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Article

A Note on Kadec-Klee Property

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Abstract

The objective of this paper is to rigorously define the Kadec-Klee property for modular spaces endowed with a sequential convergence structure, and to demonstrate that this property leads to the normal structure in such spaces. Consequently, we establish that the Kadec-Klee property defined herein implies the corresponding fixed point property for these spaces. These results are new in the modular space setting. Furthermore, given that the examined class of spaces encompasses Banach spaces, modular function spaces, and various other types, our theory offers a comprehensive, unified framework for exploring the interconnections between the Kadec-Klee property, normal structure, and the fixed point property.

Keywords: Kadec-Klee property; normal structure; fixed point property; opial property; banach space; modular space; modulated *LTI*-space; sequential convergence space

MSC: 47H09; 47H10; 46A80; 46E30

1. Introduction

In the Banach space setting, a norm is called weak uniformly Kadec-Klee if for every $\varepsilon > 0$ there exists a $0 < \eta < 1$ such that for every sequence $\{x_n\}$ in the unit ball, weakly converging to x , we have $\|x\| \leq 1 - \eta$ provided that

$$\text{sep}\{x_k\} := \inf\{\rho(x_n - x_m) : n \neq m\} > r\varepsilon.$$

Since the early 1980s, it has been established that the weak uniform Kadec-Klee property is linked to the fixed point property through the normal structure property [1–4]. Since the Kadec-Klee property is fundamentally sequential rather than topological, it naturally prompted an extension of the definition to include a variety of sequence convergence types, extending beyond merely weak or weak*-topologies. For examples, refer to [5–12].

Simultaneously, the pioneering work in [13] launched a thriving field of fixed point theory within modular function spaces (see, for example, [10], the related literature, and numerous subsequent results). This area focuses on spaces of measurable functions where norms are replaced by the more general concept of modulars. Within this framework, the concept of ρ almost everywhere convergence—often equivalent to almost everywhere convergence with respect to a measure—plays a crucial role. It then became a natural question to explore whether certain forms of the uniform Kadec-Klee property exist in relation to these types of convergence, as well as, to investigate potential relationships with fixed point theory in modular function spaces. Indeed, some results of this type can be found in the literature, see e.g., [[10] Theorems 4.9, 4.10]. They are, however very modular function space specific, and are not directly comparable with the Kadec-Klee theory in general norm spaces.

Although the diverse Kadec-Klee properties designed for various types of sequence convergence, considered in either norm or modular settings, can be advantageous for applications, they have led to considerable fragmentation in this field of research. To address this issue, this paper seeks to establish a unified framework for examining the Kadec-Klee property across a broad class of spaces, including Banach and modular function spaces. This framework will be grounded in a clearly defined

sequence convergence that encompasses a wide array of scenarios, such as weak convergence and almost everywhere convergence.

For our investigation, we adopt the framework of modulated (LTI)-spaces (redefined in the paper as super-regular spaces, see Definition 10), which are defined as modular spaces endowed with a sequential convergence structure, as introduced by the author in [14,15], and based on the framework of L -spaces, proposed by Kisyński in [16] (see also [17]), building upon earlier concepts by Fréchet [18] and Urysohn [19]. This choice of setting facilitates the use of convergence types that are not inherently linked to a topology, with convergence almost everywhere serving as a significant example.

The primary inquiries revolve around whether, within this broad framework, the Kadec-Klee property necessitates the presence of normal structure and, by extension, the associated fixed point property, hence extending known for normed spaces results to modular spaces equipped with the sequential convergence. Our Theorems 1 and 3 affirmatively address these questions, underscoring the novelty of our results.

Theorem 5 addresses a particular case of modular function spaces, showing that a variant of the Kadec-Klee condition applies to a broad range of spaces, including L^p , $L^{p(t)}$, Orlicz, Musielak-Orlicz, and Orlicz-Sobolev spaces. In Theorem 6, we establish a close connection between this variant of the Kadec-Klee property and the strong Opial property, which is widely used in the theory of modular function spaces. We also pose open questions regarding the conditions under which this version of the Kadec-Klee property implies normal structure and the associated fixed point property.

2. Preliminaries

Throughout this paper, X will always denote a real vector space. Recall the following concepts of the general modular space theory.

Definition 1 ([20]). A functional $\rho : X \rightarrow [0, \infty]$ is called a convex modular if

1. $\rho(x) = 0$ if and only if $x = 0$
2. $\rho(-x) = \rho(x)$
3. $\rho(\alpha x + \beta y) \leq \alpha \rho(x) + \beta \rho(y)$ for any $x, y \in X$, and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$

The vector space $X_\rho = \{x \in X : \rho(\lambda x) \rightarrow 0, \text{ as } \lambda \rightarrow 0\}$ is called a modular space.

The concepts outlined in Definitions 2 and 9 are well known in the theory of modular spaces., see, for example, [10,21]).

Definition 2. Let ρ be a modular defined on a vector space X .

1. We say that $\{x_n\}$, a sequence of elements of X_ρ , is ρ -convergent to x , and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \rightarrow 0$.
2. A sequence $\{x_n\}$ where $x_n \in X_\rho$ is called ρ -Cauchy if $\rho(x_n - x_m) \rightarrow 0$ as $n, m \rightarrow \infty$.
3. X_ρ is called ρ -complete if every ρ -Cauchy is ρ -convergent to an $x \in X_\rho$.
4. A set $B \subset X_\rho$ is called ρ -closed if for any sequence of $x_n \in B$, the convergence $x_n \xrightarrow{\rho} x$ implies that x belongs to B .
5. A set $B \subset X_\rho$ is called ρ -bounded if its ρ -diameter $\text{diam}_\rho(B) = \sup\{\rho(x - y) : x, y \in B\}$ is finite.
6. A set $K \subset X_\rho$ is called ρ -compact if for any sequence $\{x_n\}$ in K , there exists a subsequence $\{x_{n_k}\}$ and an $x \in K$ such that $\rho(x_{n_k} - x) \rightarrow 0$.
7. A sequence $\{y_n\}$ of elements of X_ρ is called strongly bounded if there exists a $\beta > 1$ such that $\sup\{\rho(\beta y_n) : n \in \mathbb{N}\} < \infty$.
8. A ρ -ball $B_\rho(x, r)$ is defined by $B_\rho(x, r) = \{y \in X_\rho : \rho(x - y) \leq r\}$.

Let us recall the standard definitions of ρ -Lipschitzian mappings, ρ -contractions, and ρ -nonexpansive mappings in the framework of modular spaces.

Definition 3. Let X_ρ be a modular space and let $C \subset X_\rho$ be convex, nonempty, ρ -closed and ρ -bounded. A mapping $T : C \rightarrow C$ is called

(i) ρ -Lipschitzian if there exists $\alpha > 0$ such that

$$\rho(T(x) - T(y)) \leq \alpha \rho(x - y) \text{ for any } x, y \in C.$$

(ii) a ρ -contraction if it is ρ -Lipschitzian with $\alpha < 1$.

(iii) ρ -nonexpansive if it is ρ -Lipschitzian with $\alpha = 1$.

An element $x \in C$ is called a fixed point of T whenever $T(x) = x$. The set of fixed points of T will be denoted by $F(T)$.

Let us revisit some fundamental concepts related to sequential convergence and modulated convergence spaces as discussed in [14,15].

Definition 4. Let X be any nonempty set. A relation ζ between sequences $\{x_n\}_{n=1}^\infty$ of elements of X and elements x of X , denoted by $x_n \xrightarrow{\zeta} x$, is called a sequential convergence on X if

1. if $x_n = x$ for all $n \in \mathbb{N}$ then $x_n \xrightarrow{\zeta} x$,
2. if $x_n \xrightarrow{\zeta} x$ and $\{x_{n_k}\}$ is a proper subsequence of $\{x_n\}$, then $x_{n_k} \xrightarrow{\zeta} x$.

The pair (X, ζ) (or shortly X) is called a convergence space.

Given a sequential convergence ζ on X we can introduce notions of closed and sequentially compact sets.

Definition 5. Let (X, ζ) be a convergence space. A set $K \subset X$ is called closed if whenever $x_n \in K$ all $n \in \mathbb{N}$ and $x_n \xrightarrow{\zeta} x$, then $x \in K$. Similarly, K is called sequentially compact if from every sequence $\{x_n\}$ of elements of K we can choose a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \xrightarrow{\zeta} x$ for an $x \in K$.

Definition 6. A sequential convergence ζ is called an L -convergence on X if

1. if $x_n \xrightarrow{\zeta} x$ and $x_n \xrightarrow{\zeta} y$, then $x = y$.

The pair (X, ζ) (or shortly X) is called an L -space.

Let us define LTI -convergence, LTI -spaces and modulated LTI -spaces.

Definition 7. Let X be a real vector space and let ζ be an L -convergence on X . We say that ζ is an LTI -convergence (translation invariant convergence) if $x_n \xrightarrow{\zeta} x$ implies that $x_n - y \xrightarrow{\zeta} x - y$ for any $y \in X$. In this case, the pair (X, ζ) is called an LTI -space.

Definition 8. Let ρ be a modular defined on X and let ζ be an L -convergence on X_ρ . The triplet (X_ρ, ρ, ζ) is called a modulated LTI -space if (X_ρ, ζ) is an LTI -space and the following two conditions are satisfied

1. $x_n \xrightarrow{\zeta} x \Rightarrow \rho(x) \leq \liminf_{n \rightarrow \infty} \rho(x_n)$,
2. if $x_n \xrightarrow{\rho} x$ then there exists a sub-sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \xrightarrow{\zeta} x$, where $x, x_n \in X$.

The statements in the following Proposition are straightforward consequences of the relevant definitions.

Proposition 1. Let (X_ρ, ρ, ζ) be a modulated LTI -space. Then the following assertions are true.

1. Every ζ -closed set is also ρ -closed.
2. Every ρ -compact set is also sequentially ζ -compact.

3. Every ρ -ball $B_\rho(x, r)$ is ζ -closed (and hence also ρ -closed).
4. Every sequentially ζ -compact set is ζ -closed.
5. Every ζ -closed subset of a sequentially ζ -compact set is sequentially ζ -compact.

The following definition is frequently used in the literature.

Definition 9 ([21] Definition 2.4). A modular space X_ρ is called a regular modular space if ρ is a convex, X_ρ is ρ -complete, and every ρ -ball $B_\rho(x, r)$ is ρ -closed, where $x \in X_\rho$, and $r > 0$.

In view of item 3 of Proposition 1, every ρ -complete modulated LTI-space is regular. For the sake of simplicity let us introduce the following definition.

Definition 10. By a super-regular modular space (X_ρ, ζ) we will understand a ρ -complete modulated LTI-space (X_ρ, ρ, ζ) .

Example 1. Typical examples of super-regular modular spaces include:

- (a) Banach spaces where ρ is a norm and ζ represents convergence in the weak topology;
- (b) Modular function spaces with the Fatou property and ζ being ρ -almost everywhere convergence;
- (c) Lebesgue spaces, variable exponent Lebesgue spaces, Orlicz spaces, Musielak-Orlicz spaces with ζ corresponding to almost everywhere convergence with respect to a measure;
- (d) Orlicz-Sobolev spaces with ζ corresponding to almost everywhere convergence with respect to a measure of all involved generalized derivatives.

The following example warrants special attention, as it illustrates that the class of super-regular modular spaces includes certain classical modular spaces that are neither normed nor classified as modular function spaces. It revisits the concept of φ -variation, introduced in [22] as a generalization of the classical quadratic variation defined by Wiener over a century ago [23]. The ρ convergence of this example, known in the literature as the convergence in φ -variation, has found numerous applications.

Example 2 ([24] Example 2 [20,22]). Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a convex function such that $\varphi(t) = 0$ if and only if $t = 0$. Let X be a space of all real-valued functions defined in the interval $[a, b]$ and vanishing at $t = a$. Musielak and Orlicz defined in [22] a φ -variation of a function $x \in X$ as follows,

$$\rho(x) = \sup_{\Pi} \sum_{i=1}^{\infty} \varphi(|x(t_i) - x(t_{i-1})|),$$

where the supremum is taken over all partitions $\Pi : a = t_0 < t_1 < \dots < t_m = b$ of the interval $[a, b]$. It is easy to see that ρ is a convex modular on X and that the value of $\rho(x)$ may be infinite. Using results of [22] and [[20] 10.7], it is straightforward to demonstrate that (X_ρ, ζ) is a super-regular modular space, where ζ is the pointwise convergence over the interval $[a, b]$. The space X_ρ is not a modular function space because φ -variation is not monotone.

3. Kadec-Klee Property in Super-Regular Modular Spaces

In this section, (X_ρ, ζ) will always denote a super-regular modular space, as defined in the previous section. Let us begin by providing a precise definition of the Kadec-Klee property for such spaces.

Definition 11. We say that (X_ρ, ζ) possesses the uniform ζ -KK1 property if for every $\varepsilon > 0$ and every $r > 0$ there exists an $\eta_1 > 0$ such that for every sequence $\{x_n\}$ in $B_\rho(0, r)$, ζ -convergent to x , such that

$$\text{sep}\{x_k\} := \inf\{\rho(x_n - x_m) : n \neq m\} > r\varepsilon,$$

we have

$$\rho(x) \leq r(1 - \eta_1).$$

As typical for modular space settings, a single entity used in normed spaces splits up into two distinct yet interrelated entities. Therefore, we present the following variant of the Kadec-Klee property, which will be utilised later in this section.

Definition 12. We say that (X_ρ, ζ) possesses the uniform ζ -KK2 property if for every $\varepsilon > 0$ and every $r > 0$ there exists an $\eta_2 > 0$ such that for every strongly bounded sequence $\{x_n\}$ in $B_\rho(0, r)$, ζ -convergent to x , such that

$$\text{sep}_2\{x_k\} := \inf \left\{ \rho\left(\frac{x_n - x_m}{2}\right) : n \neq m \right\} > r\varepsilon$$

we have

$$\rho(x) \leq r(1 - \eta_2).$$

Our next definition extends the standard definition of normal structure.

Definition 13. We say that (X_ρ, ζ) possesses the ζ -normal structure if every nonempty, convex, ρ -bounded, ζ -sequentially compact set $C \subset X_\rho$, $\text{diam}_\rho(C) > 0$, has a ρ -nondiametral point $x_0 \in C$, that is,

$$r_\rho(x_0, C) := \sup\{\rho(x_0 - y) : y \in C\} < \text{diam}_\rho(C).$$

The following result illustrates the relationship between the ζ -KK1 property and the ζ -normal structure, serving as a fundamental component of the theory discussed in this note. The construction of the sequence $\{x_n\}$ in our proof generally follows the approach established for normed spaces in [25], albeit in a different context of modulated LTI -spaces.

Theorem 1. Assume that (X_ρ, ζ) possesses the uniform ζ -KK1 property. Then, (X_ρ, ζ) possesses the ζ -normal structure.

Proof. Assume to the contrary that (X_ρ, ζ) does not have the ζ -normal structure. Then, there exists a nonempty, convex, ρ -bounded, ζ -sequentially compact set $C \subset X_\rho$, $\text{diam}_\rho(C) > 0$, that is ρ -diametral, i.e.,

$$r_\rho(z, C) = \text{diam}_\rho(C) \quad (3.1)$$

for every $z \in C$. Fix any $x_1 \in C$. Because of (3.1) applied to $z = x_1$, we can find an element $x_2 \in C$ satisfying the inequality

$$\text{diam}_\rho(C) \left(1 - \frac{1}{2^2}\right) \leq \rho(x_1 - x_2).$$

Proceeding by induction, assume that all $x_1, \dots, x_n$ have been constructed. Since C is a convex set, it follows that $\sum_{j=1}^n \frac{1}{n} x_j \in C$. Arguing as above, we conclude that there exists an element $x_{n+1} \in C$ such that

$$\text{diam}_\rho(C) \left(1 - \frac{1}{(n+1)^2}\right) \leq \rho\left(x_{n+1} - \sum_{j=1}^n \frac{1}{n} x_j\right). \quad (3.2)$$

Fix $i \in \mathbb{N}$ and take any natural number $n > i$. Observe that

$$\begin{aligned}
 \rho\left(x_{n+1} - \sum_{j=1}^n \frac{1}{n} x_j\right) &\leq \sum_{j=1}^n \frac{1}{n} \rho(x_{n+1} - x_j) \\
 &= \frac{1}{n} \rho(x_{n+1} - x_i) + \sum_{j=1, j \neq i}^n \frac{1}{n} \rho(x_{n+1} - x_j) \\
 &\leq \frac{1}{n} \rho(x_{n+1} - x_i) + \sum_{j=1, j \neq i}^n \frac{1}{n} \text{diam}_\rho(C) \\
 &= \frac{1}{n} \rho(x_{n+1} - x_i) + \left(1 - \frac{1}{n}\right) \text{diam}_\rho(C),
 \end{aligned} \tag{3.3}$$

where in the first line we used convexity of the modular ρ . From (3.2) and (3.3) it follows that

$$\text{diam}_\rho(C) \left(1 - \frac{1}{(n+1)^2}\right) \leq \frac{1}{n} \rho(x_{n+1} - x_i) + \left(1 - \frac{1}{n}\right) \text{diam}_\rho(C)$$

and, consequently, that

$$\text{diam}_\rho(C) \left(1 - \frac{n}{(n+1)^2}\right) \leq \rho(x_{n+1} - x_i) \leq \text{diam}_\rho(C),$$

which implies that

$$\lim_{n \rightarrow \infty} \rho(x_{n+1} - x_i) = \text{diam}_\rho(C).$$

Thus, after passing to a subsequence, if necessary, we can assume that

$$\text{sep}\{x_n\} > \frac{1}{2} \text{diam}_\rho(C).$$

Since C is ζ -sequentially compact, it follows that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and an element $x \in C$ such that $x_{n_k} \xrightarrow{\zeta} x$ as $k \rightarrow \infty$. Clearly,

$$\text{sep}\{x_{n_k}\} > \frac{1}{2} \text{diam}_\rho(C)$$

for any $n_k > i$. Given any $y \in C$, we have

$$\rho(x_{n_k} - y) \leq \text{diam}_\rho(C).$$

It follows from the definition of the uniform ζ -KK1 property applied to $x - y$, $\{x_{n_k} - y\}$, and $r = \text{diam}_\rho(C) > 0$ that there exists an $0 < \eta_1 < 1$ such that

$$\rho(x - y) \leq \text{diam}_\rho(C)(1 - \eta).$$

Since $y \in C$ was arbitrarily chosen, we conclude that,

$$r_\rho(x, C) = \sup\{\rho(x - y) : y \in C\} \leq \text{diam}_\rho(C)(1 - \eta) < \text{diam}_\rho(C),$$

which contradicts (3.1), completing the proof. $\square$

In order to show the linkage between the uniform ζ -KK1 property and the ζ -fpp, we need to recall the following concepts.

Definition 14 ([27] Definition 4.2). Let C be a ρ -bounded subset of X_ρ such that $\text{diam}_\rho(C) > 0$. A class $\mathcal{A}$ of subsets of C is said to be ρ -normal if for each $A \in \mathcal{A}$, such that $\text{diam}_\rho(A) > 0$, we have $R_\rho(A) := \inf\{r_\rho(x, A) : x \in A\} < \text{diam}_\rho(A)$. We will say that $\mathcal{A}$ is countably compact if any decreasing sequence $\{A_n\}$ of nonempty elements of $\mathcal{A}$, has a nonempty intersection.

The following result, being a modular version of Kirk's fixed point [26], provides the essential step in the proof of Theorem 3.

Theorem 2 ([27] Theorem 4.4). Let C be a ρ -bounded subset of X_ρ . Let $\mathcal{A}$ be a class of subsets of C which is stable under arbitrary intersections and contains all sets of the form $C \cap B_\rho(x, p)$, where $x \in C$ and $p > 0$. In addition, let us assume that $\mathcal{A}$ is normal and countably compact. If $T : C \rightarrow C$ is ρ -nonexpansive, then T has a fixed point in C .

Theorem 3. Assume that (X_ρ, ζ) possesses the uniform ζ -KK1 property. Then, (X_ρ, ζ) possesses the ζ -fpp.

Proof. Let $C \subset X_\rho$ be a nonempty, convex, ρ -bounded, ζ -sequentially compact set. Assume that $T : C \rightarrow C$ is ρ -nonexpansive. Define $\mathcal{A}$ as the class of all convex, ρ -bounded and ζ -sequentially compact subsets of C . Clearly, $\mathcal{A}$ is stable under arbitrary intersections and contains all sets of the form $C \cap B_\rho(x, p)$, where $x \in C$ and $p > 0$ (recall that ρ -balls $B_\rho(x, p)$ are convex and ζ -closed). It follows from Theorem 1 that (X_ρ, ζ) possesses the ζ -normal structure. Hence, every nonempty, convex, ρ -bounded, ζ -sequentially compact set $A \in \mathcal{A}$, such that $\text{diam}_\rho(A) > 0$, has a ρ -nondiametral point $x_0 \in C$. We conclude then that $\mathcal{A}$ is normal. It follows from Theorem 2 that T has a fixed point in C , proving that (X_ρ, ζ) possesses the ζ -fpp. $\square$

It is evident that the uniform ζ -KK1 property is equivalent to the standard definition of the uniform Kadec-Klee property for a Banach space $(X, \|\cdot\|)$ provided that ρ represents the norm $\|\cdot\|$ and that ζ stands for convergence in the weak topology of X . Therefore, Theorems 1 and 3 generalise well known results that provide linkage between the Kadec-Klee property, the normal structure property, and the weak fixed point property in Banach spaces.

Before delving deeper into the modular aspects of our theory, we must revisit a few important facts from the theory of modular function spaces. The reader is referred to [10] for comprehensive background information.

Definition 15. A function modular ρ is said to be orthogonally additive if $\rho(f, 1_{A \cup B}) = \rho(f 1_A) + \rho(f 1_B)$ whenever $A \cap B = \emptyset$.

Note that many important function modulars, including Lebesgue, Orlicz, Musielak-Orlicz modulars, are orthogonally additive.

Definition 16 ([10] Definition 3.7). We say that ρ satisfies the Δ_2 -type condition if there exists a finite constant M_2 such that $\rho(2f) \leq M_2 \rho(f)$ for every $f \in X_\rho$.

The class of modular spaces satisfying the Δ_2 -type condition contains spaces such as Lebesgue variable exponent spaces $L^p(t)$ for $1 \leq p(t) < \infty$ and Orlicz spaces L^φ for φ satisfying the Δ_2 -condition.

Theorem 4 ([10] Theorem 4.7). Let ρ be an orthogonally additive function modular. Let $\{f_n\}$ be strongly bounded, ρ -convergent to zero sequence of elements of L_ρ . For any $g \in E_\rho$ it holds

$$\liminf_{n \rightarrow \infty} \rho(f_n + g) = \liminf_{n \rightarrow \infty} \rho(f_n) + \rho(g).$$

The next result discusses how Theorems 1 and 3 can be utilised when ρ is a function modular but is not necessarily a norm.

Theorem 5. Let L_ρ be a modular function space, where ρ is orthogonally additive. Let ζ denotes the ρ -a.e. convergence. If ρ satisfies the Δ_2 -type property with constant M_2 , then the modulated LTI-space (L_ρ, ζ) possesses the uniform ζ -KK1 property. Consequently, via Theorem 3, (L_ρ, ζ) possesses the ζ -fpp.

Proof. Fix any $\varepsilon' > 0$ and $r > 0$. Assume that $\{x_n\}$ in $B_\rho(0, r)$ and $x \in L_\rho$ are such that

$$x_n \rightarrow x \text{ } \rho\text{-a.e.}$$

and

$$\text{sep}\{x_n\} > r\varepsilon' \geq r\varepsilon,$$

where $\varepsilon \leq \min\{\varepsilon', M_2\}$. Set $f_n := x_n - x$ and note that $f_n \rightarrow 0$ ρ -a.e. Since ρ satisfies the Δ_2 -type property, the following holds

$$\begin{aligned} \sup\{\rho(2f_n) : n \in \mathbb{N}\} &\leq M_2 \sup\{\rho(x_n - x) : n \in \mathbb{N}\} \\ &\leq M_2 \sup\left\{\rho\left(2\frac{x_n}{2} + 2\frac{-x}{2}\right) : n \in \mathbb{N}\right\} \\ &\leq M_2 \sup\left\{\frac{1}{2}\rho(2x_n) + \frac{1}{2}\rho(2x) : n \in \mathbb{N}\right\} \\ &\leq M_2^2\left(\frac{r}{2} + \frac{1}{2}\rho(x)\right) \\ &\leq M_2^2\left(\frac{r}{2} + \frac{1}{2}\liminf_{n \rightarrow \infty} \rho(x_n)\right) \\ &\leq M_2^2 r < \infty. \end{aligned}$$

Note that $x \in E_\rho$, because $E_\rho = L_\rho$ whenever ρ satisfies the Δ_2 -type property. It follows then from [10, Theorem 4.7] that

$$\liminf_{n \rightarrow \infty} \rho(f_n + x) = \liminf_{n \rightarrow \infty} \rho(f_n) + \rho(x),$$

which, using our notation, translates to

$$\rho(x) = \liminf_{n \rightarrow \infty} \rho(x_n) - \liminf_{n \rightarrow \infty} \rho(x_n - x) \leq r - \liminf_{n \rightarrow \infty} \rho(x_n - x). \quad (3.4)$$

On the other hand, for any $k \neq m$,

$$r\varepsilon < \text{sep}\{x_n\} \leq \rho(x_k - x_m) \leq \frac{1}{2}M_2\rho(x_k - x) + \frac{1}{2}M_2\rho(x_m - x),$$

which implies that

$$\frac{r\varepsilon}{M_2} \leq \liminf_{n \rightarrow \infty} \rho(x_n - x). \quad (3.5)$$

By substituting (3.5) into (3.4) we conclude that

$$\rho(x) \leq r(1 - \eta_1),$$

where $\eta_1 = \frac{\varepsilon}{M_2} < 1$. Since η_1 is independent of the choice of the sequence $\{x_n\}$, we can conclude that the proof is complete. $\square$

Although there exists a substantial array of examples of modular spaces that meet the assumptions of Theorem 5 – including Lebesgue spaces L^p , Orlicz spaces L^φ where φ satisfies the Δ_2 condition, and variable exponent Lebesgue spaces $L^{p(\cdot)}$ with $1 \leq p(t) \leq M < \infty$ – it is crucial to also examine the scenario in which ρ lacks the Δ_2 -type property. Keeping this task in mind, let us now introduce the concept of the strong ζ -Opial property in super-regular modular spaces.

Definition 17. We say that the $mLTI$ -space (X_ρ, ζ) has the strong ζ -Opial property if

$$\rho(x_0) + \liminf_{n \rightarrow \infty} \rho(x_n - x_0) \leq \liminf_{n \rightarrow \infty} \rho(x_n), \quad (3.6)$$

provided $x_n \xrightarrow{\zeta} x_0$ and the sequence $\{x_n - x_0\}$ is strongly bounded.

Remark 1. Let us recall that, by the assumed Fatou property, it is always true that

$$\rho(x_0) \leq \liminf_{n \rightarrow \infty} \rho(x_n), \quad (3.7)$$

whenever $x_n \xrightarrow{\zeta} x_0$. In spaces exhibiting the strong ζ -Opial property, the inequality (3.6) offers a more robust control over the estimate 3.7.

Theorem 6. If (X_ρ, ζ) has the strong ζ -Opial property then (X_ρ, ζ) possesses the uniform ζ -KK2 property.

Proof. Let us fix arbitrarily $0 < \varepsilon < 1$ and $r > 0$. Let $\{x_n\}$ and x_0 be as in Definition 12. Take any two natural numbers $n \neq m$ and calculate

$$r\varepsilon < \rho\left(\frac{x_n - x_m}{2}\right) \leq \frac{1}{2}\rho(x_n - x_0) + \frac{1}{2}\rho(x_m - x_0),$$

which implies that

$$\liminf_{n \rightarrow \infty} \rho(x_n - x_0) \geq r\varepsilon. \quad (3.8)$$

By combining (3.8) with (3.6), we conclude that

$$\rho(x_0) \leq \liminf_{n \rightarrow \infty} \rho(x_n) - r\varepsilon \leq r(1 - \varepsilon),$$

which completes the proof. $\square$

Observe that Theorem 4 ensures the existence of numerous examples of super-regular modular spaces exhibiting the strong ζ -Opial property.

4. Discussion

Theorem 1 represents the principal result of this paper. It establishes that the ζ -KK1 property for super-regular modular spaces ensures the existence of modular normal structure. This finding is further utilized in Theorem 3 to demonstrate that the ζ -KK1 property implies the ζ -fpp. These novel results extend similar discoveries made in the context of Banach spaces to a broader class of super-regular modular spaces. Additionally, we examined the practicality of our results in super-regular modular spaces that do not fit into the categories of Banach or modular function spaces. Notably, in Example 2, we focused on super-regular modular spaces defined by φ -variations, which arise from Wiener's quadratic variation.

In our investigation, we utilized the framework of modulated (LTI)-spaces, which are modular spaces equipped with a sequential convergence structure, as introduced by the author in [14,15], and built upon the concepts of L -spaces proposed by Kisy'nski in [16]. This choice of framework enabled us to incorporate various types of convergence, including weak convergence in Banach spaces, ρ -almost everywhere convergence in modular function spaces, and other convergence types that are not necessarily tied to a topology, with convergence almost everywhere being a notable example.

We would like to emphasize that super-regular modular spaces extend the concept of regular modular spaces (where ρ -balls are ρ -closed), as introduced by the author in [21], through the introduction of ζ -convergence as a counterpart to weak or almost everywhere convergence. In the Discussion section of [24], the author emphasized the urgent need to enrich the regular framework with such a

construct. It is the author's belief that the framework of super-regular spaces introduced in this paper meets this need and will therefore play a significant role in the further development of fixed point theory in modular spaces.

We would like to finish this section with the following open questions:

- (Q1) What is the relationship between ζ -KK1 and ζ -KK2 properties?
- (Q2) Does ζ -KK2 imply some form of normal structure?
- (Q3) Does ζ -KK2 directly imply some fixed point property?

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