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Aeshah Abdullah Muhammad Al-Dosari *

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# Contraction Results For Solvability And Stability to Hilfer Fuzzy Fractional Control System With Infinite Continuous Delay

**Aeshah Al-Dosari**

Department of Mathematics, Faculty of Science, Prince Sattam University, Saudi Arabia, Al-Kharj,16273; aa.aldosari@psau.edu.sa

**Abstract**

Consider the same problem given in [1] and we assume the existence results according to contraction theorem. Furthermore, we explore the necessary conditions to attain Ulam stability teste with respect to Mittage–Lefller functions.

**Keywords:** Hilfer-fuzzy multi-valued operator;Ulam-Mittage-Leffler stability test; controllability; infinite continuous delay; mild solution

**MSC:** 26A33; 34A08; 34A12

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## 1. Introduction

By the literature [1], A. Al-Dosari explored solvability conditions to the following control system according to Leray–Schauder Nonlinear Alternative Type theory together with different algorithms of FMQH-inequalities solutions.

$$ {}^{\delta}_{GH}D^{\beta,\theta}_{a^+}x(t) \in \mathcal{A}x(t) + \Pi(t, x(t), x_t, \mathbb{H}^u), \qquad t \in [a, T],\ a > 0\ \theta, \beta \in [0, 1], \tag{1.1} $$

$$ {}_{\delta}\mathcal{H}_a^{(1-\theta)(1-\beta)}x(a) = \frac{c\Gamma(\gamma)}{\Gamma(\omega+\gamma)}\left(\frac{t^{\delta}-a^{\delta}}{\delta}\right)^{\omega}, \tag{1.2} $$

$$ 0 < \omega < 1,\ \beta + \omega = 1,\ \gamma = \beta + \theta(1-\beta) $$

$$ x(t) = \psi(t), \qquad t \in [a-\sigma, a], \tag{1.3} $$

where $\psi(a) = 0$, $\delta = \tau + \alpha$, $\tau \in \mathbb{R}$, $\alpha \in [0, 1]$ and $\tau + \alpha \neq 0$,

$$ x_t(r) = x(t+r), \qquad r \in [-\sigma, 0],\ \sigma \in [a, T), $$

${}^{\delta}_{GH}D^{\beta,\theta}_{a^+}$, ${}_{\delta}\mathcal{H}^{\gamma}_a$ denote the generalized Hilfer-type fractional derivative and integral, respectively, that their definitions are given later, in Section . $\mathcal{A}$ denotes a generator of compact $C_0$ semi-groups and $\mathbb{H}^u$ defines solutions collection of the minty type FMQHI-controlled system written as follows

**FMQHI**: Find $w_1 \in K \cap S(w_1)_\beta$ such that

$$ h(v, w_2 - w_1, u) + \jmath^0(w_1, w_2 - w_1) + f(w_2, w_1) \geq 0, \qquad \forall v \in P(w_2)_\theta,\ \forall w_2 \in S(w_1)_\beta, \tag{1.4} $$

where $\jmath^0$ denotes the generalized directional derivative of Clarke type for the function $\jmath$ at the point $w_1 \in K$ in the direction of $w_2 - w_1$ given by the relation

$$ \jmath^0(u, v) = \lim_{k\to u}\ \sup_{\lambda\to 0^+} \frac{\jmath(k+\lambda v) - \jmath(k)}{\lambda},\ u = w_1,\ v = w_2 - w_1. $$

$u(t)$ is a control function and $S(w_1)_\beta$, $P(w_2)_\theta$ are defined, respectively, by

**a:** $S(w_1)_\beta = \Big\{g \in K | \mu_{S(w_1)}(g) \geq \beta\Big\}$,

**b:** $P(w_2)_\theta = \Big\{g \in K | \mu_{P(w_2)}(g) \geq \theta\Big\}$.

In this paper, we introduce a new extent to the literature [1] via contraction methods of solvability and stability to Hilfer fuzzy type fractional differential inclusion defined similarly by (1.1)-(1.4).

In fact, there are so many litterateurs (see [2–6])studying stability to heterogeneous dynamic models in different scientific fields. It is important to study the behavior of heterogeneous systems when they tend to be stable specially the ones affecting on our live. For examples, studying stability of heterogeneous DNA, populations, media, car traffic and so many and may others (see [8–12]).

We suggest this problem to open a way in applied sciences field for the sake of new effective results to make control in the behaviors of heterogeneous modelings.

## 2. Preliminaries

**Definition 2.1.** *[Generalized Conformable (GC) Integrable Function] For an order $\beta > 0$, the left-side GC–fractional integral ${}_\delta\mathcal{H}^\beta_{a_1^+}$ of with $0 < \alpha \leq 1$, $\tau \in \mathbb{R}$ and $\delta = \tau + \alpha \neq 0$ is defined by*

$${}_\delta\mathcal{H}^\beta_{a_1^+}(x)(t) = \frac{1}{\Gamma(\beta)} \int_{a_1}^t \left(\frac{t^\delta - \rho^\delta}{\delta}\right)^{\beta-1} \rho^{\delta-1} x(\rho) d\rho,$$

*for all conformable type–integrable functions $x$ on the interval $[a_1, a_2] \subset [0, \infty)$.*

**Definition 2.2.** *[Generalized Hilfer–type (GH) fractional derivative] Let $\beta \in (0,1)$, $\theta \in [0,1]$, $\tau \in \mathbb{R}$ and $0 < \alpha \leq 1$ such that $\delta = \tau + \alpha \neq 0$. For a conformable integrable function $x$ on the interval $[a_1, a_2] \subset [0, \infty]$, the left–sided GH–fractional derivative operator of order $\beta$ and type $\theta$ is defined by*

$${}^{\delta}_{GH}D^{\beta,\theta}_{a_1^+}(x)(t) = \left[{}_\delta\mathcal{H}^{\theta(1-\beta)}_{a_1^+}\left(t^{1-\delta}\frac{d}{dt}\right){}_\delta\mathcal{H}^{(1-\theta)(1-\beta)}_{a_1^+}\right](x)(t).$$

**Lemma 2.1.** *Let $\beta$, $\theta$, $\tau$, $\alpha$, $\delta$ and $x$ are all defined as in Definition* **??**. *Then, we have the following statements*

**(1)** *For all $\nu > 0$,*
$${}_\delta\mathcal{H}^\beta_{a_1^+}\left(\frac{t^\delta - a^\delta}{\delta}\right)^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu+\beta)}\left(\frac{t^\delta - a^\delta}{\delta}\right)^{\nu+\beta-1};$$

**(2)** *for $x \in C^1[a_1, a_2]$,*
$${}^{\delta}_{GH}D^{\beta,\theta}_{a_1^+}\,{}_\delta\mathcal{H}^\beta_{a_1^+}(x)(t) = x(t)$$

**(3)** *for $x \in C^1[a_1, a_2]$,*
$${}_\delta\mathcal{H}^\beta_{a_1^+}\,{}^{\delta}_{GH}D^{\beta,\theta}_{a_1^+}(x)(t) = x(t) - \left(\frac{t^\delta - a^\delta}{\delta}\right)^{\gamma-1} {}_\delta\mathcal{H}^{(1-\theta)(1-\beta)}_a x(a),$$

*wher $\gamma = \beta + \theta(1-\beta)$*

**Definition 2.3.** *Let $W$, $W_0$, and $W_1$ be given Banach spaces. Then,*

**(a) Compatible couple of Banach Spaces** *consists of two Banach spaces $W_0$, and $W_1$ continuously embedded in the same Housdroff topological vector space $V$. The spaces $W_0 \cap W_1$ and $W_0 + W_1$ are both Banach spaces equipped respectively with norms*

- $\|x\|_{W_0 \cap W_1} = \max\big(\|x\|_{W_0}, \|x\|_{W_1}\big)$
- $\|x\|_{W_0 + W_1} = \inf\{\|x_0\|_{W_0} + \|x_1\|_{W_1},\ x = x_0 + x_1,\ x_0 \in W_0,\ and\ x_1 \in W_1\}$

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**(b)Interpolation** *is the family of all intermediate spaces W between $W_0$, and $W_1$ in sense that*

$$W_0 \cap W_1 \subset W \subset W_0 + W_1,$$

*where the two included maps are continuous.*

**Remark 2.1.** *We can understand that*

- *the couple $\left(L^\infty, L^1\right)(\mathbb{R})$ is a compatible couple since $L^\infty$ and $L^1$ are both embedded in the space of measurable functions on the real line, equipped with topology convergence in measure.*
- *For all $1 < p < \infty$ the spaces $L^p(\mathbb{R})$ are intermediate spaces between $L^\infty(\mathbb{R})$ and $L^1(\mathbb{R})$. Hence,*

$$L^{1,\infty}(\mathbb{R}) = L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \subset L^p(\mathbb{R}) \subset L^\infty(\mathbb{R}) + L^1(\mathbb{R}).$$

Denote by $\mathcal{B}$ the space of all continuous function mapping $[-\sigma, 0]$ to $\mathbb{R}$. For $-\infty < a < T$, let $x : [a-\sigma, T] \to \mathbb{R}$ is defined in $(a-\sigma, T)$ and continuous on $[a, T]$. For all $r \in [-\sigma, 0]$, $t \in [a, T]$, define $x_t : C[-\sigma, 0] \to \mathbb{R}$ by $x_t(r) = x(t+r)$, $\forall$. Note that $x_t$ translates $x$ from $[t-\sigma, t]$ back to $[-\sigma, 0]$ and $x_a = x|_{[a-\sigma,a]}$.

Consider we have two Banach spaces $(W, \|.\|)$ and $(O, \|.\|)$. We say that $\phi : W \to P_{cl}(W)$ is convex (closed) multi–valued mapping if $\phi(w)$ is convex (closed) for all $w \in W$. If $\phi(B)$ is relatively compact for every $B \in P_b(W)$, then $\phi$ is completely continuous.

$\phi$ is said to be an upper semi-continuous if $E \subset W; \phi^{-1}(E)$ is a closed subset of $W$ for each closed subset (i.e., the set $\{w \in W : \phi(w) \subseteq H\}$ is open whenever $H \subset W$ is open). In return, it is a lower semi-continuous if $\forall Z \subset W; \phi^{-1}(Z)$ is an open subset of $W$. By another meaning, $\phi$ is a lower semi-continuous whenever the set $\{w \in W : \phi(w) \cap H \neq \emptyset\}$ is open for all open sets $H \subset W$.

We say that a multi–valued map $\phi : [0, \tau] \to P_{cl}(W)$ is a measurable if for every $w \in W$, the function $s \to d(w, A(s)) = \inf\{d(w, a) : a \in \phi(s)\}$ is $\mathcal{L}-$measurable function.

Given $U, V \in P_{cl}(W)$, then the Pompeiu–Housdorff distance of $U, V$ is defined by

$$h(U, V) = H_d(U, V) = d_H(U, V) = \max\left\{\sup_{u \in U} d(u, V), \sup_{v \in V} d(U, v)\right\}.$$

Moreover, the diameter distance of $V$ is given by

$$\hat{\delta}(V) = \sup_{v_1, v_2 \in V} d(v_1, v_2).$$

Note that there exists $M > 0$ such that $\hat{\delta}(V) \leq M$ if $V$ is bounded.

Suppose we adopt $\phi$ as a nonempty compact valued–completely continuous function. In that case, the sentence [ $\phi$ is upper semi-continuous] is equivalent to [ $\phi$ has a closed graph (i.e., if $\nu_n \to \nu_*$ *and* $y_n \to y_*$, then $y_n \in \phi(\nu_n)$ implies to $y_* \in \phi(\nu_*)$)].

**Definition 2.4.** *Consider a multi-valued map $\Theta : [a, b] \times \mathbb{R}^n \to P(\mathbb{R})$. Then, $\Theta$ is said to be a Caratheodory if*

*(1) $\tau \to \Theta(\tau, \{v_i\})$ is measurable, $\forall v_i \in \mathbb{R}, n \in \mathbb{N}$.*

*(2) $(\{v_i\}) \to \Theta(\tau, \{v_i\})$ a.e $\tau \in [a, b]$ is upper semi-continuous.*

*Adding to the assumptions (1) and (2), the map $\Theta$ is $L^1$- Caratheodory if for each $k > 0$, there exist $\phi_k \in L^\infty[a, b]$ satisfying $\sup_{\tau \geq 0}|\phi_k(\tau)| < +\infty$ and $\phi_k > 0$ and a nondecreasing map $Ł \in L^1[a, b]$ for which*

$$\|\Theta(\tau, \{v_i\})\| = \sup\{|\theta| : \theta(\tau) \in \Theta(\tau, \{v_i\})\} \leq \phi_k(\tau)Ł(\{|v_i\|\}),$$

*for all $\|v_i\| < k$, $i = 1, .., n$, $n \in \mathbb{N}$, $\tau \in [a, b]$.*

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**Definition 2.5** ((Lipschitz contiuity)). *Let $J \subset \mathbb{R}$ is compact interval and $\mathbb{X}$ and $\mathbb{Y}$ are both real Banach spaces. Define the multi-valued map $F : J \times \mathbb{X} \to P(\mathbb{Y})$. $F$ is said to be continuous if there exists a positive constant $K$ for which*

$$\|F(t, x_1(t)) - F(t, x_2(t))\|_{\mathbb{Y}} \leq K\|x_1 - x_2\|_{\mathbb{X}}.$$

Consider for all $\varepsilon > 0$, $t \in [a, T]$ that

$$\mathcal{H}_d\left({}^{\delta}_{GH}D^{\beta,\theta}_{a^+}x(t) - \mathcal{A}x(t),\ \Pi(t, x(t), x_t, \mathbb{H}^u)\right) \leq \varepsilon\mathbb{E}_{\beta,\beta}\left(t^{\delta} - a^{\delta}\right). \tag{2.1}$$

**Remark 2.2.** *An arbitrary function $x(t) \in L^p([a, T], \mathbb{R})$ is a solution of the inequality (2.1) if and only if there exists a function $\iota \in L^p([a, T], \mathbb{R})$ (which depends onx) such that*

$$(i)\ |\iota(t)| \leq \varepsilon\mathbb{E}_{\beta,\beta}\left(t^{\delta} - a^{\delta}\right) \text{for all } t \in [a, T];$$

$$(ii)\ {}^{\delta}_{GH}D^{\beta,\theta}_{a^+}x(t) \in \mathcal{A}x(t) + \Pi(t, x(t), x_t, \mathbb{H}^u) + \iota(t), \qquad t \in [a, T],$$

$$\mathbb{E}_{\gamma,\beta}(\eta) = \sum_{j=0}^{\infty} \frac{\eta^j}{\Gamma(\gamma j + \beta)}$$

**Definition 2.6.** *The problem (1.1)-(1.4) is said to be Ulam–Hyers stable if there exists a positive constant $A_{\Pi,\psi}$ such that for each mild solution $x(t) \in L^{1,\infty}(\mathbb{R})$ of the inequlity*

$$\mathcal{H}_d\left({}^{\delta}_{GH}D^{\beta,\theta}_{a^+}x(t) - \mathcal{A}x(t),\ \Pi(t, x(t), x_t, \mathbb{H}^u)\right) \leq \varepsilon, t \in J,$$

*there exists a solution $\nu(t) \in L^{1,\infty}(\mathbb{R})$ of the problem (1.1)-(1.4) such as*

$$\|x(t) - \nu(t)\| \leq \varepsilon A_{\Pi,\psi}$$

**Definition 2.7.** *The problem (1.1)-(1.4) is said to be Generalized Ulam–Hyers stable if there exists $A_{\Pi,\psi} : C[0, \infty) \to C[0, \infty)$ subject to $A_{\Pi,\psi}(0) = 0$ such that for each mild solution $x(t) \in L^{1,\infty}(\mathbb{R})$ of the inequlity*

$$\mathcal{H}_d\left({}^{\delta}_{GH}D^{\beta,\theta}_{a^+}x(t) - \mathcal{A}x(t),\ \Pi(t, x(t), x_t, \mathbb{H}^u)\right) \leq \varepsilon a_{\Pi,\psi}, t \in J,$$

*there exists a solution $\nu(t) \in L^{1,\infty}(\mathbb{R})$ of the problem (1.1)-(1.4) such as*

$$\|x(t) - \nu(t)\| \leq A_{\Pi,\psi}(\varepsilon).$$

**Definition 2.8.** *The problem (1.1)-(1.4) is said to be Ulam–Hyers-Rassias stable with respect to Mittag-Leffler $\mathbb{E}_{\beta,\beta}$ if there exists a positive constant $A_{\Pi,\psi,\beta}$ such that for each mild solution $x(t) \in L^{1,\infty}(\mathbb{R})$ of the inequlity (2.1) $t \in J$, there exists a solution $\nu(t) \in L^{1,\infty}(\mathbb{R})$ of the problem (1.1)-(1.4) such as*

$$\|x(t) - \nu(t)\| \leq \varepsilon A_{\Pi,\psi,\beta}(\mathbb{E}_{\beta,\beta}).$$

**Definition 2.9.** *The problem (1.1)-(1.4) is said to be Generalized Ulam–Hyers-Rassias stable with respect to Mittag-Leffler $\mathbb{E}_{\beta,\beta}$ if there exists a positive constant $A_{\Pi,\psi,\beta}$ such that for each mild solution $x(t) \in L^{1,\infty}(\mathbb{R})$ of the inequlity*

$$\mathcal{H}_d\left({}^{\delta}_{GH}D^{\beta,\theta}_{a^+}x(t) - \mathcal{A}x(t),\ \Pi(t, x(t), x_t, \mathbb{H}^u)\right) \leq \mathbb{E}_{\beta,\beta}\left(t^{\delta} - a^{\delta}\right),\ t \in J,$$

*there exists a solution $\nu(t) \in L^{1,\infty}(\mathbb{R})$ of the problem (1.1)-(1.4) such as*

$$\|x(t) - \nu(t)\| \leq \varepsilon A_{\Pi,\psi,\beta}(\mathbb{E}_{\beta,\beta}).$$

**Remark 2.3.** *Def(2.6)$\to$ Def(2.7),*

*Def(2.8)$\to$ Def(2.9),*

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*Def(2.8)→ Def(2.6) if* $\|\mathbb{E}_{\beta,\beta}\| \leq 1$.

**Lemma 2.2.** *The problem (1.1)-(1.4) is equivalent to the inclusion*

$$x(t) \in \aleph(x)(t), \tag{2.2}$$

$$\aleph : K \to P\Big(L^{1,\infty}[a-\sigma, T]\Big) \ defined\ by$$

$$\aleph(x)(t) = \Big\{e(t) \in L^{1,\infty}[a-\sigma, T] | e(t) = \underline{\Delta}^{\psi}_{\eta}(t),\ \eta(t) \in \overline{S^{u,(1,\infty)}_{\Pi,x}},\ \psi \in L^{1,\infty}[a-\sigma, a]\Big\}, \tag{2.3}$$

$$where\ \underline{\Delta}^{\psi}_{\eta}(t) = \begin{cases} \psi(t), & t \in [a-\sigma, a], \\ \Delta_{\eta}(t), & t \in [a, T] \end{cases} \tag{2.4}$$

$$Hence,\ \aleph_J(x)(t) = \Big\{e_J(t) \in L^{1,\infty}[a, T] | e_J(t) = \Delta_{\eta}(t),\ \eta(t) \in \overline{S^{u,(1,\infty)}_{\Pi,x}}\Big\}, \tag{2.5}$$

$$\Delta_{\eta}(t) = cQ^{\xi}_{\beta}\Big(t^{\delta} - a^{\delta}\Big) + \int_a^t \left(\frac{t^{\delta} - \rho^{\delta}}{\delta}\right)^{\beta-1} \hat{Q}_{\beta}\Big(t^{\delta} - \rho^{\delta}\Big)\eta(\rho)d\rho^{\delta}, \qquad \forall t \in [a, T], \tag{2.6}$$

$$Q^{\xi}_{\beta}\Big(t^{\delta} - a^{\delta}\Big) = \left(\frac{t^{\delta} - a^{\delta}}{\delta}\right)^{\xi-1} \mathbb{E}_{\beta,\xi}\left(\mathcal{A}(t)\left(\frac{t^{\delta} - a^{\delta}}{\delta}\right)^{\beta}\right), \tag{2.7}$$

$$\hat{Q}_{\beta}\Big(t^{\delta} - \rho^{\delta}\Big) = \mathbb{E}_{\beta,\beta}\left(\mathcal{A}(t)\left(\frac{t^{\delta} - \rho^{\delta}}{\delta}\right)^{\beta}\right). \tag{2.8}$$

**Lemma 2.3.** *Consider* $\mathcal{A}$ *is a generator of compact* $C_0-$*semi groups. Then,*

- *there exists* $M_{\beta} > 0$ *such that* $\big\|\mathbb{E}_{\beta,\beta}\big\| \leq M_{\beta}$,
- $\Big\|Q^{\xi}_{\beta}(t^{\delta} - a^{\delta})\Big\| \leq \frac{M_{\beta}\Gamma(\beta)}{\Gamma(\xi)} \sup_{t\in[a,T]}\left|\left(\frac{t^{\delta}-a^{\delta}}{\delta}\right)^{\xi-1}\right|$

**Proof.** see [1]:Proposition 2. □

**Lemma 2.4.** *The operators* $\aleph$ *and* $\aleph_J$ *are closed and bounded operators, where*

1. $\frac{R}{\psi^* + M_{\beta}\Gamma(\beta)G\big(\delta,\xi,\beta,R,\hat{\delta}(\mathbb{H}^u)\big)} \geq 1, \qquad \psi^* = \|\psi\|$ *or,*
2. $\frac{R}{\psi^* + M_{\beta}\Gamma(\beta)G(\delta,\xi,\beta,R,g^*)} \geq 1, \qquad g^* = \|g\|,$

*and*

$$G_0(R, w) = \|\phi_R\|[L_1(R) + L_2((A_1 + N^*)R)] + \|\hat{\phi}_R\|L_3(w);$$

$$G(\delta, \xi, \beta, R, w) = \frac{|c|}{\Gamma(\xi)} \sup_{t\in[a,T]}\left|\left(\frac{t^{\delta} - a^{\delta}}{\delta}\right)^{\xi-1}\right| + \frac{1}{\Gamma(\beta+1)}\left(\frac{T^{\delta} - a^{\delta}}{\delta}\right)^{\beta} G_0(R, w)$$

**Proof.** To prove 1, apply Lemma 2.3 and see [1]:proofs of Theorem 3[Step 2:$(I_1)$ and Step 3]
To prove 2, apply Lemma 2.3 and see [1]:proofs of Theorem 4. □

**Theorem 2.1** ( Covitz and Nadler)**.** *Let* $\aleph : A \to P_{cl}(A)$*. If* $\aleph$ *is a contraction, where* $(A, d)$ *is compelete metric space , then* $\aleph$ *has fixed points at least one.*

**Definition 2.10.** *Let* $(H, d)$ *be a metric space and let* $\psi : [0, \infty) \to [0, \infty)$ *be a monotone increasing function which is continuous at 0 and* $\psi(0) = 0$*. Then, we say that* $F :\ H \to P_{cl}(H)$ *is a* $\psi$*–multi–valued weakly Picard* *operator ( $\psi$–MWP operator) if it is a MWP operator and there exists a selection $f^{\infty} : Graph(F) \to Fix(F)$ of $F^{\infty}$ such that $d(h, f^{\infty}(h, w)) \le \psi(d(h, w))$; for all $(h, w) \in Graph(F)$, where*

$$F^{\infty}(h, w) := \{z \in Fix(F) : \text{ there exists a sequence of successive approximations } (h_n) \text{ of } F \text{ starting from } (h, w) \; (i.e., h_0 = h,\ h_1 = w,\ h_{n+1} \in F(h_n)) \text{ that converges to } z\}$$

.

*F is called a c-multivalued weakly Picard operator (c-MWP operator) if there exists $c > 0$ such that $\psi(\tau) = c\tau$, for each $\tau \in [0, \infty)$.*

**Theorem 2.2.** *Let $F : H \to P_{cl}(H)$ be a $\phi-$ contraction multi-valued map, where H is a compelete metric space. Then, F is said to be a MWP operator.*

**Theorem 2.3.** *Let $(H, d)$ be a metric space. Consider a $\psi$-MWP operator $F : H \to P_{cp}(H)$, then inclusion $h \in F(h)$ is generalized Ulam-Hyers stable. In case that F is c-MWP operator, then inclusion $h \in F(h)$ is Ulam-Hyers stable.*

To see more explainings see [1–7] and the references therein.

## 3. Existence and Stability Results

Consider that the statements $(a_1) - (a_4)$ in Definition 7 and $(J_1) - (J_4)$ in [1] are fulfilled. Moreover, consider the statements

$J_5$ There exists a constant $Ł_{\Pi} > 0$ such that

$$H_d(\Pi(t; u, u', z); \Pi(t; v, v', z)) \le Ł_{\Pi}(\|u - v\| + \|u' - v'\|);$$

$J_6$ $0 < Ł = \frac{2M_{\beta}\Gamma(\beta)Ł_{\Pi}}{\Gamma(\beta+1)}\left[\frac{T^{\delta}-a^{\delta}}{\delta}\right]^{\beta} \le 1$

**Theorem 3.1.** *Assume that all hypotheses $(a_1) - (a_4)$ and $(J_1), (J_2), (J_5)$ and $(J_6)$, the the following statements are valid*

*A The problem (1.1)-(1.4) is solvable.*

*B The problem (1.1)-(1.4) is Generalized Ulam–Hyers–Rassias stable with respect to Mittag-Leffler $\mathbb{E}_{\beta,\beta}$.*

**Proof. Proof of A:**

**Step1:** Using Lemma 2.4: part 1 we can see $\aleph : K \to P_{cl}(K)$.

**Step2:** In view of Covitz and Nadler Theorem 2.1, we still have to claim that $\aleph$ is contraction. For that, assume $x, x' \in K$ and $y \in \aleph(x)$, then there exists $\eta(t) \in \overline{S^{u,(1,\infty)}_{\Pi,x}}$ such that

$$y(t) = \Delta^{\psi}_{\eta}(t),$$

where $\Delta^{\psi}_{\eta}$ is defined by

From $J_5$ there exists $e \in \overline{S^{u,(1,\infty)}_{\Pi,x}}$, where

$$\|\eta(t) - e(t)\| \le Ł_{\Pi}(\|x - x'\| + \|x_t - x'_t\|).$$

Define the map $W(t) : [a, T] \to P(K)$ by

$$W(t) = \{e : \ \|\eta - e\| \le Ł_{\Pi}(\|x - x'\| + \|x_t - x'_t\|)\}.$$

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Then, we have $\eta' \in \overline{S^{u,(1,\infty)}_{\Pi,x}}$ that is measurable and satisfies

$$\|\eta(t) - \eta'(t)\| \leq Ł_{\Pi}\left(\|x - x'\| + \|x_t - x'_t\|\right).$$

Take $y'(t) = \underline{\Delta}^{\psi}_{\eta'}(t)$, then we get

$$\begin{aligned}
&\|y(t) - y'(t)\| = 0, \qquad t \in [a-\sigma, a];\\
&\|y(t) - y'(t)\| \leq \left| \int_a^t \left(\frac{t^\delta - \rho^\delta}{\delta}\right)^{\beta-1} \hat{Q}_\beta\left(t^\delta - \rho^\delta\right)[\eta(\rho) - \eta'(\rho)]d\rho^\delta \right|, \qquad t \in [a, T]\\
&\leq \frac{M_\beta \Gamma(\beta) Ł_\Pi}{\Gamma(\beta+1)} \left[\frac{T^\delta - a^\delta}{\delta}\right]^\beta \left(\|x - x'\| + \|x_t - x'_t\|\right)
\end{aligned}$$

Take

$$0 < Ł = \frac{2M_\beta \Gamma(\beta) Ł_\Pi}{\Gamma(\beta+1)} \left[\frac{T^\delta - a^\delta}{\delta}\right]^\beta \leq 1$$

and use the analoge relation by exchainging $x, x'$ we get that

$$H_d(\aleph(x), \aleph(x')) \leq Ł\|x - x'\|_K.$$

Step1,2 conclude that $\aleph$ has solutions at least one.

**Proof of B:** To prove the point B, using the fact A we see that there is a fixed point $\nu$ for the problem (1.1)-(1.4). Thus, there exists $\eta_\nu \in \overline{S^{u,(1,\infty)}_{\Pi,\nu}}$ such as

$$\nu(t) = \underline{\Delta}^{\psi}_{\eta_\nu}(t).$$

Let $y \in K$ be a solution of the inequality (2.1). By Remark 2.2 there exists $\eta(t) \in \overline{S^{u,(1,\infty)}_{\Pi,y}}$ where the following statements is held

$$y(t) = \underline{\Delta}^{\psi}_{\eta+\iota}(t),$$

which implies that

$$y(t) = \Delta_{\eta+\iota}(t), \qquad t \in [a, T].$$

Hence,

$$\begin{aligned}
&\forall\, t \in [a, T],\\
\|y(t) - \Delta_\eta(t)\| &= \left| \int_a^t \left(\frac{t^\delta - \rho^\delta}{\delta}\right)^{\beta-1} \hat{Q}_\beta\left(t^\delta - \rho^\delta\right)\iota(\rho)d\rho^\delta \right|\\
&\leq \frac{M_\beta \Gamma(\beta)}{\Gamma(\beta+1)} \left[\frac{T^\delta - a^\delta}{\delta}\right]^\beta \varepsilon \mathbb{E}_{\beta,\beta}\left(t^\delta - a^\delta\right)
\end{aligned}$$

This tends to

$$\begin{aligned}
\|y(t) - \nu(t)\| &= \|y(t) - \Delta_\eta(t) + \Delta_\eta(t) - \Delta_{\eta_\nu}(t)\|\\
&\leq \|y(t) - \Delta_\eta(t)\| + \|\Delta_\eta(t) - \Delta_{\eta_\nu}(t)\|\\
&\leq \frac{M_\beta \Gamma(\beta)}{\Gamma(\beta+1)} \left[\frac{T^\delta - a^\delta}{\delta}\right]^\beta \varepsilon \mathbb{E}_{\beta,\beta}\left(t^\delta - a^\delta\right) + Ł\|y - \nu\|
\end{aligned}$$

So, we have

$$\|y - \nu\| \leq \frac{Ł}{2Ł_{\Pi}(1-L)} \varepsilon \mathbb{E}_{\beta,\beta}\left(t^{\delta} - a^{\delta}\right)$$

Take

$$\frac{Ł}{2Ł_{\Pi}(1-L)} = A_{\Pi,\psi,\beta}.$$

We conclude that for all $t \in [a-\sigma, T]$,

$$\|y - \nu\| \leq A_{\Pi,\psi,\beta} \varepsilon \mathbb{E}_{\beta,\beta}\left(t^{\delta} - a^{\delta}\right)$$

which means that the problem (1.1)-(1.4) is Generalized Ulam–Hyers-Rassias stable with respect to Mittag-Leffler $\mathbb{E}_{\beta,\beta}$ □

**Theorem 3.2.** *Assume that all hypotheses $(a_1) - (a_4)$ and $(J_2) - (J_4), (J_5)$ and $(J_6)$, the the following statements are valid*

*A The problem (1.1)-(1.4) is solvable.*
*B The problem (1.1)-(1.4) is Generalized Ulam–Hyers–Rassias stable with respect to Mittag-Leffler $\mathbb{E}_{\beta,\beta}$.*

**Proof. Proof of A:**
**Step1:** Using Lemma 2.4: part 2 we can see $\aleph : K \to P_{cl}(K)$.
**Step2:** similarly to the proof of Theorem 3.1 but w.r.t Step 1.
**Proof of B:** Similarly to the proof of Theorem 3.1 but w.r.t Step 1. □

## 4. Conclusions

This paper explained contraction methods to prove solvability of fractional inclusions via the collection set of MQH-inequalities. Also, we put the needed conditions to see the stable stations according to Lemma2.4. All presented results were given in the vision of compact $c_0$-semi groups, infinite continuous delay,algorithm of fuzzy sets and phase space theory. we hope that we made a new extent on stability studying field to add more explainings for natural dynamic systems.

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