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*Article*

# Ranked Soft Groups and Its Applications

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**Abstract**

In this article, we extend the concept of soft groups to ranked soft groups, which also serve as a generalization of fuzzy soft groups, addressing some of their limitations. We begin by introducing the definition of ranked soft groups, accompanied by illustrative examples and applications. Building on this foundation, we develop the notions of normalistic ranked soft groups and their homomorphisms. Furthermore, we explore various related properties and examine the structures preserved under normalistic ranked soft group homomorphisms.

**Keywords:** ranked soft set; ranked soft groups; ranked soft subgroups; normalistic ranked soft groups; ranked soft mapping

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## 1. Introduction

Classical mathematical methods are not always effective in solving complex problems in the fields of economics, engineering, environmental sciences, medicine, and social sciences because these problems often involve a number of uncertainties. As a result, there have been numerous alternative studies and applications of some specialized techniques, including probability theory, in the literature. Fuzzy and rough sets theories [1–4], vague and interval set s theories [5,6], while the interval set theory is also known asand interval mathematics . As Molodtsov noted, each of these theories has significant challenges even though they are all helpful methods for describing uncertainty [7]. To overcome most of the problems in the above theories Molodtsov [7] suggested a brand-new method for simulating ambiguity and uncertainty known as soft set theory. The smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory, and measurement theory are just a few of the domains in which soft set theory may find use. The majority of these uses were previously illustrated in Molodtsov's essay [7].

Soft set theory research is currently advancing quickly. Maji et al. [8] examined how to apply soft set theory to a problem involving decision-making. Garcia [9] defined ranked soft sets and developed their core theory. Soft sets are powerfully improved by this novel, non-numerical paradigm of uncertain information. The fundamental parameterized description provided by soft sets is qualitatively improved by the model. They also defined relationships between ranked soft sets and a few current models that use extra quantities to improve the soft set spirit (N-soft set's or fuzzy soft sets, and probabilistic soft sets). At the theoretical level, set-theoretic operations and representation theorems were major contributors to their growth; at the practical level, scoring and aggregation operators were key. Finally, they created a multi-person decision-making approach that makes use of these components for information in the form of ranked soft sets. In their study of the fundamental ideas of soft set theory, Aktas and Cagman [10,11] contrasted soft sets with fuzzy and rough sets, giving examples to illustrate the distinctions between the two. Additionally, they defined and investigated normal soft subgroups, soft homomorphisms, soft groups, and soft subgroups. Feng et al [12] presented and looked into soft semirings, soft sub-semirings, soft ideals, idealistic soft semi rings, and soft semi ring homomorphisms. Feng et al. [12] also presented the most basic concepts of soft semirings and defined soft sub-semirings, soft ideals, idealistic soft semirings, and soft semiring homomorphisms, along with the properties that are associated with them. Yang et al [13] continued the study of soft set toward fuzzy soft set in 2009.

To deepen the analytical understanding of the structure, we have extended the concept of soft groups to ranked soft groups (RSGs) by integrating a ranking function. This advancement from soft groups to ranked soft groups enables a more precise classification of group elements according to their significance or priority. While soft groups offer a flexible structural framework, ranked soft groups enhance this flexibility by introducing a hierarchical organization among the soft elements.

## 2. Soft Set

This section introduces the fundamental concept of soft sets, which was first proposed by D. Molodtsov in 1999 as a general mathematical framework for dealing with uncertainty. Traditional mathematical tools such as fuzzy sets, rough sets, and probability theory often face limitations when applied to problems that involve vague, incomplete, or parameter-dependent information. Soft set theory provides a flexible alternative by allowing uncertainty to be modeled through a parameterized family of subsets over a given universe.

A soft set is essentially a mapping from a set of parameters to the power set of a universe. Each parameter is associated with a subset of elements, capturing the idea that the truth or membership of elements can vary with respect to different criteria or viewpoints. This parameterized structure allows soft sets to be applied effectively in various domains, such as decision-making, data analysis, pattern recognition, and optimization.

In this section, we formally define soft sets, present basic operations, and discuss some illustrative examples. These foundational concepts pave the way for advanced structures such as soft groups, fuzzy soft sets, and the ranked soft group framework discussed in later sections.

**Definition 2.1.** *A pair $(\mathbb{F}, C)$ is called a soft set over $M$ (here $M$ is called universal set), where $\mathbb{F}$ is a mapping given by $\mathbb{F} : C \to P(M)$. Where $P(M)$ is a power set of $M$.*

In other words, a soft set over $M$ is a parameterized family of subsets of the universal set $M$. For $\varepsilon \in C, \mathbb{F}(\varepsilon)$ may be considered as the set of $\varepsilon$-elements of the soft set $(\mathbb{F}, C)$, or as the set of $\varepsilon$ approximate elements of the soft set.

**Definition 2.2.** *The intersection of two soft sets $(\mathbb{F}, C)$, $(\mathbb{L}, D)$ over a common universe set $M$ is the soft set $(\mathbb{K}, E)$, where $E = C \cap D$, and for all $a \in E$, $\mathbb{K}(a) = \mathbb{F}(a)$ or $\mathbb{L}(a)$, (as both are the same set). We write $(\mathbb{F}, C)\tilde{\cap}(\mathbb{L}, D) = (\mathbb{K}, E)$.*

**Definition 2.3.** *Let $(\mathbb{F}, C)$ and $(\mathbb{L}, D)$ be two soft sets over a common universe $M$. The extended intersection of $(\mathbb{F}, C)$ and $(\mathbb{L}, D)$ is defined to be the soft set $(\mathbb{K}, E)$, where $E = C \cup D$ and for all $a \in E$, $\mathbb{K}(a) =$*
$$\left\{ \begin{array}{ll} \mathbb{F}(a) & \text{if } a \in C\backslash D \\ \mathbb{L}(a) & \text{if } a \in D\backslash C \\ \mathbb{F}(a) \cap \mathbb{L}(a) & \text{if } a \in C \cap D \end{array} \right\}$$

This relation is denoted by $(\mathbb{F}, C) \sqcap_{\varepsilon} (\mathbb{L}, D) = (\mathbb{K}, E)$.

**Definition 2.4.** *Let $(\mathbb{F}, C)$ and $(\mathbb{L}, D)$ be two soft sets over a common universe $M$ such that $C \cap D \neq \emptyset$. The restricted intersection of $(\mathbb{F}, C)$ and $(\mathbb{L}, D)$ is denoted by $(\mathbb{F}, C) \cap (\mathbb{L}, D)$, and is defined as $(\mathbb{F}, C) \cap (\mathbb{L}, D) = (\mathbb{K}, E)$, where $E = C \cap D$ and for all $q \in E, \mathbb{K}(q) = \mathbb{F}(q) \cap \mathbb{L}(q)$.*

**Definition 2.5.** *If $(\mathbb{F}, C)$ and $(\mathbb{L}, D)$ are two soft sets over a common universe $M$, then $(\mathbb{F}, C)$ and $(\mathbb{L}, D)$ denoted by $(\mathbb{F}, C) \wedge (\mathbb{L}, D)$ is defined by $(\mathbb{F}, C) \wedge (\mathbb{L}, D) = (\mathbb{K}, C \times D)$, where $\mathbb{K}(x), y) = \mathbb{G}(x) \cap \mathbb{L}(y)$ for all $(x, y) \in C \times D$.*

**Definition 2.6.** *If $(\mathbb{F}, C)$ and $(\mathbb{L}, P)$ are two soft sets over a common universe $M$, then $(\mathbb{F}, C)$ or $(\mathbb{L}, D)$ denoted by $(\mathbb{F}, C) \vee (\mathbb{L}, D)$ is defined by $(\mathbb{F}, C) \vee (\mathbb{L}, D) = (\mathbb{K}, C \times D)$, where $\mathbb{K}(x), y) = \mathbb{F}(x) \cup \mathbb{L}(y)$ for all $(x, y) \in C \times D$.*

**Definition 2.7.** *Let $(\mathbb{F}, C)$ and $(\mathbb{L}, D)$ be two soft sets over a common universe $M$. The union of $(\mathbb{F}, C)$ and $(\mathbb{L}, D)$ is defined to be the soft set $(\mathbb{K}, E)$ satisfying the following conditions:*
*(i) $E = C \cup D$*
*(ii) for all $a \in E$,* $\mathbb{K}(a) = \left\{ \begin{array}{ll} \mathbb{F}(a) & \text{if } a \in C \backslash D \\ \mathbb{L}(a) & \text{if } a \in D \backslash C \\ \mathbb{F}(a) \cup \mathbb{L}(a) & \text{if } a \in C \cap D \end{array} \right\}.$

For more definitions and results about soft set see [7]. While for ranked soft set see [9].

## 3. Ranked Soft Groups

This section is devoted to the study of ranked soft groups, including their definitions, illustrative examples, and potential applications. The concept of soft groups was first introduced by H. Aktaş and N. Ca~gman in 2007 [10], while the notion of ranked soft sets was proposed more recently by G. S. Garcia in 2023 [9]. The present work synthesizes these two foundational concepts to introduce a more expressive structure: the ranked soft group.

In practical scenarios, not all parameters (or symmetries) hold the same level of importance. To address this variability, a ranking function is introduced to assign a weight or importance score to each parameter. For example, in a hiring process, "technical skills" may be considered more critical than "hobby preferences," so the rank assigned to technical skills would be higher. This ranking mechanism reflects real-world prioritization more effectively than traditional approaches.

Ranked soft groups provide a meaningful extension of fuzzy soft groups, especially in contexts where assigning precise membership degrees in the interval [0,1] is either infeasible or lacks interpretive value. In such cases, the ranked soft group model offers a more practical and semantically appropriate alternative.

Overall, this work blends the ideas from [9,10], leading to new theoretical results. Remarkably, the proposed framework also generalizes fuzzy soft groups while addressing their limitations, as demonstrated through examples.

**Definition 3.1.** *Let $\mathbb{G}$ be a group with binary operation " $\cdot$ ", $E$ be a set of parameters, $F : E \to \mathcal{P}(\mathbb{G})$ be a soft set over $\mathbb{G}$, i.e., for each parameter $e \in E$, $F(e)$ is subgroup of $\mathbb{G}$, $r : E \to R$ be a ranking function assigning a real-valued importance to each parameter. The triplet $(F, E, r)$ is called ranked soft group (RSG) over the group $\mathbb{G}$.*

**Example 3.1.** *Let $\mathbb{G} = (\mathbb{Z}_6, +)$, the group of integers modulo 6. Let $E = \{e_1, e_2, e_3, e_4\}$ be a set of parameters which measure the order of the subgroup with*

- *$F(e_1) = \{0\}$, a subgroup of $\mathbb{Z}_6$,*
- *$F(e_2) = \{0, 3\}$, also a subgroup of $\mathbb{Z}_6$,*
- *$F(e_3) = \{0, 2, 4\}$, also a subgroup of $\mathbb{Z}_6$,*
- *$F(e_4) = \mathbb{Z}_6$, also a subgroup of $\mathbb{Z}_6$,*
- *Ranking function: $r(e_1) = 1$, $r(e_2) = 2$, , $r(e_3) = 3$, $r(e_4) = 6$.*

*Then the triplet $(F, E, r)$ is a RSG over $\mathbb{Z}_6$.*

Note: The above example illustrates a ranked soft group (RSG) that is not a fuzzy soft group. This distinction highlights that ranked soft groups form a broader class of structures, capable of modeling scenarios where fuzzy membership degrees are either unavailable or inappropriate.

In the next section, we explore the substructures of ranked soft groups, such as sub-ranked soft groups and normalistic ranked soft subgroups. These concepts allow us to analyze the internal composition of ranked soft groups and understand how structural properties are preserved within their subsets.

**Definition 3.2.** *A triple $(H, B, r')$ is called a subRSG of RSG $(F, E, r)$ over group $\mathbb{G}$ if:*

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1. $B \subseteq E$
2. $H(a)$ *is a subgroup of* $F(a)$ *for all* $a \in B$,
3. $r'(a) \leq r(a)$ *for all* $a \in B$.

**Definition 3.3.** *Let* $(H_1, B_1, r'_1)$ *and* $(H_2, B_2, r'_2)$ *be two sub-ranked soft groups (subRSGs) of a ranked soft group* $(F, E, r)$ *over the group* $\mathbb{G}$. *Their intersection is defined as the triple* $(H, B, r')$, *where:*

- $B = B_1 \cap B_2$,
- $H(a) = H_1(a) \cap H_2(a)$ *for all* $a \in B$,
- $r'(a) = \min\{r'_1(a), r'_2(a)\}$ *for all* $a \in B$.

**Definition 3.4.** *Let* $(H_1, B_1, r'_1)$ *and* $(H_2, B_2, r'_2)$ *be two sub-ranked soft groups (subRSGs) of a ranked soft group* $(F, E, r)$ *over the group* $\mathbb{G}$. *Their union is defined as the triple* $(H, B, r')$, *where:*

- $B = B_1 \cup B_2$,
- $H(a) = H_1(a) \cup H_2(a)$ *for all* $a \in B$,
- $r'(a) = \max\{r'_1(a), r'_2(a)\}$ *for all* $a \in B$.

**Definition 3.5.** *Let* $(F, E, r)$ *be an RSG over a group* $\mathbb{G}$, *the support of the RSG is defined as:* $\mathrm{supp}(F, E, r) = \{F(e) \neq \emptyset,\ r(e) > 0,\ for\ all\ e \in E\}$.

Remarks: If $r(e, x) = 0$, it means the element $x$ has no relevance or weight in the group context under parameter e. Where $supp(F, E, r)$ gives the active or effective part of the RSG the pairs where the ranked soft structure matters. The null RSG is an RSG with an empty support, and an RSG $(F, E, r)$ is non-null if $\mathrm{supp}(F, E, r) \neq 0$.

**Theorem 3.1.** *Let* $(H_1, B_1, r'_1)$ *and* $(H_2, B_2, r'_2)$ *be two sub-ranked soft groups (subRSGs) of a ranked soft group* $(F, E, r)$ *over the group* $\mathbb{G}$. *Then their intersection is also subRSG over the group* $\mathbb{G}$, *if it is non-null.*

Proof: To prove that $(H, B, r')$ is a subRSG of $(F, E, r)$, we verify the three conditions of a subRSG:

1. Since $B_1 \subseteq E$ and $B_2 \subseteq E$, it follows that $B = B_1 \cap B_2 \subseteq E$.
2. For all $a \in B$, we have $a \in B_1$ and $a \in B_2$. Since $(H_1, B_1, r'_1)$ and $(H_2, B_2, r'_2)$ are subRSGs, we have: $H_1(a) \leq F(a)$ and $H_2(a) \leq F(a)$. Thus, $H(a) = H_1(a) \cap H_2(a) \subseteq F(a)$. Moreover, the intersection of two subgroups is also a subgroup, so $H(a)$ is a subgroup of $F(a)$.
3. For all $a \in B$, we have $r'_1(a) \leq r(a)$ and $r'_2(a) \leq r(a)$, so $r'(a) = \min\{r'_1(a), r'_2(a)\} \leq r(a)$.

Therefore, the triple $(H, B, r')$ satisfies all the conditions of a subRSG of $(F, E, r)$.

**Theorem 3.2.** *Let* $(H_i, B_i, r'_i)$ $i \in I$ *is index of sub-ranked soft groups (subRSGs) of a ranked soft group* $(F, E, r)$ *over the group* $\mathbb{G}$. *Then their intersection is also subRSG over the group* $\mathbb{G}$, *if it is non-null.*

Proof: The proof is an immediate consequence of the previous theorem and is therefore omitted.

Note: It can be observed that the intersection of two sub-ranked soft groups (sub-RSGs) is also a sub-RSG. However, the same is not generally true for the union of two sub-RSGs. This is primarily due to the second condition in the definition of a sub-RSG, which requires that for each parameter, the soft group value must be a subgroup of the corresponding value in the original RSG. In the case of union, this condition may fail to hold.

## 4. Ranked Soft Groups Homomorphism and Normalistic Ranked Soft Groups

In this section, we extend the classical notion of group homomorphisms to the ranked soft setting by introducing the concept of ranked soft group homomorphisms. These mappings preserve both the algebraic structure of groups and the layered soft information associated with ranked parameters. Such homomorphisms allow us to study how ranked soft groups relate to one another under structurepreserving transformations, enabling a deeper analysis of their morphic behavior and categorical properties.

Additionally, we have investigated a special class of ranked soft groups, namely normalistic ranked soft groups, where each soft component is a normal subgroup of the base group. These structures inherit symmetry properties from their classical counterparts and are central to understanding kernel-like structures and quotient-like constructions within the ranked soft context.

Together, the study of homomorphisms and normalistic ranked soft groups forms a foundation for exploring advanced algebraic operations and relationships in ranked soft group theory. This section introduces formal definitions, key properties, and fundamental results related to these concepts.

**Definition 4.1.** *Let $(F_1, E_1, r_1)$ be a ranked soft group over a group $\mathbb{G}_1$, and $(F_2, E_2, r_2)$ be a ranked soft group over a group $\mathbb{G}_2$. A triple $(\phi, \psi, \rho)$ is called a ranked soft group homomorphism from $(F_1, E_1, r_1)$ to $(F_2, E_2, r_2)$ if:*

1. $\phi : \mathbb{G}_1 \to \mathbb{G}_2$ *is a group homomorphism,*
2. $\psi : E_1 \to E_2$ *is a mapping between parameter sets,*
3. $\rho : \mathbb{R} \to \mathbb{R}$ *is a function satisfying* $\rho(r_1(e)) \le r_2(\psi(e))$ *for all* $e \in E_1$,
4. *For all* $e \in E_1$, $\phi(F_1(e)) \subseteq F_2(\psi(e))$.

Let $(F, E, r)$ be a ranked soft group over a group $\mathbb{G}$. A sub-ranked soft group $(H, B, r')$ of $(F, E, r)$ is called a ranked soft normal subgroup (RSNS) of $(F, E, r)$ if for every $a \in B$, $H(a)$ is a normal subgroup of $F(a)$, i.e., $gH(a)g^{-1} \subseteq H(a) \quad$ for all $g \in F(a)$. Equivalently, $gH(a)g^{-1} = H(a)$ for all $g \in F(a)$ and $a \in B$.

**Definition 4.2.** *Let $(F, E, r)$ be a ranked soft group over a group $\mathbb{G}$. Then $(F, E, r)$ is called a normalistic ranked soft group (NRSG) over $\mathbb{G}$ if $F(e) \trianglelefteq \mathbb{G} \quad$ for all $e \in Supp(F, E, r)$, where $\trianglelefteq$ stands for normal subgroup.*

**Theorem 4.1.** *Let $\mathbb{G}$ be a group, and let $(F, E, r)$ be a NRSG over $\mathbb{G}$. Let $B \subseteq E$ and define $r_B = r|_B$ (the restriction of $r$ to $B$). If $(F, B, r_B)$ is non-null, i.e., there exists $b \in B$ such that $F(b) \neq \emptyset$, then $(F, B, r_B)$ is also a normalistic ranked soft group over $\mathbb{G}$.*

proof: Since $(F, E, r)$ is a normalistic ranked soft group, we have: $F(e) \trianglelefteq \mathbb{G}, \quad$ for all $e \in \mathrm{Supp}(F, E) = \{e \in E \mid F(e) \neq \emptyset\}$.

Let $B \subseteq E$ and define $r_B = r|_B$. Suppose that $(F, B, r_B)$ is non-null, i.e., there exists $b \in B$ such that $F(b) \neq \emptyset$. Then:

- $b \in B \cap \mathrm{Supp}(F, E) \subseteq \mathrm{Supp}(F, B)$.
- Since $F(b) \trianglelefteq \mathbb{G}$ (as $b \in \mathrm{Supp}(F, E)$), and this holds for all $b \in \mathrm{Supp}(F, B) \subseteq \mathrm{Supp}(F, E)$, we conclude that: $F(b) \trianglelefteq \mathbb{G}, \quad$ for all $b \in \mathrm{Supp}(F, B)$.

Hence, $(F, B, r_B)$ satisfies the definition of a normalistic ranked soft group.

In other words the statement of the theorem can be illustrated as: $(F, E, r)$ be a normalistic ranked soft group NRSG over a group $\mathbb{G}$, and let $B \subseteq E$. Then the restriction $(F, B, r|_B)$ is also a normalistic ranked soft group over $\mathbb{G}$, provided that $(F, B, r|_B)$ is non-null, i.e., there exists some $b \in B$ such that $F(b) \neq \emptyset$.

**Definition 4.3.** *Let $(F_1, E_1, r_1)$ and $(F_2, E_2, r_2)$ be normalistic ranked soft groups over the groups $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively. Define their product as the triple $(F, E_1 \times E_2, r)$ over the group $\mathbb{G}_1 \times \mathbb{G}_2$, where:*

- $F(e_1, e_2) = F_1(e_1) \times F_2(e_2)$ *for all* $(e_1, e_2) \in E_1 \times E_2$,
- $r(e_1, e_2) = \min\{r_1(e_1), r_2(e_2)\}$.

*Then $(F, E_1 \times E_2, r)$ is called the product ranked soft group of $(F_1, E_1, r_1)$ and $(F_2, E_2, r_2)$.*

*If each $F_1(e_1) \trianglelefteq \mathbb{G}_1$ and $F_2(e_2) \trianglelefteq \mathbb{G}_2$, then $F(e_1, e_2) = F_1(e_1) \times F_2(e_2) \trianglelefteq \mathbb{G}_1 \times \mathbb{G}_2$, so the product we can see in the next theorem is also a normalistic ranked soft group.*

**Theorem 4.2.** *Let $(F, A, r_1)$ and $(H, B, r_2)$ be two normalistic ranked soft groups over the groups $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively.*

*Define their product as $(F \times H, A \times B, r)$ over $\mathbb{G}_1 \times \mathbb{G}_2$, where:*

- $(F \times H)(a, b) = F(a) \times H(b)$ *for all* $(a, b) \in A \times B$,
- $r(a, b) = \min\{r_1(a), r_2(b)\}$.

*If the product soft set is non-null, i.e., there exists $(a, b) \in A \times B$ such that $F(a) \neq \emptyset$ and $H(b) \neq \emptyset$, then $(F \times H, A \times B, r)$ is a normalistic ranked soft group over $\mathbb{G}_1 \times \mathbb{G}_2$.*

Proof: Let $(F, A, r_1)$ and $(H, B, r_2)$ be normalistic ranked soft groups over groups $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively.

This means that for all $a \in \text{Supp}(F, A)$ and $b \in \text{Supp}(H, B)$, we have: $F(a) \trianglelefteq \mathbb{G}_1$ and $H(b) \trianglelefteq \mathbb{G}_2$.

Define a new soft set $F \times H$ on the Cartesian product $A \times B$ by $(F \times H)(a, b) = F(a) \times H(b), \quad \forall (a, b) \in A \times B$.

Also define the ranking function $r : A \times B \to \mathbb{R}$ by $r(a, b) = \min\{r_1(a), r_2(b)\}$.

Now suppose the product soft set is non-null, i.e., there exists some $(a_0, b_0) \in A \times B$ such that $F(a_0) \neq \emptyset$ and $H(b_0) \neq \emptyset$. Then the support of the product soft set is: $\text{Supp}(F \times H, A \times B) = \{(a, b) \in A \times B \mid F(a) \neq \emptyset \text{ and } H(b) \neq \emptyset\}$.

We now show that $(F \times H)(a, b)$ is a normal subgroup of $\mathbb{G}_1 \times \mathbb{G}_2$ for each $(a, b) \in \text{Supp}(F \times H, A \times B)$.

Since $F(a) \trianglelefteq \mathbb{G}_1$ and $H(b) \trianglelefteq \mathbb{G}_2$ (by normalistic property of the original soft groups), it is a well-known group-theoretic fact that: $F(a) \times H(b) \trianglelefteq \mathbb{G}_1 \times \mathbb{G}_2$.

Hence, $(F \times H, A \times B, r)$ satisfies:

- $(F \times H)(a, b) \trianglelefteq \mathbb{G}_1 \times \mathbb{G}_2$ for all $(a, b)$ in its support,
- $r(a, b) = \min\{r_1(a), r_2(b)\}$ is a well-defined ranking function.

Therefore, $(F \times H, A \times B, r)$ is a normalistic ranked soft group over $\mathbb{G}_1 \times \mathbb{G}_2$.

**Theorem 4.3.** *Let $(F_i, A_i, r_i)$ be normalistic ranked soft groups over groups $\mathbb{G}_i$ for $i = 1, 2, 3$. Then the product is associative up to isomorphism: $((F_1, A_1, r_1) \times (F_2, A_2, r_2)) \times (F_3, A_3, r_3) \cong (F_1, A_1, r_1) \times ((F_2, A_2, r_2) \times (F_3, A_3, r_3))$ as normalistic ranked soft groups over $\mathbb{G}_1 \times \mathbb{G}_2 \times \mathbb{G}_3$.*

Proof: We define the product of normalistic ranked soft groups component-wise, both in terms of soft sets and ranking functions.

Let $(F_1, A_1, r_1)$, $(F_2, A_2, r_2)$, and $(F_3, A_3, r_3)$ be N-RSGs over $\mathbb{G}_1$, $\mathbb{G}_2$, and $\mathbb{G}_3$, respectively.

Define: $\mathbb{G} = \mathbb{G}_1 \times \mathbb{G}_2 \times \mathbb{G}_3 \quad$ and $\quad A = A_1 \times A_2 \times A_3$.

Let us construct both group structures:

- **LHS:** $((F_1 \times F_2) \times F_3, (A_1 \times A_2) \times A_3, r)$
- **RHS:** $(F_1 \times (F_2 \times F_3), A_1 \times (A_2 \times A_3), r')$

We define a bijection between the index sets: $\phi : ((A_1 \times A_2) \times A_3) \to (A_1 \times (A_2 \times A_3)), \quad \phi((a_1, a_2), a_3) = (a_1, (a_2, a_3))$. This map is a natural isomorphism of sets, and similarly, the group isomorphism: $\psi : ((\mathbb{G}_1 \times \mathbb{G}_2) \times \mathbb{G}_3) \to (\mathbb{G}_1 \times (\mathbb{G}_2 \times \mathbb{G}_3)), \quad \psi(((g_1, g_2), g_3)) = (g_1, (g_2, g_3))$, is a well-known group isomorphism (since direct product is associative up to isomorphism). Now consider the soft set: $((F_1 \times F_2) \times F_3)((a_1, a_2), a_3) = (F_1(a_1) \times F_2(a_2)) \times F_3(a_3)$, and on the RHS: $(F_1 \times (F_2 \times F_3))(a_1, (a_2, a_3)) = F_1(a_1) \times (F_2(a_2) \times F_3(a_3))$. But these are isomorphic subgroups of $\mathbb{G}$ under the group isomorphism $\psi$. Furthermore, the ranking functions are: $r((a_1, a_2), a_3) = \min\{r_1(a_1), r_2(a_2), r_3(a_3)\}$, $r'(a_1, (a_2, a_3)) = \min\{r_1(a_1), r_2(a_2), r_3(a_3)\}$. So $r = r'$ under $\phi$. Hence, the two ranked soft structures are isomorphic via the natural isomorphisms on groups and parameter spaces. Since each $F_i(a_i)$ is normal in $\mathbb{G}_i$, their products are normal in the group products, and thus both structures define normalistic RSGs over the same group.

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**Theorem 4.4.** *Let $(F, A, r)$ be a normalistic ranked soft group over a group $\mathbb{G}_1$, and let $\phi : \mathbb{G}_1 \to \mathbb{G}_2$ be a group homomorphism. Define a new soft set $\phi(F) : A \to \mathcal{P}(\mathbb{G}_2)$ by $\phi(F)(a) = \phi(F(a)) = \{\phi(g) \mid g \in F(a)\}, \quad$ for all $a \in A$. Then the triple $(\phi(F), A, r)$ is a normalistic ranked soft group over $\mathbb{G}_2$.*

Proof: Since $(F, A, r)$ is a normalistic ranked soft group over $\mathbb{G}_1$, by definition, for every $a \in \text{Supp}(F, A)$, we have: $F(a) \trianglelefteq \mathbb{G}_1$. That is, $F(a)$ is a normal subgroup of $\mathbb{G}_1$ for all such $a$. Let $\phi : \mathbb{G}_1 \to \mathbb{G}_2$ be a group homomorphism. Then it is a standard result in group theory that the image of a normal subgroup under a group homomorphism is a subgroup of the image, but not necessarily a normal subgroup in general. However, if $\phi$ is surjective, then $\phi(F(a))$ is a normal subgroup of $\mathbb{G}_2$. To ensure that $(\phi(F), A, r)$ defines a normalistic ranked soft group over $\mathbb{G}_2$, we assume that $\phi$ is a surjective group homomorphism. Under this assumption, for each $a \in \text{Supp}(F, A)$: $F(a) \trianglelefteq \mathbb{G}_1 \quad \Rightarrow \quad \phi(F(a)) \trianglelefteq \mathbb{G}_2$. Thus, for each $a \in \text{Supp}(\phi(F), A)$, $\phi(F)(a)$ is a normal subgroup of $\mathbb{G}_2$. The ranking function $r : A \to \mathbb{R}$ remains unchanged and is well-defined. Therefore, $(\phi(F), A, r)$ is a normalistic ranked soft group over $\mathbb{G}_2$.

**Theorem 4.5.** *Let $(F, A, r)$ and $(H, B, s)$ be ranked soft groups over the groups $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively, and let $\phi : \mathbb{G}_1 \to \mathbb{G}_2$ be a group homomorphism. Suppose that for each $a \in A$, $\phi(F(a)) \subseteq H(a)$ and $r(a) \leq s(a)$. Define the kernel soft set $\ker(\phi) : A \to \mathcal{P}(\mathbb{G}_1)$ by: $\ker(\phi)(a) = \{g \in F(a) \mid \phi(g) = e_{\mathbb{G}_2}\}$, where $e_{\mathbb{G}_2}$ is the identity in $\mathbb{G}_2$. Then the triple $(\ker(\phi), A, r)$ is a normalistic ranked soft group over $\mathbb{G}_1$.*

Proof: Let $(F, A, r)$ be a ranked soft group over $\mathbb{G}_1$. For each $a \in A$, $F(a)$ is a subgroup of $\mathbb{G}_1$. Let $\phi : \mathbb{G}_1 \to \mathbb{G}_2$ be a group homomorphism and define $\ker(\phi)(a) = \{g \in F(a) \mid \phi(g) = e_{\mathbb{G}_2}\}$. We claim that for each $a \in A$, $\ker(\phi)(a)$ is a normal subgroup of $\mathbb{G}_1$. Since $\phi$ is a group homomorphism, the kernel of $\phi$ is a normal subgroup of $\mathbb{G}_1$. Also, the intersection of a subgroup with a normal subgroup is a subgroup. Note that: $\ker(\phi)(a) = F(a) \cap \ker(\phi)$, where $\ker(\phi)$ denotes the kernel of the homomorphism $\phi$ as a subgroup of $\mathbb{G}_1$. Since:

- $F(a) \leq \mathbb{G}_1$ (because $(F, A, r)$ is a soft group),
- $\ker(\phi) \trianglelefteq \mathbb{G}_1$ (standard group theory),
- the intersection of a subgroup and a normal subgroup is a subgroup (not necessarily normal),

it follows that $\ker(\phi)(a)$ is a subgroup of $\mathbb{G}_1$. To show that $\ker(\phi)(a) \trianglelefteq \mathbb{G}_1$, we assume that $F(a)$ is not only a subgroup but is normal in $\mathbb{G}_1$ (i.e., $(F, A, r)$ is a normalistic ranked soft group). Then: $\ker(\phi)(a) = F(a) \cap \ker(\phi)$ is the intersection of two normal subgroups of $\mathbb{G}_1$, and thus is itself a normal subgroup: $\ker(\phi)(a) \trianglelefteq \mathbb{G}_1$. Hence, for each $a \in \text{Supp}(\ker(\phi), A)$, $\ker(\phi)(a)$ is a normal subgroup of $\mathbb{G}_1$. The ranking function $r : A \to \mathbb{R}$ remains unchanged from the original ranked soft group. Therefore, the triple $(\ker(\phi), A, r)$ is a normalistic ranked soft group over $\mathbb{G}_1$.

**Theorem 4.6.** *Let $\mathbb{G}$ be a group, and let $N \trianglelefteq \mathbb{G}$ be a fixed normal subgroup of $\mathbb{G}$. Let $A$ be a nonempty set of parameters, and let $F : A \to \mathcal{P}(\mathbb{G})$ be defined by $F(a) = N, \quad$ for all $a \in A$. Let $r : A \to \mathbb{R}$ be any ranking function. Then the triple $(F, A, r)$ is a normalistic ranked soft group over $\mathbb{G}$.*

Proof: By the definition of the soft set $F$, for each $a \in A$, we have: $F(a) = N$, where $N \trianglelefteq \mathbb{G}$ is a fixed normal subgroup of $\mathbb{G}$. Therefore, for every $a \in A$: $F(a) \trianglelefteq \mathbb{G}$. This directly satisfies the condition for a normalistic ranked soft group, which requires that for all $a \in \text{Supp}(F, A)$, the value $F(a)$ must be a normal subgroup of $\mathbb{G}$. Since $F(a) = N \neq \emptyset$ for all $a \in A$, we have $\text{Supp}(F, A) = A$. The ranking function $r : A \to \mathbb{R}$ is arbitrary but valid, and assigns a real-valued rank to each parameter $a \in A$. Hence, the triple $(F, A, r)$ satisfies all the conditions of a normalistic ranked soft group:

- For all $a \in A$, $F(a) \trianglelefteq \mathbb{G}$.
- $r$ is a real-valued function on $A$.

Therefore, $(F, A, r)$ is a normalistic ranked soft group over $\mathbb{G}$.

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## 5. Application: Decision-Making Using Normalistic Ranked Soft Groups

In many real-world scenarios, especially in multi-criteria decision-making problems, not all evaluation parameters carry equal significance. Normalistic Ranked Soft Groups (NRSGs) provide a mathematical framework to model such problems by incorporating both parameter-specific subsets and their relative importance.

### *5.1. Scenario: Candidate Selection in Hiring Process*

Suppose a company is hiring for a software engineering position. Let $\mathbb{G}$ denote the set of all applicants. Each applicant is to be evaluated based on a set of parameters $E = \{e_1, e_2, e_3, e_4, e_5\}$, where:

- $e_1$: Technical Skills
- $e_2$: Communication Skills
- $e_3$: Work Experience
- $e_4$: Hobbies
- $e_5$: Cultural Fit

We define a soft set $F : E \to \mathcal{P}(\mathbb{G})$ such that $F(e_i)$ is the subset of candidates who meet the criterion $e_i$. For example: $F(e_1)$ = set of candidates who passed the technical interview, $F(e_3)$ = set of candidates with at least 3 years experience. Assume that each $F(e_i)$ is a normal subgroup of $\mathbb{G}$ under some group operation (e.g., equivalence under role transitions or transformations). This makes $(F, E, r)$ a normalistic ranked soft group provided we also define a ranking function $r : E \to \mathbb{R}$ to quantify the importance of each parameter. An example ranking is given below:

| Parameter | Description | Rank $r(e_i)$ |
|---|---|---|
| $e_1$ | Technical Skills | 0.9 |
| $e_2$ | Communication Skills | 0.7 |
| $e_3$ | Work Experience | 0.8 |
| $e_4$ | Hobbies | 0.2 |
| $e_5$ | Cultural Fit | 0.6 |

### *5.2. Interpretation of NRSG Structure*

The triple $(F, E, r)$ forms a normalistic ranked soft group over $\mathbb{G}$ because:

- For each $e_i \in E$, $F(e_i) \trianglelefteq \mathbb{G}$.
- The ranking function $r$ captures the relative importance of each criterion.

Such a structure allows us to perform operations such as:

- Intersection of sub-RSGs: Identifying candidates who satisfy multiple high-priority criteria.
- Restriction: Focusing on a subset of parameters relevant for a specialized role (e.g., backend vs frontend).
- Kernel-based analysis: Extracting core capabilities by homomorphic mapping to role-specific requirement spaces.

## Comparison: Fuzzy Sets vs Soft Sets vs RSGs

**Table 1.** Comparison of Fuzzy Sets, Soft Sets, and Ranked Soft Groups.

| Feature | Fuzzy Sets | Soft Sets | RSGs |
|---|---|---|---|
| Handles Uncertainty | ✓ | ✓ | ✓ |
| Parameter-Based Subsets | × | ✓ | ✓ |
| Prioritizes Parameters | × | × | ✓ |
| Structure within Subsets | × | × | ✓ |

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*5.3. Conclusion*

The concept of normalistic ranked soft groups provides a rigorous tool to model and solve real-world problems involving parameterized decision-making with varying levels of importance. This example demonstrates the applicability of NRSGs in intelligent filtering, human resource systems, and expert-based evaluations.

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