Theorem 4.1, if $x_2 = 1$, then $z = 1/2$,

$$\begin{aligned}
&H_3^* = -5/8,\ H_4^* = -17/16,\ H_6^* = -89/64,\\
&H_5^* = 41/32,\ H_7^* = -461/2^7,\ H_9^* = 6481/2^9,\\
&-C_1 = f_1 = -5/12,\ p_{24}^* = -17/32,\ p_{26}^* = -89/288,\ f_2 = -121/144,\\
&p_{35}^* = 41/80,\ p_{37}^* = -461/384,\ p_{39}^* = 6481/1728,\ f_3 = 52921/17280,\\
&C_2 = 83/72,\ C_3 = -39571/17280.
\end{aligned}$$

We worked to 8 significant figures, but display less. If $x = (1,1)'$, then

$$\begin{aligned}
&D_1 = -113/324 = -.349,\ D_2 = 120199/2^3 3^8 = 2.290,\\
&D_3 = 8896102087/2^7 3^{12} 5 = 26.156.
\end{aligned}$$

So to $O(n^{-2})$ the relative conditional density of (3.19) for $n = 4, 16, 64$, is

$$\begin{aligned}
&(1,1,1)' - (2^{-1},4^{-1},8^{-1})'.349 + (4^{-1},16^{-1},64^{-1})'2.290\\
&+ (8^{-1},64^{-1},2^{-9})'26.156 = \begin{pmatrix} 1 & -.174 & +.573 & +3.269\\ 1 & -.087 & +.143 & +.409\\ 1 & -.044 & +.036 & +.051 \end{pmatrix},
\end{aligned}$$

so that for $n = 4$ and 16 we can only include two terms, and for $n = 64$, only three terms. We now give the 1st 3 g_r, G_r, needed by (3.23) for the conditional distribution to $O(n^{-2})$. By (3.36),
$\sigma^2 = 3/2$, $\sigma^2 = 1.225$. *By (3.3),* $\mu_{1.2} = x_2/2$.

$$
\begin{aligned}
&\text{For } x = (1,1)',\ \mu_{1.2} = 1/2, \text{ and by (3.4)},\ v = 6^{-1/2} = .408\\
&G_0 = g_0 = \gamma_0 = \Phi(v) = .658,\ \gamma_1 = -\phi(v) = -.367,\ \gamma_2 = .509,\ \gamma_3 = -.795,\\
&\gamma_4 = 1.501,\ \gamma_5 = -3.191,\ \gamma_6 = 7.500,\ \gamma_7 = -19.150,\ \gamma_8 = 52.500,\ \gamma_9 = -153.200.\\
&K_{0s} = (-6^{-1/2})^s \gamma_s \Rightarrow K_{01} = .150,\ K_{02} = .0848,\ K_{03} = .0541,\ K_{04} = .0417,\\
&K_{05} = .0362,\ K_{06} = .03472,\ K_{07} = .0362,\ K_{08} = .0405,\ K_{09} = .0483.\\
&\text{So by (4.11)}, J_{01} = .479,\ J_{02} = .0203,\ ,\ J_{03} = .372,\ J_{04} = .0738,\\
&J_{05} = .0731,\ J_{06} = .0713, J_{07} = .0718,\ J_{08} = .441,\ J_{09} = .269.\\
&\text{So for } x = (1,1)',\ b_{13} = -13/27 = -.481,\ g_1 = -.246,\\
&b_{24} = -47/54,\ b_{26} = 2726/2107,\ g_2 = -.696,\ b_{35} = 10/27, b_{37} = -9371/4374,\\
&b_{39} = 163806/59049 = 2.774,\ g_3 = 3.375.\\
&\text{By (3.24)},\ G_1 = g_1 + C_1 g_0 = .0281,\ G_2 = -.040,\ G_3 = .762.
\end{aligned}
$$

For example for $n = 4, 16, 64$, *to* $O(n^{-2})$, $P(X_{n1} < 1 | X_{n2} = 1) =$

$$
\begin{aligned}
&.658 + .0141 - .01000 + .0952,\ n = 4,\\
&.658 + .00703 - .00250 + .0119,\ n = 16,\\
&.658 + .00351 - .000625 + .00149,\ n = 64,
\end{aligned}
$$

so that divergence begins with the 4th term.

$$
\begin{aligned}
&\text{If } x_2 = 2 \text{ then } z = 1, H_3^* = -1/2,\ H_4^* = -23/4,\ H_6^* = 23/8,\ H_5^* = -1/4,\\
&H_7^* = 29/8,\ H_9^* = -175/16,\ -C_1 = f_1 = -1/3,\ p_{24}^* = -23/4,\ p_{26}^* = 23/36,\\
&f_2 = 161/72,\ p_{35}^* = -1/10,\ p_{37}^* = 29/24,\ p_{39}^* = -175/54,\\
&f_3 = -2303/1080 = -2.132,\ C_2 = -17/8 = -2.125,\ C_3 = 733/1080 = .679.\\
&\text{If } x = (2,2)', \text{ then } D_1 = -37/81 = -.457,\ D_2 = .387,\ D_3 = 13.313.
\end{aligned}
$$

So to $O(n^{-2})$ *the relative conditional density of (3.19) for* $n = 4, 16, 64$, *is*

$$
\begin{aligned}
&(1,1,1)' - (2^{-1}, 4^{-1}, 8^{-1})'.457 + (4^{-1}, 16^{-1}, 64^{-1})'.387\\
&+ (8^{-1}, 64^{-1}, 2^{-9})'13.313 = \begin{pmatrix} 1 & -.228 & +.0969 & +1.664 \\ 1 & -.114 & +.0242 & +.208 \\ 1 & -.0571 & +.00605 & +.0260, \end{pmatrix},
\end{aligned}
$$

so that we can only include three terms. Finally, we now give the 1st three g_r, G_r, needed by (3.23) for the conditional distribution to $O(n^{-2})$.

$$\begin{aligned}
&\text{For } x = (2,2)',\ \mu_{1.2} = 1,\ v = (2/3)^{1/2} = .816,\ G_0 = \gamma_0 = \Phi(v) = .793,\\
&\gamma_1 = -\phi(v) = -.286,\ \gamma_2 = .559,\ \gamma_3 = -.762,\ \gamma_4 = 1.522,\\
&\gamma_5 = -3.176, \gamma_6 = 7.511,\ \gamma_7 = -19.142,\ \gamma_8 = 52.505,\ \gamma_9 = -153.190.\\
&K_{0s} = (-6^{-1/2})^s \gamma_s \Rightarrow K_{01} = .117,\ K_{02} = .0932,\ K_{03} = .0519,\ K_{04} = .0423,\\
&K_{05} = .0360,\ K_{06} = .0348,\ K_{07} = .0362,\ K_{08} = .0405,\ K_{09} = .0483.\\
&\text{So by (4.11)}, J_{01} = .910,\ J_{02} = 1.0028,\ J_{03} = 1.055,\ J_{04} = 1.097,\\
&J_{05} = 1.133,\ J_{06} = 1.168,\ J_{07} = 1.204,\ J_{08} = 1.249,\ J_{09} = 1.293.\\
&I_{03} = -.764,\ I_{04} = -1.877,\ I_{05} = -3.877,\ I_{06} = 6.731,\ I_{07} = 6.263,\\
&I_{08} = -276.110,\ I_{09} = -848.735,\\
&b_{13} = 11/27,\ g_1 = -.605,\ b_{24} = -26/27,\ b_{26} = 1660/2187,\ g_2 = .771,\\
&b_{35} = -138/405,\ b_{37} = 20128/4374,\ b_{39} = 1795048/3^{10},\ g_3 = -224.802.\\
&\text{By (3.24)},\ G_1 = g_1 + C_1 g_0 = .0180,\ G_2 = -2.463,\ G_3 = 4.204.
\end{aligned}$$

For example for $n = 4, 16, 64$, to $O(n^{-2})$, $P(X_{n1} < 2 | X_{n2} = 2) =$

$$\begin{aligned}
&.793 + .00902 - .616 + .525,\ n = 4,\\
&.793 + .00451 - .154 + .131,\ n = 16,\\
&.793 + .00226 - .0385 + .0164,\ n = 64,
\end{aligned}$$

so that divergence begins with the 3rd term.

Example 4.3. *Conditioning when the distribution of $\hat{w}$ is symmetric about w. Then for r odd, $C_r = D_r = g_{rk} = g_r = 0$. By (3.19), the conditional density is*

$$p_{n1.2}(\mathbf{x}_1) = \sigma^{-1}\phi(v)\ [1 + n^{-1}D_2 + O(n^{-2})],\ \textit{where } D_2 = \tilde{p}_2(x) - p_2^*(x_2),$$

for $\tilde{p}_2(x)$ of Example 1 of [30], H_k^* of (4.4), and

$$p_2^*(x_2) = k_{022}H_2^*/2 + k_{043}H_4^*/24.$$

By (3.38), the conditional distribution of $X_{n1.2}$ is

$$\begin{aligned}
&\Phi(v) + n^{-1}G_2 + O(n^{-2}),\ \textit{where } G_2 = g_2 - p_2^*(x_2)\ \Phi(v),\\
&g_2 = \sum_{k=2,4} [P_2(0k)I_2(0k) - b_{2k}\,\sigma^{-1}\phi(v)],
\end{aligned}$$

for b_{2k} of (4.16) and (4.17).

Example 4.4. *Discussions of pivotal statistics advocate using the distribution of a sample mean, given the sample variance. So $q = 2$. Let $\hat{w}_2, \hat{w}_2$ be the usual unbiased estimates of the mean and variance from a univariate random sample of size n from a distribution with rth cumulant κ_r. So $w_1 = \kappa_1$, $w_2 = \kappa_2$. By the last 2 equations*

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of Section 12.15 and (12.35)–(12.38) of [26], the cumulant coefficients needed for $\bar{P}_r^{1-k}$ of (2.3) for $r \leq 3$, – the coefficients needed for the conditional density to $O(n^{-2})$, in terms of $(i_1^{j_1} i_2^{j_2} \dots) = \kappa_{i_1}^{j_1} \kappa_{i_2}^{j_2} \dots$, are

$$
\begin{aligned}
& k_{201} = \kappa_2,\ k_{111} = \kappa_3,\ k_{021} = \kappa_4 + 2\kappa_2^2,\ \Rightarrow V = \begin{pmatrix} \kappa_2 & \kappa_3 \\ \kappa_3 & \kappa_4 + 2\kappa_2^2 \end{pmatrix}, \\
& k_{101} = k_{011} = 0,\ k_{302} = \kappa_3,\ k_{212} = k_{122} = 0,\ k_{032} = (6) + 12(24) + 4(3^2) + 8(2^3), \\
& k_{202} = k_{112} = 0,\ k_{022} = 2(2^2),\ k_{403} = (4),\ k_{313} = (5), \\
& k_{223} = k_{133} = 0,\ k_{043} = (8) + 24(26) + 32(35) + 32(4^2) + 144(2^2 4) + 96(23^2) \\
& \ + 48(2^4),\ k_{102} = k_{012} = k_{303} = k_{213} = k_{123} = 0,\ k_{033} = 12(24) + 16(2^3), \\
& k_{504} = k_{324} = k_{234} = k_{144} = 0,\ k_{414} = (6), k_{054} = (10) + 40(28) + 80(37) \\
& + 200(46) + 96(5^2) + 480(2^2 6) + 1280(235) + 1280(24^2) + 960(3^2 4) + 1920(2^3 4) \\
& + 1920(2^2 3^2) + 384(2^5).
\end{aligned}
$$

(3.19) gives D_r in terms of $\tilde{p}_r$ and p_r^*, that is, in terms of $\tilde{p}_{rk}$ and p_{rk}^* of (3.14) in terms of $P_r(ab)$. In this example, many of these are 0. The non-zero $P_r(ab)$ are in order needed,

$$
\begin{aligned}
& P_1(30) = \kappa_3/6,\ P_1(03) = k_{032}/6,\ P_2(02) = \kappa_2^2,\ P_2(40) = \kappa_4/24, \\
& P_2(04) = k_{043}/24,\ P_2(32) = \kappa_5/6,\ P_2(60) = \kappa_3^2/72, \\
& P_2(06) = k_{032}^2/72,\ P_2(33) = \kappa_3 k_{032}/36.\ P_3(03) = k^{033}/6. \\
& P_3(05) = k_{054}/120 + k_{022}k_{032}/12.\ P_3(70) = \kappa_3\kappa_4/144,\ P_3(07) = k^{032}k_{042}/144, \\
& P_3(62) = \kappa_3 k_{313}/36,\ P_3(43) = k_{032}\kappa_4/144,\ P_3(34) = (\kappa_3 k_{043} + k_{022}k_{313})/144,
\end{aligned}
$$

$$
\begin{aligned}
& P_3(90) = \kappa_3/6^3,\ P_3(09) = (k_{032}/6)^3,\ P_3(63) = 3\kappa_3^2 k_{032}/6^3, \\
& P_3(36) = 3\kappa_3 k_{032}^2/6^3. \\
& \textit{So, } \tilde{p}_{11} = 0,\ \tilde{p}_{13} = P_1(30)\ H_{30} + P_1(03)\ H_{03},\ ,\ \tilde{p}_{22} = P_2(02)H_{02}, \\
& \tilde{p}_{24} = P_2(40)H_{40} + P_2(04)H_{04} + P_2(32)H_{31},\ \tilde{p}_{26} = P_2(60)H_{60} + P_2(06)H_{06} \\
& \ + P_2(33)H_{33},\ \tilde{p}_{31} = 0,\ \tilde{p}_{33} = P_3(30)H_{30} + P_3(03)H_{03},\ \tilde{p}_{35} = P_3(05)H_{05}, \\
& \tilde{p}_{37} = P_3(70)H_{70} + P_3(2^7)H_{07} + P_3(62)H_{61} + P_3(43)H_{43} + P_3(34)H_{34} \\
& \ + P_3(25)H_{25} + P_3(16)H_{16} + P_3(07)H_{07}, \\
& \tilde{p}_{39} = P_3(90)H_{90} + P_3(63)H_{63} + P_3(36)H_{36} + P_3(09)H_{09}. \\
& \textit{Also, } b_{13} = P_1(30)H_{20},\ b_{22} = P_2(20)\ H_{10} + P_2(11)\ H_{01}, \\
& b_{24} = P_2(40)\ H_{30} + P_2(31)\ H_{21},\ b_{26} = P_2(60)H_{50} + P_2(33)\ H_{23}, \\
& b_{31} = 0,\ b_{33} = P_3(30)\ H_{20},\ b_{35} = 0, \\
& b_{37} = P_3(70)\ H_{60} + P_3(61)\ H_{51} + P_3(43)\ H_{33} + P_3(34)\ H_{24}, \\
& b_{39} = P_3(90)\ H_{80} + P_3(63)\ H_{53} + P_3(36)\ H_{26}.
\end{aligned}
$$

For $r = 1,2,3$, $\tilde{p}_r(x)$ is now given by (2.13), $p_r^*(x)$, and Section 2 of [30]. By (2.4) and (3.19), this gives the conditional density $p_{n1.2}(\mathbf{x}_1)$ to $O(n^{-2})$. And (4.14) gives g_r needed for the conditional distribution $P_{n1.2}(\mathbf{x}_1)$ to $O(n^{-2})$.

5. Conclusions

[30] gave the density and distribution of $X_n = n^{1/2}(\hat{w} - w)$ to $O(n^{-2})$, for $\hat{w} \in R^q$ any standard estimate, in terms of functions of the cumulant coefficients $\bar{k}_j^{1-r}$ of (2.1), called the Edgeworth coefficients, $\bar{P}_r^{1-k}$.

Most estimates of interest are standard estimates, including smooth functions of sample moments, like the sample skewness, kurtosis, correlation, and any multivariate function of k-statistics. (These are unbiased estimates of cumulants and their products, the most common example being that for a variance.) Unbiased estimates are not needed for Edgeworth expansions, although this does simplify the Edgeworth coefficients, as seen in Examples 4.1, 4.2, 4.4. However unbiased estimates are not available for most parameters or functions of them, such as the ratio of two means or variances, except for special cases of exponential families. [29] gave the cumulant coefficients for smooth functions of standard estimates.

As noted, conditioning is a very useful and basic way to use correlated information to reduce the variability of an estimate. Section 3 gave the conditional density and distribution of $\mathbf{X}_{n1}$ given $\mathbf{X}_{n2}$ to $O(n^{-2})$ where $\binom{\mathbf{X}_{n1}}{\mathbf{X}_{n2}}$ is any partition of $X_n = n^{1/2}(\hat{w} - w)$. The expansion (3.19) gave the conditional density of any multivariate standard estimate. Our main result, an explicit expansion for the conditional distribution (3.23) to $O(n^{-2})$, is given in terms of the leading $\bar{I}^{1-k}$ of (3.28). These are given explicitly by Theorems 3.3 and 3.4.

When $q_1 = q_2 = 1$, Theorem 4.1 simplified the conditional density expansion, and Theorem 4.3 gave a huge simplification, and the coefficients of the conditional *distribution* expansion in terms of $I_{0k} = I^{2^k}$ of Theorem 4.4.

Cumulant coefficients can also be used to obtain estimates of bias $O(n^{-k})$ for $k \geq 2$: see [34,35,37].

6. Discussion

A good approximation for the distribution of an estimate, is vital for accurate inference. It enables one to explore the distribution's dependence on underlying parameters. Our analytic method avoids the need for simulation or jack-knife or bootstrap methods while providing greater accuracy than any of them. [13] used the Edgeworth expansion to show that the bootstrap gives accuracy to $O(n^{-1})$. [12] said that "2nd order correctness usually cannot be bettered". But this is not true using our analytic method. Simulation, while popular, can at best shine a light on behaviour, only when there is a small number of parameters, and only for limited values of their range.

Estimates based on a sample of independent, but not identically distributed random vectors, are also generally standard estimates. For example, for a univariate sample mean $\bar{X} = n^{-1}\sum_{j=1}^n X_{jn}$ where X_{jn} has rth cumulant κ_{rjn}, then $\kappa_r(\bar{X}) = n^{1-r}\kappa_r$ where $\kappa_r = n^{-1}\sum_{j=1}^n \kappa_{rjn}$ is the average rth cumulant. For some examples, see [22], [23] and [32] , 2020). The last is for a function of a weighted mean of complex random matrices. For conditions for the validity of multivariate Edgeworth expansions, see [24] and its references, and Appendix C of [30].

While the use of Edgeworth-Cornish-Fisher expansions is widespread, few papers address how to deal with their divergence for small sample sizes. [8] and [11] avoided this question as it did not arise in their examples. In contrast we confronted this in Example 4.2, the examples of Withers (1984), and in Example 7 of [30].

We now turn to conditioning. Conditioning on $\hat{\mathbf{w}}_2$ makes inference on $\mathbf{w}_1$ more precise by reducing the covariance of the estimate. The covariance of $\hat{\mathbf{w}}_1|\hat{\mathbf{w}}_2$ can be substantially less than that of $\hat{\mathbf{w}}_1$. [3] pp34-36 argue that an ideal choice would be when the distribution of $\hat{\mathbf{w}}_2$ does not depend on $\mathbf{w}_1$. But this is generally not possible except for some exponential families. An example when it *is* true, is when w_1 and w_2 are location and scale parameters: on p54 they essentially suggest choosing $w_2 = n\ var\ \hat{w}_1$. This is our motivation for Example 4.4. For some examples, see [2]. Their (7.5) gave a form for the 3rd order expansion for the conditional density of a sample mean to $O(n^{-3/2})$, but did not attempt to integrate it.

Tilting (also known as small sample asympotics, or saddlepoint expansioins), was first used in statistics by [9]. He gave an approximation to the density of a sample mean, good for the whole line, not just in the region where the Central Limit Theorem approximation holds. A conditional distribution by tilting, was first given by [25] up to $O(n^{-1})$, for a bivariate sample mean. Compare [2].

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For some other results on conditional distributions, see [21], [5], [10], [14], Hansen (1994), [20], Chapter 4 of [6], and [17].

Future directions. 1. The results here give the first step for constructing confidence intervals and confidence regions of higher order accuracy. See [15] and [28]. What is needed next, is an application of [29] to obtain the cumulant coefficients of $\hat{\theta}_i = \hat{\mathbf{V}}_{ii}^{-1/2}(\hat{\mathbf{w}}_i - \mathbf{w}_i)$, $i = 1, 2$, or those of $\hat{\theta} = \hat{V}^{-1/2}(\hat{\mathbf{w}} - \mathbf{w})$. This should be straightforward.

2. When $q_1 = 1$, our expansion for the conditional distribution of $\mathbf{X}_{n1.2}$ of (3.1), can be inverted using the Lagrange Inversion Theorem, to give expansions for its percentiles. This should be straightforward. (The quantile expansions of [8] and Withers (1984) do not apply as Appendix A shows that conditional estimates of standard estimates are not standard estimates.)

3. Here we have only considered expansions about the normal. However expansions about other distributions can greatly reduce the number of terms by matching the leading bias coefficient. The framework for this is [32], building on [15]. For expansions about a matching gamma, see [33,36].

4. The results here can be extended to tilted (saddlepoint) expansions by applying the results of [32]. The tilted version of the multivariate distribution and density of a standard estimate are given by Corollaries 3, 4 there, and that of the conditional distribution and density follow from these. For the entangled gamma of Example 4.2, this requires solving a cubic. See also [16].

5. A possible alternative approach to finding the conditional distribution, is to use conditional cumulants, when these can be found. Section 6.2 of [18] uses conditional cumulants to give the conditional density of a sample mean to $O(n^{-3/2})$. Section 5.6 of [19] gave formulas for the 1st 4 cumulants conditional on $\mathbf{X}_2 = \mathbf{x}_2$ only when $\mathbf{X}_1$ and $\mathbf{X}_2$ are uncorrelated. He says that this assumption can be removed, but gives no details how. That is unlikely to give an alternative to our approach, for as well as giving expansions for the first 3 conditional cumulants, Appendix A shows that the conditional estimate is not a standard estimate.

6. Lastly we discuss numerical computation. We have used [27] for our calculations. Its input is V^{11}, V^{12}, V^{22} and y_1, y_2, - not V_{11}, V_{12}, V_{22} and x_1, x_2. There is a function sub2(sb1,sb2) which takes as argument the two subscripts of mu, and returns the value. If global variables mu20, mu02, mu11 are symbolic variables (defined using sympy) then it returns the answer in terms of those, but if they are numeric then it returns a numeric answer. There is another function called biHermite(n, m, y1, y2) which takes the 2 subscripts of H. If y1 and y2 are symbolic, then it returns a symbolic answer, but if they are numeric it returns a numeric answer. A numerical example is given by Example 4.2, that is, for the case $V_{11} = 2, V_{12} = 1, V_{22} = 2$ and $x_1 = x_2 = 1$ or $x_1 = x_2 = 2$.

Similar software for numerical calculations for Theorems 4.1, 4.3 and 4.4 would be invaluable, as would software for applying the Lagrange Inversion Theorem. (We mention R-4.4.1 for Windows: dmvnorm for the density function of the multivariate normal, mvtnorm for the multivariate normal, qmvnorm for quantiles, and rmvnorm to generate multivariate normal variables.) On bivariate Hermite polynomials, see cran.r-project.org/web/packages/calculus/vignettes/hermite.html

Appendix A Conditional Moments

Here we give expansions for the conditional moments of $\mathbf{X}_{n1.2}$ of (3.1), in terms of the conditional normal moments of $\mathbf{X}_{1.2}$, of (3.1). And we show that

$$\hat{\mathbf{w}}_{1.2} = \hat{\mathbf{w}}_1 | (\mathbf{X}_{n2} = \mathbf{x}_2) \tag{A1}$$

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is neither a standard estimate of w_1, nor a Type B estimate, as defined below.
Consider the case $q_1 = 1$. By (3.5),

$$
\begin{aligned}
&X_{1.2} = \mu + \sigma N, \text{ where } \mu = \mu_{1.2},\ \sigma^2 = V_{1.2} \text{ of (3.4)},\ N \sim \mathcal{N}_1(0,1).\\
&\text{Set } M_s = E\ X_{1.2}^s. \qquad \text{(A2)}\\
&\text{So } M_1 = \mu,\ M_2 = \mu^2 + \sigma^2,\ M_3 = \mu^3 + 3\mu\sigma^2,\ M_4 = \mu^4 + 6\mu^2\sigma^4 + 3\sigma^4.\\
&\text{For } r = 0,1,\ldots, \text{ set } A_r \otimes B_r = \sum_{a+b=r} A_a B_b,\ A_r \otimes B_r \otimes C_r = \sum_{a+b+c=r} A_a B_b C_c.
\end{aligned}
$$

Non-central moments.

Theorem A1. *Take C_r, D_r of Theorem 3.1. Set $\tilde{p}_0(x) = 1$. For $s > 0$, the sth conditional moment of $X_{n1.2}$ of (3.1) about $\Phi_{1.2}(x_1)$ of (3.10), has the expansion*

$$
m_{ns} = E\ X_{n1.2}^s = n^{s/2} E\ (\hat{w}_{1.2} - w_1)^s \approx \sum_{r=0}^{\infty} n^{-r/2} G_r^s \qquad \text{(A3)}
$$

where $G_r^s = C_r \otimes g_r^s$, $g_r^s = E\ X_{1.2}^s P_r$, and $P_r = \tilde{p}_r(x)$ at $x_1 = X_{1.2}$.

$$
\begin{aligned}
&\text{So, } G_0^s = g_0^s = M_s = E\ X_{1.2}^s.\\
&g_r^s = \sum_{k=1}^{3r} [g_{rk}^s :\ k - r \text{ even}], \text{ for } r \geq 1, \text{ where for } \tilde{p}_{rk} \text{ of (2.6)}, \qquad \text{(A4)}\\
&g_{rk}^s = E\ X_{1.2}^s \tilde{P}_{rk} = \bar{P}_r^{1-k} \bar{I}_s^{1-k},\ \tilde{P}_{rk} = \tilde{p}_{rk} \text{ at } x_1 = X_{1.2}, \qquad \text{(A5)}\\
&\text{and for } 1 \leq i_1, \ldots, i_k \leq q,\ \bar{I}_s^{1-k} = I_s^{i_1 \cdots i_k} = E\ X_{1.2}^s\ \bar{H}^{1-k}(X_{1.2}), \qquad \text{(A6)}\\
&\text{for } \bar{H}^{1-k}(X_{1.2}) = \bar{H}^{1-k} \text{ at } x_1 = X_{1.2}.
\end{aligned}
$$

PROOF This follows from Theorem 3.1 □

So by (A3), the sth conditional moment of $X_{n1.2}$ is

$$
\begin{aligned}
&m_{ns} = M_s + n^{-1/2} G_1^s + n^{-1} G_2^s + O(n^{-3/2}), \text{ where}\\
&G_1^s = C_1 M_s + g_1^s,\ G_2^s = C_2 M_s + C_1 g_1^s + g_2^s,\\
&g_1^s = g_{11}^s + g_{13}^s,\ g_2^s = g_{22}^s + g_{24}^s + g_{26}^s,
\end{aligned}
$$

of (A5) and (A6). For example,

$$
g_{11}^s = \bar{k}_1^1\ E\ X_{1.2}^s \bar{H}^1(X_{1.2}), \text{ and } g_{13}^s = \bar{k}_2^{1-3}\ E\ X_{1.2}^s \bar{H}^{1-3}(X_{1.2})/6.
$$

So $\hat{w}_{1.2}$ of (A1) *is not a standard estimate*, as by (A3), the expansion for its mean is a power series in $n^{-1/2}$, not n^{-1}. Is it a Type B estimate? These are defined as for a standard estimate, but with cumulant expansions being series in $n^{-1/2}$, not n^{-1}. We shall see. Take $q_2 = q_1 = 1$. By Theorem 4.2, for $P_r(ab)$ of (2.3), g_{rk}^s of (A4) is given by

$$
\begin{aligned}
&g_{rk}^s = \sum_{b=0}^{k} P_r(k-b, b)\ I_{k-b,b}^s, \text{ where } I_{ab}^s = E\ X_{1.2}^s\ H_{ab}(X_{1.2}),\\
&\text{and } H_{ab}(X_{1.2}) = H_{ab} \text{ at } x_1 = X_{1.2}.
\end{aligned}
$$

For example,

$$
\begin{aligned}
&g_{r1}^s = P_r(10) I_{10}^s + P_r(01) I_{01}^s\\
&g_{r3}^s = P_r(30) I_{30}^s + P_r(21) I_{21}^s + P_r(12) I_{12}^s + P_r(03) I_{03}^s.
\end{aligned}
$$

Finding the I^s_{ab}.
The H_{ab} needed are given in Appendix B of [30] in terms of

$$y = V^{-1}x: \; y_i = V^{i1}x_1 + V^{i2}x_2, \; y_1 = \mu_{20}x_1 + \mu_{11}x_2, \; y_2 = \mu_{11}x_1 + \mu_{02}x_2.$$

For example,

$$\begin{aligned}
&H_{10} = y_1 = \mu_{20}x_1 + \mu_{11}x_2, \; H_{01} = y_2 = \mu_{11}x_1 + \mu_{02}x_2,\\
&H_{30} = y_1^3 - 3y_1\mu_{20} = (\mu_{20}x_1 + \mu_{11}x_2)^3 - 3(\mu_{20}x_1 + \mu_{11}x_2)\mu_{20},\\
&H_{03} = y_2^3 - 3y_2\mu_{02} = (\mu_{11}x_1 + \mu_{02}x_2)^3 - 3(\mu_{11}x_1 + \mu_{02}x_2)\mu_{20},\\
&H_{21} = y_2(y_1^2 - \mu_{20}) - 2y_1\mu_{11}, \; H_{12} = y_1(y_2^2 - \mu_{02}) - 2y_2\mu_{11}.
\end{aligned}$$

Let us write H_{ab} in terms of M_s of (A2), as

$$\begin{aligned}
&H_{ab} = \sum_{k=0}^{a+b} H^k_{ab}x_1^k. \text{ Then, } I^s_{ab} = \sum_{k=0}^{a+b} [H^k_{ab}M_{s+k} : \; s+k \text{ even}].\\
&\text{So, } I^s_{10} = H^0_{10}\, M_s + H^1_{10}\, M_{s+1}, \; I^s_{01} = H^0_{01}\, M_s + H^1_{01}\, M_{s+1} :\\
&\text{for odd } s, \; I^s_{10} = H^1_{10}M_{s+1} = \mu_{20}M_{s+1}, \; I^s_{01} = H^1_{01}M_{s+1} = \mu_{02}x_2M_{s+1},\\
&\text{and for even } s, \; I^s_{10} = H^0_{10}M_s = \mu_{11}x_2M_s, \; I^s_{01} = H^0_{01}M_s = \mu_{02}x_2M_s.
\end{aligned}$$

$$\begin{aligned}
&\text{So, } H^0_{10} = \mu_{11}x_2, \; H^1_{10} = \mu_{20}, \; H^0_{01} = \mu_{02}x_2, \; H^1_{01} = \mu_{11},\\
&H^0_{30} = (\mu_{11}x_2)^3 - 3\mu_{11}x_2\mu_{20}, \; H^1_{30} = 3\mu_{20}\,[(\mu_{11}x_2)^2 - \mu_{20}],\\
&H^2_{30} = 3\mu_{20}^2\mu_{11}x_2, \; H^3_{30} = \mu_{20}^3,\\
&H^0_{03} = (\mu_{02}x_2)^3 - 3\mu_{02}x_2\mu_{20}, \; H^1_{03} = 3\mu_{11}\,[(\mu_{02}x_2)^2 - \mu_{11}],\\
&H^2_{03} = 3\mu_{11}^2\mu_{02}x_2, \; H^3_{03} = \mu_{11}^3,\\
&H^0_{21} = \mu_{02}x_2[(\mu_{11}x_2)^2 - \mu_{20}] - 2\mu_{11}^2x_2,\\
&H^1_{21} = \mu_{11}\,[(\mu_{11}x_2)^2 - \mu_{20}] + 2\mu_{20}\mu_{11}(\mu_{02}x_2^2 - 1),\\
&H^2_{21} = \mu_{20}\mu_{22}x_2 \text{ since } \mu_{22} = \mu_{20}\mu_{02} + 2\mu_{11}^2, \; H^3_{21} = \mu_{11}\mu_{20}^2,\\
&H^0_{12} = \mu_{11}x_2\,[(\mu_{02}x_2)^2 - \mu_{02}] - 2\mu_{02}\mu_{11}x_2,\\
&H^1_{12} = \mu_{20}\,[(\mu_{02}x_2)^2 - \mu_{02}] + 2\mu_{11}^2(\mu_{02}x_2^2 - 1), \; H^2_{12} = \mu_{11}\mu_{22}x_2, \; H^3_{12} = \mu_{20}\mu_{11}^2.
\end{aligned}$$

To get a general formula for H^k_{ab}, set

$$\begin{aligned}
&c_1 = V_{11}x_1, \; c_2 = IV_{11}X_1, \; c_3 = V_{12}x_2, \; c_4 = IV_{12}X_2.\\
&\text{So, } y_1 + IY_1 = c_1 + c_2, \; y_2 + IY_2 = c_3 + c_4,\\
&H_{ab} = E\,(c_1 + c_2)^a(c_3 + c_4)^b = \sum_{j=0}^{a} \binom{a}{j} c_1^{a-j} \sum_{k=0}^{b} \binom{b}{k} c_3^{b-k} C_{jk}\\
&\text{where } C_{jk} = E\, c_2^j c_4^k = I^{j+k}V_{11}^j V_{12}^k\, \mu^{jk}, \; \mu^{jk} = E\, X_1^j X_2^k.\\
&\text{So, } H^{a-j}_{ab} = \binom{a}{j} V_{11}^{a-j} \sum_{k=0}^{b} \binom{b}{k} c_3^{b-k} C_{jk},
\end{aligned}$$

where $C_{jk} = 0$ if $j+k$ is odd. μ^{jk} is just μ_{jk} of Appendix B of [30] with V replaced by V^{-1}.

Central moments. Set $m_s(X) = E\ X^s$ and $\mu_s(X) = E\ (X - E\ X)^s$.
For $m_s = m_{s1.2}$ of (A3), set

$$\mu_s = \mu_{s1.2} = \mu_s(X_{n1.2}) = n^{s/2}\mu_s(\hat{w}_{1.2}).$$

So by (A3), $\mu_2 = m_2 - m_1^2 \approx \sum_{r=0}^{\infty} n^{-r/2}\mu_{2r}$, where $\mu_{2r} = G_r^2 - G_r^1 \otimes G_r^1$,

and $\mu_3 = m_3 - 3m_1m_2 + 2m_1^3 \approx \sum_{r=0}^{\infty} n^{-r/2}\mu_{3r}$

where $\mu_{3r} = G_r^3 - 3G_r^1 \otimes G_r^2 + 2G_r^1 \otimes G_r^1 \otimes G_r^1$.

Is the conditional estimate $\hat{w}_{1.2}$ a Type B estimate? This requires its rth cumulant to have magnitude $O(n^{1-r})$ for $r \geq 1$. This is true for $r = 1$ and 2 but not for $r = 3$, as $\mu_r(\hat{w}_{1.2})$ has magnitude $O(n^{-r/2})$, since $\mu_{s1.2} = O(1)$.

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$$l_\nu = O(1), \text{ where } a(r) = r/2 - I(r \geq 3) \text{ as } n \to \infty. \tag{7.3}$$
This is". p226. Replace κ_r on LHS of 4th displayed equation by k_r. p226. Replace k_r on RHS of 6th displayed equation by K_r. p227. Replace r in (7.5) and the following equation by $|\nu|$. p227 Replace "variance" in (7.6) by "covariance".
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