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Not peer-reviewed version

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Posted Date: 8 March 2024

doi: 10.20944/preprints202403.0470.v1

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A New Computational Algorithm for Assessing Overdispersion in Machine Learning Count Models with Python

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Abstract: Count data analysis presents unique challenges due to its discrete nature, often exhibiting excessive zeros and overdispersion. To address these complexities, count models, such as Poisson regression and Negative Binomial regression, have been developed, enabling modeling and prediction of count-based phenomena. Additionally, it's important to notice that zero inflation, a phenomenon commonly observed in count data, requires specialized techniques for robust analysis. This article provides an overview of count data and count models, explores zero inflation, introduces Likelihood Ratio Tests, and explains how the Vuong Test can be used as a model selection criteria. Furthermore, we created a Vuong Test implementation from scratch using the Python programming language. This implementation enhances the accessibility and applicability of the Vuong Test in real-world scenarios, providing a valuable contribution to the academic community, since Python didn't have an implementation of this statistical test.

Keywords: count data; machine learning; negative binomial regression; overdispersion; Poisson regression; Python; Vuong test; zero inflation

1. Introduction

Count data analysis is a statistical approach used to analyze data that consists of non-negative integer values, representing the number of occurrences of a specific event within a given context. This data type is commonly encountered in various fields, such as biology, economics, social sciences, and engineering. However, count data poses unique challenges due to its discrete nature, often exhibiting characteristics such as excessive zeros and overdispersion [11].

Excessive zeros refer to an unusually high number of observations with a count of zero, which cannot be adequately explained by standard statistical models. It requires specialized techniques to be appropriately accounted for in the analysis to avoid biased model estimates [27,31]. Overdispersion occurs when the variance of the count data is higher than expected, indicating that additional sources of variation need to be accounted for beyond what the basic models can handle [20].

To address these complexities and effectively model count-based phenomena, specialized statistical models known as count models have been developed. Two commonly used count models are Poisson regression and Negative Binomial regression. Poisson regression is suitable for count data with low variability, assuming that the mean and variance of the data are equal. However, in cases where overdispersion is present, Poisson regression may not provide accurate results. Negative Binomial regression, on the other hand, allows for overdispersion and provides a more flexible approach to model count [25].

In this article, we aim to provide a comprehensive overview of count data analysis and count models. We explore the concept of zero inflation and its impact on count models, demonstrating how it can lead to model misspecification if not properly addressed. In addition, we introduce two statistical tests: Likelihood Ratio Tests (LRT) and the Vuong Test [8]. LRT is used to compare nested models and assess the improvement in model fit when additional parameters are added. On the other hand, the Vuong Test is a model selection criterion that compares non-nested models, making it ideal for comparing count models with and without zero inflation.

Furthermore, the article emphasizes the practicality of our approach by providing the complete code we created that implements the Vuong Test in the widely used Python programming language. Python is renowned for its ease of use and extensive libraries, making it an excellent choice for data analysts and researchers [44]. By creating a Python implementation of the Vuong Test, we enhance the accessibility and applicability of this essential model selection technique in real-world scenarios, enabling readers to make informed model selection decisions and perform robust analysis of count-based phenomena in their projects.

2. Count data

Count data refers to the observations made about events or items that are enumerated. More specifically, in the context of statistics, count data refers to the number of occurrences of an event within a fixed period. It only contains positive integer values that go from zero to some greater value, because an event cannot occur a negative number of times [11,12]. Examples of count data include the number of speeding tickets received in a year, the number of crimes on campus per semester, or the number of trips per year a person makes.

Following the concept of count data, it's possible to define count variable as a list or array of count data [11]. Count variables indicate how many times something has happened, and can be used as response variables in statistical modeling when the goal is to understand or predict the factors influencing the count [10].

3. Count models

In simple terms, a statistical model helps us understand how different variables are related to each other. It provides a mathematical representation of how one variable or a set of variables can explain or predict another variable. Specifically, when we have a count variable, the statistical model helps us explain the counts using one or more explanatory variables [11,18].

Statistical models are considered stochastic because they are based on probability functions. This means that they take into account the uncertainty and variability inherent in real-world data. Instead of providing exact predictions or explanations, the model provides a probabilistic framework that describes the likelihood or probability of different outcomes [13]. By utilizing statistical models, we can analyze and quantify the relationships between variables, make predictions, and assess the statistical significance of explanatory factors. These models allow us to gain insights into the underlying patterns and processes driving the count variable, accounting for the inherent randomness and variability observed in real-world phenomena. In the scope of count data, those statistical models have applications that exploit the full probability distribution of counts to provide comprehensive predictions. This capability enhances our ability to interpret and utilize count data effectively in various applications, such as forecasting, risk assessment, resource allocation, and decision-making [18].

When using count variables in Ordinary Least Squares (OLS) regression, there are potential issues to consider. OLS regression, which has minimal assumptions about predictors, allows count variables to be used as predictors with one caveat. If the count variable has very little variability, which can happen with count data having a small range, the regression coefficient associated with that predictor becomes unstable and has a large standard error [9]. It's important to note that this instability is not unique to count predictors - any predictor with low variability would result in an unstable regression coefficient.

However, different problems arise when a count variable is used as the outcome or dependent variable in OLS regression. When the average count value of the outcome variable is relatively high, OLS regression can generally be applied without significant difficulty. But when the mean of the outcome variable is low, OLS regression produces undesirable outcomes, including biased standard errors and significance tests [12].

So, while count variables can be used as predictors in OLS regression with caution, using count variables as outcome variables in OLS regression can result in inefficient, inconsistent, and biased estimates. Even though there are situations in which the Linear Regression Model (LRM) provides reasonable results, it is much safer to employ regression models specifically designed for count outcomes, as they offer improved statistical power and appropriately handle the specific characteristics of count data [10,12]. Examples of such models include Poisson Regression, Negative Binomial Regression (NB), and variations of these models for zero-inflated counts (ZIP and ZINB). These models consider the discrete nature of count variables, their variability, and the specific statistical assumptions needed for accurate estimation and inference [10,20].

Generalized linear models offer a versatile method for addressing a wide variety of response modeling issues. The most frequently used responses are Normal, Poisson, and Binomial, but other distributions may also be employed [22,23]. When dealing with count data, we typically begin by estimating the parameters using a Poisson regression model because of its simplicity. In this scenario, the dependent variable in a Poisson regression model should adhere to a Poisson distribution with the mean equaling the variance [24]. However, this property is frequently violated in empirical research, as it is common for there to be overdispersion, where the variance of the dependent variable exceeds its mean. In such instances, we will estimate a negative binomial regression model [26].

4. Poisson and Negative Binomial Models

The Poisson distribution is distinct from the normal distribution in several ways that make it more appealing for representing the characteristics of count data. Firstly, the Poisson distribution is a discrete distribution that only assumes probability values for non-negative integers. This feature of the Poisson distribution makes it an ideal choice for modeling count outcomes, which can only take on integer values of 0 or more [12]. The Poisson distribution, for a given observation i, has the following probability of occurrence of a count m (m = 0, 1, 2...) in a given exposure:

$$p(Y_i = m) = \frac{e^{-\lambda_i}.\lambda_i^m}{m!}. \tag{1}$$

In (1), λ is the expected number of occurrences or the estimated incidence rate ratio of the phenomenon under study for a given exposure. In the Poisson distribution, the mean and variance of the variable under study must be equal to λ. This characteristic is known as the equidispersion of the Poisson distribution. If this condition is satisfied, a Poisson regression model can be calculated, which is described as follows:

$$ln(\hat{Y}_i) = ln(\lambda_i) = \alpha + \beta_1.X_{1i} + \cdots + \beta_k.X_{ki}. \tag{2}$$

In (2), α stands for the constant, β_j (j = 1, 2, . . . , k) are the calculated parameters for each explanatory variable, X_j are the explanatory variables (either metrics or dummies) and the subscript i denotes each observation in the sample (i = 1, 2, . . . , n, where n is the size of the sample).

The log-likelihood function, which must be maximized, can be written as:

$$LL = \sum_{i=1}^{n}[-\lambda_i + (Y_i).ln(\lambda_i) - ln(Y_i!)]. \tag{3}$$

As previously noted, the data is assumed to have equidispersion. If this is not the case, a negative binomial regression model could be used for estimation [11]. The NB distribution uses an additional parameter to model overdispersion. In other words, an NB-distributed random variable is equivalent to a Poisson random variable with a random, gamma-distributed mean, allowing it to model unobserved individual heterogeneity [27].

For a given observation i (i = 1, 2, … , n, where n is the sample size), the probability distribution function of the variable v_i will be given by the following:

$$p(v_i) = \frac{\delta^{\psi}.v_i^{\psi-1}.e^{-v_i.\delta}}{\Gamma(\psi)}. \tag{4}$$

In (4), ψ is called the shape parameter ($\psi > 0$), δ is called the rate parameter ($\delta > 0$) and, for $\psi > 0$ and integer, $\Gamma(\psi)$ can be approximated by $(\psi - 1)!$. The log-likelihood function for the NB2 regression, which must be maximized, can be written as:

$$LL = \sum_{i=1}^{n} \left[Y_i.ln\left(\frac{\phi.u_i}{1+\phi.u_i} - \frac{ln(1+\phi.u_i)}{\phi} \right) + ln\Gamma(Y_i + \phi^{-1}) - ln\Gamma(Y_i + 1) - ln\Gamma(\phi^{-1}) \right]. \tag{5}$$

For a more detailed discussion of both the Poisson and NB distributions, see [20,25,29].

5. Zero-inflated count data models

Simply testing for overdispersion and using a NB model may not be enough. It's important to note that overdispersion can be caused by an excess of zeros in the data, and this zero inflation may need to be accounted for in the model [27,31].

When data is zero-inflated, it is necessary to modify the Poisson or NB procedure to prevent incorrect estimation of model parameters and standard errors, as well as incorrect specification of the distribution of test statistics. Ignoring these misspecifications can lead to incorrect conclusions about the data and introduce uncertainty into research and practice. As a result, the use of zero-inflated Poisson (ZIP) and zero-inflated negative binomial (ZINB) models has grown in popularity across a wide range of fields [21].

According to [32], zero-inflated regression models are considered a combination of a count data model and a binary data model. They are used to investigate the reasons for a certain number of occurrences (counts) of a phenomenon, as well as the reasons for the occurrence (or not) of the phenomenon itself, regardless of the number of counts observed.

Facing this scenario, according to [30], the first step is to address potential overdispersion in the distribution by comparing Poisson versus NB models, using statistical tests such as the likelihood ratio test (LRT) and the Wald test. The next step is to assess whether the distribution has an excess of zeros by comparing Poisson versus ZIP or NB versus ZINB models, using the Vuong test [8]. If the zero-inflated version of the model fits the data significantly better than the standard model, it is considered evidence that the data contains an excess of zeros [27].

5.1. Zero Inflated Poisson Models

In the ZIP model, the probability p of observing zero counts for a given observation i ($i = 1, 2, \ldots, n$, where n is the sample size), or $p(Y_i = 0)$, is determined by combining a dichotomous component with a count component. As a result, the probability P_{logit_i} of observing zero counts due solely to the dichotomous component must be established. The probability of observing a specific count m ($m = 1, 2, \ldots$), or $p(Y_i = m)$, is determined by the probability expression of the Poisson distribution, multiplied by $(1 - P_{logit_i})$ [25]. To sum up, for $m = 1, 2, \ldots$, we have the following equations, on which $Y \sim ZIP\ (\lambda, P_{logit_i})$.

$$\begin{cases} P(Y_i = 0) & = P_{logit_i} + (1 - P_{logit_i}).e^{-\lambda_i}, \\ P(Y_i = m) & = (1 - P_{logit_i}).\frac{e^{-\lambda_i}.\lambda_i^m}{m!}. \end{cases} \tag{6}$$

It is evident that if $P_{logit_i} = 0$, the probability distribution in expression 6 simplifies to the Poisson distribution, even for instances where $Y = 0$. In other words, ZIP regression models have two processes that generate zeros. One is due to the binary distribution, which generates what are known as structural zeros. The other is due to the Poisson distribution, which generates count data, including what is known as sample zeros [25].

The log-likelihood function of a ZIP regression model (7) should also be maximized.

$$LL = \sum_{Y_i=0} ln\left[P_{logit_i} + (1 - P_{logit_i}).e^{-\lambda_i}\right] + \sum_{Y_i>0}\left[ln(1 - P_{logit_i}) - \lambda_i + Y_i.ln(\lambda_i) - ln(Y_i!)\right]. \tag{7}$$

5.2. Zero Inflated Negative Binomial Models

In the ZINB models, The probability of no counts for a given observation i, or $p(Y_i = 0)$, is also calculated by adding a dichotomous component to a count component. The chance of a specific count m ($m = 1, 2, \ldots$), or $p(Y_i = m)$, now follows the probability expression of the Poisson-Gamma distribution [25]. In equation (8), for $m = 0, 1, 2, \ldots, Y \sim ZINB\ (\phi, u, P_{logit_i})$, ϕ represents the inverse of the shape parameter of a given Gamma distribution.

$$P(Y_i = m) = (1 - P_{logit_i}).\left[\binom{m + \phi^{-1} - 1}{\phi^{-1} - 1}.\left(\frac{1}{1 + \phi.u_i}\right)^{\frac{1}{\phi}}.\left(\frac{\phi.u_i}{\phi.u_i + 1}\right)^{m}\right]. \tag{8}$$

The log-likelihood function of a ZINB regression model (9) should also be maximized.

$$\begin{aligned} LL = \sum_{Y_i=0} ln\left[P_{logit_i} + (1 - P_{logit_i}).\left(\frac{1}{1 + \phi.u_i}\right)^{\frac{1}{\phi}}\right] + \sum_{Y_i>0}\Bigg[ln(1 - P_{logit_i}) + \\ + Y_i.ln\left(\frac{1}{1 + \phi.u_i}\right) - \frac{ln(1 + \phi.u_i)}{\phi}\Bigg] + ln\Gamma(Y_i + \phi^{-1}) - ln\Gamma(Y_i + 1) - ln\Gamma(\phi^{-1}). \end{aligned} \tag{9}$$

6. Likelihood Ratio Tests for Model Selection: Vuong Test

The decision to use either conventional count regression or zero-inflated modeling is based on the balance between avoiding overfitting and accurately representing the empirical features of the data. One way to determine if a zero-inflated model is necessary is to compare its fit to that of a standard count model. This test is important because choosing the wrong model can have consequences. If the standard model is chosen when the zero-inflated model is more appropriate, an important part of the data-generating process may be overlooked. On the other hand, if the zero-inflated model is chosen when it's not necessary, the model becomes unnecessarily complicated by adding an extra equation [33].

With this dilemma in mind, researchers commonly use the Vuong test [8] to determine whether the zero-inflated model fits the data statistically significantly better than count regression with a single equation. The Vuong test is a method for comparing the fit of two models to the same data using maximum likelihood. Its purpose is to test the null hypothesis that the two models fit the data equally well. The models being compared need not be nested, and one of them doesn't need to be the correct specification [33].

7. Vuong Test Implementation on Python

Python is currently experiencing a surge in popularity as a programming language, and this can be attributed to several factors. Firstly, Python is known for its ease of use and accessibility, making it an attractive option for beginners. Additionally, the learning curve for Python is relatively fast, allowing new users to quickly become proficient. Another major draw for Python is the vast array of high-quality packages available for data science and machine learning applications.

A few libraries in Python already support statistical and econometric analysis, such as statsmodels [39] and pandas [40]. However, when it comes to general statistics, Python lags behind the R programming language. As a result, many scientists continue to rely on R for their statistical analysis [38].

In an effort to make Python more complete for statistical analysis, a gap was identified in the Vuong test, which currently lacks an implementation in Python. So, we present a code that implements the Vuong test in Python. In addition, we demonstrate its use on a database [41]. Other machine learning and deep learning studies have already been covered in this same journal, such as [45–47].

7.1. Corruption Database

According to [41], in a empirical approach developed for evaluating the role of both social norms and legal enforcement in corruption by studying parking violations among United Nations diplomats living in New York City, diplomatic immunity is a privilege that allows mission personnel and their families to avoid paying parking fines. This was the case until November 2002. Illegally parking can be considered an act of corruption, as it is an abuse of power for personal gain. Comparing parking violations by diplomats from different societies can serve as a measure of the extent of corruption norms or culture.

The data set contains the following information:

1. Country name
2. Code of the country
3. Number of parking violations
4. Number of UN Mission Diplomats in 1998
5. Indicator (Yes/No) that the data is before or after the law enforcement
6. Corruption Index - CI [42]

The data set is organized in a table as follows in Table 1. It contains data from 149 countries, and it has 298 rows, given that which country has 2 rows (one for data before law enforcement and the other for after law enforcement). Also, the histogram of the data (counts per number of parking violations) is shown in Figure 1.

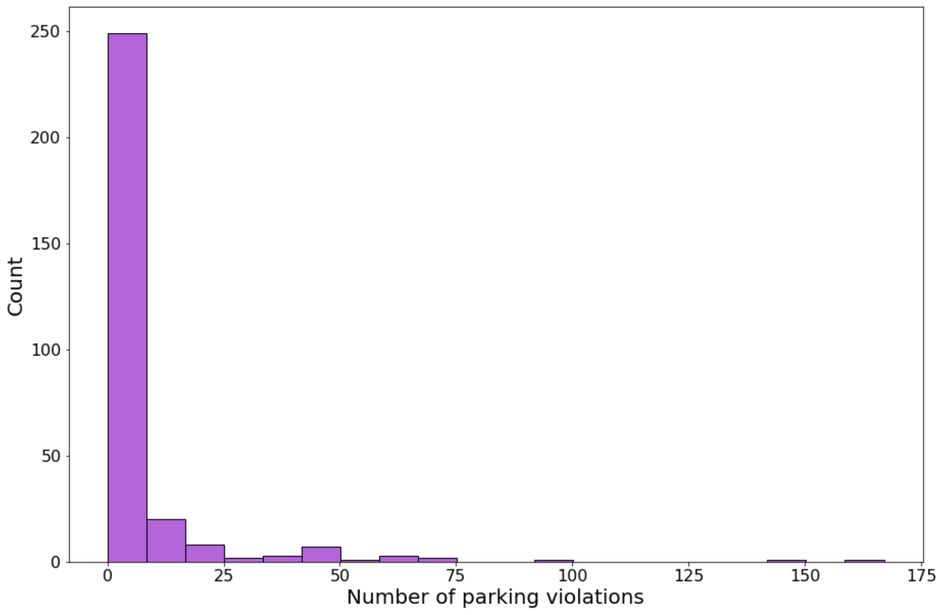


Figure 1. Histogram of the data set - counts per number of parking violations.

Table 1. Data set of corruption - First 10 rows [41].

Country	Code	Violations	Staff	Law	CI
Angola	AGO	50	9	no	1.048
Angola	AGO	1	9	yes	1.048
Albania	ALB	17	3	no	0.921
Albania	ALB	0	3	yes	0.921
UAE	ARE	0	3	no	-0.780
UAE	ARE	0	3	yes	-0.780
Argentina	ARG	5	19	no	0.224
Argentina	ARG	0	19	yes	0.224
Armenia	ARM	3	4	no	0.710
Armenia	ARM	0	4	yes	0.710

7.2. Code Availability

The Python code that was developed to implement the Vuong test [8] is available on the Appendix A - Code. It's important to execute the following command before running the code to update the statsmodel version required.

$$!pipinstallstatsmodels == 0.14.0$$

7.3. Computational Environment and Analysis Setup

The analysis was conducted on a notebook equipped with an Intel Core i5-1235U processor (1.30 GHz, 12 cores, 12 threads), 20 GB of DDR4 RAM, and a Intel IRIS Xe graphics card. The computer ran Windows 11 as the operating system. Analysis was performed using Python 3.11.3 with libraries including NumPy 1.26.3, pandas 2.2.0, statsmodels 0.14.1 and scipy 1.12.0. No parallelization was utilized, as the algorithm ran efficiently on a single processor.

7.4. Efficiency and Performance of the Algorithm

The algorithm employed in this study is not computationally demanding, as evidenced by its smooth execution and negligible runtime. It ran effortlessly on the available hardware, completing its tasks in a remarkably short amount of time.

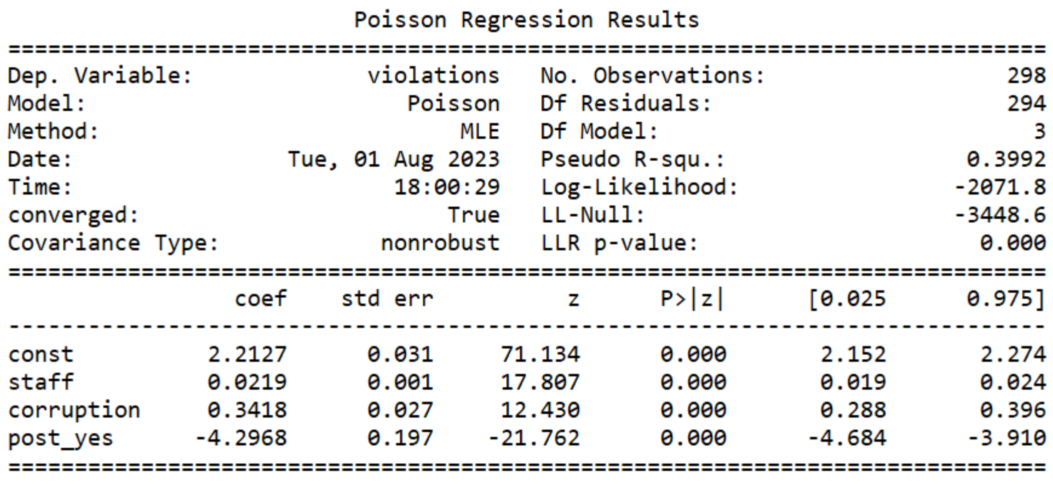
8. Results

Firstly, it was elaborated a preliminary diagnosis of equidispersion (observation of possible equality between the mean and the variance of the dependent variable 'violations'). As can be seen in Table 2, the mean (μ) and the variance (σ^2) are very distant from each other. Therefore, there is a preliminary indication of overdispersion.

Table 2. Mean (μ) and variance (σ^2) - Number of parking violations

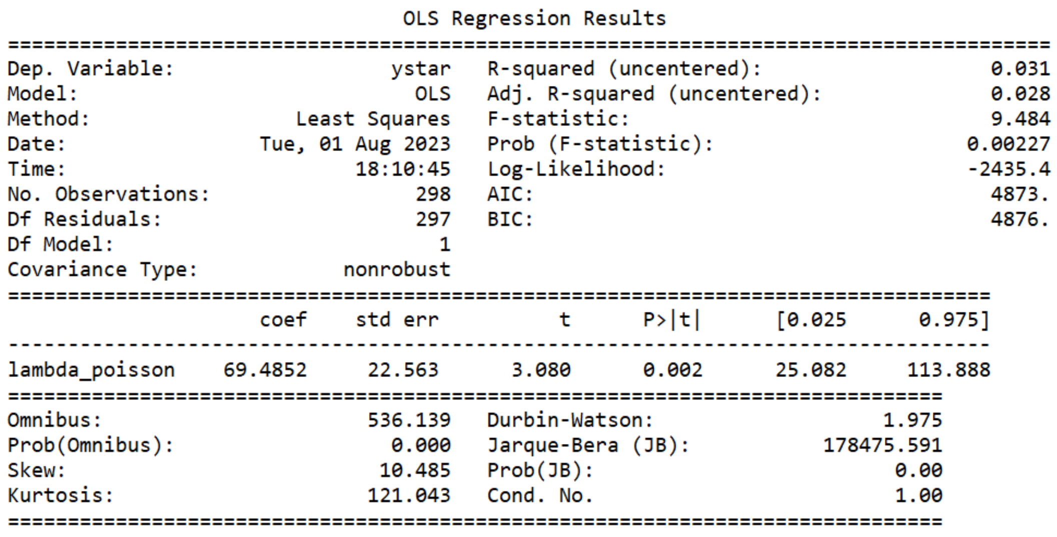
Parameter	Value
Mean (μ)	6.497
Variance (σ^2)	331.618

The next step was to estimate the Poisson model, as can be seen in Figure 2. Then, it was tested if the data has equidispersion [43]. The test can be seen in Figure 3. As the p-value of the t-test corresponding to the β parameter of λ is less than 0.05, as it is worth 0.02, it can be stated that the data of the dependent variable present overdispersion, making the estimated Poisson regression model inadequate. Then, we proceed to estimate the negative binomial model, as can be seen in Figure 4.



```
                          Poisson Regression Results
==============================================================================
Dep. Variable:             violations   No. Observations:                  298
Model:                        Poisson   Df Residuals:                      294
Method:                           MLE   Df Model:                            3
Date:                Tue, 01 Aug 2023   Pseudo R-squ.:                  0.3992
Time:                        18:00:29   Log-Likelihood:                -2071.8
converged:                       True   LL-Null:                       -3448.6
Covariance Type:            nonrobust   LLR p-value:                     0.000
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
const          2.2127      0.031     71.134      0.000       2.152       2.274
staff          0.0219      0.001     17.807      0.000       0.019       0.024
corruption     0.3418      0.027     12.430      0.000       0.288       0.396
post_yes      -4.2968      0.197    -21.762      0.000      -4.684      -3.910
==============================================================================
```

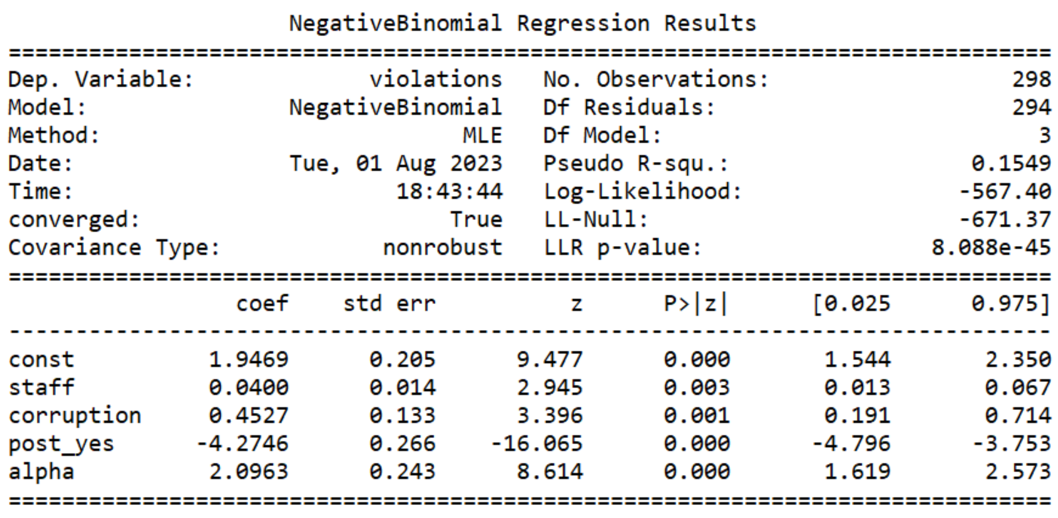
Figure 2. Poisson Regression Results.



```
                                 OLS Regression Results
=======================================================================================
Dep. Variable:                  ystar   R-squared (uncentered):                   0.031
Model:                            OLS   Adj. R-squared (uncentered):              0.028
Method:                 Least Squares   F-statistic:                              9.484
Date:                Tue, 01 Aug 2023   Prob (F-statistic):                     0.00227
Time:                        18:10:45   Log-Likelihood:                         -2435.4
No. Observations:                 298   AIC:                                      4873.
Df Residuals:                     297   BIC:                                      4876.
Df Model:                           1
Covariance Type:            nonrobust
================================================================================
                   coef    std err          t      P>|t|      [0.025      0.975]
--------------------------------------------------------------------------------
lambda_poisson  69.4852     22.563      3.080      0.002      25.082     113.888
==============================================================================
Omnibus:                      536.139   Durbin-Watson:                   1.975
Prob(Omnibus):                  0.000   Jarque-Bera (JB):           178475.591
Skew:                          10.485   Prob(JB):                         0.00
Kurtosis:                     121.043   Cond. No.                         1.00
==============================================================================
```

Figure 3. Equidispersion test.

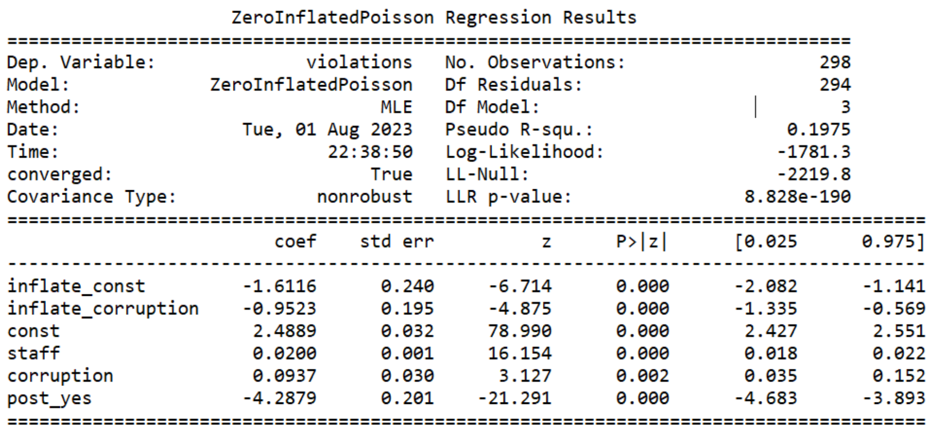
By comparing the Log-Likelihood (LL) of the Poisson and negative binomial models, as it is possible to see in Figures 2 and 4, we can see that the NB has a smaller LL than Poisson. LL is close to -567.4 for the NB model, and close to -2071.79 for the Poisson model.



```
                     NegativeBinomial Regression Results
==============================================================================
Dep. Variable:             violations   No. Observations:                  298
Model:               NegativeBinomial   Df Residuals:                      294
Method:                           MLE   Df Model:                            3
Date:                Tue, 01 Aug 2023   Pseudo R-squ.:                  0.1549
Time:                        18:43:44   Log-Likelihood:                -567.40
converged:                       True   LL-Null:                       -671.37
Covariance Type:            nonrobust   LLR p-value:                 8.088e-45
==============================================================================
                 coef    std err          z      P>|z|      [0.025      0.975]
------------------------------------------------------------------------------
const          1.9469      0.205      9.477      0.000       1.544       2.350
staff          0.0400      0.014      2.945      0.003       0.013       0.067
corruption     0.4527      0.133      3.396      0.001       0.191       0.714
post_yes      -4.2746      0.266    -16.065      0.000      -4.796      -3.753
alpha          2.0963      0.243      8.614      0.000       1.619       2.573
==============================================================================
```

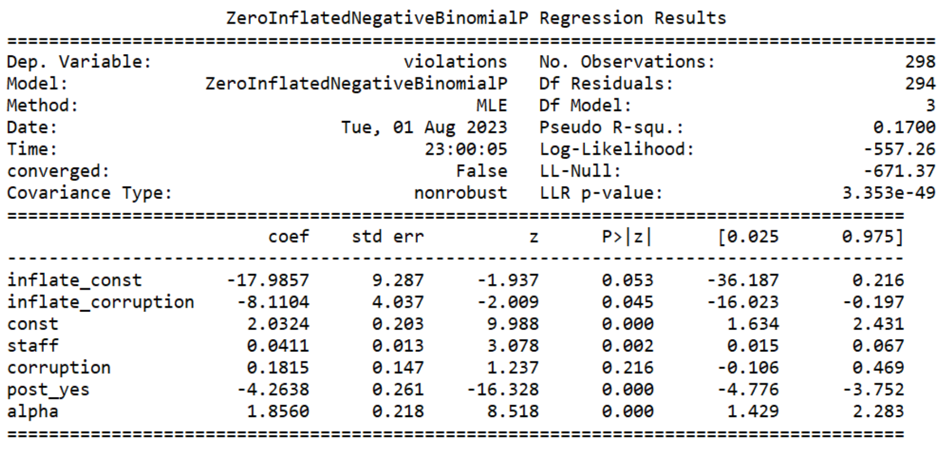
Figure 4. Negative binomial regression results.

The next step is to estimate the zero-inflated Poisson model and the zero-inflated NB model, which can be seen in Figure 5 and Figure 6, respectively.



```
                     ZeroInflatedPoisson Regression Results
===============================================================================
Dep. Variable:                violations   No. Observations:                  298
Model:               ZeroInflatedPoisson   Df Residuals:                      294
Method:                              MLE   Df Model:                            3
Date:                   Tue, 01 Aug 2023   Pseudo R-squ.:                  0.1975
Time:                           22:38:50   Log-Likelihood:                -1781.3
converged:                          True   LL-Null:                       -2219.8
Covariance Type:               nonrobust   LLR p-value:                 8.828e-190
======================================================================================
                         coef    std err          z      P>|z|      [0.025      0.975]
--------------------------------------------------------------------------------------
inflate_const         -1.6116      0.240     -6.714      0.000      -2.082      -1.141
inflate_corruption    -0.9523      0.195     -4.875      0.000      -1.335      -0.569
const                  2.4889      0.032     78.990      0.000       2.427       2.551
staff                  0.0200      0.001     16.154      0.000       0.018       0.022
corruption             0.0937      0.030      3.127      0.002       0.035       0.152
post_yes              -4.2879      0.201    -21.291      0.000      -4.683      -3.893
======================================================================================
```

Figure 5. ZIP regression results.



```
                     ZeroInflatedNegativeBinomialP Regression Results
=========================================================================================
Dep. Variable:                          violations   No. Observations:                298
Model:               ZeroInflatedNegativeBinomialP   Df Residuals:                    294
Method:                                        MLE   Df Model:                          3
Date:                             Tue, 01 Aug 2023   Pseudo R-squ.:                0.1700
Time:                                     23:00:05   Log-Likelihood:              -557.26
converged:                                   False   LL-Null:                     -671.37
Covariance Type:                         nonrobust   LLR p-value:               3.353e-49
======================================================================================
                         coef    std err          z      P>|z|      [0.025      0.975]
--------------------------------------------------------------------------------------
inflate_const        -17.9857      9.287     -1.937      0.053     -36.187       0.216
inflate_corruption    -8.1104      4.037     -2.009      0.045     -16.023      -0.197
const                  2.0324      0.203      9.988      0.000       1.634       2.431
staff                  0.0411      0.013      3.078      0.002       0.015       0.067
corruption             0.1815      0.147      1.237      0.216      -0.106       0.469
post_yes              -4.2638      0.261    -16.328      0.000      -4.776      -3.752
alpha                  1.8560      0.218      8.518      0.000       1.429       2.283
======================================================================================
```

Figure 6. ZINB regression results.

After that, it is possible to apply the Vuong test that was developed to compare the Poisson model (Figure 2) with the ZIP model (Figure 5) and to compare the NB model (Figure 4) with the ZINB model (Figure 6). The code returns the outputs on Table 3.

Table 3. Vuong z-statistic and p-value: Poisson vs. ZIP and NB vs. ZINB

Vuong Test [8]	Poisson x ZIP	NB x ZINB
Vuong z-statistic:	≈ -2.993	≈ -1.947
p-value	≈ 0.0014	≈ 0.0258

By comparing the Log-Likelihood (LL) of the ZIP and ZINB models, it is possible to see, on the Figure 7, that the ZINB has a smaller LL than ZIP. LL is close to -557.26 for the ZINB model, and close to -1781.31 for the ZIP model.

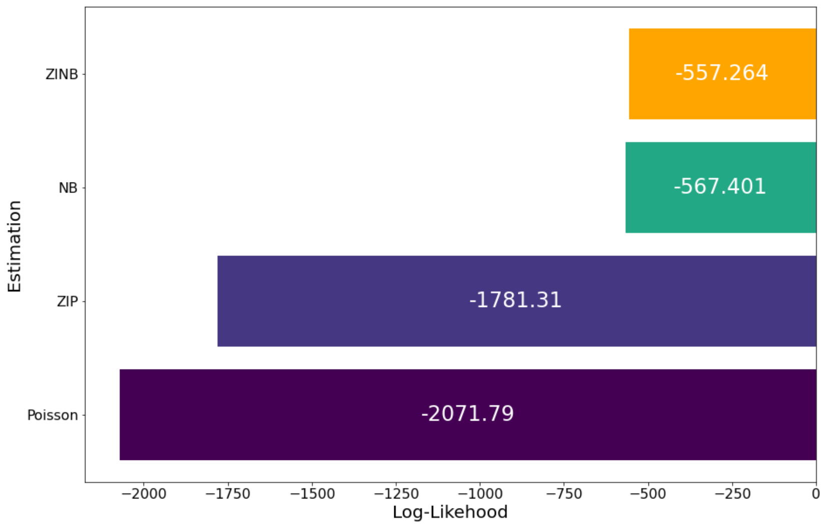


Figure 7. Log-likelihoods: Poisson and Negative binomial models.

9. Discussion

In the Vuong test, positive and statistically significant values indicate the adequacy of the ZIP or ZINB model. Negative and statistically significant values indicate the adequacy of the traditional Poisson model or negative binomial model [25].

As it was previously discussed, the data is overdispersed, making the estimated Poisson model inadequate. In this case, the Vuong test is used to choose between the NB model and the ZINB model. Considering that the Vuong test resulted in -2.993, with a p-value < 0.05 (Table 3), the traditional NB model is a better option than the ZINB at a significance level of 95%.

In Figure 8, there is a comparison of the models. It includes the observed, predicted, and fitted values and confidence intervals for Poisson, NB, ZIP, and ZINB.

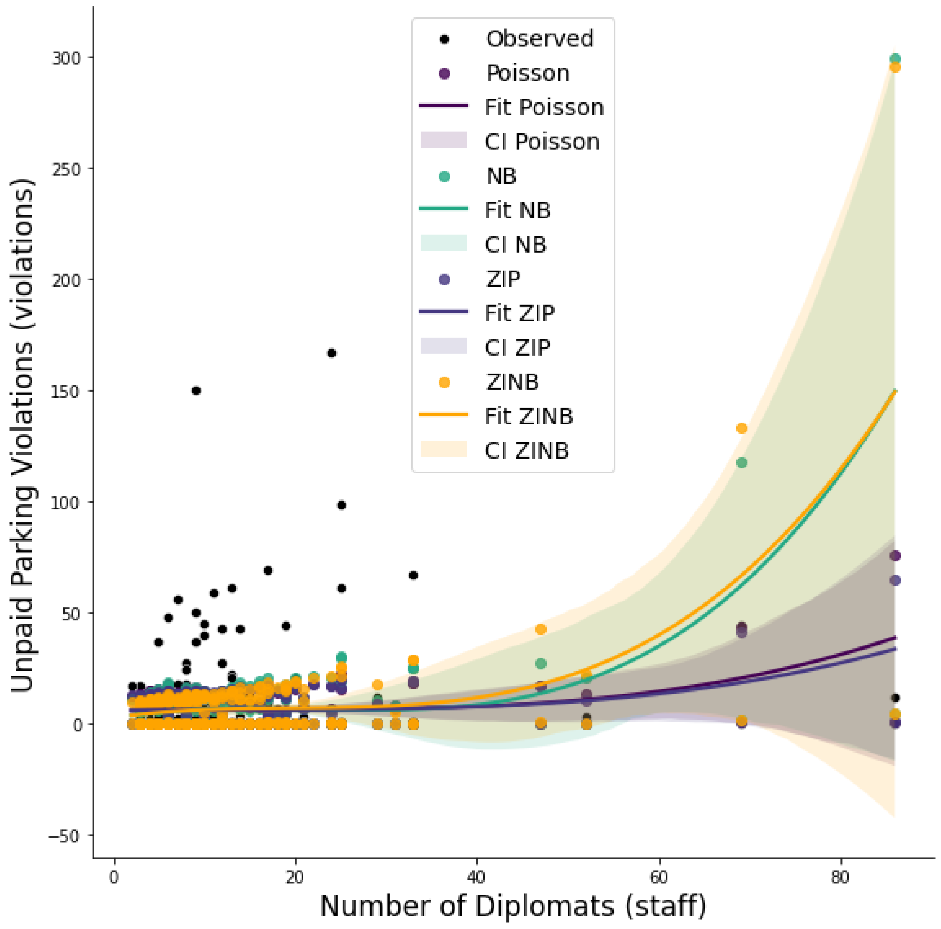


Figure 8. Comparison of the models - Observed, predicted, fit and confidence interval.

10. Conclusion

In conclusion, the Vuong Test that we developed from scratch using the Python programming language has proven to be a useful tool for choosing between regular or zero-inflated models. Prior to our work, this test did not exist in the Python environment. Now, other researchers can benefit from our contribution by using the code we have made available in Appendix A of this paper to apply the Vuong Test in their research using Python.

Funding: This research received no external funding

Data Availability Statement: The full code and the data that was used to generate tables, graphs, histograms and the Vuong Test is available on GitHub (see https://github.com/DuarteAlexandre/VuongTest.git).

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

ZIP	Zero Inflated Poisson
ZINB	Zero Inflated Negative Binomial
LRT	Likelihood Ration Tests
OLS	Ordinary Least Squares
NB	Negative Binomial
LL	Log-Likelihood
CI	Confidence Interval

Appendix A. Vuong Test code in Python

```
def vuong_test(m1, m2):

    from statsmodels.discrete.count_model import ZeroInflatedPoisson,
        ZeroInflatedNegativeBinomialP
    from statsmodels.discrete.discrete_model import Poisson, NegativeBinomial
    from scipy.stats import norm

    supported_models = [ZeroInflatedPoisson,
                        ZeroInflatedNegativeBinomialP,
                        Poisson,
                        NegativeBinomial]

    if type(m1.model) not in supported_models:
        raise ValueError(f"Model type not supported for first parameter. List
            of supported models: (ZeroInflatedPoisson,
            ZeroInflatedNegativeBinomialP, Poisson, NegativeBinomial) from
            statsmodels discrete collection.")

    if type(m2.model) not in supported_models:
        raise ValueError(f"Model type not supported for second parameter.
            List of supported models: (ZeroInflatedPoisson,
            ZeroInflatedNegativeBinomialP, Poisson, NegativeBinomial) from
            statsmodels discrete collection.")

    m1_y = m1.model.endog
    m2_y = m2.model.endog

    m1_n = len(m1_y)
    m2_n = len(m2_y)

    if m1_n == 0 or m2_n == 0:
        raise ValueError("Could not extract dependent variables from models.")

    if m1_n != m2_n:
        raise ValueError("Models appear to have different numbers of
            observations.\n"
                         f"Model 1 has {m1_n} observations.\n"
                         f"Model 2 has {m2_n} observations.")

    if np.any(m1_y != m2_y):
        raise ValueError("Models appear to have different values on dependent
            variables.")

    m1_linpred = pd.DataFrame(m1.predict(which="prob"))
    m2_linpred = pd.DataFrame(m2.predict(which="prob"))

    m1_probs = np.repeat(np.nan, m1_n)
    m2_probs = np.repeat(np.nan, m2_n)

    which_col_m1 = [list(m1_linpred.columns).index(x) if x in
        list(m1_linpred.columns) else None for x in m1_y]
    which_col_m2 = [list(m2_linpred.columns).index(x) if x in
        list(m2_linpred.columns) else None for x in m2_y]
```

```
for i, v in enumerate(m1_probs):
    m1_probs[i] = m1_linpred.iloc[i, which_col_m1[i]]

for i, v in enumerate(m2_probs):
    m2_probs[i] = m2_linpred.iloc[i, which_col_m2[i]]

lm1p = np.log(m1_probs)
lm2p = np.log(m2_probs)

m = lm1p - lm2p

v = np.sum(m) / (np.std(m) * np.sqrt(len(m)))

pval = 1 - norm.cdf(v) if v > 0 else norm.cdf(v)

print("Vuong Non-Nested Hypothesis Test-Statistic (Raw):")
print(f"Vuong z-statistic: {v}")
print(f"p-value: {pval}")
```

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