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Posted Date: 17 June 2025

doi: 10.20944/preprints202506.1349.v1

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Generalizing Coherent States with the Fox *H* Function

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Abstract: In the present scenario, coherent states of a quantum harmonic oscillator are generalized with positive Fox *H* auxiliary functions. The novel generalized coherent states provide canonical coherent states and Mittag-Leffler or Wright generalized coherent states, as particular cases, and resolve the identity operator, over the Fock space, with a weight function that is the product of a Fox *H* function and a Wright generalized hypergeometric function. The novel generalized coherent states, or the corresponding truncated generalized coherent states, are characterized by anomalous statistics of large number of excitations: the corresponding decay laws exhibit, for determined values of the involved parameters, various behaviors that depart from exponential and inverse-power-law decays, or their product. The analysis of the Mandel *Q* factor shows that, for small values of the label, the statistics of the number of excitations becomes super-Poissonian, or sub-Poissonian, by simply choosing sufficiently large values of one of the involved parameters. The effects of the dissipative processes on the novel generalized coherent states are analyzed.

Keywords: quantum harmonic oscillator; generalized coherent states; Fox *H*-functions; Wright generalized hypergeometric functions

MSC: 81R30; 26A33; 33E12; 33E20

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1. Introduction

A quantum harmonic oscillator is fundamental for the description of the most various systems in the framework of quantum theory. In this regard, a basal example is represented by the quantization of a single electromagnetic field mode. Coherent states (CSs) are special states of a quantum harmonic oscillator that are characterized by minimum uncertainty and exhibit quasi-classical properties in the time evolution. These states find the most various applications in mathematical physics, quantum optics and quantum information, to name but a few. See Refs. [1–10], to name but a few.

During the last decades, CSs have been generalized in various ways. See Refs. [3,4,6,11–15], to name but few. Klauder has generalized CSs by requiring the constraints of normalizability, continuity in the label which characterized these states, and resolution of the identity operator over the canonical Fock space with a positive weight function [3,4,6]. Mittag-Leffler or Wright generalized coherent states (GCSs) are some examples of the Klauder' s generalization [11–14]. These states are named after the normalization factors, which are represented by Mittag-Leffler function or Wright function of the square modulus of the label, respectively [16–19]. Mittag-Leffler or Wright exhibits anomalous statistics of large number of excitations.

Truncated coherent states (TCSs) are obtained from canonical CSs by considering the Fock space to be finite-dimensional (truncated). In this regards, refer to [20–30], to name but few. If compared to canonical CSs, TCSs exhibit further properties with respect to the canonical CSs [25,30]. Truncated generalized coherent states (TGCSs) are defined by requesting the three above-reported conditions introduced by Klauder. Thus, TCSs resolve the the identity operator over the truncated Fock space with a positive weight function [20–28,30]. Mittag-Leffler and Wright TGCSs are special examples of TGCSs [14].

Mittag-Leffler and Wright functions are particular cases of the Fox H-function [31–39]. The Fox H-function is a special function that is defined via the Mellin-Barnes integrals. This function finds applications in the most various areas of mathematics, statistics and physics. In this regards, refer to [36,38–43], to name but few.

As a continuation of the above-described scenario, here, we aim to find further classes of GCSs by adopting positive Fox H functions [44]. We aim to define further classes of GCSs that provide canonical CSs and Mittag-Leffler or Wright GCSs as particular cases. Particularly, we aim to find novel forms of weight functions that resolve the identity operator over the Fock space along with novel classes of GCSs [3,4,6,11–15]. We also intend to study the distributions of the number of excitations which characterize the novel GCSs, the Mandel Q factor [45], and the effects of the dissipative processes on the novel GCSs [46–50].

The paper is organized as follows. GCSs and TGCSs are introduced in Section 2, for the sake of completeness. Section 3 is devoted to the generalization of CSs and TCSs by adopting positive Fox H functions as auxiliary functions. In Section 4, we analyze the statistics of the number of excitation of the novel GCSs via the Mandel Q factor. The effects of the dissipative processes on the novel GCSs are analyzed in Section 5. Summary of the results and conclusions are reported in Section 6. Details of the calculations are provided in Appendix A.

2. GCSs and TGCSs

For the sake of clarity and completeness, we report in the present Section the definition and the main properties of GCSs. The Fock basis $\mathcal{F}$ of a quantum harmonic oscillator is composed by the eigenstates of the quantum number operator, $\mathcal{F} \equiv \{|0\rangle, |1\rangle, \ldots, \}$. The eigenstates are mutually orthogonal and normalized to unity, $\langle j||k\rangle = \delta_{j,k}$, for every $j,k \in \mathbb{N}_0$, where $\mathbb{N} \equiv \{1,2,\ldots\}$ and $\mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$.

By definition, the class $\{|z;g\rangle \; \forall z \in \mathbb{C}\}$ of GCSs is required to fulfill the conditions of normalizability, continuity in the label and resolution of the identity with a positive weight function. This class is generated by the arithmetic function $g(n)$ of the natural variable n, is defined over the Fock basis $\mathcal{F}$ as follows:

$$|z;g\rangle = \left[N_g\left(|z|^2\right)\right]^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{g(n)}}|n\rangle. \tag{2.1}$$

for every $z \in \mathbb{C} \setminus \{0\}$. Thus, the arithmetic function $g(n)$ is required to be positive, $g(n) > 0$ for every $n \in \mathbb{N}_0$. The GCS $|0;g\rangle$ is the ground state of the quantum harmonic oscillator or, equivalently, the state with no excitation (vacuum state), $|0;g\rangle = |0\rangle$. By definition, the normalization factor $N_g(|z|^2)$, given by

$$N_g\left(|z|^2\right) = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{g(n)}, \tag{2.2}$$

is required to be positive and finite,

$$0 < N_g\left(|z|^2\right) < +\infty, \tag{2.3}$$

for every $z \in \mathbb{C} \setminus \{0\}$. Condition (2.3) of normalizability is realized for every $z \in \mathbb{C} \setminus \{0\}$ iff

$$\lim_{n\to+\infty} \sup_{n'\geq n} \left[g(n')\right]^{-1/n'} = 0. \tag{2.4}$$

Due to condition (2.4), the following power series: $\sum_{n=0}^{\infty} \zeta^n / g_n$, function of the complex variable ζ, exists and is continuous for every $\zeta \in \mathbb{C}$. Thus, the following limit: $\mathrm{Re}\langle z||z_0\rangle \to 1$, holds as $z \to z_0$, and the required continuity in the label, $|z;g\rangle \to |z_0;g\rangle$, is realized for every $z, z_0 \in \mathbb{C}$.

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By definition, the class $\{|z;g\rangle,\ \forall\, z \in \mathbb{C}\}$ of GCSs resolves the identity operator I over the Fock basis $\mathcal{F}$ if a weight function $U_g(u)$, positive on the set of the positive real numbers $\mathbb{R}^+$, exists such that the following relation holds [3,6,11,12],

$$\int_{\mathbb{R}^2} U_g\left(|z|^2\right)|z;g\rangle\langle z;g|d^2z = I, \tag{2.5}$$

where $d^2z = d\,\mathrm{Re}(z)d\,\mathrm{Im}(z)$. The resolution of the identity operator is determined by the properties of arithmetic function $g(n)$ [3,6,11,12,51]. In fact, let the auxiliary function $f(u)$ be defined on $\mathbb{R}^+$ via the weight function $U^{(g)}(u)$ and the normalization factor $N_g(u)$ as follows [3,6,11,12,51],:

$$f(u) \equiv \pi\frac{U_g(u)}{N_g(u)}, \tag{2.6}$$

for every $u > 0$. The resolution of the identity operator, Eq. (2.5), holds over the Fock space $\mathcal{F}$ if

$$\hat{f}(n+1) = g(n), \tag{2.7}$$

for every $n \in \mathbb{N}_0$. The function $\hat{f}(s)$ is the Mellin transform of the auxiliary function $f(u)$,

$$\hat{f}(s) = \int_0^\infty f(u)u^{s-1}du, \tag{2.8}$$

for every value of the complex variable s such that the involved integral exists [52–56].

The probability $P_g\left(n,|z|^2\right)$ that the GCS $|z;g\rangle$ is characterized by n excitations, i.e., the state $|n\rangle$, is

$$P_g\left(n,|z|^2\right) = \frac{|z|^{2n}}{N_g(|z|^2)g(n)}, \tag{2.9}$$

for every $n \in \mathbb{N}_0$.

The truncated Fock basis $\mathcal{F}_\mathfrak{d}$ of a quantum harmonic oscillator is defined as follows: $\mathcal{F} \equiv \{|0\rangle,\ldots,|\mathfrak{d}\rangle\}$, for every $\mathfrak{d} \in \mathbb{N}$. The class $\{|z;g;\mathfrak{d}\rangle,\ \forall\, z \in \mathbb{C}\}$ of TGCSs is defined over the truncated Fock basis $\mathcal{F}_\mathfrak{d}$ as below:

$$|z;\mathfrak{d};g\rangle = \left[N_{\mathfrak{d},g}\left(|z|^2\right)\right]^{-1/2}\sum_{n=0}^{\mathfrak{d}}\frac{z^n}{\sqrt{g(n)}}|n\rangle, \tag{2.10}$$

for every $z \in \mathbb{C}\setminus\{0\}$ and $\mathfrak{d} \in \mathbb{N}$, while $|0;\mathfrak{d},g\rangle = |0\rangle$ for every $\mathfrak{d} \in \mathbb{N}$. The normalization factor $N_{\mathfrak{d},g}(u)$ is

$$N_{\mathfrak{d},g}(u) = \sum_{n=0}^{\mathfrak{d}}\frac{u^n}{g(n)}, \tag{2.11}$$

for every $u > 0$ and $\mathfrak{d} \in \mathbb{N}$. TGCSs are required to resolve the identity operator I over the truncated Fock basis $\mathcal{F}_\mathfrak{d}$,

$$\int_{\mathbb{R}^2} U_{\mathfrak{d},g}\left(|z|^2\right)|z;\mathfrak{d};g\rangle\langle z;\mathfrak{d};g|d^2z = I. \tag{2.12}$$

If the Mellin transform $\hat{f}(s)$ of the positive auxiliary function $f(u)$ exists for $1 \leq \mathrm{Re}\,s \leq \mathfrak{d}+1$, relations (2.6) and (2.7) hold for the weight function $U_{\mathfrak{d},g}(u)$ and the normalization factor $N_{\mathfrak{d},g}(u)$, for every $\mathfrak{d} \in \mathbb{N}$.

The probability $P_g\left(n, \mathfrak{d}, |z|^2\right)$ that the GCS $|z; \mathfrak{d}; g\rangle$ is characterized by n excitations, i.e., the state $|n\rangle$, is

$$P_g\left(n, \mathfrak{d}, |z|^2\right) = \frac{|z|^{2n}}{N_{\mathfrak{d},g}(|z|^2) g(n)}, \tag{2.13}$$

for every $n = 0, \ldots, \mathfrak{d}$.

3. GCSs Characterized by Positive Fox H Auxiliary Functions

In the present Section, we intend to investigate if canonical CSs, and Mittag-Leffler or Wright GCSs can be generalized further via Fox H auxiliary functions. For the sake of clarity and completeness, we report below the definition of the Fox H function and conditions under which Fox H functions are positive.

Briefly, the Fox H function is defined as follows [31–36]:

$$H^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} (\alpha_j, A_j)_1^p \\ (\beta_j, B_j)_1^q \end{matrix}\right] = \frac{1}{2\pi\imath} \int_{\mathcal{C}} \Xi^{m,n}_{p,q}\left[s \,\middle|\, \begin{matrix} (\alpha_j, A_j)_1^p \\ (\beta_j, B_j)_1^q \end{matrix}\right] z^{-s} ds, \tag{3.1}$$

where

$$\Xi^{m,n}_{p,q}\left[s \,\middle|\, \begin{matrix} (\alpha_j, A_j)_1^p \\ (\beta_j, B_j)_1^q \end{matrix}\right] = \frac{\prod_{j=1}^m \Gamma(\beta_j + B_j s) \prod_{j=1}^n \Gamma(1 - \alpha_j - A_j s)}{\prod_{j=m+1}^q \Gamma(1 - \beta_j - B_j s) \prod_{j=n+1}^p \Gamma(\alpha_j + A_j s)}. \tag{3.2}$$

The poles of the Gamma functions $\Gamma(\beta_1 + B_1 s), \ldots, \Gamma(\beta_m + B_m s)$, are required to differ from the poles of the Gamma functions $\Gamma(1 - \alpha_1 - A_1 s), \ldots, \Gamma(1 - \alpha_n - A_n s)$. This property is provided by the following inequality:

$$A_j\left(l + \beta_{j'}\right) \neq B_{j'}(\alpha_j - l' - 1), \tag{3.3}$$

that is requested to hold for every $j = 1, \ldots, n$, $j' = 1, \ldots, m$, and $l, l' \in \mathbb{N}_0$. The empty products coincide with unity. The allowed values of the indexes n and m are $0 \leq n \leq p$, $0 \leq m \leq q$, and $A_i, B_j \in \mathbb{R}^+$, $\alpha_i, \beta_j \in \mathbb{C}$, for every $i = 1, \cdots, p$, and $j = 1, \cdots, q$, where $\mathbb{C}$ is the set of the complex numbers. The following notation is adopted for the sake of shortness: $(x_j)_1^l \equiv x_1, \ldots, x_l$, for every $l \in \mathbb{N}$, while $(x_j)_1^l \equiv 1$, for $l = 0$; $(x_j, X_j)_1^l \equiv (x_1, X_1), \ldots, (x_l, X_l)$, for every $l \in \mathbb{N}$, while $(x_j, X_j)_1^l \equiv 1$, for $l = 0$; and $\left((x_j, X_j), (y_j, Y_j)\right)_1^l \equiv (x_1, X_1), (y_1, Y_1), \ldots, (x_l, X_l), (y_l, Y_l)$, for every $l \in \mathbb{N}$. Refer to [35,36] for the existence condition, the domain of analiticity and the contour path $\mathcal{C}$ that is adopted in the definition (3.1) of the Fox H function.

The Wright generalized hypergeometric function is a special case of the Fox H function and is defined by the below-reported power series [35,36],

$$\begin{aligned} {}_pW_q\left[z \,\middle|\, \begin{matrix} (\alpha_j, A_j)_1^p \\ (\beta_k, B_k)_1^q \end{matrix}\right] &= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p \Gamma(\alpha_j + A_j n)}{\prod_{k=1}^q \Gamma(\beta_k + B_k n)} \frac{z^n}{n!} \\ &= H^{1,p}_{p,q+1}\left[-z \,\middle|\, \begin{matrix} (1-\alpha_j, A_j)_1^p \\ (0,1), (1-\beta_k, B_k)_1^q \end{matrix}\right], \end{aligned} \tag{3.4}$$

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for every $z \in \mathbb{C} \setminus \{0\}$, $\alpha_j, \beta_k \in \mathbb{C}$, $A_j, B_k \in \mathbb{R}_+$. The Wright generalized hypergeometric function is an entire function of the complex variable z, for every $z \in \mathbb{C} \setminus \{0\}$ if $\mu > -1$, where

$$\mu = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j. \tag{3.5}$$

The Mellin transform of the Fox H function [35,36]:

$$\int_0^{+\infty} u^{s-1} H_{p,q}^{m,n}\left[u \left| \begin{matrix} (\alpha_j, A_j)_1^p \\ (\beta_j, B_j)_1^q \end{matrix} \right.\right] du = \Xi_{p,q}^{m,n}\left[s \left| \begin{matrix} (\alpha_j, A_j)_1^p \\ (\beta_j, B_j)_1^q \end{matrix} \right.\right], \tag{3.6}$$

exists for

$$-\min_{j=1,\dots,m} \left\{ \frac{\Re(\beta_j)}{B_j} \right\} < \Re(s) < \min_{j=1,\dots,n} \frac{1 - \Re(\alpha_j)}{A_j}, \tag{3.7}$$

if $\chi > 0$.

A class of Fox H functions, positive on $\mathbb{R}^+$, is obtained from the Mellin convolution product of functions that are positive on $\mathbb{R}^+$ [44]:

$$H_{p',q'}^{m',n'}\left[u \left| \begin{matrix} \left(\alpha'_j, A'_j\right)_{1,}^{p'} \\ \left(\beta'_{j'}, B'_j\right)_1^{q'} \end{matrix} \right.\right] > 0, \tag{3.8}$$

for every $u > 0$. The indexes m', n', p', q' are

$$m' = n_1 + n_2 + n_3, \tag{3.9}$$
$$n' = n_3 + n_4, \tag{3.10}$$
$$p' = n_2 + n_3 + n_4, \tag{3.11}$$
$$q' = n_1 + n_2 + n_3 + n_4, \tag{3.12}$$

where n_1, n_2, n_3, n_4 are natural numbers such that

$$n_1 \geq 1, \text{ or } n_3 \geq 1. \tag{3.13}$$

The involved parameters are

$$\left(\alpha'_j, A'_j\right)_1^{p'} = \left(1 - r_j, a''_j\right)_1^{n_3}, \left(1 - v_j, a'''_j\right)_1^{n_4}, \left(d_j, a'_j\right)_1^{n_2}, \tag{3.14}$$
$$\left(\beta'_j, B'_{j'}\right)_1^{q'} = \left(b_j, a_j\right)_1^{n_1}, \left(c_j, a'_j\right)_1^{n_2}, \left(o_j, a''_j\right)_1^{n_3}, \left(1 - w_j, a'''_j\right)_1^{n_4}. \tag{3.15}$$

By definition, the constraint below,

$$A'_j\left(\beta'_{j'} + l\right) \neq B'_{j'}\left(\alpha'_j - l' - 1\right), \tag{3.16}$$

is required to hold for every $j = 1, \ldots, n'$, $j' = 1, \ldots, m'$, and $l, l' \in \mathbb{N}_0$. Additionally, the parameters $a_1, \ldots, a_n, b_1, \ldots, b_n, a'_1, \ldots, a'_n, c_1, \ldots, c_n, d_1, \ldots, d_n, a''_1, \ldots, a''_n, o_1, \ldots, o_n, r_1, \ldots, r_n, a'''_1, \ldots, a'''_n, v_1, \ldots, v_n, w_1, \ldots, w_n$, are required to fulfill the following relations:

$$a_j > 0, \; b_j \geq 0, \quad j = 1, \ldots, n_1, \tag{3.17}$$

$$a'_j > 0, \; c_j \geq 0, \; d_j \geq c_j + 1, \quad j = 1, \ldots, n_2, \tag{3.18}$$

$$a''_j, r_j > 0, \; o_j \geq 0, \quad j = 1, \ldots, n_3, \tag{3.19}$$

$$a'''_j, v_j > 0, \; w_j \geq v_j + 1, \quad j = 1, \ldots, n_4. \tag{3.20}$$

for every $(n_1, n_2, n_3, n_4) \in \mathbb{S}$, where $\mathbb{S} \equiv \mathbb{N}_0^4 \setminus \{(0,0,0,0)\}$. The parameter χ is defined as below for the general form (3.1) of the Fox H function,

$$\chi = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j. \tag{3.21}$$

Let the parameterχ', be the value of the parameter χ, defined by Eq. (3.21), that characterizes the positive Fox H function involved in relation (3.8). The parameter χ' is positive,

$$\chi' = \sum_{j=1}^{n_1} a_j + 2 \sum_{j=1}^{n_3} a''_j > 0, \tag{3.22}$$

if condition (3.13) holds. Instead, the parameter χ' vanishes, $\chi' = 0$, for $n_1 = n_3 = 0$.

At this stage, we are equipped to process special forms of the auxiliary function $f(u)$ that are represented by Fox H functions. In fact, consider the following expression of the auxiliary function:

$$f_H(u) = H^{q',0}_{p',q'} \left[u \left| \begin{matrix} \left(\alpha'_j, A'_j\right)_1^{p'} \\ \left(\beta'_{j'}, B'_j\right)_1^{q'} \end{matrix} \right. \right], \tag{3.23}$$

for every $u > 0$. The involved indexes and parameters are defined by relations (3.9)-(3.22), with $n_3 = n_4 = 0$. Thus, the function $f_H(u)$ is a Fox H function that is positive on $\mathbb{R}^+$, i.e., $f_H(u) > 0$, for every $u > 0$. The parameter χ' of the Fox H function $f_H(u)$ is positive, $\chi' > 0$. Thus, the Mellin transform $\hat{f}_H(n+1)$ of the function $f_H(u)$, given by Eq. (3.23), exists for every $n \in \mathbb{N}_0$. The arithmetic function $g_H(n)$, corresponding to the auxiliary function $f_H(u)$, is determined via Eq. (2.7),

$$g_H(n) = \left(\prod_{j=1}^{n_1} \Gamma\left(\bar{b}_j + a_j n\right) \right) \left(\prod_{k=1}^{n_2} \frac{\Gamma\left(\bar{c}_k + a'_k n\right)}{\Gamma\left(\bar{d}_k + a'_k n\right)} \right), \tag{3.24}$$

for every $n \in \mathbb{N}_0$, where $\bar{b}_j = b_j + a_j$, for every $j = 1, \ldots, n_1$, and $\bar{c}_k = c_k + a'_k$, $\bar{d}_k \equiv d_k + a'_k$, for every $k = 1, \ldots, n_2$. Note that condition (2.4) holds due to the asymptotic behavior of the Gamma function [35]. According to the above-reported properties, the Fox H function $f_H(u)$, given by Eq. (3.23), represents an auxiliary function that is legitimate for the definition of GCSs.

The normalization factor $N_{g_H}\left(|z|^2\right)$ is given by a Wright generalized hypergeometric function,

$$N_{g_H}(u) = {}_{p'+1}W_{q'} \left[u \left| \begin{matrix} (1,1), \left(\bar{d}_j, a'_j\right)_1^{n_2} \\ \left(\bar{b}_j, a_j\right)_1^{n_1}, \left(\bar{c}_j, a'_j\right)_1^{n_2} \end{matrix} \right. \right], \tag{3.25}$$

for every $u > 0$. The weight function $U_{g_H}\left(|z|^2\right)$, corresponding to the auxiliary function $f_H(u)$ given by Eq. (3.23), results to be a product of a Wright generalized hypergeometric function and a Fox H function,

$$U_{g_H}(u) = \pi^{-1}{}_{p'+1}W_{q'}\left[u\middle|\begin{array}{c}(1,1),\left(\bar{d}_j,a'_j\right)_1^{n_1}\\ \left(\bar{b}_j,a_j\right)_1^{n_1},\left(\bar{c}_j,a'_j\right)_1^{\bar{n}_2}\end{array}\right]$$

$$\times H^{q',0}_{p',q'}\left[u\middle|\begin{array}{c}\left(d_j,a'_j\right)_1^{n_2}\\ \left(b_j,a_j\right)_1^{n_1},\left(c_j,a'_j\right)_1^{\bar{n}_2}\end{array}\right], \tag{3.26}$$

for every $u > 0$.

We are finally able to state that the set $\{|z;g_H\rangle\ \forall z \in \mathbb{C}\}$ is a legitimate class of GCSs, obtained with positive Fox H auxiliary functions. This class of GCSs resolves the identity operator over the Fock space $\mathcal{F}$. The positive weight function is the product of a Wright hypergeometric function and a Fox H function. Canonical CSs, Mittag-Leffler and Wright GCSs are special cases of the GCSs under study. In fact, Canonical CSs are obtained for $n_1 = 1$, $n_2 = 0$, $b_1 = 0$, $a_1 = 1$, i.e., $g_H(n) = \Gamma(n+1)$. Mittag-Leffler GCSs are obtained for $n_1 = 1$, $n_2 = 0$, i.e., $g_H(n) = \Gamma(a_1 n + b_1)$. Wright GCSs are obtained for $n_1 = 2$, $n_2 = 0$, $b_1 = 0$, $a_1 = 1$, i.e., $g_H(n) = n!\Gamma(a_1 n + \bar{b}_1)$.

The probability $P_{g_H}\left(n,|z|^2\right)$ that the generalized coherent state $|z;g_H\rangle$ coincides with n excitations, i.e., the state $|n\rangle$ of the Fock basis, is

$$P_{g_H}\left(n,|z|^2\right) = \frac{|z|^{2n}}{N_{g_H}(|z|^2)g_H(n)}, \tag{3.27}$$

for every $n \in \mathbb{N}_0$. For large values of the number of excitations, $n \gg 1$, the probability $P_{g_H}\left(n,|z|^2\right)$ is described by the following asymptotic form:

$$P_{g_H}\left(n,|z|^2\right) \sim \frac{\gamma'|z|^{2n}}{N_{g_H}(|z|^2)}\left(\frac{e}{n}\right)^{\mu' n}\left(\kappa'\right)^{-n} n^{-\bar{\delta}}, \tag{3.28}$$

where

$$\gamma' = (2\pi)^{-\rho'}\exp\left\{\sum_{j=1}^{q'}\bar{\beta}_j - \sum_{j=1}^{p'}\bar{\alpha}_j\right\}\frac{\prod_{j=1}^{p'}\left(A'_j\right)^{\bar{\alpha}_j-(1/2)}}{\prod_{j=1}^{q'}\left(B'_j\right)^{\bar{\beta}_j-(1/2)}}, \tag{3.29}$$

$$\rho' = \frac{q'-p'}{2}, \tag{3.30}$$

$$\kappa' = \left(\prod_{j=1}^{p'}\left(A'_j\right)^{-A'_j}\right)\left(\prod_{j=1}^{q'}\left(B'_j\right)^{B'_j}\right), \tag{3.31}$$

$$\bar{\delta} = \sum_{j=1}^{q'}\bar{\beta}_j - \sum_{j=1}^{p'}\bar{\alpha}_j + \frac{p'-q'}{2}, \tag{3.32}$$

for every allowed values of the involved parameters.

3.1. Truncated Coherent States Generalized with Positive Fox H Functions

At this stage, we consider the case where the Fock basis $\mathcal{F}_\mathfrak{d}$ of a quantum harmonic oscillator is truncated: $\mathcal{F} \equiv \{|0\rangle,\ldots,|\mathfrak{d}\rangle\}$, for every $\mathfrak{d} \in \mathbb{N}$. The truncated Fock space is $(\mathfrak{d}+1)$-dymensional.

doi:10.20944/preprints202506.1349.v1

Let the auxiliary function $f(u)$ be represented by a positive Fox H-function defined by relations (3.8)-(3.22) [44],

$$f_H(u) \equiv H^{m',n'}_{p',q'}\left[u \middle| \begin{array}{c} \left(\alpha'_j, A'_j\right)_1^{p'} \\ \left(\beta'_j, B'_j\right)_1^{q'} \end{array}\right], \tag{3.33}$$

for every $u > 0$. Let the following conditions hold: $\chi > 0$, and

$$\mathfrak{d} + 1 < \min_{j=1,\dots,n'} \left\{ \frac{1-\alpha'_j}{A'_j} \right\}, \tag{3.34}$$

with $\mathfrak{d} \in \mathbb{N}$. Then, the corresponding values $g_H(0), \dots, g_H(\mathfrak{d})$ are obtained from Eq. (2.7),

$$g_H(n) = \frac{\prod_{j=1}^{m'} \Gamma\left(\bar{\beta}_j + B'_j n\right) \prod_{j=1}^{n'} \Gamma\left(1 - \bar{\alpha}_j - A'_j n\right)}{\prod_{j=m'+1}^{q'} \Gamma\left(1 - \bar{\beta}_j - B'_j n\right) \prod_{j=n+1}^{p'} \Gamma\left(\bar{\alpha}_j + A'_j n\right)}, \tag{3.35}$$

for every $n = 0, \dots, \mathfrak{d}$. The corresponding normalization factor, $N_{\mathfrak{d},g_H}\left(|z|^2\right)$, is given by Eq. (2.11) in case the terms $g(0), \dots, g(\mathfrak{d})$, are obtained from Eq. (3.35), for every $n = 0, \dots, \mathfrak{d}$. The corresponding weight function $U_{\mathfrak{d},g_H}\left(|z|^2\right)$ is given by the form below,

$$U_{\mathfrak{d},g_H}(u) = \pi^{-1} N_{\mathfrak{d},g_H}(u) H^{m',n'}_{p',q'}\left[u \middle| \begin{array}{c} \left(\alpha'_j, A'_j\right)_1^{p'} \\ \left(\beta'_{j'}, B'_j\right)_1^{q'} \end{array}\right], \tag{3.36}$$

for every $u > 0$.

The probability $P_{g_H}\left(n, \mathfrak{d}, |z|^2\right)$ that the GCS $|z;\mathfrak{d};g_H\rangle$ is characterized by n excitations, i.e., the state $|n\rangle$, is

$$P_{g_H}\left(n, \mathfrak{d}, |z|^2\right) = \frac{|z|^{2n}}{N_{\mathfrak{d},g_H}(|z|^2) g_H(n)}, \tag{3.37}$$

for every $n = 0, \dots, \mathfrak{d}$. For large values of the number of excitations, $\mathfrak{d} \geq n \gg 1$, the probability $P_{g_H}\left(n, \mathfrak{d}, |z|^2\right)$ is properly approximated by the following asymptotic form:

$$P_{g_H}\left(n, \mathfrak{d}, |z|^2\right) \simeq \frac{\gamma'' |z|^{2n}}{N_{\mathfrak{d},g_H}(|z|^2)} \left(\frac{e}{n}\right)^{\mu' n} \left(\kappa'\right)^{-n} n^{-\bar{\delta}}, \tag{3.38}$$

where

$$\gamma'' = \left(\frac{2\pi}{e}\right)^{q'-m'-n'} \gamma',$$

for every allowed values of the involved parameters.

4. Sub- and Super-Poissonian Statistics of the Number of Excitations

The distribution of the numbers of excitations for a canonical CS is given by a purely Poissonian statistics. The deviation from this canonical condition is estimated by the Mandel Q parameter [45]. The Mandel Q parameter is defined in terms of the expectation values of $\hat{N}^2$, the square of the number operator, and $\hat{N}$, the number operator,

$$Q = \frac{\langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2}{\langle \hat{N} \rangle} - 1. \tag{4.1}$$

 doi:10.20944/preprints202506.1349.v1

The distributions of the number of excitations is super-Poissonian if the Mandel parameter is positive, $Q > 0$. In this case, the variance is larger than the mean value of the number of excitations, $\left[\langle\hat{N}^2\rangle - \langle\hat{N}\rangle^2\right] > \langle\hat{N}\rangle$. The distributions of the number of excitations is purely Poissonian if the Mandel parameter vanishes, $Q = 0$. In this case, the variance coincides with the mean value of the number of excitation, $\left[\langle\hat{N}^2\rangle - \langle\hat{N}\rangle^2\right] = \langle\hat{N}\rangle$. The distributions of the number of excitations is sub-Poissonian if the Mandel parameter is negative, $Q < 0$. In this case, the variance is smaller than the mean value of the number of excitations, $\left[\langle\hat{N}^2\rangle - \langle\hat{N}\rangle^2\right] < \langle\hat{N}\rangle$. Negative values of the Mandel parameter are related to the non-classical nature of the system.

For the GCSs generated by the arithmetic function $g(n)$, the Mandel parameter $Q\left(|z|^2, g\right)$ is given by the following form [57]:

$$Q\left(|z|^2, g\right) = |z|^2\left[\frac{\sum_{n=0}^{\infty}(n+1)(n+2)|z|^{2n}/g(n+2)}{\sum_{n=0}^{\infty}(n+1)|z|^{2n}/g(n+1)} - \frac{\sum_{n=0}^{\infty}(n+1)|z|^{2n}/g(n+1)}{\sum_{n=0}^{\infty}|z|^{2n}/g(n)}\right], \tag{4.2}$$

for every $z \in \mathbb{C}\setminus\{0\}$. For the GCSs generated by the arithmetic function $g_H(n)$, the Mandel parameter $Q\left(|z|^2, g_H\right)$ is expressed in terms of the Wright generalized hypergeometric function,

$$Q\left(|z|^2, g_H\right) = |z|^2\left\{\frac{{}_{p'+1}W_{q'}\left[u\,\middle|\,\begin{matrix}(3,1),\left(\alpha_j+3A'_j, A'_j\right)_1^{p'}\\ \left(\beta+3B'_j, B'_j\right)_1^{q'}\end{matrix}\right]}{{}_{p'+1}W_{q'}\left[u\,\middle|\,\begin{matrix}(2,1),\left(\alpha_j+2A'_j, A'_j\right)_1^{p'}\\ \left(\beta+2B'_j, B'_j\right)_1^{q'}\end{matrix}\right]} - \frac{{}_{p'+1}W_{q'}\left[u\,\middle|\,\begin{matrix}(2,1),\left(\alpha_j+2A'_j, A'_j\right)_1^{p'}\\ \left(\beta+2B'_j, B'_j\right)_1^{q'}\end{matrix}\right]}{{}_{p'+1}W_{q'}\left[u\,\middle|\,\begin{matrix}(1,1),\left(\alpha_j+A'_j, A'_j\right)_1^{p'}\\ \left(\beta+B'_j, B'_j\right)_1^{q'}\end{matrix}\right]}\right\}, \tag{4.3}$$

for every $z \in \mathbb{C}\setminus\{0\}$. The Mandel parameter of GCSs is studied for large and small values of the label in Ref. [57]. The Mandel parameter tends to the opposite of unity for large values of the label. This behavior is confirmed by the statistics of the GCSs under study,

$$Q\left(|z|^2, g_H\right) \to -1, \tag{4.4}$$

for large values of the label, $|z| \to +\infty$. In accordance with the general case, for the GCSs under study, the distributions of the number of excitations is sub-Poissonian at large values of the label.

For small nonvanishing values of the label, the Mandel parameter of the GCSs under study is positive (negative),

$$Q\left(|z|^2, g_H\right) >_{(<)} 0, \tag{4.5}$$

as $|z| \to 0^+$, with $z \neq 0$, if the following constraint holds [57]:

$$\frac{g_H(0)g_H(2)}{g_H^2(1)} <_{(>)} 2. \tag{4.6}$$

 doi:10.20944/preprints202506.1349.v1

Relation (A1) is fulfilled by the following values of the involved parameters:

$$A'_j = B'_k = 1, \quad \alpha_j = \frac{2 - 2^{\xi_j}}{2^{\xi_j} - 1}, \quad \beta_k = \frac{2 - 2^{\eta_k}}{2^{\xi_k} - 1}, \tag{4.7}$$

in case

$$\sum_{k=1}^{q'} \eta_k - \sum_{j=1}^{p'} \xi_j <_{(>)} 1, \tag{4.8}$$

for every ξ_j and η_k such that $0 < \xi_j, \eta_k \leq 1$, and every $j = 1, \ldots, p'$, $k = 1, \ldots, q'$.

The GCSs under study exhibit super-Poissonian statistics of the number of excitations, $Q\left(|z|^2, g_H\right) > 0$, for small nonvanishing values of the label, $|z| \to 0^+$, with $z \neq 0$, in case the involved parameters are given by relations (4.7) and (4.8), and $\eta_j \simeq \xi_j$, for every $j = 1, \ldots, p'$, $\eta_j \gtrsim 0$, for ever $j = p' + 1, \ldots, q'$, as $q' > p'$. Instead, the statistics of the number of excitations is sub-Poissonian , $Q\left(|z|^2, g_H\right) < 0$, for small nonvanishing values of the label, $|z| \to 0^+$, with $z \neq 0$, in case the involved parameters are given by relations (4.7) and (4.8), and $\xi_j \gtrsim 0$, for every $j = 1, \ldots, p'$, $\sum_{k=1}^{q'} \eta_k > 1$.

Super-Poissonian statistics of the number of excitations for small nonvanishing values of the label are realized, more generally, if one index j' exists, at least, such that the value of the parameter $A'_{j'}$ is sufficiently large, i.e., $A'_{j'} \gg 1$. Similarly, sub-Poissonian statistics of the number of excitations for small nonvanishing values of the label is realized if one index j'' exists, at least, such that the value of the parameter $B'_{j''}$ is sufficiently large, i.e., $B'_{j''} \gg 1$.

In summary, the GCSs under study exhibit super- or sub-Poissonian statistics of the number of excitations, according to the values of the involved parameters. The corresponding Mandel parameter is expressed in terms of Wright generalized hypergeometric functions and is negative for large values of the label. Instead, positive or negative values of the Mandel parameter are obtained for small nonvanishing values of the label and determined values of the involved parameters. In this way, the resulting statistic is super- or sub-Possonian, respectively.

5. Dissipative Effects

At this stage we evaluate the dissipative effects that a memoryless environment produces over the GCSs under study [46–50]. For the sake of clarity and consistency, we report below the dissipative evolution of a GCS interacting with such an environment, by following Refs. [48–50].

Briefly, the dissipative evolution of an initial state described by the density matrix $\rho(0)$, results to be the mixed state $\rho(t)$, given by the form below,

$$\rho(t) = \sum_{l=0}^{\infty} \Lambda_l(t)\rho(0)\Lambda_l^{\dagger}(t), \tag{5.1}$$

for every $t \geq 0$. The effect operators $\Lambda_0(t)$, $\Lambda_1(t)$, …, mimic the loss of zero, one, or more excitations, or, equivalently, decay events, and are given by the following form in the canonical Fock basis $\mathcal{F}$:

$$\Lambda_l(t) = \sum_{l'=l}^{\infty} \sqrt{\frac{l'![p(t)]^{l'-l}[1 - p(t)]^l}{l!(l' - l)!}} \, |l' - l\rangle\langle l'|, \tag{5.2}$$

for every $t \geq 0$, and $l \in \mathbb{N}_0$. The function $p(t)$ is the exponentially damped survival probability of the initial state,

$$p(t) = \exp(-\upsilon t), \tag{5.3}$$

for every $t \geq 0$.

The loss of zero, one, or more excitations by a GCS $|z;g\rangle$, produces states that, despite the loss, remain GCSs [50],

$$\Lambda_l(t)|z;g\rangle = z^l \mathfrak{K}_l\left[p(t),|z|^2,g_l\right]|z\sqrt{p(t)};g_l\rangle, \tag{5.4}$$

for every $t \geq 0$, $z \in \mathbb{C}\setminus\{0\}$ and $l \in \mathbb{N}_0$. The function $\mathfrak{K}_l\left[p(t),|z|^2,g_l\right]$ is given by the following expression:

$$\mathfrak{K}_l\left[p(t),|z|^2,g_l\right] = \sqrt{\frac{[1-p(t)]^l N_{g_l}\left[|z|^2 p(t)\right]}{l!\, N_{g_0}\left(|z|^2\right)}}. \tag{5.5}$$

The transformed GCS $|z\sqrt{p(t)};g_l\rangle$, is generated by the arithmetic function $g_l(n)$, given by

$$g_l(n) = \frac{g_0(n+l)}{(n+1)\cdots(n+l)}, \tag{5.6}$$

for every $l,n \in \mathbb{N}_0$. Note that $g_0(n) = g(n)$ for every $n \in \mathbb{N}_0$. The class $\{|z;g_l\rangle, \forall\, z \in \mathbb{C}\}$ of GCSs produced by l decay events, resolve the identity operator with the weight function $U_{g_l}(u)$, given by the expression below,

$$U_{g_l}(u) = N_{g_l}(u)\int_u^\infty \cdots \int_{u_{l-1}}^\infty \frac{U_{g_0}(u_l)}{N_{g_0}(u_l)}\, du_l \ldots du_1, \tag{5.7}$$

for every $u > 0$, and $l \in \mathbb{N}$, with $u_0 = u$.

According to Eq. (5.1), the time evolution of the initial GCS $|z;g\rangle$, i.e., $\rho(0) = |z;g\rangle\langle z;g|$, is

$$\rho(t) = \sum_{l=0}^{\infty} \Xi_l\left[p(t),|z|^2,g_l\right]|z\sqrt{p(t)};g_l\rangle\langle z\sqrt{p(t)};g_l|, \tag{5.8}$$

for every $t \geq 0$. The statistical mixture $\rho(t)$ is composed by the transformed GCSs $|z\sqrt{p(t)};g_0\rangle$, $|z\sqrt{p(t)};g_1\rangle,\ldots$, with weights $\Xi_0\left(p(t),|z|^2,g_0\right)$, $\Xi_1\left(p(t),|z|^2,g_1\right),\ldots$, given by

$$\Xi_l\left[p(t),|z|^2,g_l\right] = |z|^{2l}\mathfrak{K}_l^2\left[p(t),|z|^2,g_l\right] = \frac{|z|^{2l}[1-p(t)]^l N_{g_l}\left[|z|^2 p(t)\right]}{l!\, N_{g_0}\left(|z|^2\right)}, \tag{5.9}$$

for every $t \geq 0$, $l \in \mathbb{N}_0$, and $z \in \mathbb{C}\setminus\{0\}$.

At this stage, we are equipped to describe the dissipative effects on the CSs generalized with the Fox H function due to the unavoidable interaction with the external environment. The generalized coherent state $|z;g_H\rangle$ is transformed by l decay events into the generalized coherent state $|z;g_{H,l}\rangle$, for every $z \in \mathbb{C}\setminus\{0\}$, described by Eqs. (5.4) and (5.5), for every $l \in \mathbb{N}_0$. The involved arithmetic function $g_{H,l}(n)$ is obtained from Eqs. (3.24) and (5.6),

$$g_{H,l}(n) = \frac{\Gamma(1+n)}{\Gamma(1+l+n)}\left(\prod_{j=1}^{n_1}\Gamma\left(b_{j,l}+a_j n\right)\right)\left(\prod_{k=1}^{n_2}\frac{\Gamma\left(c_{k,l}+a'_k n\right)}{\Gamma\left(d_{k,l}+a'_k n\right)}\right), \tag{5.10}$$

for every $l, n \in \mathbb{N}_0$. The involved parameters are defined as follows:

$$b_{j,l} = b_j + (l+1)a_j, \quad c_{k,l} = c_k + (l+1)a'_k, \quad d_{k,l} = d_k + (l+1)a'_k, \tag{5.11}$$

for every $j = 1, \ldots, n_1$, $k = 1, \ldots, n_2$, $n_1, n_2 \in \mathbb{N}$, $l \in \mathbb{N}_0$. Note that $g_{H,0}(n) = g_H(n)$, for every $n \in \mathbb{N}_0$.

The (positive) auxiliary function $f_{H,l}(u)$, corresponding to the arithmetic function $g_{H,l}(n)$, is given by the following Fox H function:

$$f_{H,l}(u) = H^{q'+1,0}_{p'+1,q'+1}\left[u \left| \begin{array}{c} (l+1,1), \left(\alpha'_{j,l}, A'_j\right)_1^{p'} \\ (1,1), \left(\beta'_{j,l}, B'_j\right)_1^{q'} \end{array} \right. \right], \tag{5.12}$$

for every $u > 0$, and $l \in \mathbb{N}_0$. The involved parameters are

$$\left(\alpha'_{j,l}, A'_j\right)_1^{p'} = \left(d_{k,l}, a'_k\right)_1^{n_2}, \tag{5.13}$$

$$\left(\beta'_{j,l}, B'_{j'}\right)_1^{q'} = \left(b_{j,l}, a_j\right)_1^{n_1}, \left(c_{k,l}, a'_k\right)_1^{n_2}, \tag{5.14}$$

for every $l, n \in \mathbb{N}_0$, and $n_1, n_2 \in \mathbb{N}$.

The normalization factor $N_{g_{H,l}}\left(|z|^2\right)$ of the transformed generalized coherent state $|z; g_{H,l}\rangle$ is given by a Wright generalized hypergeometric function,

$$N_{g_{H,l}}(u) = {}_{p'+1}W_{q'}\left[u \left| \begin{array}{c} (l+1,1), \left(\alpha'_{j,l}, A'_j\right)_1^{p'} \\ \left(\beta'_{j,l}, B'_j\right)_1^{q'} \end{array} \right. \right], \tag{5.15}$$

for every $u > 0$. The weight function $U_{g_{H,l}}\left(|z|^2\right)$, corresponding to the auxiliary function $f_{H,l}(u)$ given by Eq. (5.12), results to be a product of a Wright generalized hypergeometric function and a Fox H function,

$$U_{g_{H,l}}(u) = \pi^{-1}\, {}_{p'+1}W_{q'}\left[u \left| \begin{array}{c} (l+1,1), \left(\alpha'_{j,l}, A'_j\right)_1^{p'} \\ \left(\beta'_{j,l}, B'_j\right)_1^{q'} \end{array} \right. \right] \times H^{q'+1,0}_{p'+1,q'+1}\left[u \left| \begin{array}{c} (l+1,1), \left(\alpha'_{j,l}, A'_j\right)_1^{p'} \\ (1,1), \left(\beta'_{j,l}, B'_j\right)_1^{q'} \end{array} \right. \right], \tag{5.16}$$

for every $u > 0$.

The exponentially damped survival probability of the initial state, $p(t)$, vanishes over long times, $t \gg 1/u$. Thus, every time-dependent GCS generated by the zero, one or more decay events, $|z\sqrt{p(t)}; g_{H,l}\rangle$ for every l with nonvanishing label, $z \in \mathbb{C} \setminus \{0\}$ for every $l, n \in \mathbb{N}_0$, tends to become the ground or vacuum states, $|z\sqrt{p(t)}; g\rangle \to |0\rangle$, for $t \gg 1/u$. Consequently, the amplitude damping noise ultimately reduces the initial GCS $|z; g_H\rangle$ to the vacuum state, $\rho(t) \to |0\rangle\langle 0|$, over long times, $t \gg 1/u$ [50].

6. Summary and Conclusions

CSs of a quantum harmonic oscillator are fundamental states of minimum uncertainty that exhibit Poissonian distribution of the number of excitations. CSs find applications in the most various scenarios, from quantum optics to mechanical devices.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 June 2025 doi:10.20944/preprints202506.1349.v1

Theoretical generalizations of CSs are performed in various ways. Klauder' s generalization of CSs is performed by requiring the conditions of normalizability, continuity in the label and resolution of the identity operator with a (positive) weight function [3]. This approach has led to various generalizations of CSs with special functions including Mittag-Leffler and Wright functions [3,11–15,57]. These generalized states exhibit various distributions of the number of excitations, ranging from super-Poissonian to the non-classical sub-Poissonian statistics. Additionally, Wright generalized Schrödinger cat states evolve under amplitude damping noise similarly to the canonical Scrödinger cat states, opening to possible applications in quantum information processing [48–50].

Fox H function is a special function that produces Mittag-Leffler and Wright special functions as particular cases [31–36,36–38,38,39,39–43]. Therefore, in the present scenario, we have adopted this special function to perform further generalization of CSs. We have found that the resulting GCSs provide canonical CSs, and Mittag-Leffler and Wright GCSs as particular cases. Additionally, these GCSs exhibit anomalous distributions of large numbers of excitations that result to be products of exponential and power laws and powers of the term n^{-n}. Thus, the novel GCSs are equipped to describe a large variety of anomalous distributions of number of excitations in the framework of purely quantum theory. The corresponding Mandel Q factor consists of ratios of Wright generalized hypergeometric functions that is negative for large values of the label and, therefore, witnesses (non-classical) sub-Poissonian statistics. Instead, for small nonvanishing values of the label, the Mandel Q factor is positive or negative, according to the values of the involved parameters. In these cases the statistics is super- or sub-Poissonian, respectively, according to the values of the involved parameters. Additionally, the super-Poissonian regime is obtained for small nonvanishing values of the label by simply choosing sufficiently large values of just one parameter. Same property holds for the realization of the sub-Poissonian statistics. The GCSs under study resolve the identity operator with a weight functions that is the product of a Wright generalized hypergeometric function and a Fox H function.

In conclusion, the CSs generalized via Fox H auxiliary functions provide a wide variety of behaviors ranging from classical to non-classical properties, and anomalous statistics of the number of excitations that described various decay laws. We believe that these property might be help to describe anomalous phenomena in the framework of purely quantum theory.

Appendix A. Details

The normalization factor $N_{g_H}\left(|z|^2\right)$, given by Eq. (3.25), is obtained from the general form (2.2), by considering the arithmetic function $g_H(n)$, given by Eq. (3.24). The corresponding power series provides the Wright generalized hypergeometric function via Eq. (3.4). The weight function $U_{g_H}(u)$, given by Eq. (3.26), is derived from Eqs. (2.6), (3.23), (3.25). The asymptotic form (3.28) is derived from Eq. (3.27) and the asymptotic expansion of the form $1/g_H(n)$ as $n \to +\infty$ [35]. The above-reported methods hold also for the truncated GCSs under study. In this way, Eqs. (3.36) - (3.28) are obtained from Eqs. (3.33), (3.34).

Expression (4.3) of the Mandel parameter is obtained from Eq. (4.2) by using form (3.24) of the arithmetic function $g(n)$. The asymptotic form (4.4) is evaluated in straightforward way from Eqs. (4.3), by considering the generalized hypergeometric function as a particular case of the Fox H function [35], Eq. (3.4).

Condition (4.6) is equivalent to the following relation involving products of ratios of non-negative Gamma functions:

$$\left(\prod_{j=1}^{q'} \frac{\Gamma\left(\bar{\beta}_j + 2B'_j\right)}{\Gamma\left(\bar{\beta}_j + B'_j\right)}\right)\left(\prod_{j=1}^{p'} \frac{\Gamma\left(\bar{\alpha}_j + A'_j\right)}{\Gamma\left(\bar{\alpha}_j\right)}\right)$$
$$<_{(>)} 2\left(\prod_{j=1}^{p'} \frac{\Gamma\left(\bar{\alpha}_j + 2A'_j\right)}{\Gamma\left(\bar{\alpha}_j + A'_j\right)}\right)\left(\prod_{j=1}^{q'} \frac{\Gamma\left(\bar{\beta}_j + B'_j\right)}{\Gamma\left(\bar{\beta}_j\right)}\right). \tag{A1}$$

Conditions (4.7) and (4.8) are obtained from relation (A1) and the recurrence relation involved Gamma function [58], $\Gamma(z+1) = z\Gamma(z)$, holding for $\mathrm{Re}\, z > 0$.

Condition $A'_{j'} \gg 1$, realizing the super-Poissonian statistics, is derived from the following vanishing asymptotic behavior [58]:

$$\lim_{A'_{j'} \to +\infty} \frac{\Gamma\left(\bar{\alpha}_{j'} + A'_{j'}\right)}{\Gamma\left(\bar{\alpha}_{j'}\right)} \Big/ \frac{\Gamma\left(\bar{\alpha}_{j'} + 2A'_{j'}\right)}{\Gamma\left(\bar{\alpha}_{j'} + A'_{j'}\right)} = 0^{+}, \tag{A2}$$

Similarly, condition $B'_{j''} \gg 1$, realizing the sub-Poissonian statistics, is derived from the following vanishing asymptotic behavior [58]:

$$\lim_{B'_{j''} \to +\infty} \frac{\Gamma\left(\bar{\beta}_{j''} + 2B'_{j''}\right)}{\Gamma\left(\bar{\beta}_{j''} + B'_{j''}\right)} \Big/ \frac{\Gamma\left(\bar{\beta}_{j''} + B'_{j''}\right)}{\Gamma\left(\bar{\beta}_{j''}\right)} = +\infty. \tag{A3}$$

Relations (5.10) - (5.16) are obtained in straightforward way from Eqs. (2.6), (2.7), (3.1) - (3.26). This concludes the demonstration of the present results.

Conflicts of Interest: The author declares no conflicts of interest.

References

1. E. Schrödinger, Naturwissenschaften **14**, 664 (1926).
2. R.J. Glauber, Phys. Rev. **130**, 2529 (1963); **131**, 2766 (1963).
3. J.R. Klauder, J. Math. Phys. **4**, 1058 (1963).
4. J.R. Klauder and B. Skagerstam, *Coherent States, Applications in Physics and Mathematical Physics* (World Scientific, Singapore, 1985)
5. A. Perelomov, *Generalized Coherent States and Their Applications* (Springer, Berlin, 1986)
6. J.R. Klauder, Ann. Phys.; NY **237**, 147 (1995).
7. R. Loudon, *Quantum Theory of Light* 2nd Ed. (Oxford U.P., Oxford, 1983).
8. D.F. Walls and G.J. Milbourn, *Quantum Optics* (Springer-Verlag, Berlin, 1994).
9. L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge U.P., Cambridge, 199
10. B.C. Sanders, J. Phys. A **45**, 224002 (2012).
11. J.M. Sixdeniers, K.A. Penson and A.I. Solomon, J. Phys. A **32**, 7543 (1999).
12. K.A. Penson and A.I. Solomon, J. Math. Phys. **40**, 2354 (1999).
13. R. Garra, F. Giraldi and F. Mainardi, WSEAS Transactions on Mathematics, **18**, 428 (2019).
14. F. Giraldi and F. Mainardi, J. Math. Phys. **64**, 032105 (2023).
15. R. Droghei, Mathematics **13**, 759 (2025).
16. M.G. Mittag-Leffler, Acta Math. **4**, 1-79 (1884).
17. M.G. Mittag-Leffler, C.R. Acad. Sci. Paris **137**, 554-558 (1903).
18. R. Gorenflo, A.A. Kilbas, F. Mainardi, S.V. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications* (NewYork, USA: Springer, 2020).
19. E.M. Wright, Journal London Math. Soc. **8**, 71-79 (1933).
20. T.S. Santhanam and A.R. Tekumella, Fund. Phys. **6**, 583 (1976).
21. I. Goldhirsch, J. Phys. A **13**, 3479 (1980).
22. D.T. Pegg and S.M. Barnett, Europhys. Lett. **6**, 483 (1988).
23. W.M. Zhang, D.H. Feng and R. Gilmore, Rev. Mod. Phys. **62**, 867 (1990).
24. V. Buzek *et al.*, Phys. Rev. A **45**, 8709 (1992).
25. L.-M. Kuang, F.-B. Wang and Y.-G. Zhou, J. Mod. Opt. **41**, 1307 (1994).
26. A. Miranowics, K. Piatek and R. Tanas, Phys. Rev. A **50**, 3423 (1994).
27. W. Leonski, Phys. Rev. A **55**, 3874 (1997).
28. W. Leonski, Kowalewska-Kudlaszye, Prog. Opt. **56**, 131 (2011).
29. S. Sivakumar, arXiv.1402.1487 (2014).
30. W.S. Chung and H. Hassanabadi, Eur. Phys. J. Plus **135**, 556 (2020).
31. C. Fox, Trans. Amer. Math. Soc. **98**, 395 (1961).

32. H.M. Srivastava, K.C. Gupta, S.P. Goyal, *The H-functions of one and two variables* New Delhi: South Asian Publishers Pvt. Ltd (1982).
33. A.P. Prudnikov, Y.A. Brychkov, and O.I. Marichev, *Integrals and Series. Additional Chapters* (Nauka, Moscow, 1986).
34. A.P. Prudnikov, Y.A. Brychkov, and O.I. Marichev, *Integrals and series, Vol. 3, more special functions.* Gordon and Breach Science, New York (1990).
35. A.A. Kilbas, M. Saigo, *H-transforms: Theory and applications*, Chapman & Hall/CRC (2004).
36. A.M. Mathai, R. K. Saxena H. J. Haubold, *The H-functions: Theory and applications*, Springer (2010).
37. Y. Luchko, V. Kiryakova, Fract. Calc. Appl. Anal. **16** 405 (2013).
38. Mehrez K., J. Math. Anal. Appl. **468**, 650 (2018).
39. K. Mehrez , Anal. Math. Phys. **11**, 114 (2021).
40. B.D. Carter, M.D. Springer, SIAM J. Appl. Math. **33**, 542 (1977).
41. A.M. Mathai, R.K. Saxena, *The H-function with Applications in Statistics and Other Disciplines*, Wiley Eastern Ltd.,New Delhi, 1978.
42. F. Mainardi, G. Pagnini, R.K. Saxena, J. Comput. Appl. Math. **178**, 331 (2005).
43. L. Beghin, L. Cristofaro, J.L. da Silva, J. Theor. Probab. **38**, 18 (2025).
44. F. Giraldi, arXiv:2502.10483 [math.CV]
45. L. Mandel, Opt. Lett. **4**, 205 (1979).
46. V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, J. Math. Phys. **17**, 821 (1976).
47. G. Lindblad, Comm. Math. Phys. 48, 119 (1976).
48. L. Chuang, O.W. Leung and Y. Yamamoto, Phys. Rev. A. **56**, 1114 (1997).
49. P.T. Cochrane, G.J. Milburn and W.J. Munro, Phys. Rev. A **59**, 2631 (1999).
50. F. Giraldi, Phys. Scr. **100**, 065245 (2025).
51. N.I. Akhiezer, *The classical moment problem and some related questions in analysis*, (London: Oliver ans Boyd,1965).
52. H.-J. Glaeske , A.P. Prudnikov , K.A. Skórnik, *Operational Calculus and Related Topics*, Chapman & Hall/CRC, Taylor and Francis Group, Boca Raton, London, New York (2006).
53. E.C. Titchmarsh, *Introduction to the theory of Fourier integrals*, 2nd ed. (Oxford Univ. Press, London and New York, 1948).
54. D.V. Widder, *The Laplace Transform* (Princeton Univ. Press, Princeton, NJ, 1941).
55. G. Doetsch, *Handbuch der Laplace Transformation*, Vols. 1-3 (Birkäuser, Basel, 1955).
56. O.I. Marichev, *Handbook of Integral Transforms of Higher Transcendental Functions*, Theory and Algorithmic Tables (Ellis Horwood, Chichester, 1982).
57. F. Giraldi, J. Phys. A **56**, 305301 (2023).
58. *NIST Handbook of Mathematical Functions.* Edited by Frank W.J. Olver (editor-in-chief), D.W. Lozier, R.F. Boisvert, and C.W. Clark. National Institute of Standards and Technology: Gaithersburg, Maryland, and Cambridge University Press: New York (2010).