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Posted Date: 19 March 2026

doi: 10.20944/preprints202603.1541.v1

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Existence and Uniqueness of Weak Solutions for the Stochastic Fractional Ginzburg-Landau Equation

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Abstract

This paper investigates the existence and uniqueness of weak solutions for a stochastic Ginzburg-Landau equation involving the fractional Laplacian. The primary focus is on establishing a rigorous mathematical framework to handle the coexistence of the nonlocal fractional Laplacian and stochastic perturbations. By employing the Galerkin method, we establish that the initial-boundary value problem admits a unique global weak solution for any L_a^2 initial value. This study utilizes the properties of the fractional Laplacian and fractional Sobolev spaces to provide a rigorous proof of the existence and uniqueness theorem. These results extend the analysis of Ginzburg-Landau equations to models incorporating stochastic terms and fractional Laplacian.

Keywords: stochastic ginzburg-landau equation; fractional laplacian; weak solutions; galerkin method; existence and uniqueness

MSC: 35R11; 60H15; 41A65

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1. Introduction

The Ginzburg-Landau equation has become a research focus due to its wide applications in fields such as superconductivity, fluid flow, and optical soliton transmission, see [1] and [2]. The introduction of stochastic perturbations is more in line with the analysis needs of nonlinear systems that reflect uncertainties in real-world systems, such as reaction-diffusion equations, infectious disease models, and climate system models. It also provides key support for solving practical problems.

Recently, Qin, Jiang and Chen [3] proposed a numerical scheme combining a timporal direction leap-frog method with a spatial finite difference scheme to solve this space-fractional Ginzburg-Landau-Schrödinger equation

$$(\alpha - i\beta)\varphi_t + (-\Delta)^{\frac{\gamma}{2}}\varphi - (1 - |\varphi|^2)\varphi = 0, \quad x \in \Omega,\ t > 0.$$

They proved the convergence of the scheme while keeping both the discrete mass and energy conserved. Pino, Juneman and Musso [4] investigated the linearized Ginzburg-Landau equation around a standard degree-one vortex solution in the plane. By utilizing explicit expressions for Fourier modes, they obtained precise estimates for the inverse of the linearization operator applicable to cases without orthogonality conditions. Zhang, Lu and Liu [5] researched the stability and optimal controls for time-space fractional Ginzburg–Landau systems. This paper aims to extend the aforementioned results to fractional Ginzburg-Landau equations with stochastic perturbations.

 doi:10.20944/preprints202603.1541.v1

Let $(S, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space. W be an Wiener process defined on $(S, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$. This paper considers the weak solutions of stochastic Ginzburg-Landau equation involving fractional Laplacian as follows

$$\begin{cases} \frac{\partial z(t,x,w)}{\partial t} + (-\Delta)^{\gamma} z(t,x,w) = \alpha(x,w)(z(t,x,w) - |z(t,x,w)|^{\theta} z(t,x,w)) \\ \qquad +\beta(t,x,w) + h(t,x,w)\frac{\partial W(t,x,w)}{\partial t}, \quad (t,x) \in I_T, w \in S, \\ z(t,x,w) = 0, \quad (t,x) \in Q_T, w \in S, \\ z(0,x,w) = \phi(x,w), x \in I, w \in S, \end{cases}$$

where I is a smooth bounded domain of $\mathbb{R}^n$, $T \in (0, +\infty)$, $I_T = (0, T) \times I$ and $Q_T = (0, T) \times (\mathbb{R}^n \setminus I)$. $(-\Delta)^{\gamma}$ is fractional Laplacian and $\gamma \in (0, 1)$. θ denotes the index of the nonlinear term. Briefly, by omitting w, we rewrite the initial-boundary value problem as follows

$$\begin{cases} \frac{\partial z(t,x)}{\partial t} + (-\Delta)^{\gamma} z(t,x) = \alpha(x)(z(t,x) - |z(t,x)|^{\theta} z(t,x)) \\ \qquad +\beta(t,x) + h(t,x)\frac{\partial W(t,x)}{\partial t}, \quad (t,x) \in I_T, \\ z(t,x) = 0, \quad (t,x) \in Q_T, \\ z(0,x) = \phi(x), x \in I. \end{cases} \tag{1.1}$$

Relevant nonlinear models are also a research focus in the field of Ginzburg-Landau equations. Some scholars have proven the existence and uniqueness of solutions to initial-boundary value problems in appropriate function spaces[6], [7], [8]. Juárez-Campos, Kaikina and Ruiz-Paredes [9] proved the existence and uniqueness of solutions to the stochastic nonlinear Ginzburg-Landau equation with Neumann white-noise boundary conditions, and analyzed its regularity near the origin, particularly when the boundary data is irregular. For semilinear equations with boundary noise and point noise, scholars used the fixed-point method to prove the existence and uniqueness of solutions, see [10] and [11]. Meanwhile, for real-valued stochastic Ginzburg-Landau equations driven by multiplicative noise in the Itô sense, scholars have obtained exact triangular and hyperbolic stochastic solutions, see [12] and [13]. For one-dimensional complex Ginzburg-Landau equations with multiplicative noise terms, they not only proved the existence and uniqueness of solutions but also proposed a new numerical approximation method and conducted error estimation, see [14] and [15]. For the existence of solutions to partial differential equations with fractional differential operators, see [16] and [17]. Inspired by the above research, this paper proves the existence and uniqueness of weak solutions for problem (1.1) by Galerkin method.

The structure of this paper is arranged as follows: Section 2 introduces some notations; Section 3 presents the global existence of weak solutions to problem (1.1) under the initial value $\phi \in L^2_a(I)$. The subsequent sections will elaborate on the research content of this paper in detail.

2. Some Notations

The main difficulty in studying problem (1.1) arises from the coexistence of a stochastic term and the fractional Laplacian. This section is devoted to establishing appropriate function spaces for problem (1.1) .

Let z be a function in Schwartz space. For $\gamma \in (0, 1)$, $(-\Delta)^{\gamma} z$ is defined by

$$(-\Delta)^{\gamma} z(x) = C(n, \gamma)\text{P.V.} \int_{\mathbb{R}^n} \frac{z(x) - z(y)}{|x - y|^{n+2\gamma}} \mathrm{d}y, \quad x \in \mathbb{R}^n.$$

Denote

$$H^{\gamma}(\mathbb{R}^n) = \left\{ z \in L^2(\mathbb{R}^n) : \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|z(x) - z(y)|^2}{|x - y|^{n+2\gamma}} \mathrm{d}x\mathrm{d}y < +\infty \right\}$$

and

$$B_0(I) = \{ z \in H^{\gamma}(\mathbb{R}^n) : z = 0, \text{ a.e. in } \mathbb{R}^n \backslash I \}.$$

 doi:10.20944/preprints202603.1541.v1

In addition, $B_0(I)$ is a Hilbert space equipped with the following norm and inner product

$$\|z\|_{B_0} = \left(\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|z(x) - z(y)|^2}{|x-y|^{n+2\gamma}} \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}},$$

$$(\xi, \eta)_{B_0} = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(\xi(x) - \xi(y))(\eta(x) - \eta(y))}{|x-y|^{n+2\gamma}} \mathrm{d}x\mathrm{d}y.$$

Recall that $(S, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$ is a complete filtered probability space. Let $\mathbb{H}$ denote a separable Hilbert space, and let W be a $\mathbb{H}$-valued Wiener process defined on $(S, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$. For any $p \in [1, +\infty)$, let

$$L_a^p(0, T; \mathbb{X}) = \left\{ z \text{ is a } \mathcal{F}_{t\in[0,T]} \text{ adapted random process} : \mathbb{E} \int_0^T \|z(t)\|_{\mathbb{X}}^p dt < \infty \right\}$$

endowed with the norm

$$\|z\|_{L_a^p(0,T;\mathbb{X})} = \left(\mathbb{E} \int_0^T \|z(t)\|_{\mathbb{X}}^p dt \right)^{\frac{1}{p}}.$$

If $p = +\infty$, denote

$$L_a^\infty(0, T; \mathbb{X}) = \left\{ z \text{ is a } \mathcal{F}_{t\in[0,T]} \text{ adapted random process} : \operatorname*{ess\,sup}_{t\in[0,T]} \mathbb{E}\|z(t)\|_{\mathbb{X}} < \infty \right\}$$

and

$$\|z\|_{L_a^\infty([0,T],\mathbb{X})} = \operatorname*{ess\,sup}_{t\in[0,T]} (\mathbb{E}\|z(t)\|_{\mathbb{X}}).$$

In this paper, $\mathbb{X}$ is taken to be $B_0(I)$, $L^2(I)$ or $L^{\theta+2}(I)$. Lastly, we set $C_0^1([0, T])$ to be

$$C_0^1([0, T]) = \{z \in C^1([0, T]) \text{ with } z \text{ vanished at } T\}.$$

3. Existence and Uniqueness of Weak Solutions

The fractional Laplacian $(-\Delta)^\gamma$ possesses a sequence of eigenfunctions, denoted by

$$e_1, e_2, \cdots, e_m, \cdots,$$

which simultaneously serves as an orthonormal basis for $L^2(I)$ and an orthogonal basis for $B_0(I)$. Since W is a $\mathbb{H}$-valued Wiener process and $\mathbb{H}$ is a separable Hilbert space, let

$$\iota_1, \iota_2, \cdots, \iota_m, \cdots$$

be an orthonormal basis of $\mathbb{H}$. See [18] and [19] for more details. Hence, we utilize the Galerkin method to establish the existence and uniqueness of weak solutions for problem (1.1).

Definition 3.1. *A stochastic process z is called a weak solution to initial-boundary value problem (1.1) if z satisfies*

1) $z \in L_a^2(0, T; B_0(I)) \cap L_a^{\theta+2}(I_T) \cap L_a^\infty(0, T; L^2(I))$ *and* $z(0, x) = \phi(x)$, $\mathbb{P}-$ a.s..

2) For any $\eth \in C^1(0, T; B_0(I))$ *with* $\eth(T) = 0$.

$$\begin{aligned} & -\langle \phi, \eth(t) \rangle_{L^2} - \int_0^T \left\langle z(t), \frac{\mathrm{d}\eth(t)}{\mathrm{d}t} \right\rangle_{L^2} \mathrm{d}t + \int_0^T ((-\Delta)^\gamma z(t), \eth(t))_{B_0^*, B_0} \mathrm{d}t \\ = & \int_0^T \left(\alpha(z(t) - |z(t)|^\theta z(t)), \eth(t) \right)_{(L_a^{\theta+2})^*, L_a^{\theta+2}} \mathrm{d}t + \int_0^T \langle \beta(t), \eth(t) \rangle_{L^2} \mathrm{d}t \\ & + \int_0^T \langle \eth(t), h(t) \mathrm{d}W(t) \rangle_{L^2}, \quad \mathbb{P} - \text{a.s.}. \end{aligned} \tag{3.1}$$

 doi:10.20944/preprints202603.1541.v1

In this paper, we make the following assumptions:
(A1) $\alpha \in L_a^\infty(I)$ and $\beta \in L_a^\infty(0,T;L^2(I))$.
(A2) $h \in L_a^\infty(0,T;L^2(I))$.

We recall the properties of the fractional Laplacian and fractional Sobolev spaces.

Lemma 3.1. *([20]) Let $\gamma \in (0,1)$, then for any $z \in B_0$,*

$$\langle (-\Delta)^\gamma z, z\rangle_{B_0^*,B_0} = \frac{1}{2}C(n,\gamma)\|z\|_{B_0}^2. \tag{3.2}$$

Lemma 3.2. *([20]) Let $\gamma \in (0,1)$ and $2\gamma < d$. If $p \in [2, 2_\gamma^*]$, then*

$$B_0 \hookrightarrow L^p \hookrightarrow L^2.$$

Thus, there exists $C \in (0,+\infty)$ such that

$$\|z\|_{L^2} \le C\|z\|_{B_0}. \tag{3.3}$$

Moreover, the embedding $B_0 \hookrightarrow\hookrightarrow L^p$ is compact when $p \in [2, 2_\gamma^*)$.

Next, we present the existence and uniqueness theorem of weak solutions for problem (1.1).

Theorem 3.1. *Let $\gamma \in (0,1]$, $2\gamma < n$ and $0 < \theta < \frac{2\gamma}{n}$. Provided that (A1) and (A2) are satisfied, then for any $\phi \in L_a^2(I)$, there exists a unique weak solution to the initial-boundary value problem (1.1).*

Proof. The proof strategy is to first establish the boundedness of the m-dimensional approximate solutions for problem (1.1), then prove the existence of weak solutions for problem (1.1), and finally demonstrate the uniqueness of the weak solutions.

Step 1. The m-dimensional approximate solutions for problem (1.1).

Let $m \in \mathbb{N}_+$ and set

$$z_m(t) \triangleq \sum_{i=1}^m z(t)e_i, \quad \phi_m \triangleq \sum_{i=1}^m \phi e_i, \quad \beta_m(t) \triangleq \sum_{i=1}^m (\beta(t), e_i)_{L^2} e_i,$$

$$h_m(t) \triangleq \sum_{i=1}^m (h(t), e_i)_{L^2} e_i, \quad W_m(t) \triangleq \sum_{i=1}^m (W(t), \iota_i)_{\mathbb{H}} \iota_i.$$

Assume that z_m satisfies the following equations:

$$\begin{cases} \frac{\partial z_m(t,x)}{\partial t} + (-\Delta)^\gamma z_m(t,x) = \alpha(x)(z_m(t,x) - |z_m(t,x)|^\theta z_m(t,x)) \\ \qquad +\beta_m(t,x) + h_m(t,x)\frac{\partial W_m(t,x)}{\partial t}, \quad (t,x) \in I_T, \\ z_m(t,x) = 0, \quad (t,x) \in Q_T, \\ z_m(0,x) = \phi_m(x), x \in I. \end{cases} \tag{3.4}$$

That is, z_m is the approximate solution of problem (1.1).

Applying integration by parts to both sides of (3.4), we yield

$$\begin{aligned} &\frac{1}{2}(z_m(t), z_m(t))_{L_a^2} + \int_0^t < (-\Delta)^\gamma z_m(\tau), z_m(\tau) >_{B_0^*,B_0} d\tau \\ &+ \int_0^t \Big(\alpha(x)|z_m|^\theta z_m, z_m\Big) d\tau = \frac{1}{2}(\phi_m, \phi_m)_{L_a^2} + \int_0^t (\alpha(x) z_m, z_m) d\tau \\ &+ \int_0^t (\beta_m(\tau), z_m(\tau))_{L^2} d\tau + \int_0^t (z_m(\tau), h_m(\tau) dW_m(\tau)). \end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{2}\|z_m\|_{L_a^2}^2+\int_0^t((-\Delta)^\gamma z_m,z_m)_{B_0^*,B_0}d\tau+\int_0^t\left(\alpha(x)|z_m|^\theta z_m,z_m\right)d\tau\\=&\frac{1}{2}\|\phi_m\|_{L_a^2}^2+\int_0^t(\alpha(x)z_m,z_m)d\tau+\int_0^t(\beta_m,z_m)_{L_a^2}d\tau+\int_0^t(h_m^*z_m,dW_m).\end{aligned}$$

Utilizing (A1), (A2), (3.2), (3.3), and Young's inequality, and taking expectations, we arrive at

$$\begin{aligned}&\mathbb{E}\|z_m(t)\|_{L_a^2}^2+C_1\int_0^t\mathbb{E}\|z_m(\tau)\|_{B_0}^2d\tau+C_2\int_0^t\|z_m\|_{L^{\theta+2}(I)}^{\theta+2}d\tau\\\le&\mathbb{E}\|\phi_m\|_{L_a^2}^2+C_3M_0^2+C_4\int_0^t\mathbb{E}\|\beta_m(\tau)\|_{L_a^2}^2d\tau.\end{aligned}\tag{3.5}$$

In view of $\phi\in L_a^2(S,L^2)$ and $\beta\in L_a^2([0,T],L^2)$, it follows from Gronwall's inequality and (3.5) that

$$\|z_m\|_{L_a^2(0,T;B_0)}+\|z_m\|_{L^{\theta+2}(I_T)}\le C,$$

where $C=C(\|\phi\|_{L_a^2(S,L^2)},\ \|\beta\|_{L_a^2([0,T],L^2)},\ T)$. Using (3.5) once more yields $\{z_m\}$ is bounded in $L_a^\infty([0,T],L^2)\cap L_a^2([0,T],B_0)\cap L_2^{\theta+2}([0,T],L^{\theta+2})$.

Since the embedding of $B_0(I)\hookrightarrow\hookrightarrow L^2(I)$ is compact, there exists a subsequence of $\{z_m\}$ (still denoted by $\{z_m\}$) and an element z such that

$$\begin{aligned}&z_m\to z,m\to\infty,\ \text{a.e. in } I_T\\&z_m\to z,m\to\infty,\ \text{weakly in}\ \ L_a^2([0,T],B_0),\\&z_m\to z,m\to\infty,\ \text{weakly in}\ \ L_a^{\theta+2}\left([0,T],L^{\theta+2}\right),\\&z_m\to z,m\to\infty,\ \text{weakly}*\ \text{in}\ \ L_a^\infty([0,T],L^2).\end{aligned}\tag{3.6}$$

Step 2. The existence of weak solutions for (1.1).

We claim that z is a weak solution to problem (1.1). To this end, it suffices to show that z satisfies the requirements of Definition 3.1. On one hand, It follows from (3.6) that

$$z\in L_a^2(0,T;B_0(I))\cap L_a^{\theta+2}(I_T)\cap L_a^\infty(0,T;L^2(I)),$$

$$z(0,x)=\phi(x),\ \ \mathbb{P}-\text{a.s.}.$$

For any $\xi\in C_0^1([0,T])$ and $\eta\in L^\infty(S)$, it follows from (3.4) that

$$\begin{aligned}0=&\mathbb{E}\{\eta(\phi_m,\xi(0)e_i)\}+\mathbb{E}\left\{\eta\int_0^T(z_m(t),\dot{\xi}e_i)_{L^2}dt\right\}\\&-\mathbb{E}\left\{\eta\int_0^T<(-\Delta)^\gamma z_m(t),e_i>_{B_0^*,B_0}\xi(t)dt\right\}\\&-\mathbb{E}\left\{\eta\int_0^T\left(\alpha(x)|z_m|^\theta z_m,\xi(t)e_i\right)dt\right\}+\mathbb{E}\left\{\eta\int_0^T(\alpha(x)z_m,\xi(t)e_i)dt\right\}\\&+\mathbb{E}\left\{\eta\int_0^T(\beta_m(t),\xi(t)e_i)_{L^2}dt\right\}+\mathbb{E}\left\{\eta\int_0^T(\xi(t)e_i,h_m(t)dW_m(t))\right\}\\\triangleq&E_1+E_2-E_3-E_4+E_5+E_6+E_7,\end{aligned}$$

where $i=1,2,\cdots,m$. Since ξe_i is a deterministic function, we apply (3.6)and the Dominated Convergence Theorem to establish the assertion.

Starting with the term $E_1=\mathbb{E}\{\eta(\phi_m,\xi(0)\bar{e}_i)\}$, and taking into account that $z_m(0)$ is the m-dimensional truncation of ϕ, we arrive at

$$\phi_m\to\phi,m\to\infty,\ \ \text{in}\ \ L_a^2(S,L^2).$$

 doi:10.20944/preprints202603.1541.v1

Furthermore, since there exists a constant $c_1 > 0$, depending on $\|\eta\|_{L^\infty(S)}$ and $\xi(0)$,

$$\mathbb{E}\{\eta\phi_m, \xi(0)e_i)\} \le c_1\|\phi\|^2_{L^2_a(S,L^2)}$$

we obtain the following convergence as $m \to \infty$

$$\mathbb{E}\{\eta(\phi_m, \xi(0)e_i)\} \longmapsto \mathbb{E}\{(\eta\phi, \xi(0)e_i)\}.$$

As for $E_2 = \mathbb{E}\Big\{\eta \int_0^T (z_m(t), \dot{\xi}e_i)_{L^2} dt\Big\}$. Hölder inequality yields the bound

$$\begin{aligned}
&\mathbb{E}\Big\{\eta \int_0^T (z_m(t), \dot{\xi}e_i)_{L^2} dt\Big\} \\
\le&\|\eta\|_{L^\infty(S)} \int_0^T (\mathbb{E}\|z_m(t)\|_{L^2})^{\frac{1}{2}} (\mathbb{E}\|\dot{\xi}(t)e_i\|_{L^2})^{\frac{1}{2}} dt \\
\le&c_2\|z_m\|_{L^\infty_a([0,T],L^2)}
\end{aligned}$$

with $c_2 > 0$, depending on $\|\eta\|_{L^\infty(S)}$ and $\|\dot{\xi}\|_{C([0,T])}$. Consequently, by invoking (3.6) and the Dominated Convergence Theorem, we arrive at

$$\mathbb{E}\Big\{\eta \int_0^T (z_m(t), \dot{\xi}e_i)_{L^2} dt\Big\} \longrightarrow \mathbb{E}\Big\{\eta \int_0^T (z(t), \dot{\xi}e_i)_{L^2} dt\Big\},\ m \to \infty.$$

Next, we turn to $E_3 = \mathbb{E}\Big\{\eta \int_0^T < (-\Delta)^\gamma z_m(t), e_i >_{B_0^*, B_0} \xi(t) dt\Big\}$. In view of the facts that $p \in [1, 2^*_\gamma/2)$ and $\xi\bar{e}_i \in L^2([0,T], B_0)$, applying Lemma 3.2 and (3.6) yields the following result as $m \to \infty$,

$$\begin{aligned}
&\int_0^T < (-\Delta)^\gamma z_m(t), e_i >_{B_0^*, B_0} \xi(t) dt \\
=&\frac{C(n,\gamma)}{2} \int_0^T \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(z^i_m(t,x) - z^i_m(t,y))(\xi(t)e_i(x) - \xi(t)e_i(y))}{|x-y|^{n+2\gamma}} dxdydt \\
\to&\frac{C(n,\gamma)}{2} \int_0^T \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(z^i(t,x) - z^i(t,y))(\xi(t)e_i(x) - \xi(t)e_i(y))}{|x-y|^{n+2\gamma}} dxdydt.
\end{aligned}$$

Hence, the term is bounded by

$$\mathbb{E}\Big\{\eta \int_0^T < (-\Delta)^\gamma z_m(t), e_i >_{B_0^*, B_0} \phi(t) dt\Big\} \le c_3\|\eta\|_{L^\infty(S)}.$$

for some $c_3 > 0$. From this estimate, it follows immediately that

$$\begin{aligned}
&\mathbb{E}\Big\{\eta \int_0^T < (-\Delta)^\gamma z_m(t), e_i >_{B_0^*, B_0} \xi(t) dt\Big\} \\
\longrightarrow&\mathbb{E}\Big\{\eta \int_0^T < (-\Delta)^\gamma z(t), e_i >_{B_0^*, B_0} \xi(t) dt\Big\},\ m \to \infty.
\end{aligned}$$

In view of $E_4 = \mathbb{E}\Big\{\eta \int_0^T \Big(\alpha(x)|z_m|^\theta z_m, \xi(t)e_i\Big) dt\Big\}$ and $E_5 = \mathbb{E}\Big\{\eta \int_0^T (\alpha(x) z_m, \xi(t)e_i) dt\Big\}$, the application of assumption (A1) and (3.6) ensure the convergence

$$\mathbb{E}\Big\{\eta \int_0^T \Big(\alpha(x)|z_m|^\theta z_m, \xi(t)e_i\Big) dt\Big\} \longrightarrow \mathbb{E}\Big\{\eta \int_0^T \Big(\alpha(x)|z|^\theta z, \xi(t)e_i\Big) dt\Big\},\ m \to \infty.$$

and

$$\mathbb{E}\Big\{\eta \int_0^T (\alpha(x) z_m, \xi(t)e_i) dt\Big\} \longrightarrow \mathbb{E}\Big\{\eta \int_0^T (\alpha(x) z, \xi(t)e_i) dt\Big\},\ m \to \infty.$$

 doi:10.20944/preprints202603.1541.v1

Next, we consider the term $E_6 = \mathbb{E}\Big\{\eta\int_0^T(\beta_m(t),\xi(t)e_i)_{L^2}dt\Big\}$. Since β_m is the m-dimensional truncation of β, it follows that

$$\beta_m \to \beta \text{ in } L^2_a([0,T],L^2),\ m\to\infty.$$

Consequently, an application of the Dominated Convergence Theorem yields

$$\mathbb{E}\Big\{\eta\int_0^T(\beta_m(t),\xi(t)e_i)_{L^2}dt\Big\}\longrightarrow\mathbb{E}\Big\{\eta\int_0^T(\beta(t),\xi(t)e_i)_{L^2}dt\Big\},\ m\to\infty.$$

Regarding $E_7 = \mathbb{E}\Big\{\eta\int_0^T(\xi(t)e_i,h_m(t)dW_m(t))\Big\}$, the definition of W_m as the m-dimensional truncation of W implies that

$$W_m \to W,\ m\to\infty, \text{ in } \mathbb{H}.$$

By virtue of assumption (A2), we deduce the convergence of the stochastic integral as follows:

$$\mathbb{E}\Big\{\eta\int_0^T(\xi(t)h_m^*(t)e_i,dW_m(t))\Big\}\longrightarrow\mathbb{E}\Big\{\eta\int_0^T(\xi(t)h^*(t)e_i,dW(t))\Big\}.$$

as $m\to\infty$.

In summary, the fact that $\{e_i\}$ is a basis of B_0 ensures that

$$\begin{aligned}
&-\mathbb{E}\{\eta(\phi,\xi(0)\mu)\}-\mathbb{E}\Big\{\eta\int_0^T(z(t),\dot{\xi}\mu)_{L^2}dt\Big\}\\
&+\mathbb{E}\Big\{\eta\int_0^T<(-\Delta)^\gamma z(t),\mu>_{B_0^*,B_0}\xi(t)dt\Big\}\\
=&\mathbb{E}\Big\{\eta\int_0^T\Big(\alpha(x)|z|^\theta z,\xi(t)\mu\Big)dt\Big\}+\mathbb{E}\Big\{\eta\int_0^T(\alpha(x)z,\xi(t)\mu)dt\Big\}\\
&+\mathbb{E}\Big\{\eta\int_0^T(\beta(t),\xi(t)\mu)_{L^2}dt\Big\}+\mathbb{E}\Big\{\eta\int_0^T(\xi(t)h^*(t)\mu,dW(t))\Big\}
\end{aligned}$$

for any $\mu\in B_0$, $\xi\in C_0^1([0,T])$ and $\eta\in L^\infty(S)$. Consequently, for any $\eth\in C^1(0,T;B_0(I))$ with $\eth(T)=0$, z satisfies

$$\begin{aligned}
&-\langle\phi,\eth(t)\rangle_{L^2}-\int_0^T\Big\langle z(t),\frac{\mathrm{d}\eth(t)}{\mathrm{d}t}\Big\rangle_{L^2}\mathrm{d}t+\int_0^T((-\Delta)^\gamma z(t),\eth(t))_{B_0^*,B_0}\mathrm{d}t\\
=&\int_0^T\Big(\alpha(z(t)-|z(t)|^\theta z(t)),\eth(t)\Big)_{(L_a^{\theta+2})^*,L_a^{\theta+2}}\mathrm{d}t+\int_0^T\langle\beta(t),\eth(t)\rangle_{L^2}\mathrm{d}t\\
&+\int_0^T\langle\eth(t),h(t)\mathrm{d}W(t)\rangle_{L^2},\quad \mathbb{P}-\text{a.s.}.
\end{aligned}$$

Therefore, we have demonstrated that z is indeed a weak solution to problem (1.1).

Step 3. The uniqueness of weak solutions for (1.1).

Let $\hat{z}$ and $\check{z}$ be two weak solutions to problem (1.1) in the spaces $L_a^\infty([0,T],L^2)\cap L_a^2([0,T],B_0)\cap L_a^{\theta+2}([0,T],L^{\theta+2})$, corresponding to the initial states $\hat{\phi}$, $\check{\phi}$, and $\hat{\beta},\check{\beta}$ respectively. Since $\hat{z}$ and $\check{z}$ satisfy problem (1.1) in the weak sense with probability one, an application of the integration-by-parts formula leads to

$$\begin{aligned}
&\frac{1}{2}\|\hat{z}-\check{z}\|_{L^2}^2+\int_0^t((-\Delta)^\gamma(\hat{z}-\check{z}),\hat{z}-\check{z})_{B_0^*,B_0}d\tau+\int_0^t\Big(\alpha(x)|\hat{z}|^\theta\hat{z}-\alpha(x)|\check{z}|^\theta\check{z},\hat{z}-\check{z}\Big)d\tau\\
=&\frac{1}{2}\|\hat{\phi}-\check{\phi}\|_{L^2}^2+\int_0^t(\alpha(x)(\hat{z}-\check{z}),\hat{z}-\check{z})d\tau+\int_0^t(\hat{\beta}-\check{\beta},\hat{z}-\check{z})_{L^2}d\tau+\int_0^t(h^*\hat{z}-h^*\check{z},dW).
\end{aligned}$$

By invoking Simon's inequality, it is shown that

$$\begin{aligned}\left(\alpha(x)|\hat{z}|^{\theta}\hat{z}-\alpha(x)|\check{z}|^{\theta}\check{z},\hat{z}-\check{z}\right)&=\int_I\alpha(x)\left(|\hat{z}|^{\theta}\hat{z}-|\check{z}|^{\theta}\check{z}\right)(\hat{z}-\check{z})dx\\&\geq C_1\int_I|\hat{z}-\check{z}|^{\theta+2}dx\\&\geq 0\end{aligned}$$

Furthermore, based on assumptions (A1) and (A2) and estimates (3.2), (3.3), Young's inequality implies

$$\begin{aligned}&\|\hat{z}-\check{z}\|_{L^2}^2+C(n,\gamma)\int_0^t\|\hat{z}-\check{z}\|_{B_0}^2d\tau\\\leq&\|\hat{\phi}-\check{\phi}\|_{L^2}^2+c\int_0^t\|\hat{\beta}-\check{\beta}\|_{L^2}^2d\tau+\int_0^t(h^*\hat{z}-h^*\check{z},dW)\end{aligned}$$

Taking the expectation, we arrive at the following estimate:

$$\begin{aligned}&\mathbb{E}\|\hat{z}-\check{z}\|_{L^2}^2+C(n,\gamma)\mathbb{E}\int_0^t\|\hat{z}-\check{z}\|_{B_0}^2d\tau\\\leq&\mathbb{E}\|\hat{\phi}-\check{\phi}\|_{L^2}^2+c\mathbb{E}\int_0^t\|\hat{\beta}-\check{\beta}\|_{L^2}^2d\tau\end{aligned}$$

where the stochastic integral term vanishes in expectation, i.e., $\mathbb{E}\int_0^t(h^*\hat{z}-h^*\check{z},dW)=0$. By setting the initial data and parameters such that $\hat{\phi}=\check{\phi},\hat{\beta}=\check{\beta}$, it follows immediately that $\hat{z}=\check{z}$ almost surely. □

4. Conclusions

This paper investigates the well-posedness of the stochastic fractional Ginzburg-Landau equation. First, we introduced a framework for weak solutions within adapted solution spaces. The existence of these solutions was established using a Galerkin-type approximation and the Dominated Convergence Theorem, supported by detailed energy estimates and compactness arguments. Furthermore, the uniqueness of the weak solution was confirmed through the application of Simon's and Young's inequalities, ensuring the stability of the system.

The results presented in this work contribute to the theoretical understanding of nonlocal stochastic partial differential equations (SPDEs). Future work could extend these results to investigate the long-term asymptotic behavior, such as the existence of random attractors, or explore optimal control problems for this fractional stochastic system.

Author Contributions: Conceptualization, J. X. Li and L. X. Yan; methodology,J. X. Li and L. X. Yan; software, D. G. Ma, X. Y. Li and X. J. Zhang ; validation, J. X. Li, D. G. Ma, X. Y. Li, X. J. Zhang and L. X. Yan; investigation, J. X. Li, D. G. Ma, X. Y. Li and X. J. Zhang; writing—original draft preparation, J. X. Li; writing—review and editing, J. X. Li and L. X. Yan; funding acquisition, J. X. Li and L. X. Yan. All authors have read and agreed to the published version of the manuscript.

Funding: The authors are supported by the Northeast Forestry University College Student Innovation and Entrepreneurship Training Program Project: S202510225077.

Institutional Review Board Statement: Not applicable.

Data Availability Statement: Dataset available on request from the authors.

Acknowledgments: The authors wish to express their sincere gratitude to the editors and anonymous reviewers for their valuable comments and suggestions.

Conflicts of Interest: The authors declare no conflicts of interest.

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