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# A Full-Newton Step Interior-Point Method for Weighted Quadratic Programming Based on the Algebraic Equivalent Transformation

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**Abstract:** In this paper, a new full-Newton step feasible interior point method for convex quadratic programming is presented and analyzed. The idea behind this method is to replace the complementarity condition with a non-negative variable weight vector and use the algebraic equivalent transformation for the obtained equation. Under the selection of appropriate parameters, the quadratic rate of convergence of the new algorithm is established. In addition, the iteration complexity of the algorithm is obtained. Finally, some numerical results are presented to demonstrate the practical performance of the proposed algorithm.

**Keywords:** quadratic programming; interior-point methods; full-Newton step; weight vector

## 1. Introduction

Since Karmarkar's seminal paper [1], a large amount of research has been devoted to the study of interior-point methods (IPMs). IPMs are one of the most efficient methods for solving linear optimization (LO). At the same time, these methods have been extended to other optimization problems, including convex quadratic programming (CQP), semidefinite optimization (SDO) and etc.

Full-Newton step IPMs for solving LO were initiated by Roos et al. [2]. The main advantage of these methods is that they use only full-Newton steps, no line searches are required. Furthermore, the iterates lie always in the quadratic convergence neighborhood, under some mild assumptions. In 2003, Darvay [3] proposed an algebraic equivalent transformation (AET) technique to determine search directions in IPMs for LO. He applies a continuously differentiable and monotone function $\psi : (0, \infty) \to \mathbb{R}$ on both sides of the nonlinear equation of the central path, and then uses Newton's method to derive the search directions. In addition, he introduced a full-Newton step primal-dual path-following interior-point algorithm for LO using the square root function in the AET technique. Later on, Achache [4], Wang and Bai [5–7] and Wang et al. [8] respectively extended Darvay's algorithm for LO to CQP, second-order cone optimization (SOCO), SDO and symmetric cone optimization (SCO) and $P_*(\kappa)$ linear complementarity problem ($P_*(\kappa)$-LCP).

The weighted linear complementarity problem (WLCP) has been introduced by Potra [9]. In this paper, Potra defined a smooth central path for the WLCP and proposed two interior-point algorithms to solve the WLCP, both of which follow the smooth central path. Asadi et al. [10] extended the full-Newton step IPM introduced in [2] to the monotone WLCP and proved the quadratic rate of convergence to the target points on the smooth central path. Recently, Kheirfam [11] extended the full-Newton step IPM using the $\psi(t) = \sqrt{t}$ function in the AET technique for the monotone WLCP. Very recently, Boudjellal and Benterki [12] extended the full-Newton step feasible IPM to solve convex quadratic programming (CQP) based on replacing the complementarity condition by a non-negative weight vector.

Inspired by the works mentioned above, we extend the full-Newton step IPM to CQP. We replace the complementarity condition with a non-negative weight vector and then use the $\psi(t) = \sqrt{t}$ function in the AET technique. We apply the Newton method to the system defining the weighted central path to get search directions and take full steps along these search directions. We prove the feasibility of

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the full steps and quadratic rate of convergence to the target points on the weighted central path. By choosing appropriate values for the parameters, we derive an iteration bound for WCQP with the same complexity as the one obtained for this type of problem.

The paper is organized as follows. In Section 2, we recall the primal-dual pair of CQPs and then state the weighted central path for the CQP. In Section 3, we describe the AET technique on the weighted central path and define a norm-based proximity measure. A generic framework of the algorithm is presented. Section 4 is devoted to the analysis of the algorithm. In Section 5 we derive an iteration bound for the proposed algorithm. Some numerical results are presented in Section 6. Concluding remarks are given in Section 7.

## 2. CQP and Its Weighted Central Path

Consider the primal-dual CQP problem pair in the following standard form:

$$\begin{array}{ll} \min\ c^T x + \frac{1}{2}x^T Q x & \max\ b^T y - \frac{1}{2}x^T Q x \\ s.t.\ Ax = b & s.t.\ A^T y + s - Qx = c \\ \qquad x \geq 0, & \qquad s \geq 0, \end{array} \tag{1}$$

where $Q \in \mathbb{R}^{n\times n}$ is a symmetric and positive semidefinite matrix, $A \in \mathbb{R}^{m\times n}$ is a full row rank matrix, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. Let $\mathcal{F}^0$ denote the set of strictly feasible solutions of the primal-dual pair (1); i.e.,

$$\mathcal{F}^0 = \{(x, y, s) :\ Ax = b,\ \ A^T y + s - Qx = c,\ \ x > 0,\ s > 0\}.$$

It is well known that finding an optimal solution for the primal-dual pair (1) is equivalent to solve the following Karush-Khun-Tucker (KKT) optimality conditions:

$$\begin{array}{c} Ax = b,\ \ x \geq 0, \\ A^T y + s - Qx = c,\ \ s \geq 0, \\ xs = 0. \end{array} \tag{2}$$

Let an initial point $(x^0, s^0, y^0) \in \mathcal{F}^0$ be given, we define

$$t^0 = \frac{(x^0)^T s^0}{n},\ \ cc = x^0 s^0,\ \gamma = \frac{\min cc}{t^0},\ w(t) = (1 - \tfrac{t}{t^0})w^+ + \tfrac{t}{t^0}cc,\ w^+ = (1-\theta)w,$$

where $t \in ]0, t^0]$, $w \in \mathbb{R}^n_+$ and $\theta \in (0, 1)$. We assume that the complementarity condition $xs = 0$ in (2) is replaced by the parameterized equation $xs = w(t)$. In this way, we get the following perturbed system

$$\begin{array}{c} Ax = b,\ \ x \geq 0, \\ A^T y + s - Qx = c,\ \ s \geq 0, \\ xs = w(t). \end{array} \tag{3}$$

It is shown that, under our assumptions, the system (3) has a unique solution for each $t \in ]0, t^0]$. This solution is denoted as $(x(w(t)), y(w(t)), s(w(t)))$. The set of all these solutions forms the so-called weighted path for (1). If $t \to 0$ and $w \to 0$, then $w(t) \to 0$ and the limit of the weighted path exists and the limit point satisfies the complementarity condition. Therefore, the limit gives an optimal solution of (1).

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## 3. New Search Direction and Algorithm

According to the idea of algebraic equivalent transformation presented by Darvay [3], we write the system (3) in the following equivalent form:

$$\begin{aligned} Ax = b,\ \ x \geq 0, \\ A^T y + s - Qx = c,\ \ s \geq 0, \\ \psi(xs) = \psi(w(t)), \end{aligned} \tag{4}$$

where $\psi : (\xi, \infty) \to \mathbb{R}$ is a continuously differentiable function with $\psi'(t) > 0$ for $t \in (\xi, \infty)$ and $\xi \in [0, 1)$. It is worth noting that the transformed system (4) does not change the weighted path and only specifies different directions depending on the $\psi$ function.

Let us be at the point $(x, s, y) \in \mathcal{F}^0$, then by applying Newton's method in system (4), the search direction $(\Delta x, \Delta y, \Delta s)$ is the solution of the following linear system:

$$\begin{aligned} A\Delta x = 0, \\ A^T \Delta y + \Delta s - Q\Delta x = 0, \\ s\Delta x + x\Delta s = \tfrac{\psi(w(t)) - \psi(xs)}{\psi'(xs)}. \end{aligned} \tag{5}$$

Considering the function $\psi(t) = \sqrt{t}$ for system (5), we obtain the following system

$$\begin{aligned} A\Delta x = 0, \\ A^T \Delta y + \Delta s - Q\Delta x = 0, \\ s\Delta x + x\Delta s = 2\sqrt{xs}\big(\sqrt{w(t)} - \sqrt{xs}\big). \end{aligned} \tag{6}$$

The new iterates are then given as:

$$x^+ = x + \Delta x,\quad y^+ = y + \Delta y,\quad s^+ = s + \Delta s.$$

This means that a full Newton step is taken along the search directions.

For ease of analysis, we consider a scaled version of (6). For this purpose, we introduce the vector

$$v = \sqrt{\frac{xs}{t}}$$

and the scaled search directions $d_x$ and $d_s$ as follows:

$$d_x = \frac{v\Delta x}{x},\ \ d_s = \frac{v\Delta s}{s}.$$

With these notations, one easily checks that the system (6) can be rewritten as follows:

$$\begin{aligned} \overline{A} d_x = 0, \\ \overline{A}^T \tfrac{\Delta y}{t} + d_s - \overline{Q} d_x = 0, \\ d_x + d_s = 2\Big(\sqrt{\tfrac{w(t)}{t}} - v\Big), \end{aligned} \tag{7}$$

where $\overline{A} = \sqrt{t}AD,\ \overline{Q} = DQD,\ D = \mathrm{diag}\big(\sqrt{\tfrac{x}{s}}\big)$.

We define the norm-based proximity $\delta(v)$ to measure the distance between the current iterate $(x, y, s)$ and the weighted centre $(x(w(t)), y(w(t)), s(w(t)))$ for given $t > 0$, as follows:

$$\delta(v) := \delta(x, s; t) = \Big\| \sqrt{\frac{w(t)}{t}} - v \Big\|. \tag{8}$$

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Note that

$$xs = w(t) \Leftrightarrow v^2 = \frac{w(t)}{t} \Leftrightarrow v = \sqrt{\frac{w(t)}{t}} \Leftrightarrow \delta(v) = 0.$$

Let $p_v = d_x + d_s$ and $q_v = d_x - d_s$. Then, we have

$$d_x = \frac{p_v + q_v}{2}, \quad d_s = \frac{p_v - q_v}{2}, \quad d_x d_s = \frac{p_v^2 - q_v^2}{4}.$$

Furthermore, we have

$$\|q_v\|^2 = \|d_x - d_s\|^2 = \|d_x + d_s\|^2 - 4d_x^T d_s \leq \|p_v\|^2 = 4\delta^2, \tag{9}$$

where the inequality comes from the fact that

$$d_x^T d_s = d_x^T (\overline{Q} d_x - \overline{A}^T \frac{\Delta y}{t}) = d_x^T \overline{Q} d_x \geq 0.$$

We now give a generic framework of our new weighted interior point algorithm.

**Algorithm** : Full − Nwton step IPM for WCQP

**Input**: $A \in \mathbb{R}^{m\times n}, Q \in \mathbb{R}^{n\times n}, c \in \mathbb{R}^n, b \in \mathbb{R}^m$;
the accuracy parameter $\varepsilon > 0$;
the threshold parameter $0 < \tau < 1$;
the barrier update parameter $0 < \theta < 1$;
An initial point $(x^0, s^0, y^0) \in \mathcal{F}^0$ with $\delta(x^0, s^0; t^0) \leq \tau$, where $t^0 = \frac{(x^0)^T s^0}{n}$;
$cc = x^0 s^0, w^0 \geq cc$;
**begin**
$x = x^0,\ y = y^0,\ s = s^0,\ t = t^0,\ w = w^0$;
**while** $\max\{\|\sqrt{w} - \sqrt{xs}\|, \|w\|\} > \varepsilon$ **do**
Set $t = (1-\theta)t,\ w = (1-\theta)w,\ w(t) = (1 - \frac{t}{t^0})w + \frac{t}{t^0}cc$;
Determine $(\Delta x, \Delta y, \Delta s)$ according to (6);
Set $(x, y, s) = (x, y, s) + (\Delta x, \Delta y, \Delta s)$;
**end**
**end**.

## 4. Analysis of the Algorithm

In the next lemma, we give a condition which guarantees the strictly feasible of the full-Newton step.

**Lemma 1.** *A new iterate $(x^+, y^+, s^+)$ is strictly feasible if*

$$\delta := \delta(v) < \sqrt{\gamma}.$$

**Proof.** We introduce a step length $\alpha \in [0, 1]$ and define

$$x(\alpha) = x + \alpha\Delta x, \quad s(\alpha) = s + \alpha\Delta s.$$

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Therefore,

$$
\begin{aligned}
\frac{x(\alpha)s(\alpha)}{t} &= \frac{xs}{t} + \alpha\frac{x\Delta s + s\Delta x}{t} + \alpha^2\frac{\Delta x\Delta s}{t} \\
&= v^2 + \alpha v(d_x + d_s) + \alpha^2 d_x d_s \\
&= v^2 + \alpha v p_v + \alpha^2\frac{p_v^2 - q_v^2}{4} \\
&= (1-\alpha)v^2 + \alpha(v^2 + vp_v) + \alpha^2\frac{p_v^2 - q_v^2}{4} \\
&= (1-\alpha)v^2 + \alpha\left(\frac{w(t)}{t} - (1-\alpha)\frac{p_v^2}{4} - \alpha\frac{q_v^2}{4}\right),
\end{aligned} \tag{10}
$$

where the last equality is obtained from the following

$$
v^2 + vp_v = v^2 + 2v\left(\sqrt{\frac{w(t)}{t}} - v\right) = 2v\sqrt{\frac{w(t)}{t}} - v^2 = \frac{w(t)}{t} - \frac{p_v^2}{4}.
$$

Furthermore, for $0 \le \alpha \le 1$, we have

$$
\begin{aligned}
\left\|(1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right\|_\infty &\le (1-\alpha)\frac{\|p_v^2\|_\infty}{4} + \alpha\frac{\|q_v^2\|_\infty}{4} \\
&\le (1-\alpha)\frac{\|p_v\|^2}{4} + \alpha\frac{\|q_v\|^2}{4} \\
&\le (1-\alpha)\delta^2 + \alpha\delta^2 = \delta^2 < \gamma,
\end{aligned}
$$

where the third inequality is due to (9) and the last inequality comes from the assumption $\delta < \sqrt{\gamma}$. Moreover, we have

$$
\begin{aligned}
\min\left(\frac{w(t)}{t} - (1-\alpha)\frac{p_v^2}{4} - \alpha\frac{q_v^2}{4}\right) &\ge \min\frac{w(t)}{t} - \left\|(1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right\|_\infty \\
&\ge \min\frac{cc}{t^0} - \left\|(1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right\|_\infty \\
&= \gamma - \left\|(1-\alpha)\frac{p_v^2}{4} + \alpha\frac{q_v^2}{4}\right\|_\infty > 0.
\end{aligned}
$$

Thus for all $\alpha \in [0,1]$, we have

$$
(1-\alpha)v^2 + \alpha\left(\frac{w(t)}{t} - (1-\alpha)\frac{p_v^2}{4} - \alpha\frac{q_v^2}{4}\right) > 0,
$$

which, by due to (10), implies that $x(\alpha)s(\alpha) > 0$ for $\alpha \in [0,1]$. Hence, none of the entries of $x(\alpha)$ and $s(\alpha)$ vanish $\alpha \in [0,1]$. Since $x > 0$ and $s > 0$ and $x(\alpha)$ and $s(\alpha)$ are linearly function on $\alpha$, this implies that $x(\alpha) > 0$ and $s(\alpha) > 0$ for $\alpha \in [0,1]$. Hence $x(1) = x^+ > 0$ and $s(1) = s^+ > 0$. This completes the proof. $\square$

**Lemma 2.** *Let $\delta = \delta(v)$. After a full Newton step*

$$
\min_i(v_i^+) \ge \sqrt{\gamma - \delta^2}, \quad \text{where} \quad v^+ = \sqrt{\frac{x^+s^+}{t}}.
$$

**Proof.** From (10) with $\alpha = 1$, it follows that

$$
\frac{x^+s^+}{t} = \frac{w(t)}{t} - \frac{q_v^2}{4}. \tag{11}
$$

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Now from (11), the definition of $v^+$ and $w(t) \geq \frac{t}{t^0} cc$, we have

$$\min_i (v_i^+)^2 = \min_i \Big( \frac{w_i(t)}{t} - \frac{(q_v^2)_i}{4} \Big) \geq \min_i \frac{w_i(t)}{t} - \frac{\|q_v^2\|_\infty}{4}$$
$$\geq \frac{\min_i cc_i}{t^0} - \frac{\|q_v\|^2}{4} \geq \gamma - \delta^2.$$

Taking the square root on both sides of the above inequality gives the desired inequality in lemma and the proof is complete. □

**Lemma 3.** *Let $\delta := \delta(v) < \sqrt{\gamma}$. Then, we have*

$$\delta^+ := \delta(v^+) \leq \frac{\delta^2}{\sqrt{\gamma} + \sqrt{\gamma - \delta^2}}.$$

*Thus $\delta^+ \leq \frac{1}{\sqrt{\gamma}} \delta^2$, which shows the quadratic convergence of the algorithm.*

**Proof.** We have

$$(\delta^+)^2 = \Big\| \sqrt{\frac{w(t)}{t}} - v^+ \Big\|^2 = \sum_{i=1}^n \Big( \sqrt{\frac{w_i(t)}{t}} - v_i^+ \Big)^2$$
$$= \sum_{i=1}^n \Big( \sqrt{\frac{w_i(t)}{t}} - v_i^+ \Big)^2 \frac{\Big( \sqrt{\frac{w_i(t)}{t}} + v_i^+ \Big)^2}{\Big( \sqrt{\frac{w_i(t)}{t}} + v_i^+ \Big)^2}$$
$$\leq \frac{1}{\big( \sqrt{\frac{\min_i cc_i}{t}} + \min_i (v_i^+) \big)^2} \sum_{i=1}^n \Big( \frac{w_i(t)}{t} - (v_i^+)^2 \Big)^2$$
$$\leq \frac{1}{\big( \sqrt{\gamma} + \sqrt{\gamma - \delta^2} \big)^2} \Big\| \frac{w(t)}{t} - (v^+)^2 \Big\|^2$$
$$= \frac{1}{\big( \sqrt{\gamma} + \sqrt{\gamma - \delta^2} \big)^2} \Big\| \frac{q_v^2}{4} \Big\|^2$$
$$\leq \frac{1}{\big( \sqrt{\gamma} + \sqrt{\gamma - \delta^2} \big)^2} \Big( \frac{\|q_v\|^2}{4} \Big)^2$$
$$\leq \Big( \frac{\delta^2}{\sqrt{\gamma} + \sqrt{\gamma - \delta^2}} \Big)^2,$$

where the second inequality follows from the fact that $\frac{w_i(t)}{t} \geq \frac{cc_i}{t^0} \geq \frac{\min_i cc_i}{t^0} = \gamma$ and Lemma 2, the fourth equality is due to (11), the third inequality results from the fact that $\|x^2\| \leq \|x\|^2$ for $x \in \mathbb{R}^n$ and the last inequality is due to (9). By taking the square root of both sides of the above inequality, the proof is complete. □

In the following lemma, we present an upper bound of the duality gap after a full-Newton step.

**Lemma 4.** *After a full-Newton step, we have*

$$(x^+)^T s^+ \leq e^T w(t).$$

**Proof.** From (11), we obtain

$$\frac{(x^+)^T s^+}{t} = e^T \frac{w(t)}{t} - e^T \frac{q_v^2}{4} = e^T \frac{w(t)}{t} - \frac{1}{4} \|q_v\|^2 \leq \frac{1}{t} e^T w(t).$$

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This proves the lemma. □

We estimate an upper bound on the value of the proximity measure when $t$ is updated in each iteration.

**Lemma 5.** *Let $\delta := \delta(x,s;\mu) < \sqrt{\gamma}$ and $t^+ := (1-\theta)t$, where $0 < \theta < 1$. Then, we have*

$$\delta(x^+,s^+;t^+) \leq \frac{1}{\sqrt{1-\theta}\left(\sqrt{\gamma - \frac{\theta}{t^0}\|w-cc\| - \frac{\theta^2}{t^0}\|w\|} + \sqrt{\gamma-\delta^2}\right)}\left(\delta^2 + \frac{\theta}{t^0}\|w-cc\| + \frac{\theta^2}{t^0}\|w\|\right).$$

**Proof.** Let $\bar{v}^+ := \sqrt{\frac{x^+s^+}{t^+}}$. Furthermore, from the definition of $w(t)$, we have

$$w(t^+) = w(t) + \frac{\theta t}{t^0}(w^+ - cc) \geq \frac{t}{t^0}cc + \frac{\theta t}{t^0}(w^+ - cc) = (1-\theta)\frac{t}{t^0}cc + \frac{\theta t}{t^0}w^+ > 0.$$

Therefore, we have

$$\begin{aligned}
\delta^2(x^+,s^+;t^+) &= \left\|\sqrt{\frac{w(t^+)}{t^+}} - \bar{v}^+\right\|^2 = \left\|\sqrt{\frac{w(t) + \frac{\theta t}{t^0}(w^+ - cc)}{(1-\theta)t}} - \frac{v^+}{\sqrt{1-\theta}}\right\|^2 \\
&= \frac{1}{1-\theta}\left\|\sqrt{\frac{w(t)}{t} + \frac{\theta}{t^0}(w^+ - cc)} - v^+\right\|^2 \\
&\leq \frac{1}{1-\theta}\sum_{i=1}^{n}\frac{1}{\left(\sqrt{\frac{w_i(t)}{t} + \frac{\theta}{t^0}(w_i^+ - cc_i)} + v_i^+\right)^2}\left(\frac{w_i(t)}{t} + \frac{\theta}{t^0}(w_i^+ - cc_i) - (v_i^+)^2\right)^2 \\
&\leq \frac{1}{(1-\theta)\left(\sqrt{\gamma - \frac{\theta}{t^0}\|w^+ - cc\|} + \sqrt{\gamma-\delta^2}\right)^2}\left(\left\|\frac{w(t)}{t} - (v^+)^2\right\| + \frac{\theta}{t^0}\|w^+ - cc\|\right)^2 \\
&\leq \frac{1}{(1-\theta)\left(\sqrt{\gamma - \frac{\theta}{t^0}\|w^+ - cc\|} + \sqrt{\gamma-\delta^2}\right)^2}\left(\frac{\|q_v\|^2}{4} + \frac{\theta}{t^0}\|w^+ - cc\|\right)^2 \\
&\leq \frac{1}{(1-\theta)\left(\sqrt{\gamma - \frac{\theta}{t^0}\|w^+ - cc\|} + \sqrt{\gamma-\delta^2}\right)^2}\left(\delta^2 + \frac{\theta}{t^0}\|w^+ - cc\|\right)^2 \\
&\leq \frac{1}{(1-\theta)\left(\sqrt{\gamma - \frac{\theta}{t^0}\|w - cc\| - \frac{\theta^2}{t^0}\|w\|} + \sqrt{\gamma-\delta^2}\right)^2}\left(\delta^2 + \frac{\theta}{t^0}\|w - cc\| + \frac{\theta^2}{t^0}\|w\|\right)^2.
\end{aligned}$$

Taking square roots of both sides of the inequality above gives the desired inequality. The proof is complete. □

## 5. Iteration Bound

We obtain an upper bound on the number of iterations of the proposed algorithm. Before doing so, we determine the values for the barrier parameter $\theta$ and the threshold parameter $\tau$ guaranteeing that the iterates are in the $\tau$-neighborhood of the central path; i.e, if $\delta := \delta(x,s;t) \leq \tau$ then $\delta(x^+,s^+;t^+) \leq \tau$. By Lemma 5, we have

$$\delta(x^+,s^+;t^+) \leq \frac{\delta^2 + \frac{\theta}{t^0}\|w-cc\| + \frac{\theta^2}{t^0}\|w\|}{\sqrt{1-\theta}\left(\sqrt{\gamma - \frac{\theta}{t^0}\|w-cc\| - \frac{\theta^2}{t^0}\|w\|} + \sqrt{\gamma-\delta^2}\right)},$$

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because $\delta \leq \tau$, we have

$$\delta(x^+,s^+;t^+) \leq \frac{\tau^2 + \frac{\theta}{t^0}\|w-cc\| + \frac{\theta^2}{t^0}\|w\|}{\sqrt{1-\theta}\left(\sqrt{\gamma - \frac{\theta}{t^0}\|w-cc\| - \frac{\theta^2}{t^0}\|w\|} + \sqrt{\gamma-\tau^2}\right)}.$$

Substituting $t^0 = \frac{\min cc}{\gamma}$ in the latter inequality, we obtain

$$\delta(x^+,s^+;t^+) \leq \frac{\tau^2 + \frac{\gamma\theta}{\min cc}\|w-cc\| + \frac{\gamma\theta^2}{\min cc}\|w\|}{\sqrt{1-\theta}\left(\sqrt{\gamma - \frac{\gamma\theta}{\min cc}\|w-cc\| - \frac{\gamma\theta^2}{\min cc}\|w\|} + \sqrt{\gamma-\tau^2}\right)}.$$

If we take $\tau = \sqrt{\frac{\gamma}{2}}$, we get

$$\delta(x^+,s^+;t^+) \leq \frac{\tau^2 + \frac{2\tau^2\theta}{\min cc}\|w-cc\| + \frac{2\tau^2\theta^2}{\min cc}\|w\|}{\sqrt{1-\theta}\left(\sqrt{2\tau^2 - \frac{2\tau^2\theta}{\min cc}\|w-cc\| - \frac{2\tau^2\theta^2}{\min cc}\|w\|} + \tau\right)}$$

The iterates $(x^+,y^+,s^+)$ are in the $\tau$-neighborhood of the central path; i.e., the iterates that satisfy $\delta(x^+,s^+;t^+) \leq \tau$. So

$$\frac{\tau^2 + \frac{2\tau^2\theta}{\min cc}\|w-cc\| + \frac{2\tau^2\theta^2}{\min cc}\|w\|}{\sqrt{1-\theta}\left(\sqrt{2\tau^2 - \frac{2\tau^2\theta}{\min cc}\|w-cc\| - \frac{2\tau^2\theta^2}{\min cc}\|w\|} + \tau\right)} \leq \tau,$$

or

$$\frac{1 + \frac{2\theta}{\min cc}\|w-cc\| + \frac{2\theta^2}{\min cc}\|w\|}{\sqrt{1-\theta}\left(\sqrt{2 - \frac{2\theta}{\min cc}\|w-cc\| - \frac{2\theta^2}{\min cc}\|w\|} + 1\right)} \leq 1.$$

If we take

$$\theta = \frac{\min cc}{4(\min cc + \|w\|)}, \text{ and } w > cc,$$

we obtain $\theta \leq \frac{1}{4}$, which yields $\frac{1}{\sqrt{1-\theta}} \leq \frac{2}{\sqrt{3}}$ and

$$\frac{1 + \frac{2\theta}{\min cc}\|w-cc\| + \frac{2\theta^2}{\min cc}\|w\|}{\sqrt{1-\theta}\left(\sqrt{2 - \frac{2\theta}{\min cc}\|w-cc\| - \frac{2\theta^2}{\min cc}\|w\|} + 1\right)} \leq \frac{2}{\sqrt{3}}\left(\frac{6}{6+\sqrt{53}}\right)\left(\frac{55}{36}\right)$$
$$\leq 0.7970 < 1.$$

The main result of the paper is given in the following theorem.

**Theorem 1.** *If $\theta = \frac{\min cc}{4(\min cc + \|w\|)}$ and $\tau = \sqrt{\frac{\gamma}{2}}$, then the algorithm achieved an $\varepsilon$-approximate solution $(x,s,y) \in \mathcal{F}^0$ after at most*

$$\left\lceil \frac{8(\min cc + \|w\|)}{\min cc} \log \frac{\max\{\sqrt{\frac{\min cc}{2}} + \left\|\frac{cc-w}{2\sqrt{w+e}}\right\|, \|w\|\}}{\varepsilon} \right\rceil$$

*iterations.*

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**Proof.** We have

$$\begin{aligned}\|\sqrt{w}-\sqrt{xs}\| &\leq \|\sqrt{w(t)}-\sqrt{xs}\|+\|\sqrt{w(t)}-\sqrt{w}\| \\ &= \sqrt{t}\Big\|\sqrt{\frac{w(t)}{t}}-v\Big\|+\|\sqrt{w(t)}-\sqrt{w}\| \\ &= \sqrt{t}\delta+\|\sqrt{w(t)}-\sqrt{w}\| \\ &\leq \sqrt{\frac{t\gamma}{2}}+\|\sqrt{w(t)}-\sqrt{w}\|. \end{aligned} \tag{12}$$

On the other hand, from the definition of $w(t)$, we have

$$\begin{aligned}\|\sqrt{w(t)}-\sqrt{w}\| &= \Big\|\sqrt{w+\frac{t}{t^0}(cc-w)}-\sqrt{w}\Big\| \\ &\simeq \Big\|\sqrt{w}+\frac{\frac{t}{t^0}(cc-w)}{2\sqrt{w}+e}-\sqrt{w}\Big\| \leq \sqrt{\frac{t}{t^0}}\Big\|\frac{cc-w}{2\sqrt{w}+e}\Big\|.\end{aligned}$$

Substitution this bound into (12), and after $k$ iterations, yields

$$\|\sqrt{w}-\sqrt{xs}\| \leq \Big(\sqrt{\frac{t^0\gamma}{2}}+\Big\|\frac{cc-w}{2\sqrt{w}+e}\Big\|\Big)\sqrt{\frac{t}{t^0}} \leq \Big(\sqrt{\frac{t^0\gamma}{2}}+\Big\|\frac{cc-w}{2\sqrt{w}+e}\Big\|\Big)(1-\theta)^{\frac{k}{2}}.$$

Using the definition of $\gamma$, we deduce that $\|\sqrt{w}-\sqrt{xs}\| \leq \varepsilon$ is satisfied if

$$\Big(\sqrt{\frac{\min cc}{2}}+\Big\|\frac{cc-w}{2\sqrt{w}+e}\Big\|\Big)(1-\theta)^{\frac{k}{2}} \leq \varepsilon.$$

Taking logarithms of both sides and using the inequality $\log(1+\theta) \leq \theta$ for $\theta > -1$, we obtain

$$k \geq \frac{8\big(\min cc+\|w\|\big)}{\min cc}\log\frac{\sqrt{\frac{\min cc}{2}}+\Big\|\frac{cc-w}{2\sqrt{w}+e}\Big\|}{\varepsilon}.$$

Moreover, since at each iteration the norm of the $w$ vector is also reduced by the factor $1-\theta$,so the result is obtained. The proof is completed. □

## 6. Numerical Results

In this section, we present computational results under the MATLAB environment to demonstrate the effectiveness of the proposed algorithm. We used the value of the accuracy parameter $\varepsilon = 10^{-4}$. In the implementation, we take different values of the weight vector $w^0$ such that $w^0 \in \{cc+10^{-3}, \sqrt{3}cc, 3cc, ncc\}$. We reduced the value of the parameter $t$ and the weight vector $w$ by the factor $1-\theta$ with $\theta = 0.2$. Table 1 shows the number of iterations (iter) and the time produced by the algorithm to obtain the optimal solution. The optimal values of the primal and dual objective functions are denoted by *pri* and *dua*, respectively. In the following, we give the standard test problems of CQP problems [12].

**Example 1.** $A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $c = \begin{pmatrix} -2 \\ -4 \\ 0 \end{pmatrix}$, $Q = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

The initial primal-dual interior point is:

$$x^0 = (0.3262, 1.3261, 0.3477)^T,\ y^0 = (0, -2.0721)^T,\ s^0 = (0.7247, 0.7247, 2.0722)^T.$$

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**Example 2.** $A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{pmatrix}$, $c = \begin{pmatrix} -4 \\ -6 \\ 0 \\ 0 \end{pmatrix}$, $Q = \begin{pmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

The initial primal-dual interior point is:

$$x^0 = (0.9683, 0.5775, 0.4543, 1.1444)^T, \; y^0 = (-0.9184, -1.1244)^T,$$

$$s^0 = (0.7612, 0.9141, 0.9185, 1.1244)^T.$$

**Example 3.** $A = \begin{pmatrix} 1 & 1.2 & 1 & 1.8 & 0 \\ 3 & -1 & 1.5 & -2 & 1 \\ -1 & 2 & -3 & 4 & 2 \end{pmatrix}$, $c = \begin{pmatrix} 1 \\ -1.5 \\ 2 \\ 1.5 \\ 3 \end{pmatrix}$, $b = \begin{pmatrix} 9.31 \\ 5.45 \\ 6.60 \end{pmatrix}$

$$Q = \begin{pmatrix} 20 & 1.2 & 0.5 & 0 & -1 \\ 1.2 & 32 & 1 & 1 & 1 \\ 0.5 & 1 & 14 & 1 & 1 \\ 0.5 & 1 & 1 & 15 & 1 \\ -1 & 1 & 1 & 1 & 16 \end{pmatrix}.$$

The initial primal-dual interior point is:

$$x^0 = (2.4539, 0.7875, 1.5838, 2.4038, 1.3074)^T, \; y^0 = (20.5435, 9.4781, 4.3927)^T,$$

$$s^0 = (7.1215, 7.9763, 8.3150, 6.8686, 7.9750)^T.$$

**Table 1.** The numerical results of Examples 1, 2 and 3 with different values of $w^0$.

| Exam | $w^0$ | iter | time | *pri* | *dua* |
|---|---|---|---|---|---|
| Exam. 1 | $cc + 10^{-3}$ | **43** | 4.1313 | -4.4999 | -4.4995 |
| Exam. 1 | 3cc | 48 | 4.2173 | -4.4999 | -4.4994 |
| Exam. 1 | $\sqrt{3}$cc | 45 | 3.5224 | -4.4999 | -4.4994 |
| Exam. 1 | (n+1)cc | 46 | 3.6981 | -4.4999 | -4.4994 |
| Exam. 2 | cc+$10^{-3}$ | **44** | 4.4588 | -7.1614 | -7.1610 |
| Exam. 2 | 3cc | 49 | 4.2526 | -7.1614 | -7.1609 |
| Exam. 2 | $\sqrt{3}$cc | 46 | 4.0459 | -7.1614 | -7.1610 |
| Exam. 2 | n cc | 50 | 3.7132 | -7.1614 | -7.1609 |
| Exam. 3 | cc+$10^{-3}$ | **57** | 5.6786 | 172.7165 | 172.7169 |
| Exam. 3 | 3cc | 62 | 3.2721 | 172.7165 | 172.7170 |
| Exam. 3 | $\sqrt{3}$cc | 59 | 4.2797 | 172.7165 | 172.7169 |
| Exam. 3 | n cc | 61 | 3.7649 | 172.7165 | 172.7170 |

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## 7. Concluding Remarks

In this paper, we have presented a full-Newton step IPM based on the AET for weighted convex quadratic programming. We used the square root function in order to obtain a new search direction. By appropriate choosing the barrier parameter $\theta$ and threshold parameter $\tau$, we have shown that the proposed algorithm has a complexity bound of $\left\lceil \frac{8\left(\min cc+\|w\|\right)}{\min cc} \log \frac{\max\{\sqrt{\frac{\min cc}{2}}+\left\|\frac{cc-w}{2\sqrt{w}+e}\right\|,\|w\|\}}{\varepsilon} \right\rceil$. Some numerical results illustrate the efficiency of the algorithm for solving CQP.

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